Pontifícia Universidade Católica
do Rio de Janeiro

Bruno Fânzeres dos Santos

Robust Strategic Bidding in Auction-Based Markets

Tese de Doutorado

Thesis presented to the Programa de Pós-Graduação em Engenharia Elétrica of PUC–Rio, in partial fulfillment of the requirements for the degree of Doutor em Engenharia Elétrica.

Advisor: Prof. Alexandre Street de Aguiar

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Abstract

Fânzeres dos Santos, Bruno; Street de Aguiar, Alexandre (Advisor). **Robust Strategic Bidding in Auction-Based Markets**. Rio de Janeiro, 2017. 68p. Tese de Doutorado — Departamento de Engenharia Elétrica, Pontifícia Universidade Católica do Rio de Janeiro.

We propose an alternative methodology to devise profit-maximizing strategic bids under uncertainty in markets endowed with a sealed-bid uniform-price auction with multiple divisible products. The optimal strategic bid of a price maker agent largely depends on the knowledge (information) of the rivals' bidding strategy. By recognizing that the bid of rival competitors may deviate from the equilibrium and are of difficult probabilistic characterization, we proposed a two-stage robust optimization model with equilibrium constraints to devise an risk-averse strategic bid in the auction. The proposed model is a trilevel optimization problem that can be recast as a particular instance of a bilevel program with equilibrium constraints. Reformulation procedures are proposed to construct a single-level-equivalent formulation suitable for column and constraint generation (CCG) algorithm. Differently from previously reported works on two-stage robust optimization, our solution methodology does not employ the CCG algorithm to iteratively identify violated scenarios for the uncertain factors, which in this thesis are obtained through continuous variables. In the proposed solution methodology, the CCG is applied to identify a small subset of optimality conditions for the third-level model capable of representing the auction equilibrium constraints at the optimum solution of the master (bidding) problem. A numerical case study based on short-term electricity markets is presented to illustrate the applicability of the proposed robust model. Results show that even for the case where an impression of 1% on the rivals' offer at the Nash equilibrium is observed, the robust solution provides a non-negligible risk reduction in out-of-sample analysis.

Keywords

Sealed-Bid Uniform-Price Auction; Strategic Bidding; Mathematical Programming with Equilibrium Constraints; Column-and-Contraint Generation; Day-Ahead Electricity Markets.

Resumo

Fânzeres dos Santos, Bruno; Street de Aguiar, Alexandre . **Estratégia de Ofertas Robusta em Mercados Baseados em Leilão**. Rio de Janeiro, 2017. 68p. Tese de Doutorado — Departamento de Engenharia Elétrica, Pontifícia Universidade Católica do Rio de Janeiro.

Nesta de tese de doutorado é proposta uma metodologia alternativa para obter estratégias ótimas de oferta sob incerteza que maximizam o lucro de um agente em mercados dotados de um leilão de preço uniforme e envelope fechado com multiplos produtos divisíveis. A estratégia ótima de um agente *price maker* depende amplamente da informação conhecida dos agentes rivais. Reconhecendo que a oferta dos agentes rivais pode desviar do equilíbrio de mercado e é de difícil caracterização probabilística, nós propomos um modelo de otimização robusta dois estágios com restrições de equilíbrio para obter estratégias de oferta ótimas avessas a risco. O modelo proposto é um modelo de otimização de três níveis passível de ser reescrito como uma instância particular de um programa binível com restrições de equilíbrio. Um conjunto de procedimentos é proposto a fim de construir uma formulação equivalente de de nível único adequado para aplicação de algoritmos de Geração de Coluna e Restrição (GCC). Diferentemente de trabalhos publicados anteriormente em modelos de otimização dois estágios, nossa metodologia de solução não aplica o método de GCC para iterativamente identificar os cenários mais violados dos fatores de incerteza, variáveis que são identificadas através de variáveis contínuas. Na metodologia de solução proposta, o algorítmo GCC é aplicado para identificar um pequeno subconjunto de condições de otimalidade para o modelo de terceiro nível capaz de representar as restrições de equilíbrio do leilão na solução ótima do problema master (problema de oferta). Um estudo de caso numérico baseado em mercados de energia de curto prazo é apresentado para ilustrar a aplicabilidade do modelo robusto proposto. Resultados indicam que mesmo em um caso em que é observada uma imprecissão de 1% na oferta de equilíbrio de Nash dos agentes rivais, a solução robusta provê uma redução significativa de risco em uma análise fora da amostra.

Palavras–chave

Leilões de Envelope Fechado e Preço Uniforme; Ofertas Estratégicas; Programação Matemática com Restrições de Equilíbrio; Geração de Coluna e Restrição; Mercados de Energia de Curto Prazo.

– What is jazz, Mr. Armstrong?
– My dear lady, as long as you have to ask that question, you will never know it.

Louis Armstrong, *Musician.*

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1
Introduction

A key challenge of the majority of economy sectors is to determine how to trade goods efficiently. Desirable transaction designs seek for fair trading prices with maximum overall welfare. With the aim of achieving such efficiency, auctions have been studied since antiquity and have been used in a variety of industrial applications [1, 2]. A substantial portion of the GDP of most countries comes from transactions using some form of auction. The products auctioned are spread over public and private sectors and vary over a wide range, from flowers and art objects to financial products and various categories of commodities such as electricity and gold. More recently, auctions have also become the main mechanism for online trading.

Auctions are an economic resource allocation mechanism that aim to meet supply and demand through a competitive bidding process. Auction processes are generally recognized as transparent schemes and, because of that, are one of the main mechanisms used to induce competitive markets. By publicly setting the auction rules, agents are not only able to derive their strategy based on common (available) information, but also to reproduce the auction process. In addition, by fostering competition, products are typically more fairly valued, thus reducing unilateral discrepancies and increasing overall welfare [3].

Since the seminal works [4, 5], the auction theory has been broadly studied in technical literature from various angles. An important area of the research consists of analyses of the behavior of agents within auction-based markets. The extensive literature in this area can be divided into two main groups, specifically, *game theory* and *decision theory*. Roughly speaking, game theory aims at identifying a set of bids for all competitors, which results in some form of equilibrium in the auction (e.g., Nash equilibrium [6]). It largely relies on a set of assumptions regarding the rationality of competing agents. In addition, computing equilibriums are usually NP-hard problems [7, 8, 9, 10], with several strong requirements such as the full knowledge regarding the operational, financial, and risk-averse structure of all agents and stable market conditions with full information availability of the system that the bidders are embedded in. Therefore, equilibrium strategies are, in general, used for market monitoring, as a tool for assessing market power opportunities rather than for strategic bidding in practical applications [11]. In this thesis, we concentrate on assessing optimal bidding strategies using a decision theory perspective. The decision theory approach focuses on devising optimal strategic bids with the ultimate goal of maximizing some measure of value for a particular set of agents, which is hereinafter called *strategic player*.

The extensive literature on optimal bidding under the decision theory framework dates back to Friedmand [5]. In his pioneering work, the author analyzes a competitive bidding environment for government contracts in which each competitor submits a single price bid for each contract. Under uncertainty regarding the “true” cost for fulfilling the contracts and the number of rival competitors, a profit-maximizing strategic bidding methodology is discussed. Following Friedmand’s ideas, [12, 13, 14] proposed extensions to the profit-maximizing model for addressing various aspects such as workforce, money, material constraints, and contract size. More recently, [15] presented a more “practical” approach for estimating cost uncertainty and evaluating the probability of winning for various levels of bids. In addition, [16] presented a cost-minimization model for optimizing resource utilization and devising optimal bidding strategies in highway projects. In electronic markets, [17] constructed a profit-maximizing strategic bid method for pay-per-click auctions, and [18] proposed a bidding procedure that combines integer and approximate dynamic programming to devise strategic bids in online auctions.

In this thesis, we focus on sealed-bid uniform-price auctions with multiple divisible products [3]. A direct application of the framework proposed is to devise bidding strategies in short-term electricity markets (usually called day-ahead market). As part of the restructuring process that took place in most power systems globally in the 1980s [19], competitive electricity markets were profoundly fostered and a daily auction, in the format of a sealed-bid uniform-price auction [20, 21] became the main cash flow stream for generating companies. Since then, a multitude of technical works have devoted special attention to the problem of optimal electricity trading in day-ahead markets [22, 23]. Closely related to the general idea explored in this thesis, we highlight [24], which makes use of binary expansion techniques to construct a mixed-integer linear programming (MILP) problem for assessing a profit-maximizing strategy in day-ahead markets under uncertainty regarding rival bids. Similarly, [25] proposed an extension in the solution technique by fixing the quantity bids as the power producer generation capacity.

A deeply relevant issue for practical implementation of any bidding method is how to represent the behavior of rival competitors. For instance, in [24, 25], a set of scenarios is assumed to be available and the standard risk-neutral stochastic programming approach is applied. However, a poor probability description of the states of nature (e.g., sampled scenarios) may result in a poor or even meaningless decision policy, consequently exposing the agent to unforeseen and undesirable risks [26]. This issue is of particular relevance in constructing the probability description that drives the bidding behavior of a group of agents owing to its complex nature. For instance, historical records may not accurately describe the future behavior of rival competitors (e.g., changes in economic *status quo* of rival companies certainly alter their aversion towards to risk). This situation is worsened when new players enter the business, as no information is usually available to provide a reliable evaluation of their strategies. In addition, even in a business in which auctions take place frequently and periodically, the auction rules/designs and business conditions may constantly change over time, thus altering the competitors' strategy. Therefore, we argue that estimating the underlying stochastic process that drives rival behavior is a difficult task, which, combined with a scarcity of risk-averse bidding models existing in technical literature, challenges the practical application of current state-of-the-art methods.

In this context, robust optimization [27] emerges as an alternative tool for addressing general decision-making problems involving uncertain parameters the true probability distribution of which is difficult to evaluate/estimate. In this thesis, we leverage on robust optimization techniques to characterize the uncertainty regarding rival competitors' behavior and estimation imprecision. On exploring the modeling of a general polyhedral uncertainty set, a two-stage robust optimization model with second-state variables representing the auction equilibrium conditions, such as clearing prices and accepted offers, is proposed for devising a revenue-maximizing strategic offer under uncertainty in sealed-bid uniform-price auctions. Structurally, the proposed mathematical model is a trilevel optimization problem. In the first level, a unique bidding strategy — in the form of price and quantity bids for each auctioned product — is defined. The second-level problem identifies, within a pre-specified polyhedral set, a vector of bids for the rival players that creates the worst adversity for the strategic player's net revenue. The auction process, modeled as a third-level optimization problem, is then evaluated to determine the optimal allocation of products among competitors and the corresponding clearing (uniform) prices.

It should be noted that the hierarchical structure of the robust optimization problem proposed in this thesis is slightly different from the "standard" trilevel optimization problem widely studied in technical literature [28, 29, 30, 31]. In general, two-stage robust optimization models follow a hierarchical "min-max-min" system of optimization problems. However, in this thesis, only first- and second-level problems are linked through the objective function. The third-level problem (solution of the auction-clearing problem) affects the two upper-level problems by means of its primal and dual optimal decision variables. Therefore, the mathematical formulation proposed in this thesis can be seen as a particular instance of a bilevel program with equilibrium constraints (BPEC) [32, 33]. The decision process, however, follows a two-stage robust optimization rationale. As a result, the proposed formulation can be classified as a two-stage robust mathematical programming with equilibrium constraints (TSR-MPEC) that differs structurally from previously reported classical two-stage robust models in the manner in which the third-level solution affects second- and first-level problems. As a consequence, the current state-of-the-art literature on theory and algorithms developed for two-stage robust optimization models cannot be directly applied to the model proposed in this thesis. A solution methodology based on a set of reformulation procedures and column-and-constraint generation (CCG) algorithms is thus proposed, allowing practitioners for the use of commercial MILP solvers to find near-global optimal solutions for the robust-bidding problem. In the proposed solution methodology, the CCG is applied to identify a small set of optimality conditions for the third-level problem capable to represent the auction equilibrium constraints at the optimum solution of the master (bidding) problem. In the next section, the objectives and contributions of this thesis are summarized.

1.1 Objectives and Contributions Regarding Existing Literature

The objective of this Ph.D. thesis is to present an alternative methodology to devise optimal strategic bids in markets endowed with a sealed-bid uniform-price auction of multiple divisible products. A key source of uncertainty inherent to this problem is the behavior of rival competitors in the auction. We recognize that a broad range of uncertainty factors of difficult probabilistic characterization largely affect the strategies of rival players, thus challenging the estimation of a precise distribution for rival behavior. In this context, we make use of robust optimization with polyhedral uncertainty set to tackle this modeling issue.

As an example of a practical application of the general framework proposed in this thesis is the optimal strategic bidding is short-term electricity markets (commonly referred to as day-ahead electricity market). Since the beginning of the deregulation process that took place in most power systems globally in mid 80s, a vast number of technical works on optimal bidding in day ahead electricity markets appeared in literature. The contents of research widely spread from a modeling point-of view to efficient algorithms to solve the problem.

One important stream of research concerns how the strategic player is modeled within the day-ahead electricity auction. Although it is recognized that all players has an impact in the auction solution, the magnitude of such impact varies depending on the "size" of the player. More specifically, small-scale power generators usually have a marginal or even an absent power to alter market solution towards their own good. Such players are usually called *price takers* in the literature. An important result under the assumption that the market is composed only by price taker agents (e.g., small-scale power generators) is due to Gross and Finlay [34]. They showed that an optimal bid strategy of a price taker agent is to bid the unit capacity (as quantity bid) at the marginal cost (price bid). In general, the key challenge faced by a price taker agent is how to characterize the market price uncertainty and obtain an optimal self-scheduling solution [35]. Among the multitude of research devoted to this particular modeling framework, we highlight [36, 37, 38] which applied robust optimization techniques to characterize the market price uncertainty. On the opposite direction of the price-taker agent hypothesis, we classify agents that have enough power (ability) to manipulate the market solution as *price makers*. Since these players have the possibility to alter the market result towards their own good, the key challenge faced by price-maker agents is how to model and obtain an optimal bidding strategy taking into account the impact of their own offer into the auction solution. Due to this intrinsic difficulty, technical literature has devoted much less attention to this problem. Two main lines of research address the price maker optimal bidding problem. The first one assumes available a price-responsive function (also referred to as price/quota curve) that links the market price to the amount offered by the strategic player. Such approach has been studied in [39, 40, 41, 42] and represents a simplification of the auction solution in order to be computationally feasible to assess an optimal bid of a price maker agent. The other main line of research directly embeds the auction solution into the optimal bidding problem, representing thus a more realistic modeling approach. We refer to [24, 25, 43] as some of the few works on this modeling structure. In this thesis, we consider a price maker strategic player in latter modeling approach, i.e., we directly embed the auction solution into the optimal bidding problem.

In this modeling context, an important source of uncertainty comes from the rival player's behavior in the auction. The standard modeling framework to represent this uncertainty in the optimal bidding problem is the scenario-based two-stage stochastic programming [44, 45], as applied in [24, 25, 43]. However, we recognize that the bid of rival competitors are of difficult probabilistic characterization owing to its complex nature. Therefore, instead of a scenario-based approach, we leverage on robust optimization techniques. As far as this author is aware, robust optimization has been applied to the optimal bidding problem only in the context of a price-taker agent to characterize the market price uncertainty (e.g., [36, 37, 38]). In a price maker context with the auction dynamics mathematically embedded into the bidding problem, we identify a gap in the literature and this thesis aims at addressing this gap.

From a mathematical point-of-view, the structure of the previously reported optimal bidding strategy models [24, 25, 43] that is close-related to this thesis falls into the class of mathematical programming with equilibrium constraints (MPEC) [32]. Many techniques has been studied to handle this type of optimization problems (see, for instance, [33, 46, 47, 48, 49] and references therein). However, because of the inclusion of the worst-case metric into the MPEC problem, a new class of optimization problems emerges in this thesis: the two-stage robust mathematical programming with equilibrium constraints (TSR-MPEC). As a consequence, the current state-of-the-art literature on theory and algorithms cannot be directly applied to address the model proposed in this thesis. Therefore, we construct a solution approach for this new class of problems based on reformulation procedures and column-and-constraint generation (CCG) algorithms.

In summary, the main objectives and contributions of this thesis are threefold:

1. To provide a novel risk-averse model, based on robust optimization, to devise the optimal bidding strategy under uncertainty on rivals' offer in sealed-bid uniform-price auctions with multiple divisible products. The model considers an infinite set of scenarios for the rivals' bid through a user-defined polyhedral uncertainty set. Within this modeling framework, the decision maker is allowed to control its conservativeness level by means of the uncertainty set topology without the need of specifying the full probability distribution for

the rivals' bidding strategy.

2. To provide a single-level-equivalent formulation suitable for decomposition techniques based on available commercial solvers. The complementarity conditions used to ensure auction equilibrium constraints are expressed through binary relations between dual and slack variables of the third-level problem. After a series of transformations on the trilevel model and enumeration of the binary complementarity relations, a single-level-equivalent formulation with an exponential number of constraints, hereinafter, named single-level-equivalent exponential formulation, is devised.

3. To develop an efficient solution methodology for the robust bidding model based on column-and-constraint generation (CCG) algorithm applied to the single-level-equivalent exponential formulation. The CCG explores the binary nature identified for the complementarity conditions to avoid the full enumeration of the exponential set of constraints. In this methodology, a reduced set of binary relations, sufficient to represent the complementarity conditions that ensure the auction equilibrium constraints at the optimal bidding strategy, is identified by an oracle and added to a relaxed version of the problem that can be solved through off-the-shelf commercial solvers.

It is worth stressing that, differently from previously reported works on two-stage robust optimization, our solution methodology does not employ the CCG algorithm to iteratively find violated scenarios for the uncertainty factors, which in this thesis are obtained through continuous variables. In the proposed solution methodology, the CCG is applied to identify a small subset of optimality conditions for the third-level model capable to represent the auction equilibrium constraints at the optimum solution of the master (bidding) problem.

1.2 Organization of this Ph.D. Thesis

This thesis is laid out as follows. Chapter 2 introduces the problem tackled in this thesis. We begin by introducing the general mathematical formulation of the auction problem. Then, an optimal bidding strategy model under perfect information (full knowledge regarding the uncertain values) is presented and its disadvantages for practical implementation is highlighted. To conclude the chapter, we extend the perfect information model to consider uncertainty on rival behavior and introduce the robust strategic bidding model proposed in this thesis as an instance of the TSR-MPEC framework.

By recognizing the computational difficulty and non-tractability of the proposed robust bidding model, in Chapter 3, a solution methodology is discussed. Taking advantage of the particular properties of the auction problem, the trilevel program is recast as a two-level system of non-linear optimization problems using Karush-Kuhn-Tucker (KKT) optimality conditions. The sources of non-linearity existent in the model are handled by combining a set of equations of the KKT system and with an exact relaxation of the complementarity conditions using binary variables. Then, taking advantages of the strong duality theorem, an exponentially-large single-level optimization problem is derived by enumerating all binary variables associated with the complementarity conditions of the KKT system. We conclude this chapter discussing a CCG-based solution methodology, in witch a sufficient subset of the binary variables are iteratively identified, avoiding thus the full enumeration of the exponential set of constraints.

In Chapter 4, a numerical study based on electricity trading in day-ahead markets is presented to highlight the applicability of the proposed robust model. We begin by exploring the particular characteristics of the day ahead trading problem and present a detailed outline of the proposed algorithm. Then, we assume a particular format for the rivals' bid uncertainty set in which a reference bid is considered available and deviations from this reference are allowed towards the worst-case strategic player's net revenue. Such deviations are controlled by an *a priori* defined conservativeness parameter, measuring the uncertainty observed by the strategic player. As the reference rival bids, we consider a widely-discussed bidding strategy, the Nash equilibrium. Within this modeling framework, two case studies are presented. In the first one, we recognize the difficulty on precisely assess the Nash equilibrium solution and assume the existence of an imprecision on the equilibrium estimation. Finally, to conclude the chapter, the second study assumes that the rival players may not bid the Nash equilibrium and act strategically.

Chapter 5 concludes this thesis and discusses extensions and future research.

2
Bidding Problem in Auction-Based Markets

Roughly speaking, auctions are constructed under three basic pillars [3]. The first pillar is its bidding rule. It defines the format of the bids (e.g., open or closed/ sealed bids [50]) and how agents submit them (for example, a single price bid per product [5] or a general curve linking price to quantity [51, 52]). The second pillar is the definition of a clearing mechanism. Fundamentally, it is specified how the bids are compared with each other in order to determine the auction winners and the resource allocation among competitors. In most auction designs, this comparison is performed by simply ordering the price bids and the most "cheap" players win the auction. However, in some complex businesses, such as short-term electricity markets, some constraints may be included in the auction mechanism (e.g., minimum profit requirement, power plant characteristics, or the system topology), thus distorting the price comparison, which may lead to more "expensive" players winning the auction [19]. Finally, the third pillar is related to the pricing scheme to be adopted. It establishes the rule to obtain the price that settles the auction (commonly used schemes are pay-as-bid/discriminatory or uniform price [3, 53, 54]).

In this thesis, the foundation of the business environment is a competitive market endowed with a sealed-bid uniform-price auction of multiple divisible products. In this environment, agents compete to buy/sell a set of products in a given market through an auction procedure that settles the bids by defining both the (uniform/marginal) price of each product and the corresponding amount due to each participant. Structurally, each auction participant submits a price and quantity bid for one of the products to a central agent (known as the *auctioneer*), and the auction rules are then applied to settle the bids. Typically, such procedures are aimed at realizing the optimal distribution of products among competitors, which results in the greatest social welfare, and are generally implemented through the solution of optimization problems or supply and demand matching rules [55].

Let $(\boldsymbol{p}_D, \boldsymbol{q}_D) \in \mathbb{R}_+^{N_D} \times \mathbb{R}_+^{N_D}$ be the price and quantity bids for each of the N_D buyer competitors for a given product, $(\boldsymbol{p}_S, \boldsymbol{q}_S) \in \mathbb{R}_+^{N_S} \times \mathbb{R}_+^{N_S}$ the N_S price and quantity bids from the set of strategic players for various products, and $(\boldsymbol{p}_R, \boldsymbol{q}_R) \in \mathbb{R}_+^{N_R} \times \mathbb{R}_+^{N_R}$ the price and quantity bids from the rest of the sellers, hereinafter called rivals, for related products. It is important to clarify that in this framework, each of the entries of the aforementioned offer vectors is associated with a specific bid of a given agent for a given product that is being auctioned. Because the focus of this thesis is to devise an optimal bidding strategy for the strategic player, such an agent is explicitly allowed to make different bids for various products through different entries of vectors $(\boldsymbol{p}_S, \boldsymbol{q}_S)$. Nevertheless, the proposed framework is also capable of considering multi-product bids for rivals and buyers by choosing appropriated dimensions for the decision vectors and coefficients for the constraints (2-1)–(2-6). Therefore, the proposed framework comprises a simultaneous multi-product auction.

The auction format considered in this thesis follows the social welfare maximization problem presented in (2-1)–(2-6).

$$\max_{\mathbf{x}_D,\mathbf{x}_S,\mathbf{x}_R,\mathbf{y}} \quad \boldsymbol{p}_D^\top \mathbf{x}_D - \boldsymbol{p}_S^\top \mathbf{x}_S - \boldsymbol{p}_R^\top \mathbf{x}_R + \boldsymbol{p}_\mathbf{y}^\top \mathbf{y} \tag{2-1}$$

subject to:

$$\boldsymbol{A}_S \mathbf{x}_S + \boldsymbol{A}_R \mathbf{x}_R - \boldsymbol{A}_D \mathbf{x}_D + \boldsymbol{A}_\mathbf{y} \mathbf{y} = \mathbf{0} \qquad : \boldsymbol{\lambda} \tag{2-2}$$

$$\underline{\boldsymbol{h}}_S \leq \boldsymbol{I}_S \mathbf{x}_S \leq \boldsymbol{H}_S \boldsymbol{q}_S + \overline{\boldsymbol{h}}_S \qquad : \left(\underline{\boldsymbol{\lambda}}_S, \overline{\boldsymbol{\lambda}}_S\right) \tag{2-3}$$

$$\underline{\boldsymbol{h}}_R \leq \boldsymbol{I}_R \mathbf{x}_R \leq \boldsymbol{H}_R \boldsymbol{q}_R + \overline{\boldsymbol{h}}_R \qquad : \left(\underline{\boldsymbol{\lambda}}_R, \overline{\boldsymbol{\lambda}}_R\right) \tag{2-4}$$

$$\underline{\boldsymbol{h}}_D \leq \boldsymbol{I}_D \mathbf{x}_D \leq \boldsymbol{H}_D \boldsymbol{q}_D + \overline{\boldsymbol{h}}_D \qquad : \left(\underline{\boldsymbol{\lambda}}_D, \overline{\boldsymbol{\lambda}}_D\right) \tag{2-5}$$

$$\underline{\boldsymbol{h}}_\mathbf{y} \leq \boldsymbol{I}_\mathbf{y} \mathbf{y} \leq \overline{\boldsymbol{h}}_\mathbf{y} \qquad : \left(\underline{\boldsymbol{\lambda}}_\mathbf{y}, \overline{\boldsymbol{\lambda}}_\mathbf{y}\right) \tag{2-6}$$

In (2-1)–(2-6), the decision vectors $\mathbf{x}_D$, $\mathbf{x}_S$, and $\mathbf{x}_R$ represent the number of products effectively purchased by each buyer (elements of vector $\mathbf{x}_D$), as a composite of the sellers' strategic and rival accepted bids (elements of vectors $\mathbf{x}_S$ and $\mathbf{x}_R$, respectively). For the sake of generality, we introduce an auxiliary variable $\mathbf{y} \in \mathbb{R}^{N_\mathbf{y}}$ to account for all the extra variables required to model the various types of auction formats, e.g., an item's deliverability through a constrained network can be accounted for through such auxiliary variables and related constraints. Moreover, for clarity and future reference, the Lagrange multipliers of each set of constraints are specified after colons.

The objective function (2-1) comprises social welfare maximization and is defined as the difference between the buyer $(\boldsymbol{p}_D^\top \mathbf{x}_D)$ and seller (strategic $(\boldsymbol{p}_S^\top \mathbf{x}_S)$ plus rival $(\boldsymbol{p}_R^\top \mathbf{x}_R)$) surpluses [55]. Equation (2-2) represents the products' balance constraint. Roughly speaking, constraint (2-2) establishes a relation between the sold and bought volume of each product in the auction. For instance, as usually considered in most auction designs, all products sold through the auction process by strategic and rival players must be bought by a subset of the buyers, i.e., the offer must meet the demand. Therefore, the matrices $\boldsymbol{A}_S$, $\boldsymbol{A}_R$, and $\boldsymbol{A}_D$ are of particular importance as they link each product (rows) to the corresponding quantity cleared (columns) by the strategic, rivals, and buyers, respectively. In this context, the Lagrange multipliers ($\boldsymbol{\lambda} \in \mathbb{R}^M$) of each balance constraint (2-2) represent the clearing uniform prices (or marginal prices) of each product in the auction[1]. The three blocks of constraints (2-3)–(2-5) define the bounds on the buyers'/sellers' auctioned amounts. When considering the strategic player constraint (2-3) as an example, it should be noted that the auctioned amounts effectively sold $\mathbf{x}_S$ are bounded by the submitted quantity bid vector $\boldsymbol{q}_S$. Lastly, the constraints (2-6) describe the feasible region of the auxiliary variable $\mathbf{y}$.

Figure 2.1 depicts the auction process considered in this thesis. Buyers and sellers submit a set of price and quantity sealed bids to a central auctioneer, which runs an algorithm to solve the linear programming problem (2-1)–(2-6). The amount effectively bought and sold by each participant as well as the uniform clearing price of each product auctioned is then defined and reported back to each participant.

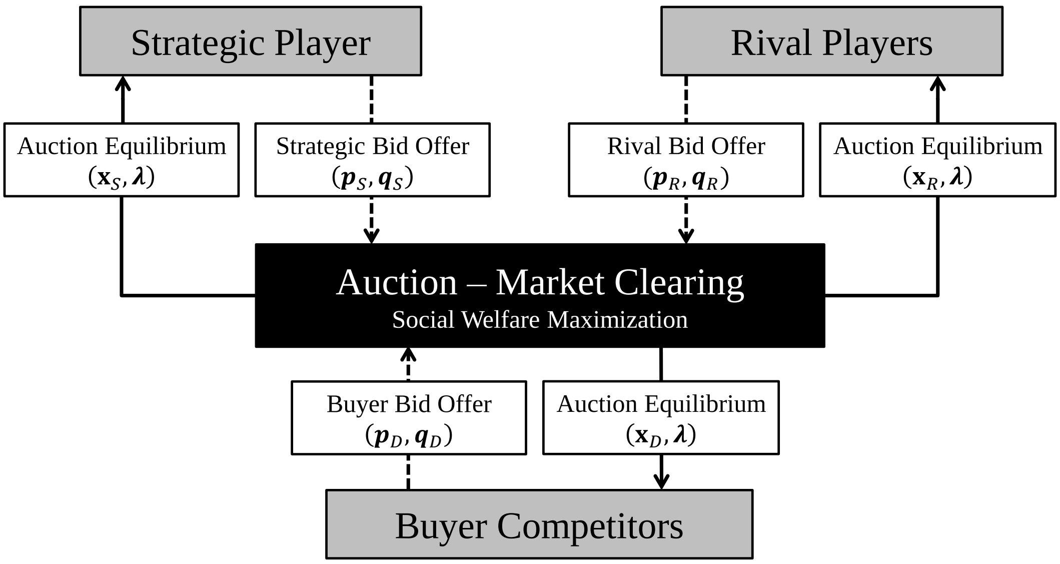

Fig. 2.1: Auction scheme considered in this thesis. Buyers and sellers submit a sealed price and quantity bid to a central agent (auctioneer) that defines the cleared amounts and a uniform clearing price for each product.

Example 1. *An example of an auction structure that fits the general framework presented in* (2-1)–(2-6) *is a single inflexible buyer* ($N_D = 1$) *willing to purchase a*

[1] We refer to [3, 57, 58] for a general theory on uniform-price auctions.

given amount d *of a single product* $(M = 1)$. *A set of* $\mathcal{N}_S = \{1, \ldots, N_S\}$ *strategic sellers and* $\mathcal{N}_R = \{1, \ldots, N_R\}$ *rivals compete to supply the buyer's demand. The auction process can thus be formulated as follows.*

$$\min_{\mathbf{x}_S, \mathbf{x}_R, x_D} \sum_{j \in \mathcal{N}_S} p_{S,j} x_{S,j} + \sum_{i \in \mathcal{N}_R} p_{R,i} x_{R,i} \tag{2-7}$$

subject to:

$$\sum_{j \in \mathcal{N}_S} x_{S,j} + \sum_{i \in \mathcal{N}_R} x_{R,i} - x_D = 0; \qquad : \lambda \tag{2-8}$$

$$0 \leq x_{S,j} \leq q_{S,j}, \qquad : (\underline{\lambda}_{S,j}, \overline{\lambda}_{S,j}) \qquad \forall\, j \in \mathcal{N}_S; \tag{2-9}$$

$$0 \leq x_{R,i} \leq q_{R,i}, \qquad : (\underline{\lambda}_{R,i}, \overline{\lambda}_{R,i}) \qquad \forall\, i \in \mathcal{N}_R; \tag{2-10}$$

$$x_D = d. \qquad : \lambda_D \tag{2-11}$$

Structurally, (2-7)–(2-11) *represent a standard supply and demand matching problem (equation* (2-8)*) where the cleared amounts* $\{x_{S,j}\}_{j \in \mathcal{N}_S}$ *and* $\{x_{R,i}\}_{i \in \mathcal{N}_R}$ *are limited to the respective quantity offers* $\{q_{S,j}\}_{j \in \mathcal{N}_S}$ *and* $\{q_{R,i}\}_{i \in \mathcal{N}_R}$*, respectively — equations* (2-9) *and* (2-10)*. Because the buyer declares inflexibility, which would be equivalent to a very high bidding price, the social welfare maximization is cast as a total cost minimization problem, where* $\{p_{S,j}\}_{j \in \mathcal{N}_S}$ *and* $\{p_{R,i}\}_{i \in \mathcal{N}_R}$ *respectively represent the price bids of the strategic player and its rivals.*

The auction structure presented in (2-7)–(2-11) *has been widely used in technical literature to formulate the short-term electricity economic dispatch problem in several power systems globally [59, 60]. We will explore this particular auction format later in this thesis.* □

2.1 Optimal Bidding with Perfect Information

The objective of this thesis is to propose an alternative bidding model for determining an optimal strategy under the uncertainty of a seller competing in a sealed-bid uniform-price auction with multiple divisible products. Firstly, for any company to construct an optimal bidding method, it is essential that its objectives be clearly defined. Several objective measures appear in technical literature, each of which naturally result in different bidding policies. In practice, private companies generally seek for profit-maximizing decisions, although other objectives such as increasing market share or total cost reduction are also commonly used. Accordingly, assuming a profit-maximizing strategic player, for a given set of price/quantity bids $\big(\boldsymbol{p}_S, \boldsymbol{q}_S, \boldsymbol{p}_R, \boldsymbol{q}_R\big)$ of both strategic and rival players, let $\mathcal{M}\big(\boldsymbol{p}_S, \boldsymbol{q}_S, \boldsymbol{p}_R, \boldsymbol{q}_R\big)$ be a non-empty[2] set of optimal points $\mathbf{x}_S$ and the respective Lagrange multipliers $\boldsymbol{\lambda}$ of the auction problem (2-1)–(2-6). More precisely,

$$\mathcal{M}\big(\boldsymbol{p}_S, \boldsymbol{q}_S, \boldsymbol{p}_R, \boldsymbol{q}_R\big) = \Big\{(\mathbf{x}_S, \boldsymbol{\lambda}) \in \mathbb{R}^{N_S} \times \mathbb{R}^M \;\Big|\; (\mathbf{x}_S, \boldsymbol{\lambda}) \text{ solves (2-1)–(2-6)} \Big\}. \tag{2-12}$$

Thus, assuming perfect information (*a priori* known information) regarding the rivals' and buyer's strategies, the optimal bidding problem of a profit-maximizing strategic player can be formulated as follows.

$$\varphi^{\text{Op}}(\boldsymbol{p}_R, \boldsymbol{q}_R) = \max_{\substack{(\boldsymbol{p}_S, \boldsymbol{q}_S) \in \mathcal{O}_S,\\ \mathbf{x}_S, \boldsymbol{\lambda}}} \; \boldsymbol{\lambda}^\top \boldsymbol{A}_S \mathbf{x}_S - \boldsymbol{c}_S^\top \mathbf{x}_S - f_S(\boldsymbol{p}_S, \boldsymbol{q}_S) \tag{2-13}$$

subject to:

$$(\mathbf{x}_S, \boldsymbol{\lambda}) \in \mathcal{M}\big(\boldsymbol{p}_S, \boldsymbol{q}_S, \boldsymbol{p}_R, \boldsymbol{q}_R\big) \tag{2-14}$$

We refer to (2-13)–(2-14) as an *optimistic approach.* Problem (2-13)–(2-14) identifies a feasible bid that maximizes the strategic player profit while taking into account that the amount effectively sold and the respective uniform clearing price are solutions of the auction problem (2-1)–(2-6). The strategic player profit is composed of three components: (i) the first term is the revenue from the sale of products on auction; (ii) the second term is a linear production cost term; and (iii) in the third term, f_S translates an additional cost related to the bid $(\boldsymbol{p}_S, \boldsymbol{q}_S)$. For instance, f_S can represent the cost of participating in the auction (entry fee), some deposit of guarantees required by the auctioneer, or a cash flow source from various financial instruments (such as long-term contracts or hedging operations). We highlight that as $\boldsymbol{A}_S$ is a matrix that links an auctioned product to its sold amount, it appears in the first term of the strategic player revenue. Finally, constraint (2-14) mathematically embeds the auction results into the optimal bidding problem, and $\mathcal{O}_S$ defines the feasible region of the strategic bids. It may comprise budget constraints or logic relations among products that must be satisfied.

Problem (2-13)–(2-14) lies in the class of mathematical programming with equilibrium constraints (MPEC) [32, 33]. Although highly intuitive, the bidding

[2] In this thesis, for convenience, we will assume that the auction problem (2-1)–(2-6) is always feasible.

problem (2-13)–(2-14) may be inadequate for practical implementation owing to its strong assumption of complete knowledge on competitors' strategies. Therefore, next, we present an extension for incorporating uncertainty regarding rival players bidding strategies.

2.2 Optimal Bidding with Imperfect Information: A Robust Approach

A key issue with the practical implementation of the optimistic approach (2-13)–(2-14) is that it is unlikely that an agent would possess perfect information regarding the strategic action of its rivals. In practice, bidding decisions are made under uncertainty. Even in rare cases where market conditions are close to equilibrium, rivals' bidding strategies are subject to uncertainty. Parameters such as fuel prices and opportunity costs are difficult to predict.

In this context, let $(\tilde{\boldsymbol{p}}_R, \tilde{\boldsymbol{q}}_R)$ represent the vector of the rivals' uncertain price/ quantity bids. Thus, an extension of the bidding problem (2-13)–(2-14) to incorporate the uncertainty regarding rivals' strategies is presented in (2-15)–(2-16).

$$\max_{\substack{(\boldsymbol{p}_S, \boldsymbol{q}_S) \in \mathcal{O}_S, \\ \tilde{\mathbf{x}}_S, \tilde{\boldsymbol{\lambda}}}} \Phi\Big(\tilde{\boldsymbol{\lambda}}^\top \boldsymbol{A}_S \tilde{\mathbf{x}}_S - \boldsymbol{c}_S^\top \tilde{\mathbf{x}}_S - f_S(\boldsymbol{p}_S, \boldsymbol{q}_S)\Big) \tag{2-15}$$

subject to:

$$(\tilde{\mathbf{x}}_S, \tilde{\boldsymbol{\lambda}}) \in \mathcal{M}\big(\boldsymbol{p}_S, \boldsymbol{q}_S, \tilde{\boldsymbol{p}}_R, \tilde{\boldsymbol{q}}_R\big) \tag{2-16}$$

The bidding problem (2-15)–(2-16) is a two-stage nonlinear optimization model under uncertainty. In this framework, Φ is a functional that measures the certainty equivalent for the strategic player. Hence, an optimal bidding policy is a point in the feasible bidding set $\mathcal{O}_S$ that maximizes the certainty equivalent of the net revenue obtained in the auction.

Figure 2.2 illustrates the uncertain two-stage bidding environment for a finite sample space Ω, the elements of which are called *scenarios* for nomenclature purposes. For a given bid $(\boldsymbol{p}_S, \boldsymbol{q}_S) \in \mathcal{O}_S$ of the strategic player, each scenario of the rivals' bid $(\boldsymbol{p}_R^{(\omega)}, \boldsymbol{q}_R^{(\omega)})$ implies in an auction outcome (market equilibrium), i.e., a uniform price and amount sold by the strategic player $(\boldsymbol{\lambda}^{(\omega)}, \mathrm{x}_S^{(\omega)})$. The uncertainty regarding the rivals' actions is then translated into an uncertain net revenue $\big(\tilde{\boldsymbol{\lambda}}^\top \boldsymbol{A}_S \tilde{\mathbf{x}}_S - \boldsymbol{c}_S^\top \tilde{\mathbf{x}}_S - f_S(\boldsymbol{p}_S, \boldsymbol{q}_S)\big)$ measured using Φ.

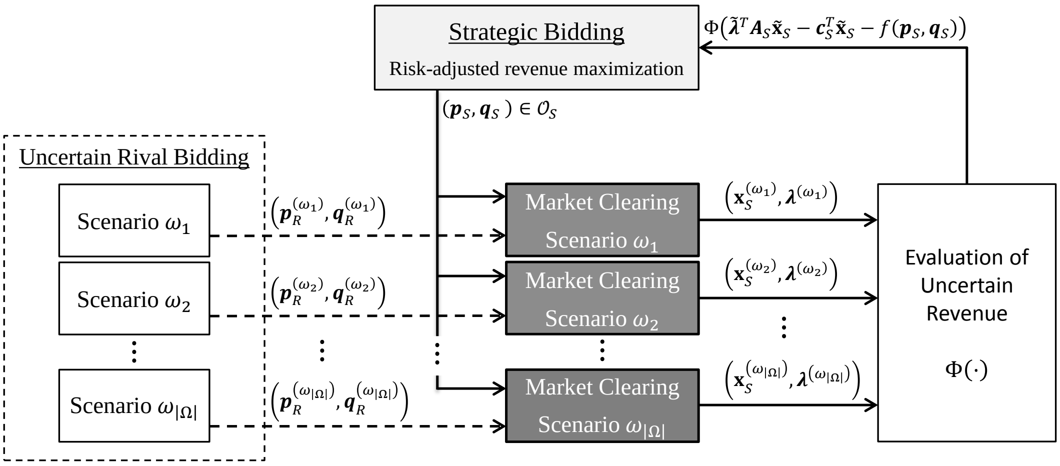

Fig. 2.2: Uncertain two-stage bidding environment with $|\Omega|$ probable scenarios of rival bids.

As is generally the case in two-stage stochastic problems, a joint probability distribution function is assumed to be available for the uncertain parameters, in this case, $(\tilde{\boldsymbol{p}}_R, \tilde{\boldsymbol{q}}_R)$. However, an adequate probabilistic description of the bidding behavior of a set of competing agents involves the characterization of the stochastic behavior of several variables that are difficult to estimate. Owing to this modeling difficulty, we propose an alternative approach that makes use of a set-based uncertainty representation for the rivals' bid, namely, robust optimization [27]. A worst-case analysis for assessing the optimal bidding strategy is thus performed with the aim of improving the robustness of the strategic player bid against unexpected rivals' strategies and estimation imprecision.

Let $\mathcal{O}_R$ denote a polyhedral uncertainty set. In the robust-optimization framework, the uncertainty set comprises all scenarios that the decision maker requires to protect against. The proposed robust bidding model is thus presented as follows:

$$\varphi^* = \max_{(\boldsymbol{p}_S, \boldsymbol{q}_S) \in \mathcal{O}_S} -f_S(\boldsymbol{p}_S, \boldsymbol{q}_S) + \left\{ \begin{array}{l} \displaystyle\min_{\substack{(\boldsymbol{p}_R, \boldsymbol{q}_R) \in \mathcal{O}_R \\ \mathbf{x}_S, \boldsymbol{\lambda}}} \boldsymbol{\lambda}^\top \boldsymbol{A}_S \mathbf{x}_S - \boldsymbol{c}_S^\top \mathbf{x}_S \\ \text{subject to:} \\ (\mathbf{x}_S, \boldsymbol{\lambda}) \in \mathcal{M}\big(\boldsymbol{p}_S, \boldsymbol{q}_S, \boldsymbol{p}_R, \boldsymbol{q}_R\big) \end{array} \right\}. \quad (2\text{-}17)$$

Problem (2-17) is a trilevel optimization problem. Structurally, the first level (outer maximization problem) defines, within a set of feasible bids, the bid that maximizes the worst-case net revenue of the strategic player in the auction. The second-level problem (inner minimization model) then finds, within the set of feasible bids for the rivals $((\boldsymbol{p}_R, \boldsymbol{q}_R) \in \mathcal{O}_R)$, the scenario that creates the worst adversity for the strategic player's net revenue. Finally, the worst-case scenario for the net revenue comprises auction equilibrium outcomes such as clearing prices and auctioned quantities, which are obtained by using the optimum set of the third-level problem (2-1)–(2-6), represented by $\mathcal{M}\big(\boldsymbol{p}_S, \boldsymbol{q}_S, \boldsymbol{p}_R, \boldsymbol{q}_R\big)$. In this context, an optimal solution to (2-17) is a bidding strategy that maximizes the worst-case net revenue in the auction.

Figure 2.3 depicts the trilevel model proposed in this thesis. It should be noted that, in general, the optimal set $\mathcal{M}\big(\boldsymbol{p}_S, \boldsymbol{q}_S, \boldsymbol{p}_R, \boldsymbol{q}_R\big)$ comprises multiple points (see [61, 62]). Therefore, by virtue, a robust bidding strategy should consider a pessimistic equilibrium point within $\mathcal{M}\big(\boldsymbol{p}_S, \boldsymbol{q}_S, \boldsymbol{p}_R, \boldsymbol{q}_R\big)$ in order to impose a certain level of robustness against degenerated clearing prices. In contrast to previously reported works, the model (2-17) identifies the worst-case auction result (clearing price and auctioned quantities) when evaluating the net revenue for the strategic player.

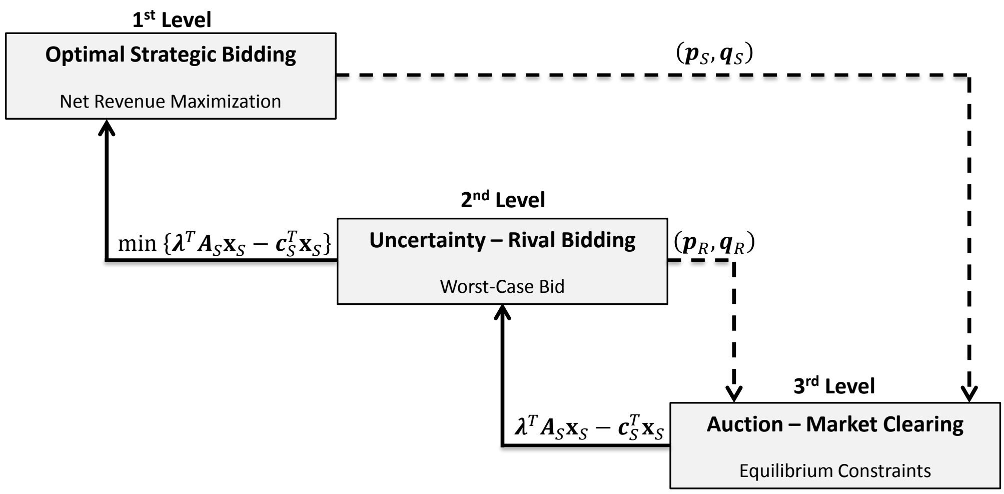

Fig. 2.3: Hierarchical structure of the proposed robust (three-level) bidding problem.

Following its mathematical structure, (2-17) is a particular instance of the TSR-MPEC that cannot be directly solved through available commercial solvers. To overcome this issue, in the next section, we present a reformulation procedure that transforms (2-17) into a large-scale single-level mathematical programming problem and an efficient solution approach based on a CCG algorithm.

3
Solution Approach: A Column and Constraint Generation Algorithm

The proposed robust bidding problem (2-17) presented in Section 2.2 is a three-level system of optimization problems not suitable for direct implementation on commercial solvers. The main goals of this chapter are thus twofold: 1) to present a single-level equivalent exponential formulation for the robust bidding problem (2-17), and 2) to develop a CCG-based algorithm capable of efficiently identifying a reduced set of complementarity conditions that are sufficient for representing equilibrium constraints at the optimal bidding strategy.

3.1 Single-level Equivalent Formulation with Exponential Number of Constraints

The auction problem (2-1)–(2-6) is a linear and continuous programming problem. Thus, we can conveniently represent the set $\mathcal{M}\big(\boldsymbol{p}_S, \boldsymbol{q}_S, \boldsymbol{p}_R, \boldsymbol{q}_R\big)$ of the solution points using its Karush–Kuhn–Tucker (KKT) optimality conditions[1]. Therefore, the trilevel problem (2-17) can be recast as a bilevel programming problem as follows:

$$\varphi^* = \max_{(\boldsymbol{p}_S,\boldsymbol{q}_S)\in\mathcal{O}_S} -f_S(\boldsymbol{p}_S,\boldsymbol{q}_S) \; + \; \left\{ \min_{\substack{(\boldsymbol{p}_R,\boldsymbol{q}_R)\in\mathcal{O}_R,\\ \mathbf{x}_S,\mathbf{x}_R,\mathbf{x}_D,\mathbf{y},\boldsymbol{\lambda},\\ \overline{\boldsymbol{\lambda}}_S,\underline{\boldsymbol{\lambda}}_S,\overline{\boldsymbol{\lambda}}_R,\underline{\boldsymbol{\lambda}}_R,\\ \overline{\boldsymbol{\lambda}}_D,\underline{\boldsymbol{\lambda}}_D,\overline{\boldsymbol{\lambda}}_\mathbf{y},\underline{\boldsymbol{\lambda}}_\mathbf{y}}} \boldsymbol{\lambda}^\top \boldsymbol{A}_S\mathbf{x}_S - \boldsymbol{c}_S^\top\mathbf{x}_S \right. \tag{3-1}$$

subject to:

$$\boldsymbol{A}_S\mathbf{x}_S + \boldsymbol{A}_R\mathbf{x}_R - \boldsymbol{A}_D\mathbf{x}_D + \boldsymbol{A}_\mathbf{y}\mathbf{y} = \mathbf{0} \tag{3-2}$$

$$\underline{\boldsymbol{h}}_S \leq \boldsymbol{I}_S\mathbf{x}_S \leq \boldsymbol{H}_S\boldsymbol{q}_S + \overline{\boldsymbol{h}}_S \tag{3-3}$$

$$\underline{\boldsymbol{h}}_R \leq \boldsymbol{I}_R\mathbf{x}_R \leq \boldsymbol{H}_R\boldsymbol{q}_R + \overline{\boldsymbol{h}}_R \tag{3-4}$$

$$\underline{\boldsymbol{h}}_D \leq \boldsymbol{I}_D\mathbf{x}_D \leq \boldsymbol{H}_D\boldsymbol{q}_D + \overline{\boldsymbol{h}}_D \tag{3-5}$$

$$\underline{\boldsymbol{h}}_\mathbf{y} \leq \boldsymbol{I}_\mathbf{y}\mathbf{y} \leq \overline{\boldsymbol{h}}_\mathbf{y} \tag{3-6}$$

[1] We refer to [63] and [64] for a complete discussion and mathematical properties regarding this transformation.

$$\boldsymbol{p}_D^\top - \boldsymbol{\lambda}^\top \boldsymbol{A}_D - \overline{\boldsymbol{\lambda}}_D^\top \boldsymbol{I}_D + \underline{\boldsymbol{\lambda}}_D^\top \boldsymbol{I}_D = \mathbf{0}^\top \quad (3\text{-}7)$$

$$-\boldsymbol{p}_S^\top + \boldsymbol{\lambda}^\top \boldsymbol{A}_S - \overline{\boldsymbol{\lambda}}_S^\top \boldsymbol{I}_S + \underline{\boldsymbol{\lambda}}_S^\top \boldsymbol{I}_S = \mathbf{0}^\top \quad (3\text{-}8)$$

$$-\boldsymbol{p}_R^\top + \boldsymbol{\lambda}^\top \boldsymbol{A}_R - \overline{\boldsymbol{\lambda}}_R^\top \boldsymbol{I}_R + \underline{\boldsymbol{\lambda}}_R^\top \boldsymbol{I}_R = \mathbf{0}^\top \quad (3\text{-}9)$$

$$\boldsymbol{p}_{\mathrm{y}}^\top + \boldsymbol{\lambda}^\top \boldsymbol{A}_{\mathrm{y}} - \overline{\boldsymbol{\lambda}}_{\mathrm{y}}^\top \boldsymbol{I}_{\mathrm{y}} + \underline{\boldsymbol{\lambda}}_{\mathrm{y}}^\top \boldsymbol{I}_{\mathrm{y}} = \mathbf{0}^\top \quad (3\text{-}10)$$

$$\overline{\boldsymbol{\lambda}}_S^\top \Big(\boldsymbol{H}_S \boldsymbol{q}_S + \overline{\boldsymbol{h}}_S - \boldsymbol{I}_S \mathbf{x}_S \Big) = 0 \quad (3\text{-}11)$$

$$\underline{\boldsymbol{\lambda}}_S^\top \Big(\boldsymbol{I}_S \mathbf{x}_S - \underline{\boldsymbol{h}}_S \Big) = 0 \quad (3\text{-}12)$$

$$\overline{\boldsymbol{\lambda}}_R^\top \Big(\boldsymbol{H}_R \boldsymbol{q}_R + \overline{\boldsymbol{h}}_R - \boldsymbol{I}_R \mathbf{x}_R \Big) = 0 \quad (3\text{-}13)$$

$$\underline{\boldsymbol{\lambda}}_R^\top \Big(\boldsymbol{I}_R \mathbf{x}_R - \underline{\boldsymbol{h}}_R \Big) = 0 \quad (3\text{-}14)$$

$$\overline{\boldsymbol{\lambda}}_D^\top \Big(\boldsymbol{H}_D \boldsymbol{q}_D + \overline{\boldsymbol{h}}_D - \boldsymbol{I}_D \mathbf{x}_D \Big) = 0 \quad (3\text{-}15)$$

$$\underline{\boldsymbol{\lambda}}_D^\top \Big(\boldsymbol{I}_D \mathbf{x}_D - \underline{\boldsymbol{h}}_D \Big) = 0 \quad (3\text{-}16)$$

$$\overline{\boldsymbol{\lambda}}_{\mathrm{y}}^\top \Big(\overline{\boldsymbol{h}}_{\mathrm{y}} - \boldsymbol{I}_{\mathrm{y}} \mathbf{y} \Big) = 0 \quad (3\text{-}17)$$

$$\underline{\boldsymbol{\lambda}}_{\mathrm{y}}^\top \Big(\boldsymbol{I}_{\mathrm{y}} \mathbf{y} - \underline{\boldsymbol{h}}_{\mathrm{y}} \Big) = 0 \quad (3\text{-}18)$$

$$\left. \underline{\boldsymbol{\lambda}}_S, \overline{\boldsymbol{\lambda}}_S, \underline{\boldsymbol{\lambda}}_R, \overline{\boldsymbol{\lambda}}_R, \underline{\boldsymbol{\lambda}}_D, \overline{\boldsymbol{\lambda}}_D, \overline{\boldsymbol{\lambda}}_{\mathrm{y}}, \underline{\boldsymbol{\lambda}}_{\mathrm{y}} \geq \mathbf{0} \right\}. \quad (3\text{-}19)$$

Problem (3-1)–(3-19) is a two-level system of optimization problems with complementarity constraints [46] in the inner-level problem. Owing to a bilinear product of continuous variables ($\boldsymbol{\lambda}^\top \boldsymbol{A}_S \mathbf{x}_S$) in the second-level objective function (3-1) and the complementarity constraints (3-11)–(3-18), the second-level problem is a nonlinear optimization model.

To address the bilinear product of continuous variables $\boldsymbol{\lambda}^\top \boldsymbol{A}_S \mathbf{x}_S$ in the second-level objective function (3-1), we follow a procedure similar to that discussed in [25]. The first step is to multiply $\mathbf{x}_S$ to the right expression (3-8), which results in

$$\boldsymbol{\lambda}^\top \boldsymbol{A}_S \mathbf{x}_S = \boldsymbol{p}_S^\top \mathbf{x}_S + \overline{\boldsymbol{\lambda}}_S^\top \boldsymbol{I}_S \mathbf{x}_S - \underline{\boldsymbol{\lambda}}_S^\top \boldsymbol{I}_S \mathbf{x}_S. \quad (3\text{-}20)$$

Then, note that from equation (3-11),

$$\overline{\boldsymbol{\lambda}}_S^\top \boldsymbol{I}_S \mathbf{x}_S = \overline{\boldsymbol{\lambda}}_S^\top \boldsymbol{H}_S \boldsymbol{q}_S + \overline{\boldsymbol{\lambda}}_S^\top \overline{\boldsymbol{h}}_S. \quad (3\text{-}21)$$

Also, from equation (3-12),

$$\underline{\boldsymbol{\lambda}}_S^\top \boldsymbol{I}_S \mathbf{x}_S = \underline{\boldsymbol{\lambda}}_S^\top \underline{\boldsymbol{h}}_S. \tag{3-22}$$

Finally, combining (3-20) with (3-21) and (3-22), we have

$$\boldsymbol{\lambda}^\top \boldsymbol{A}_S \mathbf{x}_S = \boldsymbol{p}_S^\top \mathbf{x}_S + \overline{\boldsymbol{\lambda}}_S^\top \boldsymbol{H}_S \boldsymbol{q}_S + \overline{\boldsymbol{\lambda}}_S^\top \overline{\boldsymbol{h}}_S - \underline{\boldsymbol{\lambda}}_S^\top \underline{\boldsymbol{h}}_S. \tag{3-23}$$

Therefore, for a given set of first-level variables $(\boldsymbol{p}_S, \boldsymbol{q}_S)$, the objective function (3-1) can be written as a linear function of the second-level decision variables.

Another source of nonconvexity in the second-level problem (3-1)–(3-19) can be found in the complementarity constraints (3-11)–(3-18) of the KKT conditions of the problem (2-1)–(2-6). Several works in the technical literature discuss new reformulations and algorithms for handling this type of constraint (see [46, 47, 48, 49] and references therein). In this thesis, we use the technique proposed in [47] to replace the usual bilinear complementarity conditions by a set of mixed-integer linear constraints.

Take, for instance, the complementarity condition (3-11); all other complementarity constraints (3-12)–(3-18) follow the same rationale. If $\overline{\boldsymbol{\mu}}_S \in \{0,1\}^{N_S}$ is a binary vector, the set of feasible primal and dual points defined by

$$\mathcal{A}\big(\boldsymbol{H}_S, \boldsymbol{q}_S, \overline{\boldsymbol{h}}_S, \boldsymbol{I}_S, \overline{U}_S\big) \triangleq \Bigg\{ (\overline{\boldsymbol{\lambda}}_S, \mathbf{x}_S) \in \mathbb{R}^{N_S} \times \mathbb{R}^{N_S} \;\Bigg|\; \overline{\boldsymbol{\lambda}}_S^\top \Big(\boldsymbol{H}_S \boldsymbol{q}_S + \overline{\boldsymbol{h}}_S - \boldsymbol{I}_S \mathbf{x}_S\Big) = 0, \quad \boldsymbol{I}_S \mathbf{x}_S \le \boldsymbol{H}_S \boldsymbol{q}_S + \overline{\boldsymbol{h}}_S, \quad \mathbf{0} \le \overline{\boldsymbol{\lambda}}_S \le \mathbb{1}_{N_S} \overline{U}_S \Bigg\}, \tag{3-24}$$

can be equivalently obtained as follows:

$$\mathcal{A}\big(\boldsymbol{H}_S, \boldsymbol{q}_S, \overline{\boldsymbol{h}}_S, \boldsymbol{I}_S, \overline{U}_S\big) = \Bigg\{ (\overline{\boldsymbol{\lambda}}_S, \mathbf{x}_S) \in \mathbb{R}^{N_S} \times \mathbb{R}^{N_S} \;\Bigg|\; \exists\, \overline{\boldsymbol{\mu}}_S \in \{0,1\}^{N_S}; \quad \mathbf{0} \le \boldsymbol{H}_S \boldsymbol{q}_S + \overline{\boldsymbol{h}}_S - \boldsymbol{I}_S \mathbf{x}_S \le \overline{\boldsymbol{\mu}}_S \overline{U}_S, \quad \mathbf{0} \le \overline{\boldsymbol{\lambda}}_S \le (\mathbb{1}_{N_S} - \overline{\boldsymbol{\mu}}_S) \overline{U}_S \Bigg\}. \tag{3-25}$$

Essentially, the binary vector $\overline{\boldsymbol{\mu}}_S$ "chooses" if the primal or dual constraints will be active, thus ensuring the complementarity condition. In (3-24), $\overline{U}_S$ is a constant large enough to ensure relaxed primal constraints and to comprise all the values of the Lagrangian multipliers. This requirement does not represent an issue since problem (2-1)–(2-6) is assumed always feasible, i.e., the auction is assumed to always have a solution. For instance, in practical applications, we can ensure feasibility by means of deficit variables representing the auction imbalance between supply and demand. This is a requirement for disjunctive constraints in (3-25).

By using this equivalence and the previously derived relation (3-23), the bidding problem proposed in this thesis resumes to the following mixed-integer two-level system of optimization problems:

$$\varphi^* = \max_{(\boldsymbol{p}_S,\boldsymbol{q}_S)\in\mathcal{O}_S} -f_S(\boldsymbol{p}_S,\boldsymbol{q}_S) + \left\{ \min_{\substack{(\boldsymbol{p}_R,\boldsymbol{q}_R)\in\mathcal{O}_R,\\ \mathbf{x}_S,\mathbf{x}_R,\mathbf{x}_D,\mathbf{y},\boldsymbol{\lambda},\\ \overline{\boldsymbol{\lambda}}_S,\underline{\boldsymbol{\lambda}}_S,\overline{\boldsymbol{\lambda}}_R,\underline{\boldsymbol{\lambda}}_R,\\ \overline{\boldsymbol{\lambda}}_D,\underline{\boldsymbol{\lambda}}_D,\overline{\boldsymbol{\lambda}}_\mathbf{y},\underline{\boldsymbol{\lambda}}_\mathbf{y},\\ \overline{\boldsymbol{\mu}}_S,\underline{\boldsymbol{\mu}}_S,\overline{\boldsymbol{\mu}}_R,\underline{\boldsymbol{\mu}}_R,\\ \overline{\boldsymbol{\mu}}_D,\underline{\boldsymbol{\mu}}_D,\overline{\boldsymbol{\mu}}_\mathbf{y},\underline{\boldsymbol{\mu}}_\mathbf{y}}} \boldsymbol{p}_S^\top \mathbf{x}_S + \overline{\boldsymbol{\lambda}}_S^\top \boldsymbol{H}_S \boldsymbol{q}_S + \overline{\boldsymbol{\lambda}}_S^\top \overline{\boldsymbol{h}}_S - \underline{\boldsymbol{\lambda}}_S^\top \underline{\boldsymbol{h}}_S - \boldsymbol{c}_S^\top \mathbf{x}_S \right. \tag{3-26}$$

subject to:

$$\boldsymbol{A}_S\mathbf{x}_S + \boldsymbol{A}_R\mathbf{x}_R - \boldsymbol{A}_D\mathbf{x}_D + \boldsymbol{A}_\mathbf{y}\mathbf{y} = \mathbf{0} \tag{3-27}$$

$$\boldsymbol{p}_D^\top - \boldsymbol{\lambda}^\top \boldsymbol{A}_D - \overline{\boldsymbol{\lambda}}_D^\top \boldsymbol{I}_D + \underline{\boldsymbol{\lambda}}_D^\top \boldsymbol{I}_D = \mathbf{0}^\top \tag{3-28}$$

$$-\boldsymbol{p}_S^\top + \boldsymbol{\lambda}^\top \boldsymbol{A}_S - \overline{\boldsymbol{\lambda}}_S^\top \boldsymbol{I}_S + \underline{\boldsymbol{\lambda}}_S^\top \boldsymbol{I}_S = \mathbf{0}^\top \tag{3-29}$$

$$-\boldsymbol{p}_R^\top + \boldsymbol{\lambda}^\top \boldsymbol{A}_R - \overline{\boldsymbol{\lambda}}_R^\top \boldsymbol{I}_R + \underline{\boldsymbol{\lambda}}_R^\top \boldsymbol{I}_R = \mathbf{0}^\top \tag{3-30}$$

$$\boldsymbol{p}_\mathbf{y}^\top + \boldsymbol{\lambda}^\top \boldsymbol{A}_\mathbf{y} - \overline{\boldsymbol{\lambda}}_\mathbf{y}^\top \boldsymbol{I}_\mathbf{y} + \underline{\boldsymbol{\lambda}}_\mathbf{y}^\top \boldsymbol{I}_\mathbf{y} = \mathbf{0}^\top \tag{3-31}$$

$$(\overline{\boldsymbol{\lambda}}_S, \mathbf{x}_S) \in \mathcal{A}(\boldsymbol{H}_S, \boldsymbol{q}_S, \overline{\boldsymbol{h}}_S, \boldsymbol{I}_S, \overline{U}_S) \tag{3-32}$$

$$(\underline{\boldsymbol{\lambda}}_S, \mathbf{x}_S) \in \mathcal{A}(\mathbf{0}, \mathbf{0}, -\underline{\boldsymbol{h}}_S, -\boldsymbol{I}_S, \underline{U}_S) \tag{3-33}$$

$$(\overline{\boldsymbol{\lambda}}_R, \mathbf{x}_R) \in \mathcal{A}(\boldsymbol{H}_R, \boldsymbol{q}_R, \overline{\boldsymbol{h}}_R, \boldsymbol{I}_R, \overline{U}_R) \tag{3-34}$$

$$(\underline{\boldsymbol{\lambda}}_R, \mathbf{x}_R) \in \mathcal{A}(\mathbf{0}, \mathbf{0}, -\underline{\boldsymbol{h}}_R, -\boldsymbol{I}_R, \underline{U}_R) \tag{3-35}$$

$$(\overline{\boldsymbol{\lambda}}_D, \mathbf{x}_D) \in \mathcal{A}(\boldsymbol{H}_D, \boldsymbol{q}_D, \overline{\boldsymbol{h}}_D, \boldsymbol{I}_D, \overline{U}_D) \tag{3-36}$$

$$(\underline{\boldsymbol{\lambda}}_D, \mathbf{x}_D) \in \mathcal{A}(\mathbf{0}, \mathbf{0}, -\underline{\boldsymbol{h}}_D, -\boldsymbol{I}_D, \underline{U}_D) \tag{3-37}$$

$$(\overline{\boldsymbol{\lambda}}_\mathbf{y}, \mathbf{y}) \in \mathcal{A}(\mathbf{0}, \mathbf{0}, \overline{\boldsymbol{h}}_\mathbf{y}, \boldsymbol{I}_\mathbf{y}, \overline{U}_\mathbf{y}) \tag{3-38}$$

$$(\underline{\boldsymbol{\lambda}}_{\mathbf{y}}, \mathbf{y}) \in \mathcal{A}(\mathbf{0}, \mathbf{0}, -\underline{\boldsymbol{h}}_{\mathbf{y}}, -\boldsymbol{I}_{\mathbf{y}}, \underline{U}_{\mathbf{y}}) \Bigg\}. \quad (3\text{-}39)$$

Problem (3-26)–(3-39) is a BPEC and is, thereby, not suitable for off-the-shelf commercial solvers. To complete the development and achieve a single-level equivalent formulation, (3-26)–(3-39) is presented in its compact form as follows:

$$\varphi^* = \max_{\boldsymbol{z}_U \in \mathcal{O}_S} -f_S(\boldsymbol{z}_U) + \min_{\substack{\boldsymbol{z}_L \geq \mathbf{0} \\ \boldsymbol{u} \in \mathbb{B}}} \left\{ \boldsymbol{g}^\top \boldsymbol{z}_L + \boldsymbol{z}_U^\top \boldsymbol{B} \boldsymbol{z}_L \;\middle|\; \boldsymbol{L}\boldsymbol{z}_L \geq \boldsymbol{E}\boldsymbol{z}_U + \boldsymbol{F}\boldsymbol{u} + \boldsymbol{b} \right\}. \quad (3\text{-}40)$$

The set of first-level decision variables of (3-26)–(3-39) is denoted by $\boldsymbol{z}_U$ in (3-40), and the set of binary and continuous variables of the second-level problem are denoted by $\boldsymbol{u}$ and $\boldsymbol{z}_L$, respectively. If we let $\mathbb{B}$ denote the set of all binary vectors with the dimension of vector $\boldsymbol{u}$, i.e., $\boldsymbol{u} \in \mathbb{B} = \{0,1\}^{2N_S+2N_R+2N_D+2N_{\mathbf{y}}}$, then problem (3-26)–(3-39) can be manipulated as follows:

$$\varphi^* = \max_{\boldsymbol{z}_U \in \mathcal{O}_S} -f_S(\boldsymbol{z}_U) + \min_{\substack{\boldsymbol{z}_L \geq \mathbf{0} \\ \boldsymbol{u} \in \mathbb{B}}} \left\{ \boldsymbol{g}^\top \boldsymbol{z}_L + \boldsymbol{z}_U^\top \boldsymbol{B} \boldsymbol{z}_L \;\middle|\; \boldsymbol{L}\boldsymbol{z}_L \geq \boldsymbol{E}\boldsymbol{z}_U + \boldsymbol{F}\boldsymbol{u} + \boldsymbol{b} \right\} \quad (3\text{-}41)$$

$$= \max_{\boldsymbol{z}_U \in \mathcal{O}_S} -f_S(\boldsymbol{z}_U) + \min_{\boldsymbol{u} \in \mathbb{B}} \min_{\boldsymbol{z}_L \geq \mathbf{0}} \left\{ \boldsymbol{g}^\top \boldsymbol{z}_L + \boldsymbol{z}_U^\top \boldsymbol{B} \boldsymbol{z}_L \;\middle|\; \boldsymbol{L}\boldsymbol{z}_L \geq \boldsymbol{E}\boldsymbol{z}_U + \boldsymbol{F}\boldsymbol{u} + \boldsymbol{b} \right\} \quad (3\text{-}42)$$

$$= \max_{\boldsymbol{z}_U \in \mathcal{O}_S} -f_S(\boldsymbol{z}_U) + \min_{\boldsymbol{u} \in \mathbb{B}} \max_{\boldsymbol{\theta} \geq \mathbf{0}} \left\{ \boldsymbol{\theta}^\top \Big(\boldsymbol{E}\boldsymbol{z}_U + \boldsymbol{F}\boldsymbol{u} + \boldsymbol{b} \Big) \;\middle|\; \boldsymbol{L}^\top \boldsymbol{\theta} \leq \boldsymbol{g} + \boldsymbol{B}^\top \boldsymbol{z}_U \right\} \quad (3\text{-}43)$$

$$= \max_{\eta, \boldsymbol{z}_U} -f_S(\boldsymbol{z}_U) + \eta \quad (3\text{-}44)$$

subject to:

$$\eta \leq \min_{\boldsymbol{u} \in \mathbb{B}} \max_{\boldsymbol{\theta} \geq \mathbf{0}} \left\{ \boldsymbol{\theta}^\top \Big(\boldsymbol{E}\boldsymbol{z}_U + \boldsymbol{F}\boldsymbol{u} + \boldsymbol{b} \Big) \;\middle|\; \boldsymbol{L}^\top \boldsymbol{\theta} \leq \boldsymbol{g} + \boldsymbol{B}^\top \boldsymbol{z}_U \right\}; \quad (3\text{-}45)$$

$$\boldsymbol{z}_U \in \mathcal{O}_S, \quad (3\text{-}46)$$

where $\boldsymbol{\theta}$ is the dual variable associated with the constraint $\boldsymbol{L}\boldsymbol{z}_L \geq \boldsymbol{E}\boldsymbol{z}_U + \boldsymbol{F}\boldsymbol{u} + \boldsymbol{b}$ in (3-42).

Because $\mathbb{B}$ is a binary set, the minimization problem in (3-45) can be replaced by an explicit enumeration of all the possible values of $\boldsymbol{u} \in \mathbb{B}$ as follows:

$$\varphi^* = \max_{\eta, \boldsymbol{z}_U} \; -f_S(\boldsymbol{z}_U) + \eta \tag{3-47}$$

subject to:

$$\eta \leq \max_{\boldsymbol{\theta}_{\boldsymbol{u}} \geq \boldsymbol{0}} \left\{ \boldsymbol{\theta}_{\boldsymbol{u}}^\top \Big(\boldsymbol{E} \boldsymbol{z}_U + \boldsymbol{F} \boldsymbol{u} + \boldsymbol{b} \Big) \;\Big|\; \boldsymbol{L}^\top \boldsymbol{\theta}_{\boldsymbol{u}} \leq \boldsymbol{g} + \boldsymbol{B}^\top \boldsymbol{z}_U \right\}, \qquad \forall\, \boldsymbol{u} \in \mathbb{B}; \tag{3-48}$$

$$\boldsymbol{z}_U \in \mathcal{O}_S. \tag{3-49}$$

In (3-47)–(3-49), we explicitly identify $\boldsymbol{\theta}$ with its corresponding binary vector $\boldsymbol{u}$ using the notation $\boldsymbol{\theta}_{\boldsymbol{u}}$. In other words, for each $\boldsymbol{u} \in \mathbb{B}$, a different $\boldsymbol{\theta}_{\boldsymbol{u}}$ results from solving the linear programming problem $\max_{\boldsymbol{\theta}_{\boldsymbol{u}} \geq \boldsymbol{0}} \left\{ \boldsymbol{\theta}_{\boldsymbol{u}}^\top \Big(\boldsymbol{E} \boldsymbol{z}_U + \boldsymbol{F} \boldsymbol{u} + \boldsymbol{b} \Big) \;\Big|\; \boldsymbol{L}^\top \boldsymbol{\theta}_{\boldsymbol{u}} \leq \boldsymbol{g} + \boldsymbol{B}^\top \boldsymbol{z}_U \right\}$.

Finally, because the maximization problems in (3-48) are all independent of each other, the variables $\boldsymbol{\theta}_{\boldsymbol{u}}$ can be jointly coordinated at the upper maximization problem, thus resulting in the following single-level equivalent formulation for problem (3-40):

$$\varphi^* = \max_{\eta, \boldsymbol{z}_U, \boldsymbol{\theta}_{\boldsymbol{u}}} \; -f_S(\boldsymbol{z}_U) + \eta \tag{3-50}$$

subject to:

$$\eta \leq \boldsymbol{\theta}_{\boldsymbol{u}}^\top \Big(\boldsymbol{E} \boldsymbol{z}_U + \boldsymbol{F} \boldsymbol{u} + \boldsymbol{b} \Big), \qquad \forall\, \boldsymbol{u} \in \mathbb{B}; \tag{3-51}$$

$$\boldsymbol{L}^\top \boldsymbol{\theta}_{\boldsymbol{u}} \leq \boldsymbol{g} + \boldsymbol{B}^\top \boldsymbol{z}_U, \qquad \forall\, \boldsymbol{u} \in \mathbb{B}; \tag{3-52}$$

$$\boldsymbol{\theta}_{\boldsymbol{u}} \geq \boldsymbol{0}, \qquad \forall\, \boldsymbol{u} \in \mathbb{B}; \tag{3-53}$$

$$\boldsymbol{z}_U \in \mathcal{O}_S. \tag{3-54}$$

Problem (3-50)–(3-54) is a single-level optimization problem with an exponential set of constraints. In the next section, a decomposition algorithm based on CCG is developed to circumvent the dimensionality course present in the combinatorial nature of (3-50)–(3-54) owing to the cardinality of $\mathbb{B}$.

Remark 1. *We recognize that problem* (3-50)–(3-54) *is non-convex owing to the bilinear product in equation* (3-51)*. Nevertheless, in practice, most bidding rules impose a discrete set of bids. For instance, price bids are subject to cents precision and quantities are usually discretized in lots. Therefore, a binary expansion scheme can be employed to represent the strategic bidding, and the bilinear terms can be* *recast into linear expressions using disjunctive constraints (see [24] for an application in energy). Within a linear framework, off-the-shelf commercial mixed-integer linear programming (MILP) solvers, e.g., Xpress, Cplex, and Gurobi, can be used to solve the master problem.*

Nonetheless, we also highlight that bilinear programming has been widely studied in technical literature. Several efficient reformulations and algorithms have been proposed for tackling this issue (see, for instance, [65, 66, 67, 68] and the references therein). Furthermore, several commercial solvers, e.g., Xpress, Knitro, and Mosek, have efficient implementations for different classes of mathematical problems with bilinear terms that can be used in practical applications. Therefore, there exists a wide range of available algorithms that can be used to solve problem (3-50)–(3-54).

3.2 Column-and-Constraint Generation Algorithm

Problem (3-50)–(3-54) is a large-scale optimization problem because of the combinatorial nature of $\mathbb{B}$. Nevertheless, the structure of this problem precisely matches the structure suitable for the CCG algorithm. More specifically, let $\boldsymbol{z}_U^*$ be the optimal value for $\boldsymbol{z}_U$ in (3-50)–(3-54). For future reference, we refer to (3-50)–(3-54) as the *full problem*. We then consider any subset $\mathbb{B}_k \subset \mathbb{B}$ and present the following optimization problem:

$$\overline{\varphi}_k = \max_{\eta, \boldsymbol{z}_U, \boldsymbol{\theta}_{\boldsymbol{u}}} \; -f_S(\boldsymbol{z}_U) + \eta \tag{3-55}$$

subject to:

$$\eta \leq \boldsymbol{\theta}_{\boldsymbol{u}}^\top \Big(\boldsymbol{E}\boldsymbol{z}_U + \boldsymbol{F}\boldsymbol{u} + \boldsymbol{b} \Big), \qquad \forall\, \boldsymbol{u} \in \mathbb{B}_k; \tag{3-56}$$

$$\boldsymbol{L}^\top \boldsymbol{\theta}_{\boldsymbol{u}} \leq \boldsymbol{g} + \boldsymbol{B}^\top \boldsymbol{z}_U, \qquad \forall\, \boldsymbol{u} \in \mathbb{B}_k; \tag{3-57}$$

$$\boldsymbol{\theta}_{\boldsymbol{u}} \geq \boldsymbol{0}, \qquad \forall\, \boldsymbol{u} \in \mathbb{B}_k; \tag{3-58}$$

$$\boldsymbol{z}_U \in \mathcal{O}_S. \tag{3-59}$$

Because $\mathbb{B}_k \subset \mathbb{B}$, we known that $\overline{\varphi}_k \geq \varphi^*$, thereby, an upper bound for the full problem. Hereinafter, we refer to problem (3-55)–(3-59) as the *master problem* and $\boldsymbol{z}_{U,(k)}$ as the respective optimal value for $\boldsymbol{z}_U$ in this problem. In addition, we consider the following mixed integer linear programming problem:

$$\underline{\varphi}_k = -f_S(\boldsymbol{z}_{U,(k)}) + \min_{\boldsymbol{z}_L \geq \boldsymbol{0}, \boldsymbol{u} \in \mathbb{B}} \left\{ \boldsymbol{g}^\top \boldsymbol{z}_L + \boldsymbol{z}_{U,(k)}^\top \boldsymbol{B} \boldsymbol{z}_L \;\middle|\; \boldsymbol{L}\boldsymbol{z}_L \geq \boldsymbol{E}\boldsymbol{z}_{U,(k)} + \boldsymbol{F}\boldsymbol{u} + \boldsymbol{b} \right\}. \tag{3-60}$$

Because $\boldsymbol{z}_{U,(k)} \in \mathcal{O}_S$ and may not be the optimal solution for the full problem, $\underline{\varphi}_k \leq \varphi^*$ holds. Thus, $\underline{\varphi}_k$ is a lower bound for the full problem. Moreover, problem (3-60) identifies $\boldsymbol{u}_k$, namely the optimal value of $\boldsymbol{u}$, which minimizes the second term of the expression (3-50) for a fixed vector $\boldsymbol{z}_{U,(k)}$. Therefore, henceforth, such a problem is called an *oracle*.

The CCG algorithm proposed in this thesis is summarized in algorithm 1.

Algorithm 1 Column-and-Constraint Generation Algorithm

1: **Initialization**: $UB \leftarrow +\infty$, $LB \leftarrow -\infty$, $k \leftarrow 1$ and $\varepsilon(> 0)$.
2: Choose an initial subset $\mathbb{B}_k \subset \mathbb{B}$.
3: **while** $UB - LB \geq \varepsilon$ **do**
4: Solve master problem (3-55)–(3-59) with $\mathbb{B}_k$. Store $\boldsymbol{z}_{U,(k)}$ and set $UB \leftarrow \overline{\varphi}_k$;
5: Solve oracle problem (3-60) using $\boldsymbol{z}_{U,(k)}$. Store $\boldsymbol{u}_k$ and set $LB \leftarrow \underline{\varphi}_k$;
6: Make $\mathbb{B}_{k+1} \leftarrow \mathbb{B}_k \cup \{\boldsymbol{u}_k\}$. Set $k \leftarrow k+1$;
7: **end-do**
8: **Return** $\boldsymbol{z}_{U,(k-1)}$.

4
Case Study: Bidding in Day-Ahead Markets

One of the main applications of the methodology constructed in this thesis is to devise optimal bidding strategies in electricity markets. Since the 1980s, electricity markets have been widely fostered in most countries around the globe. Vertical and government-owned companies were unbundled aiming at designing a new industry structure with significant level of competition. Although the features and regulatory properties vary in each country, its backbone comprises a short-term market (usually called day-ahead market) in which energy is traded through a sealed-bid uniform-price auction of (multiple) divisible products.

In this chapter, we analyze the effectiveness of the proposed bidding strategy in a day-ahead electricity market context. We highlight that auction designs for electricity products have been significantly studied since the 1980s reforms. Nevertheless, a common representation is a single inelastic/inflexible demand, playing the role of a buyer competitor, purchasing electricity from a group of power companies in a single-node network. In this context, the mathematical formulation of the day-ahead auction follows (2-7)–(2-11) in Example 1 and the proposed bidding model can be stated as

$$\varphi^* = \max_{(\boldsymbol{p}_S,\boldsymbol{q}_S)\in\mathcal{O}_S} -f_S(\boldsymbol{p}_S,\boldsymbol{q}_S) \; + \left\{ \begin{array}{l} \displaystyle\min_{\substack{(\boldsymbol{p}_R,\boldsymbol{q}_R)\in\mathcal{O}_R \\ \mathbf{x}_S,\lambda}} \sum_{j\in\mathcal{N}_S} (\lambda - c_{S,j}) x_{S,j} \\ \text{subject to:} \\ (\mathbf{x}_S,\lambda) \in \mathcal{M}\big(\boldsymbol{p}_S,\boldsymbol{q}_S,\boldsymbol{p}_R,\boldsymbol{q}_R\big) \end{array} \right\}, \tag{4-1}$$

where $\mathcal{M}\big(\boldsymbol{p}_S,\boldsymbol{q}_S,\boldsymbol{p}_R,\boldsymbol{q}_R\big) = \Big\{(\mathbf{x}_S,\lambda) \in \mathbb{R}^{N_S}\times\mathbb{R} \;\Big|\; (\mathbf{x}_S,\lambda) \text{ solves (2-7)–(2-11)}\Big\}$. We will assume throughout this section that $f_S(\boldsymbol{p}_S,\boldsymbol{q}_S) = 0,\ \forall\,(\boldsymbol{p}_S,\boldsymbol{q}_S)\in\mathcal{O}_S$, for convenience. In this case study, we consider the following integer box-constrained set of feasible bids for the strategic player.

$$\mathcal{O}_S = \left\{ (\boldsymbol{p}_S,\boldsymbol{q}_S) \in \mathbb{Z}^{N_S}\times\mathbb{Z}^{N_S} \;\middle|\; \begin{array}{ll} 0 \le p_{S,j} \le \bar{p}_{S,j}, & \forall\, j\in\mathcal{N}_S; \\ 0 \le q_{S,j} \le \bar{q}_{S,j}, & \forall\, j\in\mathcal{N}_S; \end{array} \right\}, \tag{4-2}$$

with $\{\bar{p}_{S,j}\}_{j\in\mathcal{N}_S}$ denoting a cap on price bids and $\{\bar{q}_{S,j}\}_{j\in\mathcal{N}_S}$ the unit's capacity.

We highlight that under this integrality assumption, the bilinear product discussed on Remark 1 can be recast as a mixed-integer linear programming problem using exact linearization schemes via disjunctive inequalities [24, 68]. For completeness, in Appendix A, we carefully adapt the solution methodology presented in Chapter 3 to this particular instance of strategic bidding in day-ahead electricity markets.

Next, we present a set of numerical studies to illustrate the applicability of the proposed robust model. All numerical results derived in this chapter were obtained using the formulation in Appendix A, using a Dell Precision® T7600 Xeon® E5-2687W 3.10 GHz with 128 GB of RAM machine, with Xpress-MP 7.9 under MOSEL.

4.1 Algorithm Outline

This section is devoted to apply the iterative algorithm presented in Section 3.2 to the day-ahead bidding problem (4-1). We assume three power producers (one strategic and two rivals) competing to meed a single demand offer. With the purpose to avoid infeasibility in the auction process, i.e., to ensure that exists enough offer to cover demand, a fourth generator, hereinafter called *deficit* generator, bidding the highest price among all competitors and offering enough power capacity to cover demand is also considered. For simplicity, we will assume perfect information on rival bids and the *a priori* known vector of rival bids will be denoted by $(\hat{\boldsymbol{p}}_R, \hat{\boldsymbol{q}}_R)$. In this framework, the proposed bidding model resumes to:

$$\max_{(\boldsymbol{p}_S, \boldsymbol{q}_S) \in \mathcal{O}_S} \left\{ \min_{\mathbf{x}_S, \lambda} \sum_{j \in \mathcal{N}_S} (\lambda - c_{S,j}) x_{S,j} \;\middle|\; (\mathbf{x}_S, \lambda) \in \mathcal{M}\big(\boldsymbol{p}_S, \boldsymbol{q}_S, \hat{\boldsymbol{p}}_R, \hat{\boldsymbol{q}}_R\big) \right\}. \quad (4\text{-}3)$$

Table 4.1 resumes the *a priori* known price and quantity bids of rival players in this 3-player competition example. For expository purposes, the strategic player can bid at most $\bar{q}_S = 100$ MWh with $\bar{p}_S = c_S = 0$ \$/MWh. A demand of $d = 100$ MWh is assumed on the buyer counterpart.

The net revenue of the strategic player as a function of its quantity bid is depicted on Figure 4.1. Note that it is a non-convex discontinuous piece-wise function. By inspection, the optimal bid is to offer 19 MWh of energy into the auction with a total revenue of \$19000. Intuitively, this solution can be interpreted by noting that 19 MWh is the largest quantity bid that makes the deficit generator the marginal one.

Tab. 4.1: A priori known price and quantity bids of rival players.

	$\hat{p}_R$ (\$/MWh)	$\hat{q}_R$ (MWh)
Rival #1	50.00	40
Rival #2	100.00	40
Deficit	1000.00	100

Thereby, the uniform settling price is precisely the deficit price bid, i.e., $\lambda = 1000$ \$/MWh.

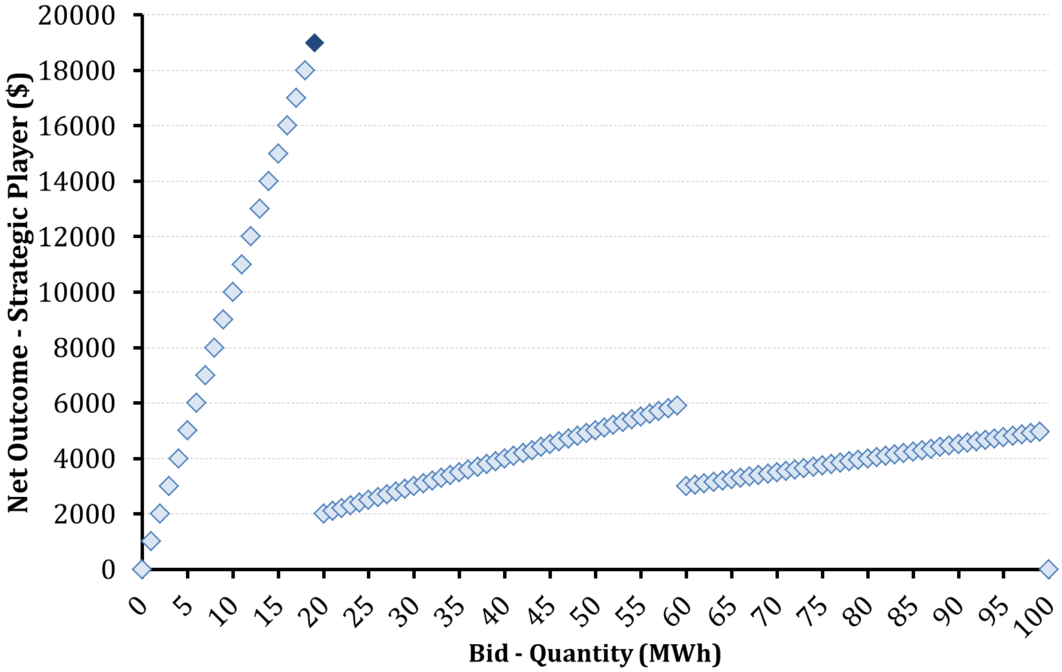

Fig. 4.1: Net revenue of the strategic player as a function of quantity bid.

To follow the algorithm's intuition it is essential to first understand the particular pattern of the net revenue function – specifically the correspondence among each piece-wise linear component and the auction solution. Fundamentally, each linear component recovers a different solution of the auction problem. For instance, the first linear piece, the steepest one, corresponds to an auction solution in which the deficit price bid defines the uniform settling price. Indeed, take the point $q_S = 10$ MWh as an example. Then, by solving the problem (2-7)–(2-11), the strategic player is fully dispatched, i.e., $x_S = 10$ MWh. Moreover, the other rivals are also fully dispatched with $x_{R,1} = x_{R,2} = 40$ MWh. However, a total demand of 10 MWh still remain unmet, thus covered by the deficit generator. In this context, the dual variable of the supply-demand equation (2-8) of the day-ahead auction problem is exactly the deficit's bidding price 1000 \$/MWh. In Fig. 4.2, an usual pictorial representation of the auction solution is presented in the format of supply-demand aggregate function [3] for the case where $q_S = 10$. A similar rationale can be applied for the other linear components. Second and third parts are associated to, respectively, rival #2 and rival #1 as marginal producers.

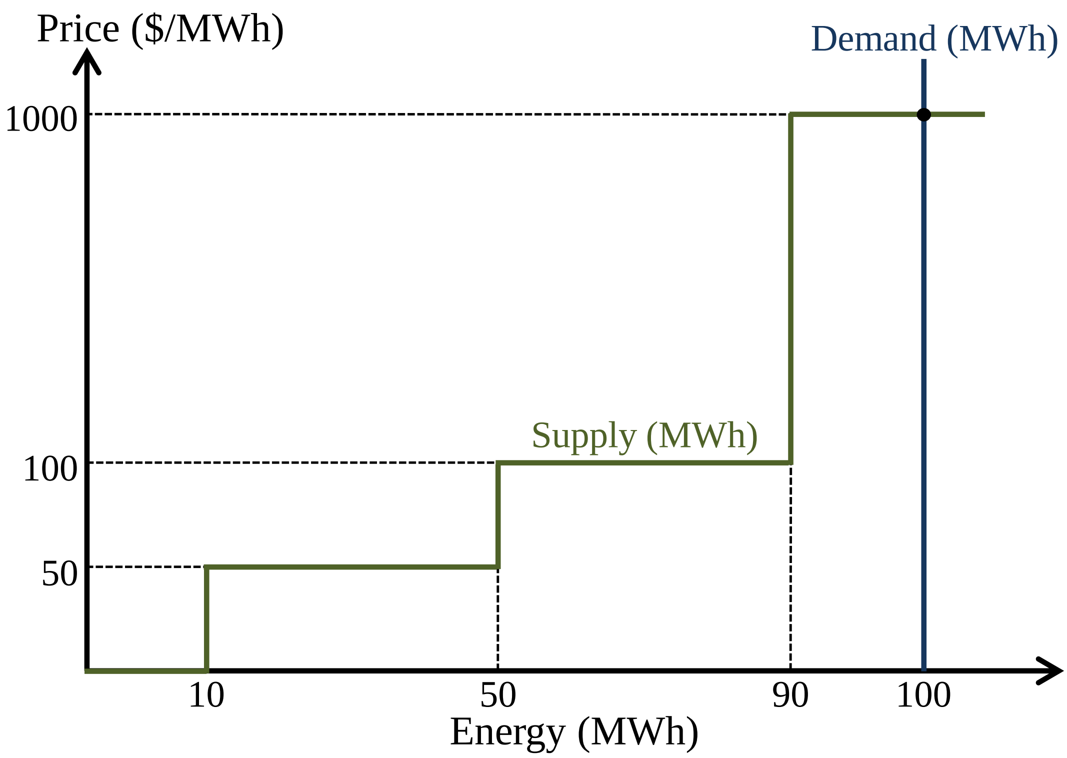

Fig. 4.2: Aggregated supply-demand function for a strategic player bid of $q_S = 10$ MWh.

Following this rationale, the algorithm iteratively "recovers" each piece of the strategic player net revenue function. More specifically, the master problem obtains a new feasible bid from a linear component that had not been accessed in previous iterations. Then, the oracle finds the (worst-case) auction equilibrium correspondent to the given strategic bidding. Such information is summarized in the binary variables $\boldsymbol{\mu}$ that translate the status of the complementarity constraints in the reformulation procedure. By introducing the set of constraints in the master problem with fixed values of $\boldsymbol{\mu}$, the correspondent linear piece is recovered and the search for a new piece resumes. The algorithm then converges when all linear components are retrieved.

For expository purposes, next we present a numerical step-by-step solution considering the competing environment of Table 4.1. Following the algorithm description, for the first iteration $k = 1$, we begin with an empty binary subset $\mathbb{B}_1 = \emptyset$ for convenience. In this case, the "recovered" strategic player net revenue function is empty (see the first picture of Fig. 4.3 – in the northwest side). Solving the master problem with $\mathbb{B}_1$ gives a strategic bid of $q_S = 0$ MWh. Then, running the oracle, we identify the deficit generator as the marginal player, defining thus the uniform settling price $\lambda = 1000$ \$/MWh. The optimal binary variables for the first iteration are shown in the second line of Table 4.2, columns 3–10. Since both rival #1 and rival #2 are fully dispatched, the respective dual variables $\overline{\lambda}_{R,1}$ and $\overline{\lambda}_{R,2}$ may assume values different from zero as indicated by $\overline{\mu}_{R,1} = \overline{\mu}_{R,1} = 0$. On the other hand, $\underline{\lambda}_{R,1} = \underline{\lambda}_{R,2} = 0$ and $\underline{\mu}_{R,1} = \underline{\mu}_{R,1} = 1$. Additionally, the deficit generator is partially dispatched. Thereby, $\overline{\lambda}_{R,\text{def}} = \underline{\lambda}_{R,\text{def}} = 0$ and $\overline{\mu}_{R,\text{def}} = \underline{\mu}_{R,\text{def}} = 1$. Finally, it is important to highlight that the solution for the strategic player is clearly degenerate. All binary combinations for $\underline{\mu}_S$ and $\overline{\mu}_S$ are optimal. However, the particular choice of $\underline{\mu}_S = \overline{\mu}_S = 0$ outcomes an interesting result with respect to the strategic player net revenue construction. Basically, this solution implies that the only feasible bid and dispatch for the strategic player in the master problem are $q_S = x_S = 0$. Thus, after one iteration, the recovered net profit function is a single point at $q_S = 0$ (see the second graph of Fig. 4.3 – in the northeast side).

Carrying on with the algorithm, we make $\mathbb{B}_2 = \emptyset \cup \{[0, 0, 0, 1, 0, 1, 1, 1]\}$. Running the master problem with $\mathbb{B}_2$, the correspondent optimal solution is $q_S = 100$ MWh. Solving the oracle with $q_S = 100$, the marginal producer is exactly the strategic player, since it offers enough energy to cover demand at the lowest price among all competitors. Thus $\lambda = 0$ \$/MWh. The optimal binary vector is presented in the third line of Table 4.2, columns 3–10. Performing a similar analysis done for the first iteration regarding the binary variables, we conclude that, along with the single point recovered in the first iteration, a new point is added to the net revenue function at $q_S = 100$ MWh (see the third graph of Fig. 4.3 – in the center-west side).

In the third iteration, the strategic bid obtained from solving the master problem is $q_S = 50$ MWh. Consequently, the marginal producer is rival #2 with $\lambda = 100$ \$/MWh. In this context, a full range of bids $q_S \in [20, 60]$ MWh is recovered from the same complementarity conditions and a linear component, as presented in the center-east side of Figure 4.3, is obtained. Moving on with the algorithm, the other two linear pieces are recovered in the following two iterations (see the two graphs in the bottom of Figure 4.3). Finally, in the sixth iteration, the algorithm converges since the net revenue function is completely constructed, resulting in an optimal bid of $q_S^* = 19$ MWh.

Tab. 4.2: Step by step solutions of the proposed algorithm applied to the day-ahead bidding problem.

Iteration	q_S	$\overline{\mu}_S$	$\underline{\mu}_S$	$\overline{\mu}_{R,1}$	$\underline{\mu}_{R,1}$	$\overline{\mu}_{R,2}$	$\underline{\mu}_{R,2}$	$\overline{\mu}_{R,\text{def}}$	$\underline{\mu}_{R,\text{def}}$	λ
1	0	0	0	0	1	0	1	1	1	1000
2	100	0	1	1	0	1	0	1	0	0
3	50	0	1	0	1	1	1	1	0	100
4	80	0	1	1	1	1	0	1	0	50
5	10	0	1	0	1	0	1	1	1	1000
6	19	0	1	0	1	0	1	1	1	1000

It is important to mention that, although we outline the algorithm under a perfect competition environment, the idea of iteratively recover the net revenue function remains when we assume uncertainty on rivals bids. Furthermore, from a computational point-of-view, note that a total of $2^8 = 256$ potential iterations may be performed by the enumeration of all complementarity constraints. However, only six of them were indeed needed. This pattern were observed in all numerical studies analyzed in this thesis and is also an intrinsic characteristic of the proposed algorithm.

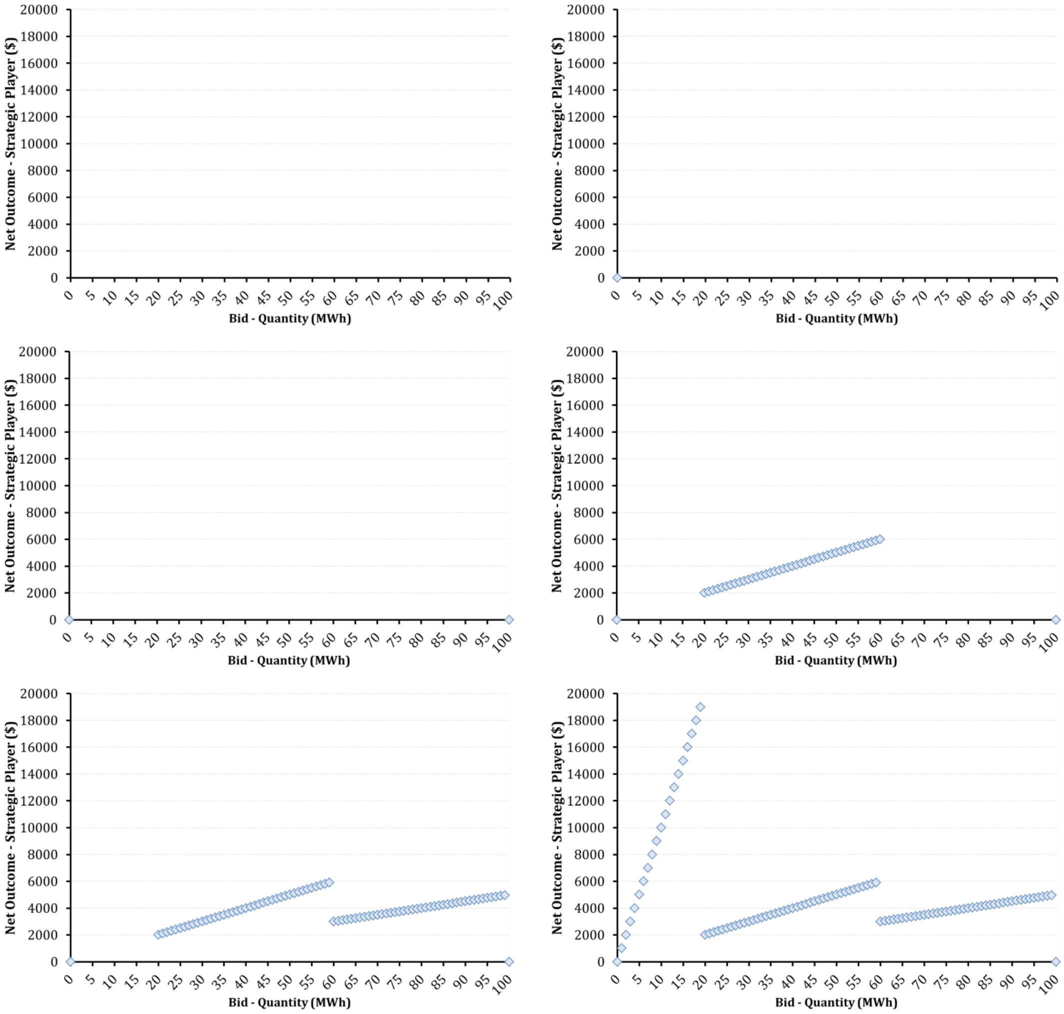

Fig. 4.3: Construction of the strategic player net revenue function.

4.2 Bidding under Imperfect Information

In this second numerical experiment, we analyze the performance of the proposed robust model under uncertainty on rival bids. We assume a strategic player owning two power units $(N_S = 2)$ whose characteristics are presented on Table 4.3.

Tab. 4.3: Characteristics (marginal costs $\boldsymbol{c}_S$ (\$/MWh); capacity $\bar{\boldsymbol{q}}_S$ (MWh); and price cap $\bar{\boldsymbol{p}}_S$ (\$/MWh)) of the units owned by the strategic player.

	Cost ($/MWh)	**Capacity** (MWh)	**Price Cap** ($/MWh)
Unit #1	10.00	50	60.00
Unit #2	30.00	20	60.00

To characterize the uncertainty on rival behavior, we start by assuming available an estimative of the rival bids – hereinafter referred to as a *nominal value*. In this section, we make use of Nash equilibrium[1] as a nominal value. However, since an accurate estimation of a joint probability distribution of market conditions that induce the equilibrium is a hard task, especially in a time-varying setting where demand, generators, fuel prices and availability, economical and climate conditions, etc., are constantly changing, deviations from the nominal equilibrium point are very likely to be observed [9, 69]. That being so, we explore two sources of deviation: (i) an imprecision over the equilibrium point evaluation and (ii) an uncertainty related to the rival players' strategic action.

Regardless of the deviation nature, the rival players' uncertainty set can be formulated as follows:

$$\mathcal{O}_R = \Big\{ (\boldsymbol{p}_R, \boldsymbol{q}_R) \in \mathbb{R}_+^{N_R} \times \mathbb{R}_+^{N_R} \;\Big|\; \tag{4-4}$$

$$\exists\, (\boldsymbol{v}^+, \boldsymbol{v}^-, \boldsymbol{w}^+, \boldsymbol{w}^-) \in \mathbb{R}^{4 \cdot N_R}; \tag{4-5}$$

$$p_{R,i} = \hat{p}_{R,i} + \Delta_{p_i}^+ v_i^+ - \Delta_{p_i}^- v_i^-, \qquad \forall\, i \in \mathcal{N}_R; \tag{4-6}$$

$$q_{R,i} = \hat{q}_{R,i} + \Delta_{q_i}^+ w_i^+ - \Delta_{q_i}^- w_i^-, \qquad \forall\, i \in \mathcal{N}_R; \tag{4-7}$$

$$\sum_{i \in \mathcal{N}_R} (v_i^+ + v_i^- + w_i^+ + w_i^-) \leq \Gamma; \tag{4-8}$$

$$0 \leq v_i^+, v_i^-, w_i^+, w_i^- \leq 1 \qquad \forall\, i \in \mathcal{N}_R; \Big\}. \tag{4-9}$$

In (4-4)–(4-9), $(\hat{\boldsymbol{p}}_R, \hat{\boldsymbol{q}}_R)$ represents nominal rivals' bid. Deviations around these values are controlled by a user-defined conservativeness (or risk-averseness) parameter Γ, that defines a joint budget for total deviation. The magnitude of positive and negative price deviation is given by $\boldsymbol{\Delta}_p^+$ and $\boldsymbol{\Delta}_p^-$, respectively. A similar structure is carried out for quantity bids. We highlight that (4-4)–(4-9) is a polyhedral set in $(\boldsymbol{p}_R, \boldsymbol{q}_R)$, thus suitable for the solution approach devised in section 3.

In this case study, we consider 14 rival players $(N_R = 14)$ and a total demand of $d = 195$ MWh on the buyer counterpart. As usual, the set of rival players is divided into *price makers* and *price takers* [70]. We assume 4 price makers and 10 price takers. For nomenclature purposes, the set of price makers will be denoted as $\mathcal{N}_R^{\text{(PM)}}$ and the set of price takers as $\mathcal{N}_R^{\text{(PT)}}$, with $\mathcal{N}_R = \mathcal{N}_R^{\text{(PM)}} \cup \mathcal{N}_R^{\text{(PT)}}$.

Typically, the influence of price maker agents on market prices is achieved by

[1] A set of bids is said to be a Nash equilibrium if no agent can improve its profits by modifying unilaterally its own bid while the remaining agents offer the equilibrium [6].

strategically defining the quantity being offered. Therefore, a widely studied format of Nash equilibrium is the so-called Nash-Cournot equilibrium [70, 71], i.e., a Nash equilibrium in which the set of strategies is composed only by quantity bids and the price bids are fixed on marginal costs (Cournot competition). Formally, the set of strategies (Π) to characterize the Nash-Cournot equilibrium considered in this case study is presented in (4-10)–(4-13).

$$\Pi = \Bigg\{ \left(\boldsymbol{q}_S^{\text{(eq)}}, \boldsymbol{q}_R^{\text{(eq)}}\right) \in \mathbb{Z}^{N_S} \times \mathbb{Z}^{N_R} \; \Bigg| \tag{4-10}$$

$$0 \leq q_{S,j}^{\text{(eq)}} \leq \bar{q}_{S,j}, \qquad \forall\, j \in \mathcal{N}_S; \tag{4-11}$$

$$\underline{q}_{R,i} \leq q_{R,i}^{\text{(eq)}} \leq \bar{q}_{R,i}, \qquad \forall\, i \in \mathcal{N}_R^{\text{(PM)}}; \tag{4-12}$$

$$q_{R,i}^{\text{(eq)}} = \bar{q}_{R,i}, \qquad \forall\, i \in \mathcal{N}_R^{\text{(PT)}}; \Bigg\}, \tag{4-13}$$

with $\underline{\boldsymbol{q}}_R$ denoting a minimum required generation. We highlight that, for the set of price taker agents, since the price bids are fixed on the respective marginal cost and the quantity bid is also fixed on the capacity, the resulting Nash equilibrium is an optimal bid. A similar method described in [10] is used to evaluate the equilibrium. Table 4.4 presents the characteristics of each rival player considered in this numerical study.

Tab. 4.4: Characteristics of each rival player: minimum generation $\underline{\boldsymbol{q}}_R$ (MWh); capacity $\bar{\boldsymbol{q}}_R$ (MWh); and marginal cost $\boldsymbol{c}_R$ (\$/MWh).

Price Maker	$\underline{\boldsymbol{q}}_R$ (MWh)	$\bar{\boldsymbol{q}}_R$ (MWh)	$\boldsymbol{c}_R$ ($/MWh)	**Price Taker**	$\underline{\boldsymbol{q}}_R$ (MWh)	$\bar{\boldsymbol{q}}_R$ (MWh)	$\boldsymbol{c}_R$ ($/MWh)
#1	10	40	60.00	**#1**	0	2	26.00
#2	20	60	40.00	**#2**	0	3	48.00
#3	5	40	45.00	**#3**	0	2	28.00
#4	10	50	15.00	**#4**	0	3	35.00
				#5	0	2	39.00
				#6	0	2	32.00
				#7	0	2	49.00
				#8	0	3	54.00
				#9	0	3	29.00
				#10	0	3	28.00

4.2.1 Imprecision on Nominal Bids Estimation

This first case study is inspired by the main discussion in [72]. The authors justify the value of a robust model by indicating a high infeasibility level if small deviations on data entry occur. In this section, we follow the same idea and analyze the impact, on the strategic player's net revenue, of an imprecision on the Nash equilibrium evaluation. Such imprecision may arise from many sources, e.g., uncertainty on the assessment of rivals' marginal cost due to fuel market prices fluctuations and opportunity costs due to technological constraints.

Formally, let $\delta > 0$ quantify the level of imprecision on the Nash equilibrium evaluation. Then, we can define the rival players' uncertainty set parameters as follows: (i) $\forall\ i \in \mathcal{N}_R$, $\big(\hat{p}_{R,i}, \Delta^+_{p_i}, \Delta^-_{p_i}\big) = \big(c_{R,i}, \delta c_{R,i}, \delta c_{R,i}\big)$; (ii) $\forall\ i \in \mathcal{N}_R^{(\text{PM})}$, $\big(\hat{q}_{R,i}, \Delta^+_{q_i}, \Delta^-_{q_i}\big) = \big(q^{(\text{eq})}_{R,i}, \delta q^{(\text{eq})}_{R,i}, \delta q^{(\text{eq})}_{R,i}\big)$ and $\forall\ i \in \mathcal{N}_R^{(\text{PT})}$, $\big(\hat{q}_{R,i}, \Delta^+_{q_i}, \Delta^-_{q_i}\big) = \big(q^{(\text{eq})}_{R,i}, 0, 0\big)$.

Figure 4.4 presents, for different levels of imprecision, the worst-case strategic player net revenue as a function of Γ in percentage of the net revenue under the equilibrium solution, i.e., for $\Gamma = 0$. Note that as expected, the strategic player net revenue is a non-increasing function of δ and Γ. We highlight however that a small imprecision on the equilibrium evaluation (e.g. $\delta = 0.01$) can create a significant impact on the strategic player net revenue, even for modest values of conservativeness level Γ. For instance, a net revenue reduction of approximately 5% is observed for $\Gamma = 3$. Furthermore, under an imprecision of $\delta = 0.10$, a non-negligible net revenue reduction of more than 20% can occur.

To better evaluate the proposed model benefits, a (sample) distribution of the strategic player net revenue is constructed assuming that the rival bids follow a Normal distribution around the nominal (equilibrium) bid with standard deviation equal to the level of imprecision. More specifically, we assume that $\tilde{p}_{R,i} \sim N\big(\hat{p}_{R,i}, (\Delta^+_{p_i})^2\big)$ and $\tilde{q}_{R,i} \sim N\big(\hat{q}_{R,i}, (\Delta^+_{q_i})^2\big)$, $\forall\ i \in \mathcal{N}_R$. For simplicity, the random vector $(\tilde{\boldsymbol{p}}_R, \tilde{\boldsymbol{q}}_R)$ is also considered pairwise independent. Under this modeling framework, a set of 1000 market conditions were simulated and the strategic player net revenue distribution is evaluated for the bids associated to various conservativeness levels Γ.

Figure 4.5 depicts an efficient frontier varying the conservativeness level Γ from 0 to 4 on a 0.25 step basis. The vertical axis represents the expected net revenue (sample average) and the horizontal axis a measure of risk evaluated as the distance between the expected net revenue and the Conditional Value-at-Risk [73]

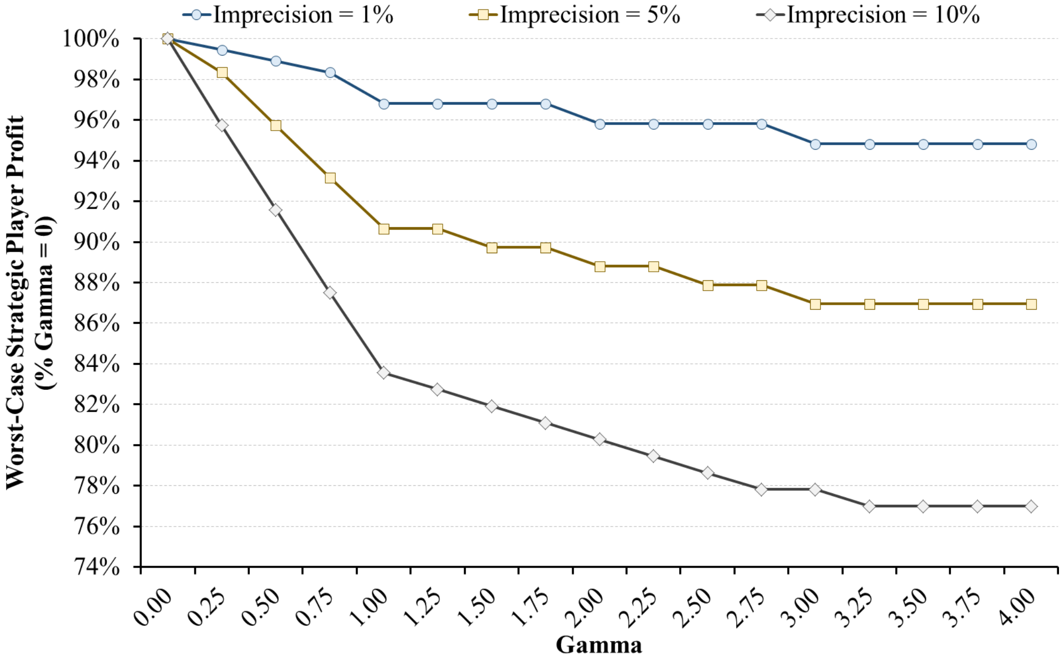

Fig. 4.4: Worst-case strategic player net revenue as a function of Γ *for different levels of imprecision* $(\delta \in \{0.01, 0.05, 0.10\})$ *in percentage of the net revenue under the equilibrium solution, i.e. for* $\Gamma = 0$.

at a confidence level of 95% $\left(\text{CVaR}_{95\%}\right)$. Roughly speaking, the $\text{CVaR}_{95\%}$ can be understood as the average of the worst 5% sampled scenarios. We refer to [74] and [75] for a further discussion on this measure of risk.

We first highlight that the equilibrium bid $(\Gamma = 0)$ is a dominated strategy under $\delta \in \{0.05, 0.10\}$. This result can be interpreted by observing that the Nash equilibrium net revenue is a highly nonlinear function of the uncertainty factors and that using $(\Gamma = 0)$ is equivalent to a deterministic solution for the biding problem using average, or nominal, values for the rivals' offer. Hence, this result suggests a positive value for the stochastic solution [44, 45] made under uncertainty. Therefore, this methodology fails to capture optimal risk-neutral strategy. This seemingly drawback notwithstanding, it is worth stressing that we are assuming that in practical situations the distribution used to assess out-of-sample results, such as those found in Figure 4.5, are not available for the strategic player. Then, the robust approach provides a interesting way to capture the imprecision in the optimal bidding strategy.

Furthermore, it is interesting to note the significant risk reduction, 77.9%, observed when raising the conservativeness level from zero to one when a 1%-

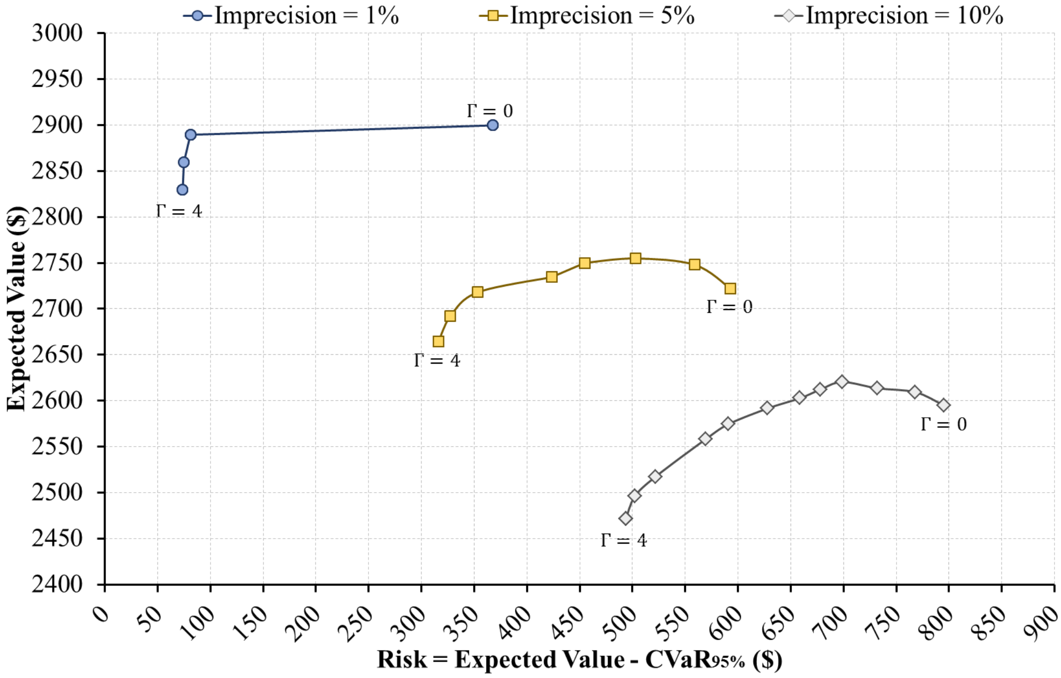

Fig. 4.5: Efficient frontier for different levels of imprecision ($\delta \in \{0.01, 0.05, 0.10\}$) *varying the budget* Γ *from 0 to 4 on a 0.25 step basis.*

standard-deviation imprecision affects the rivals' bid. This provides a important insight: even in very low imprecision environments and stable market situations, the optimal bid should not be made through a deterministic approach ($\Gamma = 0$). To better illustrate this effect, Figure 4.6 depicts the histogram for budget levels of $\Gamma \in \{0, 1\}$. Note the existence of a mass of measure (approximately) 5% under the equilibrium solution ($\Gamma = 0$), which substantially increases the risk observed in the deterministic approach. On the other hand, by introducing a level of robustness in the bidding problem, although the whole distribution is slightly shifted to the left, thus decreasing the expect net revenue, the risk is significantly reduced since the distance between the expected value and the expected value in the worst-case market conditions (CVaR) is much lower. In this situation, if the strategic player has a fixed cost of, say, 2700$, the deterministic strategy produces a 5% probability of observing a negative net revenue.

4.2.2 Uncertainty on Rival Bids

In this case study, we assume that rivals' market behavior is uncertain and rivals' quantity bid may deviate from the nominal Nash equilibrium bid towards a

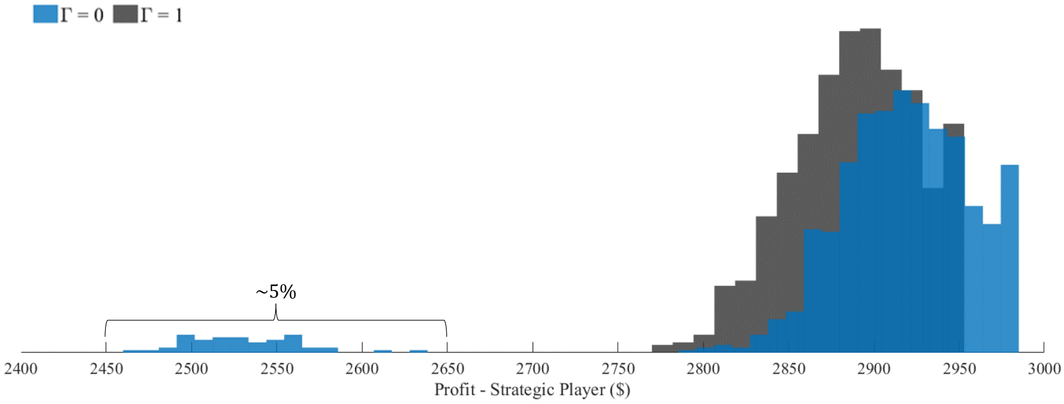

Fig. 4.6: Histogram of the sample net revenue for an level of imprecision $\delta = 0.01$ *under* $\Gamma \in \{0, 1\}$.

maximum capacity quantity bid (price taker optimal bid). An imprecision of 30% ($\delta = 0.3$) on rivals' marginal cost evaluation is also considered. In mathematical terms, let $\zeta > 0$ define a measure of uncertainty on the quantity bids. In this framework, we define the uncertainty set parameters for the rivals' bid as follows: (i) $\forall\, i \in \mathcal{N}_R$, $\left(\hat{p}_{R,i}, \Delta^+_{p_i}, \Delta^-_{p_i}\right) = \left(c_{R,i}, \delta c_{R,i}, \delta c_{R,i}\right)$; (ii) $\forall\, i \in \mathcal{N}_R^{(\text{PM})}$, $\left(\hat{q}_{R,i}, \Delta^+_{q_i}, \Delta^-_{q_i}\right) = \left(q^{(\text{eq})}_{R,i}, \zeta(\bar{q}_{R,i} - q^{(\text{eq})}_{R,i}), 0\right)$ and $\forall\, i \in \mathcal{N}_R^{(\text{PT})}$, $\left(\hat{q}_{R,i}, \Delta^+_{q_i}, \Delta^-_{q_i}\right) = \left(q^{(\text{eq})}_{R,i}, 0, 0\right)$.

Regarding the optimal price bids, in all experiments performed in this case, the solutions found for the two strategic player generators are equal to their marginal cost. This is because the only way to manipulate the spot price through price bids is being the marginal player, which is a risky situation since small deviations from nominal bids of the rivals may prevent a generator from clearing in the auction. It is worth emphasizing that this is a salient virtue of the proposed model, which by maximizing the worst-case metric mitigates the strategic player risk of failing to clear the auction.

Figure 4.7 shows the total quantity bid for the strategic player as a function of Γ and various values of ζ. For small values of Γ, the optimal quantity bid follows the same pattern for all levels of uncertainty. In order to keep the marginal price at a high level, the strategic player reduces the total amount of energy offered in the auction. However, we observe a pattern break, at $\Gamma = 3$ and 4, for the highest uncertainty-level cases, namely, $\zeta \in \{75\%, 100\%\}$. For such cases, when reaching a given threshold of conservativeness level, the optimal strategy is to increase the energy offered into the auction, bidding close to full capacity. This behavior results from the fact that, by raising the value of Γ, the strategic player is considering higher levels of uncertainty, where more rivals may deviate from their nominal bid. As a consequence, for a given uncertainty level, the worst case rivals' bid shorten the strategic player ability to manipulate the auction result towards its own good. In such case, the best strategy against the worst-case rivals' bid approaches the price taker offer, full maximum capacity at marginal cost.

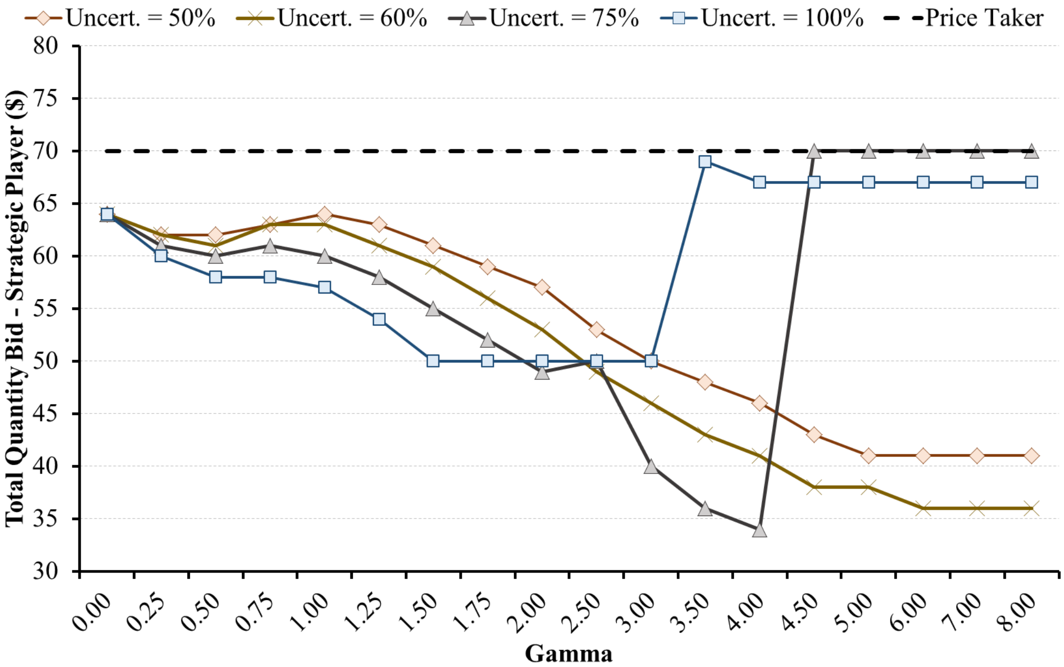

Fig. 4.7: Total quantity bid for different values of $\zeta \in \{50\%, 60\%, 75\%, 100\%\}$ *as a function of* Γ.

Next, we perform a similar out-of-sample simulation experiment carried out in the previous section. In this case, price-maker rivals' quantity bid follows a uniform distribution, taking values between the equilibrium and full capacity amounts, while prices follow a normal distribution centered at marginal costs as follows: $\tilde{p}_{R,i} \sim N\big(\hat{p}_{R,i}, (\Delta^{+}_{p_i})^2\big),\ \forall\, i \in \mathcal{N}_R$; $\tilde{q}_{R,i} \sim \mathcal{U}\big(q^{(\text{eq})}_{R,i}, \bar{q}_{R,i}\big),\ \forall\, i \in \mathcal{N}^{(\text{PM})}_R$; and $\tilde{q}_{R,i} = q^{(\text{eq})}_{R,i},\ \forall\, i \in \mathcal{N}^{(\text{PT})}_R$. A sample of 1000 bids for the rivals participating in the auction is simulated through a Monte Carlo procedure and the strategic player net revenue is evaluated for different bidding strategies parameterized in the conservativeness level Γ. In this simulation study, we analyze the solutions for $\zeta \in \{50\%, 100\%\}$.

Figure 4.8 depicts the expected net revenue (sample average) and $\text{CVaR}_{95\%}$ of the strategic player revenue for different values of Γ and Figure 4.9 presents the same metrics in percentage of $\Gamma = 0$. For $\zeta = 100\%$, the $\text{CVaR}_{95\%}$ shows an increasing pattern when the conservativeness level, Γ, is increased up to $\Gamma = 3.0$. This pattern is accompanied with a reduction on the expected value. Nevertheless, note that the CVaR$_{95\%}$ increase surpasses the expected value loss in percentage values (with respect to the expectation and CVaR$_{95\%}$ values, respectively, for $\Gamma = 0$) as shown in Figure 4.9. Above the threshold value $\Gamma = 3.0$, the CVaR$_{95\%}$ decreases and meet the value related to the solutions of the price-taker kind observed in Figure 4.7.

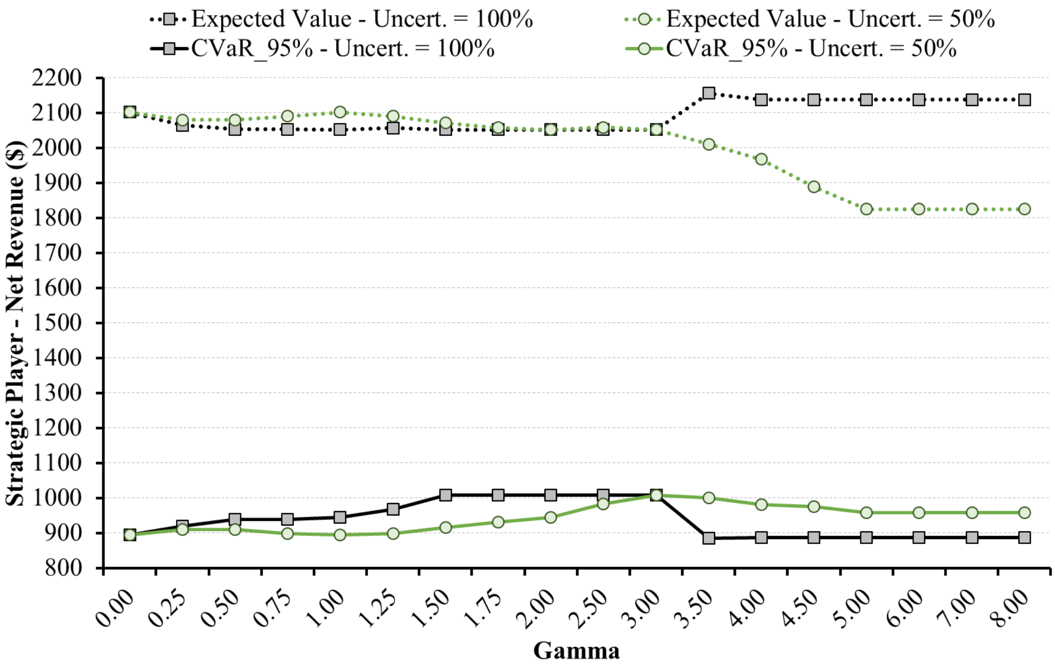

Fig. 4.8: Expected net revenue (sample average) and CVaR$_{95\%}$ of the strategic player net revenue per Γ for $\zeta \in \{50\%, 100\%\}$.

On the other hand, because the optimal bid for $\zeta = 50\%$ does not exhibits the same pattern break as observed for $\zeta = 100\%$, the expected value decrease is not interrupted for values of $\Gamma \geq 3.0$. Furthermore, the CVaR$_{95\%}$ metric also starts to decrease.

To conclude this numerical study, we extend this simulation experiment to include different assumptions over the rival's bid distribution. We now assume that the price maker quantity bids follow asymmetric distributions within the feasible bid interval $[q^{(\text{eq})}_{R,i}, \bar{q}_{R,i}]_{i \in \mathcal{N}^{(\text{PM})}_R}$. A positively and negatively skewed Beta(2,5) Distribution are assumed (see Figure 4.10 for a pictorial representation). Under both assumptions, a set of 1000 bids for the rivals is simulated through a Monte Carlo procedure and the strategic player net revenue is evaluated for different bidding

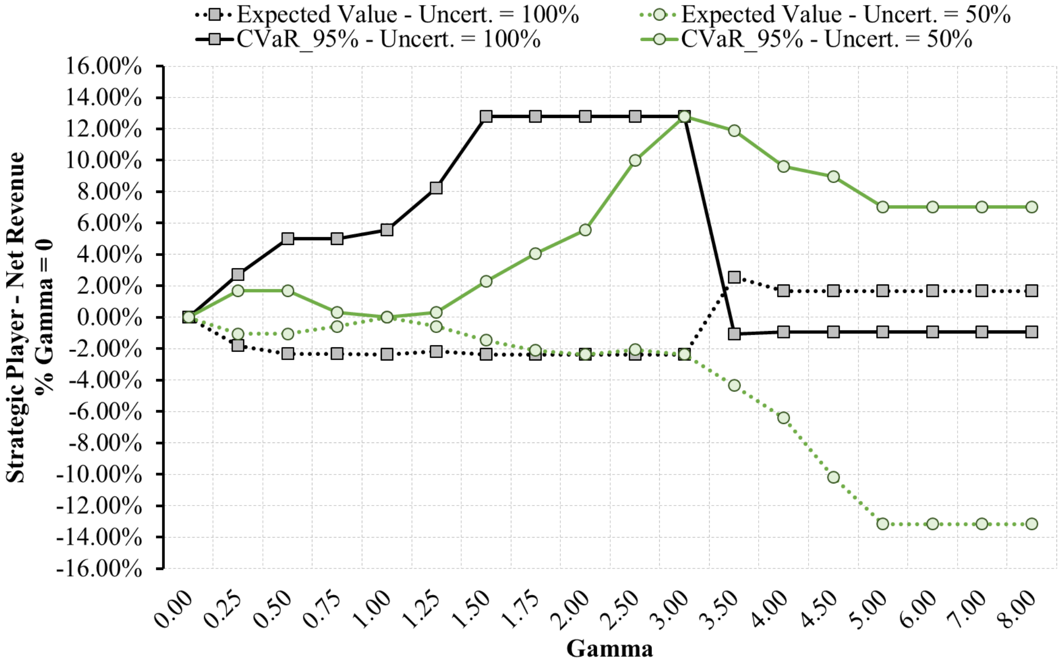

Fig. 4.9: Expected net revenue (sample average) and CVaR$_{95\%}$ of the strategic player net revenue per Γ for $\zeta \in \{50\%, 100\%\}$ in percentage of the respective values for $\Gamma = 0$ (equilibrium bid).

strategies parameterized in the conservativeness level Γ and $\zeta \in \{50\%, 100\%\}$.

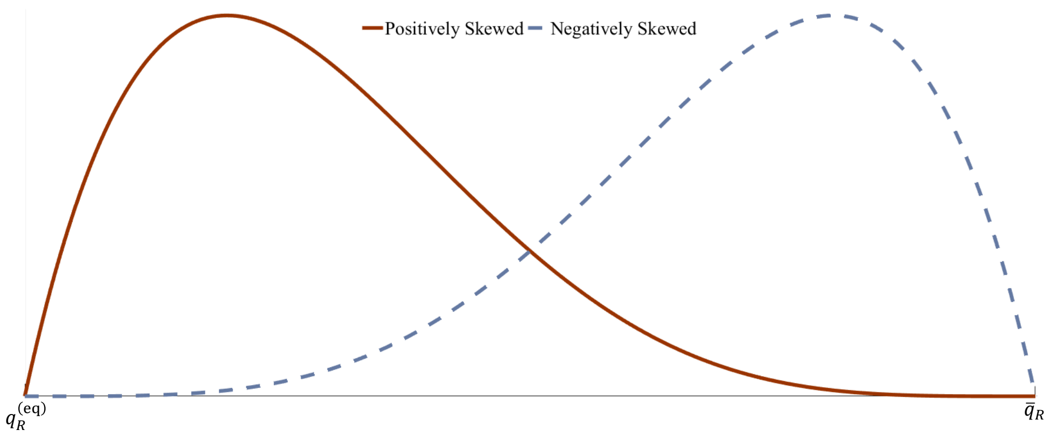

Fig. 4.10: Positive and negative skewed Beta(2,5) distribution.

In Figure 4.11, the expected net revenue (sample average) and CVaR$_{95\%}$ of the strategic player revenue for different values of Γ is presented assuming that all price-maker quantity bids follow the positively skewed distribution (red continuous curve in Figure 4.10). Note that under this assumption, the significant portion of probability mass is close to the Nash equilibrium. Nevertheless, we observe for $\Gamma \leq 3.0$ an increase in the expected value and CVaR$_{95\%}$ metrics, with respect to the equilibrium bid ($\Gamma = 0$). This result is in line with Section 4.2.1 since this market structure can be viewed as an imprecision over the equilibrium assessment. On the other hand, since for $\Gamma > 3.0$ the uncertainty observed by the strategic player (summarized in the conservativeness level Γ) induces a pattern break towards a price taker solution, a decrease in these metrics is observed. For these cases, the magnitude of the real market deviation is much lower than the one observed by the strategic player. As a consequence, the bid strategy for $\Gamma > 3.0$ is too conservative for the real market context, impacting negatively the metrics.

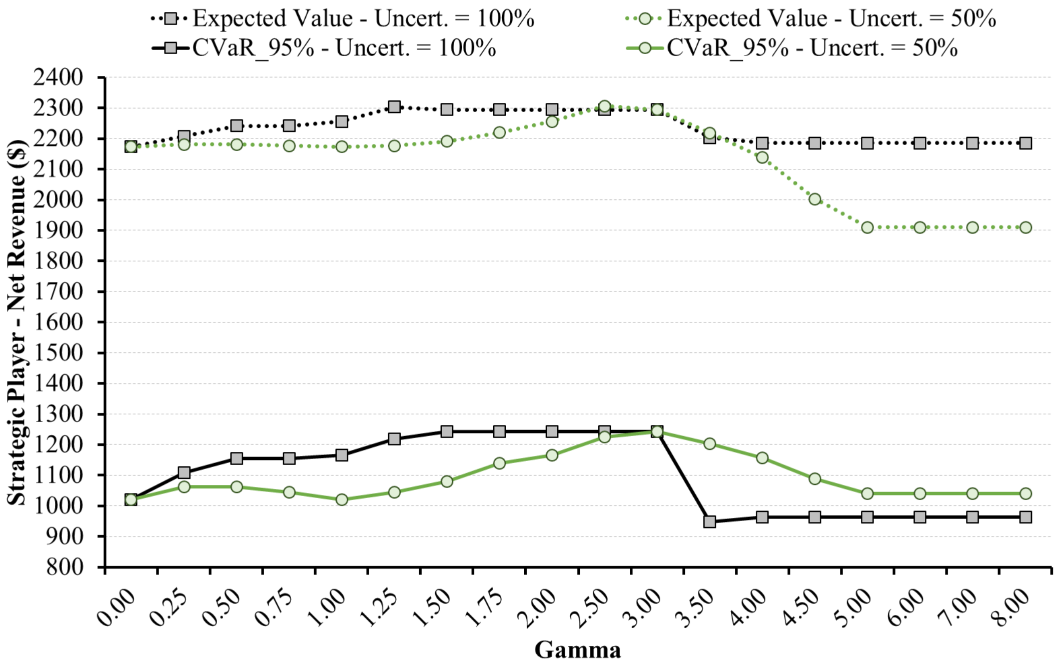

Fig. 4.11: Expected net revenue (sample average) and CVaR$_{95\%}$ per Γ for $\zeta \in \{50\%, 100\%\}$ assuming a positively skewed Beta(2,5) distribution for the quantity of all price maker agents.

An interesting result is observed for the opposite case. Figure 4.12 presents the expected net revenue (sample average) and CVaR$_{95\%}$ of the strategic player revenue for different values of Γ, assuming that all price-maker quantity bids follow the negatively skewed distribution (blue dotted curve in Figure 4.10). In this context, we represent a market structure in which all rival bids are distant from the Nash equilibrium, but close to price-taker offers, inducing thus a structural equilibrium in the market [34]. We observe in this case that the CVaR$_{95\%}$ metric is constant for all values of Γ and the expected value pattern is inverted compared to the previous case. We can interpret this result by noting that the structural equilibrium (all players acting as price takers) is more stable than the Nash equilibrium [34]. Therefore, the robust policies that act strategically $(\Gamma \leq 3.0)$ — deviating from the structural equilibrium, does not perform well due to this stability [34]. Consequently, the optimal bids with $\Gamma > 3.0$ perform much better. We can also interpret this result by noting that the magnitude of deviation from Nash equilibrium in this second market context is very high. Therefore, higher levels of conservativeness (Γ) tend to perform better than lower levels, as is observed in Figure 4.12.

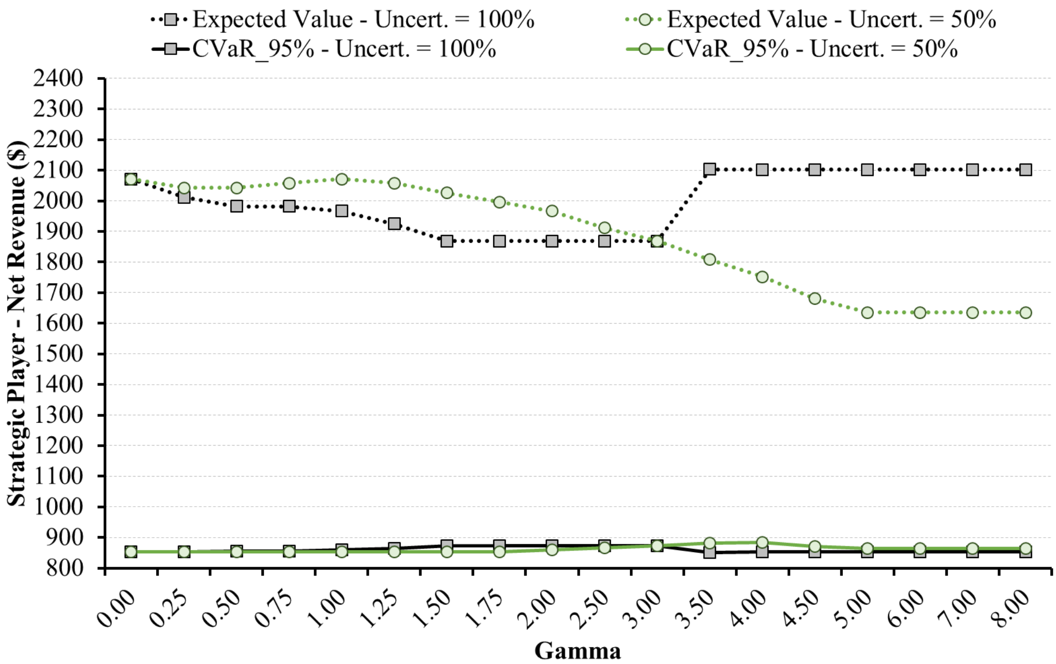

Fig. 4.12: Expected net revenue (sample average) and CVaR$_{95\%}$ per Γ for $\zeta \in \{50\%, 100\%\}$ assuming a negatively skewed Beta(2,5) distribution for the quantity of all price maker agents.

5
Conclusions

In this thesis, we propose an alternative methodology for devising profit-maximizing strategic bids in markets endowed with a sealed-bid uniform-price auction of multiple divisible products. Such a model is constructed to assess the optimal bidding policies for a subgroup of seller competitors, referred to as the strategic player, under uncertainty in the bidding behavior of its rival competitors. The standard modeling approach for this optimal bidding problem leverages on stochastic programming techniques while assuming that a probability distribution that represents the uncertain rival behavior is available. On recognizing that the characterization of such probability distributions is difficult, we propose a robust optimization model, under a polyhedral uncertainty set, in which decision makers are allowed to control their conservativeness level without the need for specifying the full probability distribution for the rivals' bidding strategy.

The proposed trilevel optimization problem can be regarded as a particular instance of the TSR-MPEC for which no algorithm is available. We present a single-level equivalent formulation with an exponential number of constraints suitable for decomposition techniques. A solution methodology based on the CCG algorithm is proposed while allowing for the use of commercial MILP solvers to obtain near-global optimal solutions for the robust-bidding problem. In contrast to previously reported works on two-stage robust optimization, our proposed model is not suitable for standard CCG algorithms, which iteratively include identified worst-case violated scenarios of the uncertainty factors in a master problem. Instead, in the proposed solution methodology, CCG is applied to identify a small set of optimality conditions for the third-level problem that can represent the auction equilibrium constraints at the optimum solution of the master (bidding) problem.

We illustrate the features and applicability of the proposed methodology in a numerical study based on short-term electricity markets. We make use of a particular polyhedral uncertainty set structure in which a reference bid, regarded as a Nash–Cournot equilibrium, is assumed to be known, and possible deviations around this reference may take place. Two sources of deviations were explored: (i) imprecision in the equilibrium estimation of the rivals' bid, and (ii) uncertainty in the rivals' behavior. In the former, we show that the impact on the net revenue of a deterministic (unprotected) Nash–Cournot equilibrium bidding strategy is non-negligible even if the imprecision in the equilibrium evaluation is considered as small as 1%. We also show that, in the former case, the use of a robust bidding strategy provides a significant risk reduction in out-of-sample tests. In the latter case, we find that when reaching a given threshold of uncertainty level considered in the robust-bidding strategy, the worst-case rivals' bid shortens the strategic player's ability to manipulate the auction result towards its own good. Therefore, the strategic player's best strategy against high levels of uncertainty turns out to be close to a price taker bid, i.e., the maximum capacity at marginal cost.

Ongoing research related to this thesis involves the development of a solution algorithm for different uncertainty set topologies, such as cones and ellipsoids. We also consider the extension of the robust formulation to account for distributionally robustness, i.e., the uncertainty on rivals' bid is modeled by a set of probability distributions with given properties. From an application point-of-view, the extension of the day-ahead problem to account for transmission constraints and its impact on the robust solution is a possible line of research.

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APPENDIX

A
Robust Bidding Model in Day-Ahead Markets

For completeness, in this appendix, we carefully adapt the robust strategic bidding model for the day-ahead market (4-1) under the uncertainty set (4-4)–(4-9) and strategic player feasible bidding set (4-2). Without loss of generality, we discard constraint (2-11) and directly replace x_D by d in (2-8). Applying the set of reformulation procedures described in Section 3.1, the robust bidding model proposed in this thesis resumes to the following two-level mixed-integer optimization problem.

$$\max_{\boldsymbol{p}_S,\boldsymbol{q}_S,\boldsymbol{\psi},\boldsymbol{\beta}} \left\{ \min_{\substack{\boldsymbol{p}_R,\boldsymbol{q}_R,\mathbf{x}_S,\mathbf{x}_R,\\ \overline{\boldsymbol{\mu}}_S,\underline{\boldsymbol{\mu}}_S,\overline{\boldsymbol{\mu}}_R,\underline{\boldsymbol{\mu}}_R,\\ \boldsymbol{v}^+,\boldsymbol{v}^-,\boldsymbol{w}^+,\boldsymbol{w}^-,\\ \lambda,\overline{\boldsymbol{\lambda}}_S,\underline{\boldsymbol{\lambda}}_S,\overline{\boldsymbol{\lambda}}_R,\underline{\boldsymbol{\lambda}}_R}} \sum_{j\in\mathcal{N}_S} p_{S,j}x_{S,j} + \overline{\lambda}_{S,j}q_{S,j} - c_{S,j}x_{S,j} \right. \tag{A-1}$$

subject to:

$$p_{R,i} = \hat{p}_{R,i} + \Delta^+_{p_i}v^+_i - \Delta^-_{p_i}v^-_i, \qquad : \kappa_{p_i} \quad \forall\, i\in\mathcal{N}_R; \tag{A-2}$$

$$q_{R,i} = \hat{q}_{R,i} + \Delta^+_{q_i}w^+_i - \Delta^-_{q_i}w^-_i, \qquad : \kappa_{q_i} \quad \forall\, i\in\mathcal{N}_R; \tag{A-3}$$

$$\sum_{i\in\mathcal{N}_R}(v^+_i + v^-_i + w^+_i + w^-_i) \le \Gamma; \qquad : \rho \tag{A-4}$$

$$0 \le v^+_i \le 1, \qquad : \sigma_{v^+_i} \quad \forall\, i\in\mathcal{N}_R; \tag{A-5}$$

$$0 \le v^-_i \le 1, \qquad : \sigma_{v^-_i} \quad \forall\, i\in\mathcal{N}_R; \tag{A-6}$$

$$0 \le w^+_i \le 1, \qquad : \sigma_{w^+_i} \quad \forall\, i\in\mathcal{N}_R; \tag{A-7}$$

$$0 \le w^-_i \le 1, \qquad : \sigma_{w^-_i} \quad \forall\, i\in\mathcal{N}_R; \tag{A-8}$$

$$p_{S,j} - \lambda + \overline{\lambda}_{S,j} - \underline{\lambda}_{S,j} = 0, \qquad : \alpha_{S,j} \quad \forall\, j\in\mathcal{N}_S; \tag{A-9}$$

$$p_{R,i} - \lambda + \overline{\lambda}_{R,i} - \underline{\lambda}_{R,i} = 0, \qquad : \alpha_{R,i} \quad \forall\, i\in\mathcal{N}_R; \tag{A-10}$$

$$\sum_{j\in\mathcal{N}_S} x_{S,j} + \sum_{i\in\mathcal{N}_R} x_{R,i} = d; \qquad : \gamma \tag{A-11}$$

$$0 \le x_{S,j} \le q_{S,j}, \qquad : \underline{\theta}_{S,j} \quad \forall\, j\in\mathcal{N}_S; \tag{A-12}$$

$$0 \le x_{R,i} \le q_{R,i}, \qquad : \underline{\theta}_{R,i} \quad \forall\, i\in\mathcal{N}_R; \tag{A-13}$$

$$q_{S,j} - x_{S,j} \le \overline{U}_S\overline{\mu}_{S,j}, \qquad : \overline{\theta}_{S,j} \quad \forall\, j\in\mathcal{N}_S; \tag{A-14}$$

$$\overline{\lambda}_{S,j} \le \overline{U}_S(1 - \overline{\mu}_{S,j}), \qquad : \psi_{S,j} \quad \forall\, j\in\mathcal{N}_S; \tag{A-15}$$

$$x_{S,j} \le \underline{U}_S\underline{\mu}_{S,j}, \qquad : \eta_{S,j} \quad \forall\, j\in\mathcal{N}_S; \tag{A-16}$$

$$\underline{\lambda}_{S,j} \leq \underline{U}_S(1 - \underline{\mu}_{S,j}), \qquad : \delta_{S,j} \quad \forall\, j \in \mathcal{N}_S; \qquad \text{(A-17)}$$

$$q_{R,i} - x_{R,i} \leq \overline{U}_R \overline{\mu}_{R,i}, \qquad : \overline{\theta}_{R,i} \quad \forall\, i \in \mathcal{N}_R; \qquad \text{(A-18)}$$

$$\overline{\lambda}_{R,i} \leq \overline{U}_R(1 - \overline{\mu}_{R,i}), \qquad : \psi_{R,i} \quad \forall\, i \in \mathcal{N}_R; \qquad \text{(A-19)}$$

$$x_{R,i} \leq \underline{U}_R \underline{\mu}_{R,i}, \qquad : \eta_{R,i} \quad \forall\, i \in \mathcal{N}_R; \qquad \text{(A-20)}$$

$$\underline{\lambda}_{R,i} \leq \underline{U}_R(1 - \underline{\mu}_{R,i}), \qquad : \delta_{R,i} \quad \forall\, i \in \mathcal{N}_R; \qquad \text{(A-21)}$$

$$\overline{\mu}_{S,j}, \underline{\mu}_{S,j}, \overline{\mu}_{R,i}, \underline{\mu}_{R,i} \in \{0,1\}, \qquad \forall\, j \in \mathcal{N}_S,\ i \in \mathcal{N}_R; \qquad \text{(A-22)}$$

$$\left.\overline{\lambda}_{S,j}, \underline{\lambda}_{S,j}, \overline{\lambda}_{R,i}, \underline{\lambda}_{R,i} \geq 0, \qquad \forall\, j \in \mathcal{N}_S,\ i \in \mathcal{N}_R; \right\} \qquad \text{(A-23)}$$

subject to:

$$p_{S,j} = \sum_{k=1}^{\lfloor \log_2(\bar{p}_{S,j}) \rfloor + 1} 2^{k-1} \beta_{k,j}, \qquad \forall\, j \in \mathcal{N}_S; \qquad \text{(A-24)}$$

$$p_{S,j} \leq \bar{p}_{S,j}, \qquad \forall\, j \in \mathcal{N}_S; \qquad \text{(A-25)}$$

$$q_{S,j} = \sum_{k=1}^{\lfloor \log_2(\bar{q}_{S,j}) \rfloor + 1} 2^{k-1} \chi_{k,j}, \qquad \forall\, j \in \mathcal{N}_S; \qquad \text{(A-26)}$$

$$q_{S,j} \leq \bar{q}_{S,j}, \qquad \forall\, j \in \mathcal{N}_S; \qquad \text{(A-27)}$$

$$\beta_{k,j} \in \{0,1\}, \qquad \forall\, k \in \left\{1, \ldots, \lfloor \log_2(\bar{p}_{S,j}) \rfloor + 1\right\},\ j \in \mathcal{N}_S; \qquad \text{(A-28)}$$

$$\chi_{k,j} \in \{0,1\} \qquad \forall\, k \in \left\{1, \ldots, \lfloor \log_2(\bar{q}_{S,j}) \rfloor + 1\right\},\ j \in \mathcal{N}_S. \qquad \text{(A-29)}$$

The objective function (A-1) represents the strategic player net outcome reformulated following equation (3-23). The set of constraints (A-2)–(A-8) depict the feasible set of rival bids, following (4-4)–(4-9). The KKT system of the day-ahead market problem (2-7)–(2-11) is presented in equations (A-9)–(A-23). More specifically, constraints (A-9)–(A-10) stands for stationarity conditions; equations (A-11)–(A-13) are primal constraints; equations (A-14)–(A-22) represents complementarity constraints reformulated using the Fortuny-Amat linearization procedure described in (3-25); and constraint (A-23) ensures dual feasibility. Finally, the set of constraints (A-24)–(A-29) represent the integer box-constrained set of feasible bids for the strategic player. Equations (A-24) and (A-26), along with the binary vectors $\boldsymbol{\beta}$ and $\boldsymbol{\chi}$, models the integrability assumption of price and quantity bidding, respectively[1]. Next, in Section A.1, we demonstrate how to construct the Master

[1] We recognize that this binary expansion representation is imprecise. If $\bar{p}_{S,j} = 0$ or $\bar{q}_{S,j} = 0$ for

Problem from (A-1)–(A-29).

A.1 Master Problem: Day-Ahead Market

Let $\boldsymbol{u} = [-\overline{\mu}_{S,j}-, -\underline{\mu}_{S,j}-, -\overline{\mu}_{R,i}-, -\underline{\mu}_{R,i}-]^\top$ denote the binary vector of the inner-problem (A-1)–(A-23). Thus, for a given feasible strategic player bid $(\boldsymbol{p}_S, \boldsymbol{q}_S) \in \mathcal{O}_S$, the dual problem of (A-1)–(A-23) is

$$\max_{\substack{\boldsymbol{\alpha}_S^{(\boldsymbol{u})}, \boldsymbol{\alpha}_R^{(\boldsymbol{u})}, \gamma^{(\boldsymbol{u})}, \overline{\boldsymbol{\theta}}_S^{(\boldsymbol{u})}, \\ \underline{\boldsymbol{\theta}}_S^{(\boldsymbol{u})}, \boldsymbol{\psi}_S^{(\boldsymbol{u})}, \boldsymbol{\eta}_S^{(\boldsymbol{u})}, \boldsymbol{\delta}_S^{(\boldsymbol{u})}, \\ \overline{\boldsymbol{\theta}}_R^{(\boldsymbol{u})}, \underline{\boldsymbol{\theta}}_R^{(\boldsymbol{u})}, \boldsymbol{\psi}_R^{(\boldsymbol{u})}, \boldsymbol{\eta}_R^{(\boldsymbol{u})}, \\ \boldsymbol{\delta}_R^{(\boldsymbol{u})}, \boldsymbol{\kappa}_p^{(\boldsymbol{u})}, \boldsymbol{\kappa}_q^{(\boldsymbol{u})}, \rho^{(\boldsymbol{u})}, \\ \boldsymbol{\sigma}_{v^+}^{(\boldsymbol{u})}, \boldsymbol{\sigma}_{v^+}^{(\boldsymbol{u})}, \boldsymbol{\sigma}_{w^+}^{(\boldsymbol{u})}, \boldsymbol{\sigma}_{w^-}^{(\boldsymbol{u})}}} \sum_{j \in \mathcal{N}_S} \Big[p_{S,j}\alpha_{S,j}^{(\boldsymbol{u})} + (q_{S,j} - \overline{U}_S \overline{\mu}_{S,j})\overline{\theta}_{S,j}^{(\boldsymbol{u})} - q_{S,j}\underline{\theta}_{S,j}^{(\boldsymbol{u})} -$$
$$\overline{U}_S(1-\overline{\mu}_{S,j})\psi_{S,j}^{(\boldsymbol{u})} - \underline{U}_S\underline{\mu}_{S,j}\eta_{S,j}^{(\boldsymbol{u})} - \overline{U}_S(1-\underline{\mu}_{S,j})\delta_{S,j}^{(\boldsymbol{u})}\Big] +$$
$$\sum_{i \in \mathcal{N}_R} \Big[\hat{p}_{R,i}\kappa_{p_i}^{(\boldsymbol{u})} + \hat{q}_{R,i}\kappa_{q_i}^{(\boldsymbol{u})} - \overline{U}_R\overline{\mu}_{R,i}\overline{\theta}_{R,i}^{(\boldsymbol{u})} -$$
$$\overline{U}_R(1-\overline{\mu}_{R,i})\psi_{R,i}^{(\boldsymbol{u})} - \underline{U}_R\underline{\mu}_{R,i}\eta_{R,i}^{(\boldsymbol{u})} - \overline{U}_R(1-\overline{\mu}_{R,i})\delta_{R,i}^{(\boldsymbol{u})} -$$
$$\sigma_{v_i^+}^{(\boldsymbol{u})} - \sigma_{v_i^-}^{(\boldsymbol{u})} - \sigma_{w_i^+}^{(\boldsymbol{u})} - \sigma_{w_i^+}^{(\boldsymbol{u})}\Big] + d\gamma^{(\boldsymbol{u})} - \Gamma\rho^{(\boldsymbol{u})} \qquad \text{(A-30)}$$

subject to:

$$\gamma^{(\boldsymbol{u})} + \overline{\theta}_{S,j}^{(\boldsymbol{u})} - \overline{\theta}_{S,j}^{(\boldsymbol{u})} - \eta_{S,j}^{(\boldsymbol{u})} \leq p_{S,j} - c_{S,j}, \qquad \forall\, j \in \mathcal{N}_S; \quad \text{(A-31)}$$
$$\gamma^{(\boldsymbol{u})} + \overline{\theta}_{R,i}^{(\boldsymbol{u})} - \underline{\theta}_{R,i}^{(\boldsymbol{u})} - \eta_{R,i}^{(\boldsymbol{u})} \leq 0, \qquad \forall\, i \in \mathcal{N}_R; \quad \text{(A-32)}$$
$$\alpha_{R,i}^{(\boldsymbol{u})} + \kappa_{p_i}^{(\boldsymbol{u})} \leq 0, \qquad \forall\, i \in \mathcal{N}_R; \quad \text{(A-33)}$$
$$\underline{\theta}_{R,i}^{(\boldsymbol{u})} - \overline{\theta}_{R,i}^{(\boldsymbol{u})} + \kappa_{q_i}^{(\boldsymbol{u})} \leq 0, \qquad \forall\, i \in \mathcal{N}_R; \quad \text{(A-34)}$$
$$-\alpha_{S,j}^{(\boldsymbol{u})} - \psi_{S,j}^{(\boldsymbol{u})} \leq q_{S,j}, \qquad \forall\, j \in \mathcal{N}_S; \quad \text{(A-35)}$$
$$\alpha_{S,j}^{(\boldsymbol{u})} - \delta_{S,j}^{(\boldsymbol{u})} \leq 0, \qquad \forall\, j \in \mathcal{N}_S; \quad \text{(A-36)}$$
$$\alpha_{R,i}^{(\boldsymbol{u})} - \psi_{R,i}^{(\boldsymbol{u})} \leq 0, \qquad \forall\, i \in \mathcal{N}_R; \quad \text{(A-37)}$$
$$-\alpha_{R,i}^{(\boldsymbol{u})} - \delta_{R,i}^{(\boldsymbol{u})} \leq 0, \qquad \forall\, i \in \mathcal{N}_R; \quad \text{(A-38)}$$
$$\sum_{j \in \mathcal{N}_S} \alpha_{S,j}^{(\boldsymbol{u})} - \sum_{i \in \mathcal{N}_R} \alpha_{R,i}^{(\boldsymbol{u})} = 0; \qquad \text{(A-39)}$$
$$-\Delta_{p_i}^{+}\kappa_{p_i}^{(\boldsymbol{u})} - \rho^{(\boldsymbol{u})} - \sigma_{v_i^+}^{(\boldsymbol{u})} \leq 0, \qquad \forall\, i \in \mathcal{N}_R; \quad \text{(A-40)}$$
$$\Delta_{p_i}^{-}\kappa_{p_i}^{(\boldsymbol{u})} - \rho^{(\boldsymbol{u})} - \sigma_{v_i^-}^{(\boldsymbol{u})} \leq 0, \qquad \forall\, i \in \mathcal{N}_R; \quad \text{(A-41)}$$
$$-\Delta_{q_i}^{+}\kappa_{R,i}^{(\boldsymbol{u})} - \rho^{(\boldsymbol{u})} - \sigma_{w_i^+}^{(\boldsymbol{u})} \leq 0, \qquad \forall\, i \in \mathcal{N}_R; \quad \text{(A-42)}$$

some $j \in \{1, \ldots, N_S\}$, then $\log_2$ is not defined. However, we argue that the simply exclusion of appropriate constraints overcomes this imprecision when this particular case happens.

$$\Delta^{-}_{q_i}\kappa^{(\boldsymbol{u})}_{q_i} - \rho^{(\boldsymbol{u})} - \sigma^{(\boldsymbol{u})}_{w_i^R} \leq 0, \qquad \forall\, i \in \mathcal{N}_R; \quad \text{(A-43)}$$

$$\overline{\theta}^{(\boldsymbol{u})}_{S,j}, \underline{\theta}^{(\boldsymbol{u})}_{S,j}, \psi^{(\boldsymbol{u})}_{S,j}, \eta^{(\boldsymbol{u})}_{S,j}, \delta^{(\boldsymbol{u})}_{S,j} \geq 0, \qquad \forall\, j \in \mathcal{N}_S; \quad \text{(A-44)}$$

$$\overline{\theta}^{(\boldsymbol{u})}_{R,i}, \underline{\theta}^{(\boldsymbol{u})}_{R,i}, \psi^{(\boldsymbol{u})}_{R,i}, \eta^{(\boldsymbol{u})}_{R,i}, \delta^{(\boldsymbol{u})}_{R,i}, \sigma^{(\boldsymbol{u})}_{v_i^+}, \sigma^{(\boldsymbol{u})}_{v_i^-}, \sigma^{(\boldsymbol{u})}_{w_i^+}, \sigma^{(\boldsymbol{u})}_{w_i^-} \geq 0, \qquad \forall\, i \in \mathcal{N}_R; \quad \text{(A-45)}$$

$$\rho^{(\boldsymbol{u})} \geq 0. \quad \text{(A-46)}$$

Then, for a given set $\mathbb{B}_k \subset \mathbb{B}$, we can write the Master Problem as the following bilinear optimization problem.

$$\max_{\substack{\boldsymbol{p}_S, \boldsymbol{q}_S, \boldsymbol{\beta}, \boldsymbol{\chi}, \varphi \\ \boldsymbol{\alpha}^{(\boldsymbol{u})}_S, \boldsymbol{\alpha}^{(\boldsymbol{u})}_R, \gamma^{(\boldsymbol{u})}, \overline{\boldsymbol{\theta}}^{(\boldsymbol{u})}_S, \\ \underline{\boldsymbol{\theta}}^{(\boldsymbol{u})}_S, \boldsymbol{\psi}^{(\boldsymbol{u})}_S, \boldsymbol{\eta}^{(\boldsymbol{u})}_S, \boldsymbol{\delta}^{(\boldsymbol{u})}_S, \\ \overline{\boldsymbol{\theta}}^{(\boldsymbol{u})}_R, \underline{\boldsymbol{\theta}}^{(\boldsymbol{u})}_R, \boldsymbol{\psi}^{(\boldsymbol{u})}_R, \boldsymbol{\eta}^{(\boldsymbol{u})}_R, \\ \boldsymbol{\delta}^{(\boldsymbol{u})}_R, \boldsymbol{\kappa}^{(\boldsymbol{u})}_p, \boldsymbol{\kappa}^{(\boldsymbol{u})}_q, \rho^{(\boldsymbol{u})}, \\ \boldsymbol{\sigma}^{(\boldsymbol{u})}_{v^+}, \boldsymbol{\sigma}^{(\boldsymbol{u})}_{v^+}, \boldsymbol{\sigma}^{(\boldsymbol{u})}_{w^+}, \boldsymbol{\sigma}^{(\boldsymbol{u})}_{w^-}}} \varphi \quad \text{(A-47)}$$

subject to:

$$\varphi \leq \sum_{j \in \mathcal{N}_S} \Big[p_{S,j}\alpha^{(\boldsymbol{u})}_{S,j} + (q_{S,j} - \overline{U}_S \overline{\mu}_{S,j})\overline{\theta}^{(\boldsymbol{u})}_{S,j} - q_{S,j}\underline{\theta}^{(\boldsymbol{u})}_{S,j} - \overline{U}_S(1 - \overline{\mu}_{S,j})\psi^{(\boldsymbol{u})}_{S,j} - \underline{U}_S \underline{\mu}_{S,j}\eta^{(\boldsymbol{u})}_{S,j} - \overline{U}_S(1 - \underline{\mu}_{S,j})\delta^{(\boldsymbol{u})}_{S,j} \Big] + \sum_{i \in \mathcal{N}_R} \Big[\hat{p}_{R,i}\kappa^{(\boldsymbol{u})}_{p_i} + \hat{q}_{R,i}\kappa^{(\boldsymbol{u})}_{q_i} - \overline{U}_R \overline{\mu}_{R,i}\overline{\theta}^{(\boldsymbol{u})}_{R,i} - \overline{U}_R(1 - \overline{\mu}_{R,i})\psi^{(\boldsymbol{u})}_{R,i} - \underline{U}_R \underline{\mu}_{R,i}\eta^{(\boldsymbol{u})}_{R,i} - \overline{U}_R(1 - \overline{\mu}_{R,i})\delta^{(\boldsymbol{u})}_{R,i} - \sigma^{(\boldsymbol{u})}_{v_i^+} - \sigma^{(\boldsymbol{u})}_{v_i^-} - \sigma^{(\boldsymbol{u})}_{w_i^+} - \sigma^{(\boldsymbol{u})}_{w_i^+} \Big] + d\gamma^{(\boldsymbol{u})} - \Gamma\rho^{(\boldsymbol{u})}, \qquad \forall\, \boldsymbol{u} \in \mathbb{B}_k; \quad \text{(A-48)}$$

Constraints (A-31)–(A-46), $\forall\, \boldsymbol{u} \in \mathbb{B}_k$; (A-49)

Constraints (A-24)–(A-29). (A-50)

Problem (A-47)–(A-50) is a non-convex optimization problem due to the bilinear products in equation (A-48). More specifically, we have the set of decision variable products $p_{S,j}\alpha^{(\boldsymbol{u})}_{S,j}$ and $q_{S,j}(\overline{\theta}^{(\boldsymbol{u})}_{S,j} - \underline{\theta}^{(\boldsymbol{u})}_{S,j})$ for each $j \in \mathcal{N}_S$. In this context, since $(\boldsymbol{p}_S, \boldsymbol{q}_S)$ are integer vectors, we can recast these bilinear terms into linear equations using well-known algebra results. Beginning with the first product, for a given $j \in \mathcal{N}_S$, we can identify $p_{S,j}\alpha^{(\boldsymbol{u})}_{S,j} \leftrightarrow \tau^{(\boldsymbol{u})}_{S,j}$. Thereby, we can construct the following set equivalence.

$$\mathcal{E}^{(p)}\Big(U_{\alpha_{S,j}^{(\boldsymbol{u})}}\Big) \triangleq \Bigg\{ \tau_{S,j}^{(\boldsymbol{u})} \in \mathbb{R} \ \Bigg| \ \exists\, (\alpha_{S,j}^{(\boldsymbol{u})}, \boldsymbol{\beta}_j) \text{ feasible in (A-49)–(A-50) and } \boldsymbol{\Lambda}_j^{(\boldsymbol{u})} \in \mathbb{R}^{\lfloor \log_2(\bar{p}_{S,j}) \rfloor + 1};$$

$$\tau_{S,j}^{(\boldsymbol{u})} = \sum_{k=1}^{\lfloor \log_2(\bar{p}_{S,j}) \rfloor + 1} 2^{k-1} \Lambda_{k,j}^{(\boldsymbol{u})};$$

$$-U_{\alpha_{S,j}^{(\boldsymbol{u})}} \beta_{k,j} \leq \Lambda_{k,j}^{(\boldsymbol{u})} \leq U_{\alpha_{S,j}^{(\boldsymbol{u})}} \beta_{k,j}, \ \ \forall\, k \in \Big\{1, \ldots, \lfloor \log_2(\bar{p}_{S,j}) \rfloor + 1\Big\};$$

$$-U_{\alpha_{S,j}^{(\boldsymbol{u})}} (1 - \beta_{k,j}) \leq \Lambda_{k,j}^{(\boldsymbol{u})} - \alpha_{S,j}^{(\boldsymbol{u})} \leq U_{\alpha_{S,j}^{(\boldsymbol{u})}} (1 - \beta_{k,j}),$$

$$\forall\, k \in \Big\{1, \ldots, \lfloor \log_2(\bar{p}_{S,j}) \rfloor + 1\Big\}; \Bigg\}$$

$$= \Bigg\{ \tau_{S,j}^{(\boldsymbol{u})} \in \mathbb{R} \ \Bigg| \ \tau_{S,j}^{(\boldsymbol{u})} = \alpha_{S,j}^{(\boldsymbol{u})} p_{S,j}, \forall\, (\alpha_{S,j}^{(\boldsymbol{u})}, p_{S,j}) \text{ feasible in (A-49)–(A-50)} \Bigg\}. \quad \text{(A-51)}$$

where $U_{\alpha_{S,j}^{(\boldsymbol{u})}}$ is an upper-bound on $|\alpha_{S,j}^{(\boldsymbol{u})}|$. Note that $\mathcal{E}\big(U_{\alpha_{S,j}^{(\boldsymbol{u})}}\big)$ is a set with linear equations, thus suitable for linear programming problems. Analogously, a similar set can be derived for the second bilinear product. For expository purposes, let $\boldsymbol{\theta}_S^{(\boldsymbol{u})} = \overline{\boldsymbol{\theta}}_S^{(\boldsymbol{u})} - \underline{\boldsymbol{\theta}}_S^{(\boldsymbol{u})}$. Thus, we can identify $q_{S,j}\theta_{S,j}^{(\boldsymbol{u})} \leftrightarrow \zeta_{S,j}^{(\boldsymbol{u})}$, $\forall\, j \in \mathcal{N}_S$ and derive the following set equivalence:

$$\mathcal{E}^{(q)}\Big(U_{\theta_{S,j}^{(\boldsymbol{u})}}\Big) \triangleq \Bigg\{ \zeta_{S,j}^{(\boldsymbol{u})} \in \mathbb{R} \ \Bigg| \ \exists\, (\overline{\theta}_{S,j}^{(\boldsymbol{u})}, \underline{\theta}_{S,j}^{(\boldsymbol{u})}, \boldsymbol{\chi}_j) \text{ feasible in (A-49)–(A-50) and } \boldsymbol{\Theta}_j^{(\boldsymbol{u})} \in \mathbb{R}^{\lfloor \log_2(\bar{q}_{S,j}) \rfloor + 1};$$

$$\zeta_{S,j}^{(\boldsymbol{u})} = \sum_{k=1}^{\lfloor \log_2(\bar{q}_{S,j}) \rfloor + 1} 2^{k-1} \Theta_{k,j}^{(\boldsymbol{u})};$$

$$-U_{\theta_{S,j}^{(\boldsymbol{u})}} \chi_{k,j} \leq \Theta_{k,j}^{(\boldsymbol{u})} \leq U_{\theta_{S,j}^{(\boldsymbol{u})}} \chi_{k,j}, \ \ \forall\, k \in \Big\{1, \ldots, \lfloor \log_2(\bar{q}_{S,j}) \rfloor + 1\Big\};$$

$$-U_{\theta_{S,j}^{(\boldsymbol{u})}} (1 - \chi_{k,j}) \leq \Theta_{k,j}^{(\boldsymbol{u})} - \theta_{S,j}^{(\boldsymbol{u})} \leq U_{\theta_{S,j}^{(\boldsymbol{u})}} (1 - \chi_{k,j}),$$

$$\forall\, k \in \Big\{1, \dots, \big\lfloor \log_2(\bar{q}_{S,j}) \big\rfloor + 1\Big\};$$

$$\left.\theta^{(\boldsymbol{u})}_{S,j} = \overline{\theta}^{(\boldsymbol{u})}_{S,j} - \underline{\theta}^{(\boldsymbol{u})}_{S,j}\right\};$$

$$= \left\{ \zeta^{(\boldsymbol{u})}_{S,j} \in \mathbb{R} \;\middle|\; \begin{array}{l} \zeta^{(\boldsymbol{u})}_{S,j} = \theta^{(\boldsymbol{u})}_{S,j} q_{S,j}, \qquad \theta^{(\boldsymbol{u})}_{S,j} = \overline{\theta}^{(\boldsymbol{u})}_{S,j} - \underline{\theta}^{(\boldsymbol{u})}_{S,j}, \\ \forall\, (\overline{\theta}^{(\boldsymbol{u})}_{S,j}, \underline{\theta}^{(\boldsymbol{u})}_{S,j}, q_{S,j}) \text{ feasible in (A-49)–(A-50)} \end{array} \right\}. \quad \text{(A-52)}$$

Again, $U_{\theta^{(\boldsymbol{u})}_{S,j}}$ is an upper-bound on $|\theta^{(\boldsymbol{u})}_{S,j}|$. The master problem (A-47)–(A-50) can be suitably written as a mixed-integer linear programming problem.

$$\max_{\substack{\boldsymbol{p}_S, \boldsymbol{q}_S, \boldsymbol{\beta}, \boldsymbol{\chi}, \varphi \\ \boldsymbol{\alpha}^{(\boldsymbol{u})}_S, \boldsymbol{\alpha}^{(\boldsymbol{u})}_R, \gamma^{(\boldsymbol{u})}, \overline{\boldsymbol{\theta}}^{(\boldsymbol{u})}_S, \\ \underline{\boldsymbol{\theta}}^{(\boldsymbol{u})}_S, \boldsymbol{\psi}^{(\boldsymbol{u})}_S, \boldsymbol{\eta}^{(\boldsymbol{u})}_S, \boldsymbol{\delta}^{(\boldsymbol{u})}_S, \\ \overline{\boldsymbol{\theta}}^{(\boldsymbol{u})}_R, \underline{\boldsymbol{\theta}}^{(\boldsymbol{u})}_R, \boldsymbol{\psi}^{(\boldsymbol{u})}_R, \boldsymbol{\eta}^{(\boldsymbol{u})}_R, \\ \boldsymbol{\delta}^{(\boldsymbol{u})}_R, \boldsymbol{\kappa}^{(\boldsymbol{u})}_p, \boldsymbol{\kappa}^{(\boldsymbol{u})}_q, \rho^{(\boldsymbol{u})}, \\ \boldsymbol{\sigma}^{(\boldsymbol{u})}_{v^+}, \boldsymbol{\sigma}^{(\boldsymbol{u})}_{v^+}, \boldsymbol{\sigma}^{(\boldsymbol{u})}_{w^+}, \boldsymbol{\sigma}^{(\boldsymbol{u})}_{w^-}, \\ \boldsymbol{\tau}, \boldsymbol{\Lambda}, \boldsymbol{\theta}_S, \boldsymbol{\zeta}, \boldsymbol{\Theta}}} \quad \varphi \qquad \text{(A-53)}$$

subject to:

$$\varphi \le \sum_{j \in \mathcal{N}_S} \Big[\tau^{(\boldsymbol{u})}_{S,j} - \overline{U}_S \overline{\mu}_{S,j} \overline{\theta}^{(\boldsymbol{u})}_{S,j} + \zeta^{(\boldsymbol{u})}_{S,j} - \overline{U}_S (1 - \overline{\mu}_{S,j}) \psi^{(\boldsymbol{u})}_{S,j} - \underline{U}_S \underline{\mu}_{S,j} \eta^{(\boldsymbol{u})}_{S,j} - \overline{U}_S (1 - \underline{\mu}_{S,j}) \delta^{(\boldsymbol{u})}_{S,j} \Big] +$$
$$\sum_{i \in \mathcal{N}_R} \Big[\hat{p}_{R,i} \kappa^{(\boldsymbol{u})}_{p_i} + \hat{q}_{R,i} \kappa^{(\boldsymbol{u})}_{q_i} - \overline{U}_R \overline{\mu}_{R,i} \overline{\theta}^{(\boldsymbol{u})}_{R,i} - \overline{U}_R (1 - \overline{\mu}_{R,i}) \psi^{(\boldsymbol{u})}_{R,i} - \underline{U}_R \underline{\mu}_{R,i} \eta^{(\boldsymbol{u})}_{R,i} - \overline{U}_R (1 - \overline{\mu}_{R,i}) \delta^{(\boldsymbol{u})}_{R,i} - \sigma^{(\boldsymbol{u})}_{v_i^+} - \sigma^{(\boldsymbol{u})}_{v_i^-} - \sigma^{(\boldsymbol{u})}_{w_i^+} - \sigma^{(\boldsymbol{u})}_{w_i^+} \Big] +$$
$$d\gamma^{(\boldsymbol{u})} - \Gamma \rho^{(\boldsymbol{u})}, \qquad \forall\, \boldsymbol{u} \in \mathbb{B}_k; \quad \text{(A-54)}$$

Constraints (A-31)–(A-46), $\forall\, \boldsymbol{u} \in \mathbb{B}_k$; (A-55)

Constraints (A-24)–(A-29); (A-56)

$$\tau^{(\boldsymbol{u})}_{S,j} \in \mathcal{E}^{(p)}\Big(U_{\alpha^{(\boldsymbol{u})}_{S,j}}\Big), \qquad \forall\, \boldsymbol{u} \in \mathbb{B}_k,\ j \in \mathcal{N}_S; \quad \text{(A-57)}$$

$$\zeta^{(\boldsymbol{u})}_{S,j} \in \mathcal{E}^{(q)}\Big(U_{\theta^{(\boldsymbol{u})}_{S,j}}\Big), \qquad \forall\, \boldsymbol{u} \in \mathbb{B}_k,\ j \in \mathcal{N}_S; \quad \text{(A-58)}$$

$$\theta^{(\boldsymbol{u})}_{S,j} = \overline{\theta}^{(\boldsymbol{u})}_{S,j} - \underline{\theta}^{(\boldsymbol{u})}_{S,j}, \qquad \forall\, \boldsymbol{u} \in \mathbb{B}_k,\ j \in \mathcal{N}_S. \quad \text{(A-59)}$$

We highlight that problem (A-53)–(A-59) is an implementable version of (A-47)–(A-50), suitable for commercial solvers.