Pontifícia Universidade Católica
do Rio de Janeiro

Lilian Cordeiro Brambila

Fibrations and Poisson structures with a finite number of leaves

Tese de Doutorado

Thesis presented to the Programa de Pós–graduação em Matemática of PUC-Rio in partial fulfillment of the requirements for the degree of Doutor em Matemática.

Advisor: Prof. David Francisco Martínez Torres

Rio de Janeiro
April 2018

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Bibliographic data

Brambila, Lilian Cordeiro

Fibrations and Poisson structures with a finite number of leaves / Lilian Cordeiro Brambila; advisor: David Francisco Martínez Torres. – Rio de janeiro: PUC-Rio, Departamento de Matemática, 2018.

v., 83 f: il. ; 30 cm

Tese (doutorado) - Pontifícia Universidade Católica do Rio de Janeiro, Departamento de Matemática.

Inclui bibliografia

1. Matemática – Teses. 2. Estruturas de Poisson;. 3. Fibrações;. 4. Grupos de Lie Poisson;. 5. Decomposição de Bruhat;. 6. Variedades tóricas..
I. Martínez Torres, David Francisco. II. Pontifícia Universidade Católica do Rio de Janeiro. Departamento de Matemática. III. Título.

CDD: 510

In memoriam of my grandfather Mário.

Acknowledgments

I am thankful for PUC-Rio for the scholarship and excellent structure provided in this journey. I also extend my sincere gratitude to Capes for the fellowship provided throughout my doctoral journey and to all DMAT staff, Carlos Henrique, Creuza, Fernanda, Kátia and Orlando notable among them.

To my advisor David, thank you for the countless discussions that guided me all these years. To my committee members Alejandro Cabrera, Alessia Mandini, Paula Balseiro and Pedro Frejlich, I appreciate your availability.

I also owe sincere thanks to my friends and colleagues that supported me in this process, in special to Alessandro, Otávio, Gabrielle, Felipe, Jana and Fernando.

I am specially indebted to my parents Joseli and João Sérgio for being supportive at all times of my life. I extend my gratitude to my brother Ricardo, my sister-in-law Camila and my nephews Maria Luiza and João Vitor for being a vital part of my life.

To my husband Ricardo, my source of inspiration and motivation, thank you for pushing me forward at all times. As he once wrote about me: "*pelos ótimos momentos que somente com ele eu poderia ter vivido. Certamente ele é uma pessoa com a qual posso contar em qualquer momento da minha vida, além de ser provavelmente a única pessoa no mundo que consiga me aguentar.*"

Abstract

Brambila, Lilian Cordeiro; Martínez Torres, David Francisco (Advisor). **Fibrations and Poisson structures with a finite number of leaves**. Rio de Janeiro, 2018. 83p. Tese de doutorado – Departamento de Matemática, Pontifícia Universidade Católica do Rio de Janeiro.

In this thesis we introduce the notion of fibered Poisson structure on a locally trivial fiber bundle. This is a Poisson structure on the total space of the fibration with natural compatibility conditions with respect to the given Poisson base and fiber. Our main result is a recipe to produce fibered Poisson structures out of appropriate (pairs of) Poisson actions of Lie groups. We apply this result to produce fibered Poisson structures with fiber and base either a toric variety or a coadjoint orbit, thus enlarging the class of compact Poisson manifolds with a finite number of symplectic leaves.

Keywords

Poisson structures; Fibrations; Poisson-Lie groups; Bruhat decomposition; Toric manifolds.

Resumo

Brambila, Lilian Cordeiro; Martínez Torres, David Francisco. **Fibrações e estruturas de Poisson com um número finito de folhas**. Rio de Janeiro, 2018. 83p. Tese de Doutorado – Departamento de Matemática, Pontifícia Universidade Católica do Rio de Janeiro.

Nesta tese introduzimos a noção de estrutura de Poisson fibrada em um fibrado localmente trivial. Isto é uma estrutura de Poisson no espaço total da fibração com condições naturais de compatibilidade com respeito as fibras e bases de Poisson dadas. Nosso resultado principal é uma receita para produzir estruturas de Poisson fibradas fora de apropriadas (pares de) ações de Poisson de grupos de Lie. Aplicamos este resultado para produzir estruturas de Poisson fibradas com fibra e base uma variedade tórica ou uma órbita coadjunta, aumentando assim a classe de variedades de Poisson compactas com um número finito de folhas simpléticas.

Palavras-chave

Estruturas de Poisson; Fibrações; Grupos de Lie Poisson; Decomposição de Bruhat; Variedades tóricas.

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1
Introduction

Fiber bundles in differential geometry are a generalization of the Cartesian product. If we have a geometric structure closed under Cartesian products, say symplectic or Poisson, there is a natural –but somewhat rigid– definition of what is a locally trivial geometric structure on a fibration. The purpose of this thesis is:

(i) to present a less rigid notion (and new, to the best of our knowledge) of fiber bundles in Poisson geometry, that we shall refer to as *fibered Poisson structures*;

(ii) to work out an effective tool to produce fibered Poisson structures;

(iii) to show that the previous tool applies to the class of Bruhat Poisson structures on compact coadjoint orbits and toric Poisson structures on projective (smooth) toric varieties.

Our guiding principle to introduce fibered Poisson structures is the following: the characteristic foliation of the Poisson structure on the total space of the fiber bundle should fiber over the characteristic foliation of the Poisson base with fiber the characteristic foliation of the Poisson fiber. Because of that and because Bruhat Poisson structures and toric Poisson structures have a finite number of leaves, our work will produce new examples of compact manifolds with Poisson structures having a finite number of symplectic leaves.

The analysis of Poisson structures on fibrations is not new. There exists a classical notion of *Poisson fibration*, first introduced in [1]: this is a fibration $P \to B$ with fiber (F, π_F), where P is endowed with a vertical Poisson structure whose Poisson fibers are Poisson diffeomorphic to (F, π_F). More elaborated Poisson structures on fibrations have also appeared in relation to coupling problems and normal forms of Poisson structures around symplectic leaves [1].

Back to the contents of this thesis, item (i) above is addressed in Chapter 2. Our discussion on Poisson structures on fibrations will start describing the obvious notion of *locally trivial Poisson structure*, where one requires the existence of local trivializations of the fiber bundle which take the Poisson structure to a product Poisson structure. This rigid class of

Poisson structures presents two features: the projection is a Poisson map and –more interestingly– the characteristic foliation fibers over the characteristic foliation of the base with base the characteristic foliation of the fiber (see Subsection 2.3 for details on fibrations for spaces endowed with singular foliations). With these preliminaries, we will move ahead an introduce in Definition 2.6 our notion of *fibered Poisson structure* π_P on a fibration P with Poisson base (B, π_B) and fiber (F, π_F). Roughly, we demand the projection to be a Poisson surjective submersion and the fibers to inherit a backward Dirac (Poisson) structure isomorphic to π_F, and we require another condition exerting additional control on the characteristic foliation. This notion allows us to keep the two aforementioned properties of locally trivial Poisson structures with much less demanding hypotheses. In particular, the bijection on leaf spaces discussed on Lemma 2.20

$$P/\mathcal{F}_{\pi_P} \simeq B/\mathcal{F}_{\pi_B} \times F/\mathcal{F}_{\pi_F}$$

will be instrumental to produce new examples of Poisson structures with a finite number of symplectic leaves.

If fact, we shall not introduce Poisson fibered structures via Dirac geometry. Our initial working definition of fibered Poisson structures will be a different one which is more appropriate for computations. Then we will address the Dirac geometry perspective. In Subsections 2.2.3 and 2.2.4 we will analyze in detail two extreme cases of Definition 2.6:

– Fibered Poisson structures with base endowed with the zero Poisson structure.

– Fibered Poisson structures with symplectic base and fiber, which will be referred to as *fibered symplectic structures*.

Under natural compactness assumptions, we show that:

– Fibered Poisson structures with zero Poisson base equal the classical Poisson fibrations.

– Fibered symplectic structures are classified via representations of the fundamental group of the base.

Chapter 3 contains the main result of this thesis, the construction of fibered Poisson structures on appropriate associated bundles. The result will require certain kinds of Poisson structures on principal bundles and certain natural types of Poisson actions, that we call *tangential* and *no-where* tangential. The conditions satisfied by these actions will be instrumental to control the characteristic foliation of the Poisson structures we want to place on our associated bundles.

Here is a rough version of our main result (Theorem 3.4):

Theorem 1.1 *Let (P, π_P) a principal G-bundle over (B, π_B) with Poisson projection map, such that the principal G-action is Poisson and no-where tangential. Let (F, π_F) be a Poisson manifold on which G-acts in a Poisson and tangential fashion.*

Then the product Poisson structure induces on the associated bundle $M := P \times F/G$ a fibered Poisson structure π over (B, π_B) with fiber (F, π_F). More specifically, each fiber F_b inherits from π a Poisson structure diffeomorphic to π_F and the characteristic foliation of π fibers over the characteristic foliation of π_B with fiber the characteristic foliation of π_F.

We shall also investigate what happens if we relax the requirements of our main construction, and we shall illustrate the difference in behaviour with examples.

Chapter 3 ends discussing the notion of cotangent modifications. This is an instance of our main construction in Theorem 1.1 which we can analyze in more detail, and which will be useful in for our applications in Chapter 4.

In Chapter 4 we apply our previous constructions to toric Poisson structures on toric varieties, and to Bruhat Poisson structures on compact coadjoint orbits (see Appendices B and C). We will use the analysis of cotangent modifications to show that both toric Poisson structures and Bruhat Poisson fit into the hypoteses of Theorem 1.1 (both playing either the role of base or the role of fiber). The corresponding fibered Poisson structures coming from Theorem 1.1 will produce the new examples of (compact) Poisson manifolds with a finite number of symplectic leaves we were seeking for. Finally, we discuss how our construction allows to construct fibered Poisson structures in topologically different fibrations (with the same base and fiber).

2
Fibrations in Poisson geometry

This chapter starts with a digression on notions of fibrations in Poisson geometry which will lead naturally to our definition of a *fibered Poisson structure* on a locally trivial fiber bundle. A fibered Poisson structure will include:

- two Poisson theoretic requirements;
- a foliation theoretic requirement.

We will describe how the two Poisson theoretic requirements are powerful enough to produce a characteristic foliation whose underlying distribution has the right fibering properties. However, they do not seem to preclude some problematic behaviour of the characteristic foliation. And, thus the reason to introduce the foliation theoretic requirement.

Next, we shall recast our definition into the more convenient Dirac language, and we shall also test it in two extreme cases. Namely, when the base is endowed with the zero Poisson structure, and when base and fiber are symplectic. The analysis of the latter case will uncover a second *horizontal* foliation present in fibered Poisson structures.

For the reader's convinience, we close the chapter with a digression in Section 2.3 on the notion of fibration for spaces endowed with possibly non-regular foliations. The reader familiar with the Stefan-Sussmann theory of singular foliations may just consult Section 2.3 for notation and definition purposes. Also, all our fibrations/fiber bundles will be locally trivial, unless otherwise stated.

2.1
Notions of fibrations in Poisson geometry

A locally trivial fiber bundle is a generalization of a Cartesian product. If we have a geometric structure closed under cartesian products –for instance a symplectic or Poisson structure, but not a contact one– there is a natural definition of what a locally trivial geometric structure on a fibration is:

Definition 2.1 *Let (F, π_F) and (B, π_B) be Poisson manifolds, let $pr : P \to B$ be a locally trivial fiber bundle with fiber F and let $\pi_P \in \mathfrak{X}^2(P)$ be a Poisson* *structure. We say that (P, π_P) is a **locally trivial Poisson fibration** with base (B, π_B) and fiber (F, π_F) if there exists an open cover $\{U_i\}_{i\in I}$ of B with trivializations*

$$\Phi_i : (pr^{-1}(U_i), \pi_P) \to (U_i, \pi_B) \times (F, \pi_F)$$

which are Poisson morphisms for all $i \in I$.

Remark 2.2 *A locally trivial Poisson structure (P, π_P) with base (B, π_B) and fiber (F, π_F) has the following immediate properties:*

1. *The map $pr : P \to B$ is a Poisson morphism.*

2. *Denote by $\mathcal{F}_{\pi_P}$, $\mathcal{F}_{\pi_B}$ and $\mathcal{F}_{\pi_F}$ the characteristic foliations associated to π_P, π_B and π_F, respectively. Then the characteristic foliation $\mathcal{F}_{\pi_P}$ fibers over $\mathcal{F}_{\pi_B}$ with fiber $\mathcal{F}_{\pi_F}$. As we will discuss in Section 2.3 this property implies that each symplectic leaf of $\mathcal{F}_{\pi_P}$ fibers over a symplectic leaf of $\mathcal{F}_{\pi_B}$, with fiber a symplectic leaf of $\mathcal{F}_{\pi_F}$. We emphasize that the fibering property is not just for the individual leaves, but for the whole characteristic foliation.*

   Moreover, if the structural group associated to the open cover $\{U_i\}_{i\in I}$ consists of diffeomorphisms of F preserving each leaf of $\mathcal{F}_{\pi_F}$, then we have a bijection

   $$P/\mathcal{F}_{\pi_P} \simeq B/\mathcal{F}_{\pi_B} \times F/\mathcal{F}_{\pi_F}$$

   where $P/\mathcal{F}_{\pi_P}$, $B/\mathcal{F}_{\pi_B}$ and $F/\mathcal{F}_{\pi_F}$ are the spaces of (symplectic) leaves of $\mathcal{F}_{\pi_P}$, $\mathcal{F}_{\pi_B}$ and $\mathcal{F}_{\pi_F}$, respectively.

It is possible to adapt a construction from differential topology to build locally trivial Poisson structures which are not Cartesian products: let (B, π_B) and (F, π_F) be two Poisson manifolds, let $\tau : F \to F$ be a Poisson diffeomorphism and let $H \subset B$ be a co-orientable (closed) hypersurface of B. Cut B open along to H to obtain the corresponding manifold with boundary [2]. Denote by $\phi \in \mathrm{Diff}(H)$ the gluing map (diffeomorphism) which allows us to recover B from the manifold with boundary [3, Theorem 3.10]. As it is customary, this gluing map can be extended to neighborhoods of H, so in our situation we can argue that this extension is a Poisson map. Now, consider the product manifold

$$(B \times F, \pi_B + \pi_F)$$

and cut $B \times F$ open along the hypersurface $H \times F$. Glue it back via the map $\phi \times \tau$ (rather than $\phi \times \mathrm{Id}$), which is a Poisson diffeomorphism. Therefore, the resulting fiber bundle P is endowed with a locally trivial Poisson structure with base (B, π_B) and fiber (F, π_F).

Example 2.1 *Let $B = \mathbb{T}$ be a torus with a Poisson structure π_B, let $F = \mathbb{S}^2$ be together with π_F a rotationally invariant bivector vanishing at linearly at a great circle and consider*

$$\tau : \mathbb{S}^2 \to \mathbb{S}^2$$

e the antipodal map. Take as hypersurface any non-separating curve in the torus, so the result of cutting it open along the curve is the annulus. If we apply the previous construction, then we obtain

$$pr : P \to \mathbb{T}$$

a sphere bundle over the torus which is not isomorphic to the trivial one (not a Cartesian product) and which supports a locally trivial Poisson structure.

The requirements in Definition 2.1 are very strong. This makes it difficult to come up with natural sources of locally trivial Poisson structures. It is therefore natural to discuss other (weaker) notions of fibration in Poisson geometry. The first natural attempt is the classical notion of a Poisson fibration:

Definition 2.3 ([Definition 4.27, [4]) *We say that a fibration $pr : P \to B$ with fiber (F, π_F) is a* ***Poisson fibration*** *if P is endowed with a vertical Poisson structure[1] π_v such that there exist fiber inclusions*

$$i_b : (F, \pi_F) \to (P, \pi_v), \quad b \in B,$$

which are Poisson morphisms.

Remark 2.4 *In the definition of Poisson fibration it is customary to require smooth (local) dependence of the embeddings $i_b : F \to P$. It is an interesting exercise to prove that if (F, π_F) is a compact symplectic manifold then both notions of Poisson fibration agree.*

We chose to drop the smooth dependence requirement because that assumes homogeneity on the points of B. In the more general situation we want to consider, if the base B carries a non-homogeneous Poisson structure –i.e., neither symplectic nor zero– then it will no longer make sense to compare fibers over base points in different symplectic leaves.

The following is a natural construction of Poisson fibrations: let (F, π_F) be a Poisson manifold, let G a Lie group and let $P \to B$ be a principal G-bundle. Suppose that we have a Poisson G-action on F and consider the zero

[1] We say that a Poisson structure π_v on a fibration $P \to B$, with fiber F, is vertical if $\pi_v^\sharp(T^*P) \subset TF$.

Poisson structure on P. The (free) diagonal G-action on $P \times F$ is a Poisson action, so the quotient (associated bundle)

$$M := (P \times F)/G$$

inherits a (vertical) Poisson structure π_v (see Theorem A.12 in Appendix A). Consider for each $p \in P$ and $b = [p] \in B$ the inclusion map:

$$\begin{array}{rcccl} i_b & : & F & \longrightarrow & M \\ & & y & \longmapsto & [p, y]. \end{array}$$

Then it follows that $(M, \pi_v) \to B$ is a Poisson fibration with fiber (F, π_F).

Example 2.2 *Consider $G = \mathbb{S}^1$ and $F = \Sigma$ any surface endowed with a circle action. Isomorphism classes of principal $\mathbb{S}^1$-bundles over any base B are in correspondence with cohomology classes $c \in H^2(B; \mathbb{Z})$ (see [5]). Then, as we know, Poisson structures on the surface invariant by the circle action are in correspondence with $\mathbb{S}^1$-invariant bivectors $\sigma \in \mathfrak{X}^2(\Sigma)^{\mathbb{S}^1}$. Applying the previous construction we conclude that a pair (c, σ) determines a Poisson fibration structure over B with fiber (Σ, σ).*

Example 2.3 *Apply the Example 2.2 to $F = \mathbb{S}^2$, the $\mathbb{S}^1$ action on $\mathbb{S}^2$ is by rotations and a rotationally invariant bivector σ vanishing linearly along the equator to produce Poisson fibrations with fiber $(\mathbb{S}^2, \sigma)$*

Example 2.4 *Let G be a Lie group and let $\mathfrak{g}$ be its Lie algebra. As we know, $\mathfrak{g}^*$ is a Poisson manifold endowed with the linear Poisson structure π_{lin} (see Appendix A). Consider the coadjoint action of G on $\mathfrak{g}^*$ and a principal G-bundle $P \to B$. Then the associated bundle*

$$(P \times \mathfrak{g}^*)/G \to B$$

is a Poisson fibration, with fiber $(\mathfrak{g}^*, \pi_{lin})$.

In a Poisson fibration $(P, \pi_v) \to B$ with fiber (F, π_F) the Poisson tensor is vertical. Therefore the fibers are saturated by symplectic leaves and the fiber embeddings

$$i_b : (F, \pi_F) \to (P, \pi_v)$$

are Poisson morphisms. Our goal in this thesis is to discuss Poisson structures π_P on P which "induce" the Poisson structure (F, π_F) on the fibers F_b, $b \in B$, but which are also linked to a given Poisson structure on the base (B, π_B). Therefore in general the Poisson structure $\pi_P \in \mathfrak{X}^2(P)$ will not be vertical, and thus there will be points $b \in B$ for which no embedding $i_b : (F, \pi_F) \to (P, \pi_P)$ will be Poisson. This raises the question of how to make sense of "inducing" a given Poisson structure π_F on a fiber F_b. There are some particular cases, such as the Poisson structures on Vorobjev's analysis of normal forms of Poisson structures around symplectic leaves [1], where one has an straightforward answer. In our language of fibrations Vorobjev's setting corresponds to a vector bundle $P \to B$ with a (typically linear) Poisson structure on the fiber (F, π_F) and a symplectic structure on the base. It so happens that every symplectic leaf of π_P is transverse to every fiber F_b. In fact, every fiber becomes a Poisson transversal [6], and thus the induced Poisson structure on F_b can be compared with the given one (F, π_F):

Definition 2.5 *Let (P, π) a Poisson manifold. We say that a embedded submanifold $Q \subset P$ is a **Poisson transversal** if Q has transverse and symplectic intersection with every symplectic leaf of π_P. Therefore, a Poisson transversal inherits a Poisson structure from π_P.*

Here is the paradigmatic example of a (linear) Poisson structure on a vector bundle (rather, around the zero section) with base a symplectic manifold [7]:

Example 2.5 *Let G a compact Lie group and $pr : P \to B$ a principal G-bundle with compact base B. Let $\theta \in \Omega^1(P; \mathfrak{g})^G$ be a principal connection 1-form on P and let $\omega \in \Omega^2(B)$ be a symplectic form on B. The product $P \times \mathfrak{g}^*$ carries the product 2-form*

$$pr^*\omega - \mathrm{d}\langle\theta, \cdot\rangle.$$

Because B is compact we may take $U \subset \mathfrak{g}^*$ a neighborhood of the origin so that the product 2-form is symplectic on $P \times U$. Because G is compact we may assume U to be G-invariant. The corresponding (inverse) Poisson structure descends to a Poisson structure π on the diagonal quotient $(P \times U)/G$, with the following property: For each $b \in B$ and $p \in P$ in the fiber over b, the inclusion

$$\begin{array}{rccc} i_b & : & F & \longrightarrow & M \\ & & y & \longmapsto & [p, y], \end{array}$$

is a Poisson transversal.

The more general situations we are interested in will have to encompass the two extreme cases of fiber behaviour discussed in Definitions 2.5 and 2.3: either fibers which are (symplectically) transverse to the symplectic leaves or fibers which are Poisson submanifolds. Our proposal of a *fibered Poisson* *structure* π_P on a locally trivial fiber bundle $P \to B$ with Poisson base (B, π_B) and Poisson fiber (F, π_F) will make precise the following properties:

– The Poisson structure π_P will "induce" a Poisson structure on each fiber F_b which is diffeomorphic to π_F.

– The Poisson structure π_P will have both properties (1) and (2) on Remark 2.2 on locally trivial Poisson structures. In particular, property (2) reflects a natural viewpoint to differential topologists: the characteristic foliation of the fibered Poisson structure should fiber over the characteristic foliation $\mathcal{F}_{\pi_B}$ of the base with fiber the characteristic foliation $\mathcal{F}_{\pi_F}$ of the fiber.

2.2 Fibered Poisson structures

In this section we introduce the principal definition of this thesis, that is, fibered Poisson structures on locally trivial fibrations $P \to B$, and study some of their properties.

Regarding the question how to induce the given Poisson structure on fibers, perhaps the cleanest approach is given by Dirac geometry. However, in our first definition we will provide a more direct approach which will be also convenient for the computations we will encounter in Chapter 3. The Dirac perspective will be addressed latter on.

Regarding the question of how to induce an appropriate foliation on the locally trivial fibration $P \to B$, our standing hypotheses will include a Poisson structure on the base, a Poisson structure on the fiber, and a bundle P whose structural group preserves the characteristic foliation of the Poisson structure of the fiber and induces a constant map on leaf spaces. Then, as will be explain in Lemma 2.20 in Section 2.3 at the end of this chapter, elementary results of differential topology imply the existence of a (possibly singular) foliation $\mathcal{F}$ which has the fibering properties described in Remark 2.2.

Definition 2.6 *Let (F, π_F), (B, π_B) be two Poisson manifolds with characteristic foliations $\mathcal{F}_{\pi_F}, \mathcal{F}_{\pi_B}$, respectively and let $pr : P \to B$ be a locally trivial fiber bundle with fiber F whose structural group consists of diffeomorphisms of $(F, \mathcal{F}_{\pi_F})$ which induce a constant map on leaf spaces. Let $\mathcal{F}$ be the induced foliation on P (cf. Lemma 2.20).*

*We say that (P, π_P) is a **fibered Poisson structure**, with base (B, π_B) and fiber (F, π_F), if π_P is a Poisson structure such that:*

1. *The map $pr : (P, \pi_P) \to (B, \pi_B)$ is a Poisson surjective submersion.;*

2. *For each $b \in B$, there exists a fiber embedding $i_b : F \to P$, $i_b(F) = F_b$, such that for any $f \in C^\infty(F)$ there exists an extension $\widetilde{f} \in C^\infty(P)$ so that the corresponding Hamiltonian vector fields satisfy:*

$$X_{\widetilde{f}}|_{F_b} = (i_b)_* X_f;$$

3. *If $\mathcal{F}_{\pi_P}$ is the characteristic foliation associated to π_P, then*

$$\mathcal{F} = \mathcal{F}_{\pi_P}.$$

Remark 2.7

1. *Condition* (2) *is one way to making precise that π_P induces π_F in the fibers. However, the subtle point to conclude that the fiber F_b "inherits" a Poisson structure is that we need to use that π_F is a Poisson bivector (rather, just a smooth bivector, as the Dirac perspective will show), since we do not require the extensions of the functions to be closed under the Poisson bracket;*

2. *We point out again that the existence of a foliation $\mathcal{F}$ on P is guaranteed by the Lemma 2.20. Let $\mathcal{O}_p$, $\mathcal{O}_y$ and $\mathcal{O}_b$ be the symplectic leaves associated to π_P, π_F and π_B, respectively, passing through the points $y \in F$, $p = i_b(y) \in P$, $b = pr(p) \in B$. Conditions* (1) *and* (2) *together immediately imply:*

$$i_b(\mathcal{O}_y) \subset \mathcal{O}_p, \quad pr(\mathcal{O}_p) \supset \mathcal{O}_b.$$

We will see that in fact these conditions imply a stronger tie among the three foliations.

2.2.1 Differentiable versus Poisson trivializations

Here we shall discuss how the Poisson bivector of a Poisson structure which satisfies conditions (1) and (2) of a fibered Poisson structure looks in a trivialization of the fiber bundle. We will draw consequences of these local formulas.

Let (P, π_P) be a fibered Poisson structure, with a base (B, π_B) and fiber (F, π_F). Fix a trivialization U of P around $b \in B$, that is

$$\mathrm{pr}^{-1}(U) \simeq U \times F.$$

Then, we can decompose $\pi_P \in \mathfrak{X}^2(P)$ as follows:

$$\pi_P = \pi_h + \pi_m + \pi_v.$$

The summands above are the horizontal, mixed and vertical components, respectively. This means that in product coordinates $(b_1, \ldots, b_d, y_1, \ldots, y_r)$ on $U \times W \subset U \times F$ we have:

$$\pi_h = \sum_{i,j} \pi^h_{ij} \frac{\partial}{\partial b_i} \wedge \frac{\partial}{\partial b_j}, \quad \pi_m = \sum_{k,l} \pi^m_{kl} \frac{\partial}{\partial b_k} \wedge \frac{\partial}{\partial y_l}, \quad \pi_v = \sum_{m,n} \pi^v_{mn} \frac{\partial}{\partial y_m} \wedge \frac{\partial}{\partial y_n}. \tag{2-1}$$

Lemma 2.8 *Let π_P be a Poisson structure on a fiber bundle $P \to B$ with fiber F which satisfies conditions (1) and (2) in the Definition 2.6 relative to (B, π_B) and (F, π_F):*

1. *For an arbitrary trivialization $pr^{-1}(U) \simeq U \times F$ condition* (1) *in Definition 2.6 is equivalent to:*

$$\pi_h|_{pr^{-1}(U)} = \pi_B;$$

2. *For a trivialization compatible with the fiber embedding $i_b : F \to P$, $i_b(F) = F_b$, condition* (2) *in Definition 2.6 is equivalent to the existence of local trivializing coordinates $y_1, \ldots, y_r : Z \subset pr^{-1}(U) \to \mathbb{R}$ such that:*

$$\pi_m|_{\{b\}\times W} = 0, \quad \pi_v|_{\{b\}\times W} = \pi_F|_{\{b\}\times W}, \qquad W = Z \cap F_b. \tag{2-2}$$

Proof. To prove (1) we use that by hypothesis (1) in Definition 2.6:

$$\mathrm{pr}_* \pi_P = \pi_B.$$

Since $\mathrm{pr}|_{\mathrm{pr}^{-1}(U)} = \mathrm{pr}_1$, then

$$\pi_B = (\mathrm{pr}_1)_*(\pi_h + \pi_m + \pi_v) = \pi_h.$$

To prove (2), first we point out that by definition that the trivialization of P be compatible with the fiber embedding means that on $U \times F$ the restriction of the inverse of the trivialization to the slice $\{b\} \times F$ is i_b. Let us choose arbitrary local coordinates $y_1, \ldots, y_r : W \to \mathbb{R}$ on the fiber $F_b \equiv F$ around y. Let us assume that π_P is a fibered Poisson structure. Then we can find extensions of the coordinates $\widetilde{y}_j$ which satisfy:

$$X_{\widetilde{y}_j}|_{\{b\}\times W} = X_{y_j}. \tag{2-3}$$

(Strictly speaking we should first extend the coordinate functions to functions on F). In a neighbourhood Z of $b \times y \in P$ these functions fit into a submersion, and thus, together with $b_1, \ldots, b_d$, they fit into coordinates around $b \times y$. For simplicity we denote the restriction of $\widetilde{y}_j$ to Z again by y_j.

By equation (2-3), the Hamiltonian vector field of y_j at any point in $\{b\} \times W$ is tangent to $\{b\} \times W$. In the coordinates $b_1, \ldots, b_d, y_1, \ldots, y_r$ the bivector $\pi_P = \pi_h + \pi_m + \pi_v$ has the local expression given by the formula (2-1). Therefore we conclude that

$$\pi_m(dy_i, db_j)|_{\{b\}\times W} = 0,$$

or, equivalently,

$$\pi_m|_{\{b\}\times W} = 0.$$

That the Hamiltonian of y_j along $\{b\} \times W$ matches the Hamiltonian of $y_j|_{\{b\}\times W}$ with respect to π_F traslates into

$$\pi_v|_{\{b\}\times W} = \pi_F|_W.$$

The converse result is proved analogously.

□

Lemma 2.8 is interesting for several reasons:

Firstly, it hints at the difficulty of recognizing/working with fibered Poisson structures in local coordinates: one needs to come out with very special coordinates where the mixed component vanishes along the fiber F_b and where the vertical component equals π_F along F_b.

Secondly, it says that the notion of fibered Poisson structure is actually much more general than that of locally trivial Poisson structure (Definition 2.1). For the latter we require a covering of P by (differentiable) trivializations which are also Poisson trivializations:

$$\pi_m|_{\mathrm{pr}^{-1}(U)} = 0, \quad \pi_v|_{\mathrm{pr}^{-1}(U)} = \pi_F.$$

Thirdly, it allows for a simple criterium to check foliation theoretic requirement in Definition 2.6:

Corollary 2.9 *Let π_P be a Poisson structure on a fiber bundle $pr : P \to B$ with fiber F which satisfies conditions (1) and (2) in the Definition 2.6 relative to (B, π_B) and (F, π_F), and whose structural group consists of diffeomorphisms of $(F, \mathcal{F}_{\pi_F})$ which induce a constant map on leaf spaces. Then the equality of* *foliations*

$$\mathcal{F}_{\pi_P} = \mathcal{F}$$

holds if and only if for every $p \in P$ there exists $H_p \subset \pi_P^\#(T_pP) = \mathcal{F}_p$ with the following properties:

$$H_p \cap \pi_F^\#(T_yF) = \{0\}, \qquad H_p \subset \mathcal{F}_p.$$

Proof. Let us choose local coordinates around p as in the proof of item (2) in Lemma 2.8. A equality of foliations is equivalent to a equality of their underlying distributions. It follows that at p the Hamiltonian directions of π_P are the direct sum

$$\pi_h^\#|_b + \pi_F^\#|_y. \tag{2-4}$$

Therefore $\mathcal{F}_p = \pi_P^\#(T_pP)$ is equivalent to the existence of $H_p \subset \mathcal{F}_p$ complementary to $\pi_F^\#(T_yF)$. □

The last consequence of Lemma 2.8 is that each leaf of the characteristic foliation $\mathcal{F}_{\pi_P}$ intersects individual fibers in the best possible way:

Corollary 2.10 *Let π_P be a Poisson structure on a fiber bundle $pr : P \to B$ with fiber F which satisfies conditions (1) and (2) in the Definition 2.6 relative to (B, π_B) and (F, π_F).*

Then each leaf of $\mathcal{F}_{\pi_P}$ intersects the fiber F_p cleanly in an immersed submanifold which is at most a countable collection of leaves of $\mathcal{F}_{\pi_F}$ of locally constant dimension.

Proof. Let $\mathcal{O}_p$ be the leaf of $\mathcal{F}$ through p. Let us choose local product coordinates around p as in the proof of item (2) in Lemma 2.8. We can assume without loss of generality that the leaf of $\mathcal{F}_{\pi_F}$ through y is given by:

$$b_1 = \cdots = b_d = y_1 = \cdots = y_s = 0.$$

In these product coordinates, consider the (linear) projection onto $b_1, \ldots, b_d, y_1, \ldots, y_s$. By (2-2) its differential at p restricts to $T_p\mathcal{O}_p$ to an isomorphism. Upon reordering of the coordinates $b_1, \ldots, b_d$, the kernel can be chose to split into factors $b_{t+1} = \cdots = b_d = 0$ and $y_{s+1} = \cdots y_r = 0$. By the implicit function theorem there is a neighborhood W of p in $\mathcal{O}_p$ (in its manifolds topology!) which is the graph of a function

$$f : V \times Z \to V' \times Z'.$$

where $V = \{b_{t+1} = \cdots = b_d = 0\}$, $Z = \{y_{s+1} = \cdots = y_r = 0\}$, $V' = \{b_1 = \cdots = b_t = 0\}$ e $Z' = \{y_1 = \cdots = y_s = 0\}$. This immediately implies that the intersection of W with the domain of the coordinates $y_1, \ldots, y_r$ in F_b is (an open subset of) the submanifold $b_1 = \cdots = b_d = y_1 = \cdots = y_s = 0$. It follows than the intersection $\mathcal{O}_p \cap F_p$ is clean (recall this is a local property).

By condition (2) we know that the whole symplectic leaf of $\mathcal{F}_{\pi_F}$ through y is contained in $\mathcal{O}_p \cap F_p$. Because $\mathcal{O}_p$ is second countable then its intersection with F_p contains at most a countable family of leaves of leaves of $\mathcal{F}_{\pi_F}$.

There is a subtle point regarding how the family of leaves looks like locally. Each individual leaf is the image of regular immersion. A collection of equidimensional leaves still fits into a regularly immersed submanifold, but we may have leaves of different dimensions. We want to argue that we cannot have a sequence of points lying into equidimensional leaves of $\mathcal{O}_p \cap F_p$ and converging to a point q in a leaf of $\mathcal{O}_p \cap F_p$ of different dimension: because these are leaves of a foliation in F_b the dimension function is lower semicontinuous. However, this dimension is also the intersection of $T\mathcal{O}_p \cap F_p$. Recall that because $\mathcal{O}_p$ is a leaf, its tangent spaces vary continuously (in the appropriate Grassmannian) with respect to the topology of P. Therefore the dimension of their intersection with the tangent space of F_b is uper semicontinuous. Thus, it is locally constant on F_p. $\square$

It is tempting to try to weaken the notion of fibered Poisson structure by removing condition (3) on the equality of foliations $\mathcal{F}_{\pi_P} = \mathcal{F}$ (so in particular one would not need to require the cocycle of P to preserve $\mathcal{F}_{\pi_F}$ and induce a constant map on its leaf space). We believe that conditions (1) and (2) alone does not imply that the (clean) intersection of a leaf of $\mathcal{F}_{\pi_P}$ with all fibers has the same dimension. To support that we present below an example of a Poisson structure π_P on a fibration P which induces on fibers different Poisson structures and whose regular symplectic leaves intersect fibers in submanifolds whose dimension changes with $b \in B$:

Example 2.6 *Let $pr : P = \mathbb{C} \times \mathbb{C}^2 \to \mathbb{C}^2$, $(z, w, \xi) \to (w, \xi)$ be a bundle endowed with the (real) Poisson structure:*

$$\pi_Q = i\frac{\partial}{\partial z} \wedge \frac{\partial}{\partial \bar{z}} + i\frac{\partial}{\partial \xi} \wedge \frac{\partial}{\partial \bar{\xi}}$$

This product Poisson structure is regular with characteristic foliation the regular (holomorphic, linear) foliation having its underlying distribution spanned by the vector fields:

$$\frac{\partial}{\partial z}, \frac{\partial}{\partial \xi}. \tag{2-5}$$

The characteristic foliation is no-where tranverse to the fibers of the projection. Now consider a diffeomorphism of P which on the (affine) planes which are the *fibers of* $(z, w, \xi) \to \xi$ *is given by the linear map with matrix with holomorphic entries:*

$$\begin{pmatrix} a(\xi) & b(\xi) \\ c(\xi) & d(\xi) \end{pmatrix}. \tag{2-6}$$

It sends the commuting frame (2-5) above to the commuting frame,

$$\begin{aligned} X &= a\frac{\partial}{\partial z} + c\frac{\partial}{\partial w} \\ Y &= \left(a'\frac{dz - bw}{\det} + b'\frac{cz + aw}{\det}\right)\frac{\partial}{\partial z} \\ &+ \left(c'\frac{dz - bw}{\det} + d'\frac{cz + aw}{\det}\right)\frac{\partial}{\partial w} + \frac{\partial}{\partial \xi}, \end{aligned} \tag{2-7}$$

where $\det$ *is the determinant of (2-6).*

Observe that $iX \wedge \overline{X} + iY \wedge \overline{Y}$ *fails to be projectable because of the dependence on* z *of the coefficient of* $\frac{\partial}{\partial w}$ *in* Y*. Nonetheless, if we choose* c *vanishing just at* $\xi = 0$*, then the leaves of the resulting Poisson structure contain the fibers for* $\xi = 0$ *and are transverse to them for* $\xi \neq 0$ *(and this automatically implies clean intersection in both cases). It is easy to prove that fibers inherit as in condition (2) the zero Poisson structure and the standard Poisson (symplectic) structure for* $\xi \neq 0$ *and* $\xi = 0$*, respectively.*

2.2.2 Dirac perspective

The condition that (P, π_P) be a fibered Poisson structure can be recast into Dirac language. For more details about Dirac structures see the Appendix A.

Proposition 2.11 *Let* (F, π_F)*,* (B, π_B) *be two Poisson manifolds with characteristic foliation* $\mathcal{F}_{\pi_F}, \mathcal{F}_{\pi_B}$*, respectively and let* $pr : P \to B$ *be a locally trivial fiber bundle with fiber* F *whose structural group consists of diffeomorphisms of* $(F, \mathcal{F}_{\pi_F})$ *which induce a constant map on leaf spaces. Let* $\mathcal{F}$ *be the induced foliation on* P*.*

Let $\pi_P \in \mathfrak{X}^2(P)$ *be a Poisson structure on* P*. Then* (P, π_P) *is a fibered Poisson structure, with base* (B, π_B) *and fiber* (F, π_F)*, if and only if the following three conditions hold:*

1. *The projection* $pr : (P, L_{\pi_P}) \to (B, L_{\pi_B})$ *is a forward Dirac map;*

2. *All fibers* F_b *are Dirac Poisson submanifolds diffeomorphic to* (F, L_{π_F}) *and all functions in* F_b *are admissible;*

3. *There is an equality of foliations:*

$$\mathcal{F} = \mathcal{F}_{\pi_P}.$$

Proof. Note that for the condition (1) we have, by the Example A.12, that $\mathrm{pr} : P \to B$ is a Poisson morphism, if, and only if, it is a forward Dirac map.

To prove item (2) recall that each fiber F_b, the graph of π_P defines via pull back the following subset of $\mathbb{T}F_b$:

$$L_{\pi_P}|_{F_b} = \left\{(\pi_P^\sharp(\alpha), \alpha^*|_{TF_b}) \mid \pi_P^\sharp(\alpha) \in TF_b\right\}. \tag{2-8}$$

This is a possibly non-smooth collection of Lagrangian subspaces of $\mathbb{T}_y F_b, y \in F_b$.

Let us assume that π_P is a fibered Poisson structure. Let $\widetilde{f} \in C^\infty(P)$ such that $X_{\widetilde{f}}|_{F_b} = (i_b)_* X_f$, $f \in C^\infty(F)$. There is a second subbundle of $\mathbb{T}F_b$ given by:

$$\left\{(\pi_P^\sharp(d\widetilde{f}), d\widetilde{f}|_{TF_b}) \mid f \in C^\infty(F)\right\}. \tag{2-9}$$

This subbundle is necessarily smooth and Lagrangian because by condition (2) for fibered Poisson structures i_b sends it isomorphically to the graph of π_F.

Taking $\alpha = d\widetilde{f}$ we deduce that the distribution in (2-9) is contained in $L_{\pi_P}|_{F_b}$ (2-8). But, since at each point they consist of Lagrangian subspaces, they must match. Therefore $L_{\pi_P}|_{F_b}$ is smooth and hence defines a Dirac structure, which is obviously diffeomorphic to L_{π_F}.

The statement of all functions in F_b be admissible is exactly the existence of global extensions whose Hamiltonians vector field are everywhere tangent to F_b.

□

2.2.3 The zero Poisson structure

We shall now test our definition of fibered Poisson structure in some key cases, starting with the zero Poisson structure. The first result relates Poisson fibrations and fibered Poisson structures.

Lemma 2.12 *Poisson fibrations coincide with Poisson structures which satisfy the conditions (1) and (2) of fibered Poisson structures.*

Proof. Let (P, π_P) be a Poisson structure on a fibration which satisfies conditions (1) and (2) of fibered Poisson structure, with base $(B, 0)$ and fiber (F, π_F). By (2-2) and because $\pi_B = 0$, in appropriate local coordinates Hamiltonian vector fields are vertical. Thus π_P is a vertical vector field.

By hypotheses for any $f \in C^\infty(F)$ and any $b \in B$, we know that there is an extension $\widetilde{f}$ such that:

$$X_{\widetilde{f}}|_{i_b(F)} = (i_b)_* X_f, \qquad (2\text{-}10)$$

Because π_P is vertical it follows that this happens for any extension of f. Therefore the fiber inclusion $i_b : (F, \pi_F) \to (P, \pi_P)$ is a Poisson morphism, and therefore (P, π_P) is a Poisson fibration.

Conversely, since π_P is a vertical structure and $\mathrm{pr} : P \to B$ is a submersion, then for any Hamiltonian vector field $X \in C^\infty(P)$ we have that

$$\mathrm{pr}_* X = 0.$$

Therefore $\mathrm{pr} : (P, \pi_P) \to (B, 0)$ is a Poisson morphism. Condition (2) follows from the requirement that $i_b : (F, \pi_F) \to (P, \pi_P)$ be a Poisson morphism. □

The next result tells us how the characteristic foliation of a fibered Poisson structure look like when the fiber supports the zero Poisson structure:

Lemma 2.13 *Let pr* $: (P, \pi_P) \to (B, \pi_B)$ *a fibered Poisson structure with fiber* $(F, 0)$. *Then, the map pr sends each leaf of* $\mathcal{F}_{\pi_P}$ *diffeomorphically into a leaf of* $\mathcal{F}_{\pi_B}$.

Proof. If the fiber F is endowed with the zero Poisson structure, then each leaf of $\mathcal{F}_{\pi_P}$ fibers over a leaf of $\mathcal{F}_{\pi_B}$ with fiber a point, which proves the result. □

2.2.4 The symplectic case

It is natural to ask what can be said about non-degenerate fibered Poisson structures:

Definition 2.14 *A* ***fibered symplectic structure*** *is a fibered Poisson structure over a symplectic base and with a symplectic fiber.*

Remark 2.15 *By Lemma 2.8 fibered symplectic structures are symplectic structures which are fibered Poisson structures (and condition* (3) *in the Definition 2.6 is void).*

Symplectic fibered structures give the first indication on the presence of (partial) flat connections associated to fibered Poisson structures:

Proposition 2.16 *Let (P, ω_P) be a fibered symplectic structure over (B, ω_B) with fiber (F, ω_F), where $\omega_P \in \Omega^2(P)$, $\omega_B \in \Omega^2(B)$, $\omega_F \in \Omega^2(F)$ are symplectic forms. The closed 2-form*

$$\omega_v := \omega_P - pr^*\omega_B$$

is symplectic on the fibers, which are symplectomorphic to $i_b^*\omega_F$, and the symplectic orthogonal w.r.t. ω_P is the kernel of ω_v. Hence, the symplectic orthogonal to the fibers defines a flat Ehresmann connection.

Proof. First of all, we need to show that

$$i_b^*\omega_P = \omega_F,$$

because then we have that, for all $b \in B$, the fiber F_b is a symplectic submanifold of (P, ω_P) and, as consequence we have that ω_v is symplectic on F_b. Indeed, let $u, v \in T_pP$ be vector tangents to the fiber F_b, and let $\widetilde{f_1}$, $\widetilde{f_2} \in C^\infty(P)$ be, such that

$$u = X_{\widetilde{f_1}}(p) = i_{b*}X_{f_1}(y), \quad v = X_{\widetilde{f_2}}(p) = i_{b*}X_{f_2}(y),$$

where

$$f_1 = i_b^*\widetilde{f_1}, \qquad f_2 = i_b^*\widetilde{f_2}.$$

Therefore

$$\begin{aligned} i_b^*\omega_y(X_{f_1}, X_{f_2}) &= (\omega_P)_p(X_{\widetilde{f_1}}, X_{\widetilde{f_2}}) = \{\widetilde{f_1}, \widetilde{f_2}\}_p \\ &= \{f_1, f_2\}_y = (\omega_F)_y(X_{f_1}, X_{f_2}). \end{aligned}$$

The symplectic orthogonal to the fibers w.r.t. ω_P is spanned by Hamiltonians of functions pulled back from the base. Thus if $w, z \in (T_pF)^\perp$, then

$$w = X_{\mathrm{pr}^*h_1}(p), \qquad z = X_{\mathrm{pr}^*h_2}(p),$$

where $h_1, h_2 \in C^\infty(B)$. Now by condition (1) in Definition 2.6 we have:

$$\begin{aligned} (\omega_P)_p(w, z) &= (\omega_P)_p(X_{\mathrm{pr}^*h_1}, X_{\mathrm{pr}^*h_2}) = \{\mathrm{pr}^*h_1, \mathrm{pr}^*h_2\}_p \\ &= \{h_1, h_2\}_b = (\omega_B)_b(X_{h_1}, X_{h_2}) \\ &= (\omega_B)_b(\mathrm{pr}_*X_{\mathrm{pr}^*h_1}, \mathrm{pr}_*X_{\mathrm{pr}^*h_2}). \end{aligned}$$

Hence we conclude that $\omega_P - q^*\omega_B$ is symplectic on fibers and vanishes on the symplectic orthogonal w.r.t. of the fibers. Since ω_v is a closed form, then

$\ker(\omega_v)$ defines a flat connection on the fibers of $\mathrm{pr} : P \to B$.

□

As a consequence of this result –under the standard compactness assumption– we can classify fibered symplectic structures by representations of the fundamental group of the base.

Corollary 2.17 *Fibered symplectic structure over the base (B, ω_B) with compact fiber (F, ω_F) are classified by representations:*

$$\pi_1(B, b_0) \to Symp(F, \omega_F).$$

Proof. Let (P, ω_P) a fibered symplectic structure, with base (B, ω_B) and fiber (F, ω_F). Then, by Proposition 2.16 we have a flat connection such that the parallel transport is by symplectomorphism. Upon fixing $b_0 \in B$ we obtain the corresponding representation of the fundamental group.

Conversely, let a representation of the fundamental group of B by symplectomorphisms be given. Consider the universal covering space of $p : \widetilde{B} \to B$ with the induced symplectic form $p^*\omega_B \in \Omega(\widetilde{B})$. Consider the diagonal action of $\pi_1(B, b_0)$ in $(\widetilde{B} \times F, p^*\omega_B + \omega_F)$, which is by symplectomorphisms. Then

$$q : (\widetilde{B} \times F, p^*\omega_B + \omega_F)/\pi_1(B, b_0) \to (B, \omega_B) \simeq (\widetilde{B}, p^*\omega_B)/\pi_1(B, b_0)$$

is a fibered symplectic structure with fiber (F, ω_F).

□

Example 2.7 *If (B, ω_B) is simply connected, then the only fibered symplectic structure with compact fiber is the product of symplectic structure.*

The following corollary shows that fibered Poisson structures have a rather rich geometry related to (partial) flat connections:

Corollary 2.18 *Let $pr : (P, \pi_P) \to (B, \pi_B)$ be a fibered Poisson structure with fiber (F, π_F). Then the following holds:*

1. *The symplectic leaf $(\mathcal{O}_p, \omega_{\mathcal{O}_p})$ is a fibered symplectic structure over the symplectic leaf $(\mathcal{O}_b, \omega_{\mathcal{O}_b})$ of π_B with fiber the symplectic leaf $(\mathcal{O}_y, \omega_{\mathcal{O}_y})$ of π_F, where $p \in P, y \in F, b \in B$ and $b = pr(p), p = i_b(y)$.*

2. *The collection of symplectic flat connections on leaves fits into a foliation $\mathcal{F}_0$ each of whose leaves is mapped by pr diffeomorphically into a leaf of $\mathcal{F}_{\pi_B}$.*

Proof. The inclusion of symplectic leaves in a Poisson manifold is a Poisson morphism, then we have the following commutative diagram:

$$\begin{tikzcd} \mathcal{O}_p \arrow[r, "i_{\mathcal{O}_p}"] \arrow[d, "\widetilde{\mathrm{pr}}"'] & P \arrow[d, "\mathrm{pr}"] \\ \mathcal{O}_b \arrow[r, "i_{\mathcal{O}_b}"'] & B, \end{tikzcd}$$

where $i_{\mathcal{O}_p} : \mathcal{O}_p \to P$, $i_{\mathcal{O}_b} : \mathcal{O}_b \to B$ and $\mathrm{pr} : P \to B$ are Poisson morphisms and

$$\mathrm{pr} \circ i_{\mathcal{O}_p} = i_{\mathcal{O}_b} \circ \widetilde{\mathrm{pr}}.$$

Because $\mathcal{O}_b$ is a regular immersed submanifold in B, then $\widetilde{\mathrm{pr}} : \mathcal{O}_p \to \mathcal{O}_b$ is a surjective submersion. Let $f_i \in C^\infty(\mathcal{O}_b)$, $i = 1, 2$, and U be a neighborhood of $b \in \mathcal{O}_b$, that

$$i^*_{\mathcal{O}_b} \widetilde{f}_i = f_i,$$

where $\widetilde{f}_i \in C^\infty(B)$ and $i = 1, 2$. Then

$$\begin{aligned} \widetilde{\mathrm{pr}}^*\{f_1, f_2\}_b &= \widetilde{\mathrm{pr}}^*\{(i_{\mathcal{O}_b})^* f_1, (i_{\mathcal{O}_b})^* f_2\}_b = (i_{\mathcal{O}_b} \circ \widetilde{\mathrm{pr}})^*\{\widetilde{f_1}, \widetilde{f_2}\}_p \\ &= (\mathrm{pr} \circ i_{\mathcal{O}_m})^*\{\widetilde{f_1}, \widetilde{f_2}\}_p = \{(\mathrm{pr} \circ i_{\mathcal{O}_m})^* \widetilde{f_1}, (\mathrm{pr} \circ i_{\mathcal{O}_m})^* \widetilde{f_2}\}_p \\ &= \{(i_{\mathcal{O}_b} \circ \widetilde{\mathrm{pr}})^* \widetilde{f_1}, (i_{\mathcal{O}_b} \circ \widetilde{\mathrm{pr}})^* \widetilde{f_2}\}_p = \{\widetilde{\mathrm{pr}}^* f_1, \widetilde{\mathrm{pr}}^* f_2\}_p. \end{aligned}$$

The previous property is referred to each single leaf with its manifold topology.

To prove item (2) let $T^v P$ denote the vertical distribution for the fiber bundle $P \to B$. Equation (2-2) implies that $\pi_P^\#(\mathrm{Ann}(T_p^v P))$ is exactly the symplectic orthogonal to $T_p\mathcal{O} \cap F_p$ inside $T_p\mathcal{O}_p$. Thus this is a smooth distribution locally generated by a finite collection of vector fields. It is an involutive distribution for the following reason: it is contained inside $\pi_P^\#(TP)$ and therefore involutivity can be checked inside each symplectic leaf, and this follows by Frobenius' Theorem. Because the (local) vector fields in the finite families which span the distribution are Hamiltonian, the property that their flows preserve the distribution can also be checked inside each symplectic leaf, where it reduces to Frobenius' Theorem. By Stefan-Sussmann theory these are sufficient conditions to deduce that the distribution integrates into a foliation, as we wanted to prove. $\square$

2.3 Fibrations and foliations

We present a brief discussion about the notion of fibration for manifolds endowed with the kind of foliations which come from Poisson geometry. This material is known to experts on Stefan-Sussmann theory of foliations. We chose to include it for the sake of having a reference to be used through this thesis, and because there are always topological subtleties that need to the checked when one works with initial immersed submanifolds (the leaves of Stefan-Sussmann foliations).

Definition 2.19 *Let P be a manifold of dimension n. A foliation $\mathcal{F}$ on P is defined by the following data:*

1. *An assignment of a subspace to each point $p \mapsto D_p \subset T_pP$, so that the corresponding dimension function is lower semicontinuous.*

2. *A a product atlas centered at every point*

$$\phi_p : U_p \to \mathbb{R}^d \times \mathbb{R}^{n-d},\ p \in P,$$

so that $d = d(p)$ is the minimum of the dimension function on U_p and there exists a local assignment D_i on $\mathbb{R}^{n-d}$, $q \to D_{p,q} \subset \mathbb{R}^{n-d}$, with the property that:

$$\phi_{p*}D_m = \mathbb{R}^d + D_{p,(\mathrm{pr}_2(\phi_p(m)))}, \quad \forall m \in U_p.$$

Foliations whose dimension function is constant are called ***regular****. Otherwise, they are called* ***singular****.*

A foliation as in Definition 2.19 defines a partition of P refining the partition given by the dimension function. Briefly:

1. an **integral submanifold** of $\mathcal{F}$ is (the image of) an immersed submanifold whose tangent space at every point equals the distribution;

2. a **leaf** of $\mathcal{F}$ is a connected integral submanifold maximal with respect to the inclusion;

3. a **plaque** of $\mathcal{F}$ is a subset of the form $\phi_p^{-1}(\mathbb{R}^d \times \{y\})$, for $y \in \mathbb{R}^{n-d}$ a point where the dimension function attains a local minimum.

As the product atlas furnishes a basis of the topology, we may assume the domain U_p to be small enough, so that each plaque is an embedded submanifold. Observe that by the property of product charts plaques are integral submanifolds of $\mathcal{F}$. Every point in P is contained in a plaque, and, therefore, through every point of P there passes a (unique) leaf of $\mathcal{F}$.

The following lemma is the result we are interested in for the purposes of this thesis:

Lemma 2.20 *Let $(B, \mathcal{F}_B)$ and $(F, \mathcal{F}_F)$ be two foliated manifolds and let $pr : P \to B$ be a locally trivial bundle with fiber F whose structural group consists of diffeomorphisms of the foliated manifold $(F, \mathcal{F}_F)$ which are (locally) constant on the leaf space. Then $pr : P \to B$ inherits a foliation $\mathcal{F}$ with the following property:*

Each leaf of $\mathcal{F}$ fibers over a leaf of $\mathcal{F}_B$ with fiber a leaf of $\mathcal{F}_F$. Moreover, if the diffeomorphisms of the structural group leave each leaf of $\mathcal{F}_F$ fixed setwise (i.e., they are the identity on the leaf space, then there is a natural bijection between the leaf space of $\mathcal{F}$ and the Cartesian product of the leaf spaces of the base and of the fiber:

$$P/\mathcal{F} \simeq B/\mathcal{F}_B \times F/\mathcal{F}_F.$$

Proof. Let $\{(U_i, \Phi_i)\}_{i\in I}$ be a system (the open subsets cover the base) of trivializations for $\mathrm{pr} : P \to B$:

$$\Phi_i : \mathrm{pr}^{-1}(U_i) \to U_i \times F.$$

It is a elementary to check that foliations as in Definition 2.19 are closed under Cartesian products. In particular on (the target of) each trivialization $U_i \times F$ we have a product foliation with factors $\mathcal{F}_{B|U_i}$ and $\mathcal{F}_F$.

By hypotheses the cocycles $U_i \cap U_j \to \mathrm{Diff}(F, \mathcal{F}_F)$ consists of diffeomorphisms which send a leaf of $\mathcal{F}_F$ to the same leaf of $\mathcal{F}$, for every point in $U_i \cap U_j$. Therefore the change of trivialization

$$\Phi_j \circ \Phi_i^{-1} : (U_i \cap U_j) \times F \to (U_j \cap U_i) \times F$$

preserves the restricted product foliations on $(U_i \cap U_j) \times F$ and on $(U_j \cap U_i) \times F$. Thus, upon patching the product foliations we induce a foliation $\mathcal{F}$ on P.

Let $\mathcal{O}$ be a leaf of $\mathcal{F}$. On the domain of a trivialization one checks that:

- the dimension of the rank (and kernel) of the differential of the restriction of pr to $\mathcal{O}$

  $$\mathrm{pr}_*|_{T\mathcal{O}}$$

  is locally constant;
- Moreover, $\mathrm{pr}_*|_{T\mathcal{O}}$ is the distribution of $\mathcal{F}_B$ at $\mathrm{pr}(\mathcal{O})$.

Because $\mathcal{O}$ is connected these dimensions are constant. Because $\mathrm{pr}|_{\mathcal{O}} : \mathcal{O} \to B$ is a map of constant rank, its image is an immersed submanifold. By the second property above, it must be an integral submanifold of $\mathcal{F}_B$. Since $\mathcal{O}$ is connected so $\mathrm{pr}(\mathcal{O})$ is. If the image would not be a leaf of $\mathcal{F}_B$ (if it failed to be maximal), then one could prove as well that $\mathcal{O}$ would not be a leaf of $\mathcal{F}$ either.

Regarding topological/smooth issues, because $\mathrm{pr}(\mathcal{O})$ is a leaf of $\mathcal{F}_B$ it is regularly immersed submanifold. Therefore the map $\mathrm{pr} : \mathcal{O} \to \mathrm{pr}(\mathcal{O})$ is smooth, i.e., a surjective submersion (the restriction map $\mathrm{pr} : \mathcal{O} \to B$ is obviously smooth, but if we restrict the codomain to $\mathrm{pr}(\mathcal{O})$ we need to use that the leaf is a regularly immersed submanifold to deduce smoothness). By working again on the domain of a trivializations one immediately sees that the fiber of $\mathrm{pr} : \mathcal{O} \to \mathrm{pr}(\mathcal{O})$ is a leaf of $\mathcal{F}_F$. $\square$

Example 2.8 *Let $B = \mathbb{S}^1$ with $\mathcal{F}_B$ be given by the circle itself, let $F = \mathbb{S}^n$, $n \geq 1$, with $\mathcal{F}_F$ given by the north and south pole, and let pr : $P \to \mathbb{S}^1$ be the mapping torus associated to the antipodal map on $\mathbb{S}^n$. By Lemma 2.20 we obtain a foliation $\mathcal{F}$ on P which has to leaves one of which is a circle.*

It is interesting to compare Example 2.8 with the Cartesian product, which is the mapping torus associated to the identity. The reason is that for n odd the antipodal map and the identity are homotopic, and therefore produce equivalent fibrations. However, the two foliations are nonequivalent since their number of leaves differs.

3
Construction of fibered Poisson structures

The main result of this chapter is a procedure to construct fibered Poisson structures on associated bundles. This will require certain kinds of Poisson structures on principal bundles and certain classes of Poisson actions. For this we introduce the natural notions of tangential actions and no-where tangential actions. The conditions satisfied by these actions will be instrumental to control the characteristic foliation of the Poisson structures we want to place on our associated bundles. We shall also illustrate with examples what happens if we relax the requirements of our main construction.

We finish with the notion of cotangent modifications, which is an instance of our main construction which we can analyze in more detail.

3.1
More on fibrations and foliations

In order to define a fibered Poisson structure on a fibration $P \to B$ with Poisson fiber (F, π_F), we need P to be given by a cocycle which preserves $\mathcal{F}_{\pi_F}$ and acts trivially on its leaf space, so we can put a foliation $\mathcal{F}$ on P in advance.

We are going to describe a situation where such cocycles appear naturally, and, in some cases, in a unique fashion. For that we need (part of) the following definition:

Definition 3.1 *Let $(P, \mathcal{F}_P)$ be a foliated manifold and let G be a Lie group acting on P by diffeomorphisms of $(P, \mathcal{F}_P)$. Denote by $G \cdot p$ the orbit passing through $p \in P$.*

We say the action is:

1. ***tangential*** *(to the leaves of $\mathcal{F}_P$) if each orbit $G \cdot p$ is contained in a single leaf of $\mathcal{F}_P$;*

2. ***no-where tangential*** *if $(T_p G \cdot p) \cap (T_p \mathcal{F}_p) = \{0\}$.*

Note that if G is connected being tangential becomes an infinitesimal property.

For Poisson manifolds we will speak about tangential and no-where tangential (Poisson) actions of Lie groups relative to the characteristic foliation. The following are natural examples of tangential actions in the Poisson setting:

Example 3.1 *Let (G, π) be a Poisson Lie group and consider the dressing transformations of the the dual lie group G^* of G, see Appendix C. Then, by Theorem C.15 we have that the orbits of the left and the right dressing transformations are exactly the symplectic leaves of π. That is, this actions are tangential.*

No-where tangent actions which are also free are very related to fibered Poisson structures with zero Poisson fiber (cf. Lemma 2.13):

Lemma 3.2 *Let (P, π_P) be a Poisson manifold on a principal G-bundle where G acts in a Poisson fashion. If (P, π_P) is a fibered Poisson structure over (B, π_B) with Poisson fiber $(G, 0)$, then the G-action is no-where tangent.*

Proof. Let us assume that (P, π_P) is a fibered Poisson structure over (B, π_B) with Poisson fiber $(G, 0)$. By Lemma 2.13 we have that the projection sends leaves of π_P diffeomorphically into leaves of π_B. This implies that the G-action is no-where tangent (and clean at intersection points).

□

Here is the technical result we are interested in:

Lemma 3.3 *Let $pr : P \to B$ be a principal G-bundle, let G act on the foliated manifold $(F, \mathcal{F}_F)$ and let $\mathcal{F}_B$ be a foliation in the base.*

If the action is tangential then it canonically defines a foliation $\mathcal{F}_M$ on the associated bundle $M := (P \times F)/G$ which fibers over $\mathcal{F}_B$ with fiber $\mathcal{F}_F$. Moreover, a cocycle defining $\mathcal{F}$ can be obtained out of any trivializing system of P.

Proof. Recall that a trivializing system for the principal bundle P is given by an open cover $\{U_i\}_{i \in I}$ of B together with sections

$$\sigma_i : U_i \to P, \quad V_i = \sigma_i(U_i).$$

The trivializing system canonically provides a system of trivializations of P,

$$\mathrm{pr}^{-1}(U_i) \simeq U_i \times G$$

whose associated cocycle is by diffeomorphisms given by the (left, say) G-action. This system of trivializations of P canonically produces a system of trivializations of the associated bundle M whose cocycle is by diffeomorphism coming from the G-action. Since a tangential action preserves $\mathcal{F}_F$ and acts trivially on its leaf space, as indicated in Lemma 2.20 out of $\mathcal{F}_F$, $\mathcal{F}_B$ and the cocycle one gets $\mathcal{F}$ on M with the desired properties.

If we choose different trivializing systems $\mathcal{V}$ and $\mathcal{V}'$, then on $\mathrm{pr}^{-1}(U_i \cap U_j')$, we have the foliations coming from the product foliations on $U_i \times F$ and $U_j' \times F$, respectively. The foliation will be the same if and only if the cocycle is by diffeomorphism which preserve each leaf setwise, which again, holds because the action of G is tangential.

□

3.2 A construction of fibered Poisson structures

In this section contains the main result of this thesis: the construction of fibered Poisson structures out of Poisson actions with some additional properties.

Theorem 3.4 *Let $pr : P \to B$ be a principal G-bundle, let F be a smooth manifold and let π_P and π_F be G-invariant Poisson structures on P and F, respectively. Let us assume that the Poisson action on (F, π_F) is tangential.*

Then the Poisson structure π induced on the associated bundle

$$q : M := (P \times F)/G \to B, \quad \pi := \widetilde{pr}_*(\pi_P + \pi_F), \quad \widetilde{pr} : P \times F \to M$$

has the following properties:

1. *It is the sum of the commuting Poisson structures $\pi_0 + \pi_v$, with π_v vertical.*
2. *It is a fibered Poisson structure over (B, π_B) with fiber (F, π_F), where π_B is the Poisson structure induced by π_P on B.*
3. *The characteristic foliation of π_0 equals the foliation $\mathcal{F}_h$ of π (see Corollary 2.18).*

Proof. Item (1) is rather straightforward: by construction

$$\pi = \pi_0 + \pi_v,$$

where $\pi_0 = \tilde{\mathrm{pr}}_* \pi_P$ and $\pi_v = \tilde{\mathrm{pr}}_* \pi_F$. The Poisson structure π_F on $P \times F$ (rather, its pullback by the second projection) is vertical with respect to the first projection. Since the bundle structure on $M \to B$ is induced by the first projection, it follows that π_v is vertical as well.

The Poisson structures π_P and π_F commute. Since for projection-related multivector fields the projection and the Schouten bracket commute, it follows that

$$[\pi_0, \pi_v] = 0,$$

The projection $q : M \to B$, is given by $q([p, y]) = [p] = b$. Note that we have the following commutative diagram:

$$\begin{tikzcd} P \times F \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\widetilde{\mathrm{pr}}"'] & P \arrow[d, "\mathrm{pr}"] \\ M \arrow[r, "q"'] & B, \end{tikzcd}$$

where $\mathrm{pr}_1 : P \times F \to P$, $\tilde{\mathrm{pr}} : P \times F \to M$ and $\mathrm{pr} : P \to B$ are surjective Poisson submersions. Then $q : M \to B$ is a Poisson morphism, which proves condition (1) of a fibered Poisson structure.

The properties above only require the actions of G on base and fiber to be Poisson.

We now check the other requirements so that π be a fibered Poisson structure.

Because the action on (F, π_F) is tangential by Lemma 3.3 M is endowed with a foliation $\mathcal{F}$ with the right fibering properties with respect to $\mathcal{F}_{\pi_B}$ and $\mathcal{F}_{\pi_F}$.

To check conditions (2) and (3) in the definition of fibered Poisson structures we will choose trivializations of P which use that the action of G on (P, π_P) is no-where tangential. In fact this trivializations will satisfy the requirements of the coordinates used in Lemma 2.8.

Because the action of G on (P, π_P) is no-where tangential at $p \in P$ it is possible to take a slice V to the G-action which contains the tangent space of the leaf of π_P through p. Associated to the slice V we have the trivializations of P and M:

$$\mathrm{pr}^{-1}(U) \simeq V \times G, \quad q^{-1}(U) \simeq V \times F.$$

Let $\mathcal{O}$ be the leaf of $\mathcal{F}$ through $[p, y] \in M$. The isomorphism $\mathrm{pr} : V \to U$ identifies locally the symplectic leaf of π_P though p, denoted by $\mathcal{O}_p$, with the symplectic leaf of π_B through $b = [p]$. Thus, in the trivialization

$$q^{-1}(U) \cong V \times F$$

the base factor of the product leaf $\mathcal{O}$ is the leaf $\mathcal{O}_p$. Now we want to argue that $T_p\mathcal{O}_p \times \{y\}$ is also identified with the leaf tangent space of the leaf of π_0 through $[p, y]$: we take any function $g \in C^\infty(V \times F)$ and we pull it back to a (diagonal) G-invariant function k on the open subset

$$V \times G \times F \subset P \times F.$$

We have to compute $\pi_P(dk)$ and then project by the differential of the diagonal action to a vector at $[p, y] \in V \times F$. To calculate $\pi_P(dk)$ we may restrict first k to $P \times \{y\}$, i.e., to calculate $\pi_P(d_P k)$. More importantly, because the tangent space of $\mathcal{O}_p$ at p is inside of the slice V, we may take the restriction of k to $V \times \{y\}$. In other words, we may assume from the very beginning that the started with a function g defined on the slice V. Finally, note that from the moment we restrict to the factor $P \times \{y\}$, we may as well consider just the action of G in P rather than the diagonal one. This shows that at $[p, y] \in V \times F$ the Hamiltonian vector fields of π_0 are those of π_B, i.e., the base directions of $\mathcal{O}$. By Corollary 2.9 this implies condition (3) for fibered Poisson structures in the presence of conditions (1) and (2).

To prove condition (2) we start with $f \in C^\infty(F)$ and define $h \in C^\infty(P \times F)$ as follows: first we pull f back by the second projection on $V \times F$ to our extension $\widetilde{f}$; next we pull it back to a G-invariant function h on $V \times G \times F$. Once more the Hamiltonian of h will have a summand coming from $d_P h$ and another coming from $d_F h$. Because h is constant in the slice V, the former summand vanishes. The last remark is that because G acts freely on P and V is a slice, then $V \times \{e\} \times F$ becomes a slice for the diagonal G-action. This means that projecting the vertical vector $\pi_F^\#(d_F h)$ by $\widetilde{\mathrm{pr}}_*$ amounts to applying the identity. Therefore

$$X_{\widetilde{f}}[p, y] = X_f(y).$$

Thus we conclude that π is a fibered Poisson structure. We also showed that functions on V and F have Hamiltonian vector fields at p contained in the slice and fiber, respectively. In other words, that the corresponding horizontal and vertical distributions are symplectically orthogonal. Since the horizontal distribution is the Hamiltonian distribution of π_0 we obtain the equality:

$$\mathcal{F}_{\pi_0} = \mathcal{F}_0.$$

□

Let $P \to B$ be a principal G bundle endowed with a Poisson structure π_P which is also a locally trivial Poisson structure over (B, π_B) with fiber $(G, 0)$ and so that G acts in a Poisson fashion. This is equivalent to saying that the action of G is Poisson and that we may take a system of slices each of which is a Poisson submanifold of (P, π_P) (so that the G-action propagates the slices into product Poisson trivializations). If G-acts as well tangent to the orbits of (F, π_F) and in a Poisson fashion, then it is straightforward to verify that the associated bundle M inherits a locally trivial Poisson structure.

In fact, for the zero Poisson fiber there is no difference between locally trivial and fibered Poisson structures. The reason is that condition (3) on the local triviality of the characteristic foliation together with condition (1) automatically furnish product Poisson structures.

By Lemma 3.2 for a fibered Poisson structure with fiber $(G, 0)$ the action is no-where tangential. So our Theorem 3.4 says that if we relax the hypothesis from locally trivial Poisson structure with Poisson G action, to Poisson no-where tangential Poisson action of G on (P, π_P), then we obtain a fibered Poisson structure on the associated bundle.

On a principal bundle getting a locally trivial Poisson structure on with zero fiber and Poisson principal action is quite demanding. The compatibility of the product Poisson trivializations with the G-action means that there is a cocycle which is given by diffeomorphism g_{ij} of G. Therefore G must support a principal flat connection. In case the structural group be $\mathbb{S}^1$, then we can relax the requirement that $g_{ij} \in \mathbb{S}^1$, as that further reduction can always be achieved by using the standard retraction from $\mathrm{Diff}(\mathbb{S}^1)$ to rotations:

Example 3.2 *If (P, π_P) is a locally trivial Poisson structure over a surface with fiber $(\mathbb{S}^1, 0)$, then P must satisfy the Milnor-Wood inequality [8, Theorem 1]. In particular no such example different from a product exists over the 2-sphere.*

Conversely, given $(\sum, \sigma)$, where σ is any bivector field, for any P is a $\mathbb{S}^1$-fiber bundle over $\sum$ satisfying the Milnor-Wood inequality, there exist a locally trivial Poisson structure on P over $(\sum, \sigma)$ with zero Poisson fiber for which the principal action is Poisson. More generally, one can consider as starting data a Poisson manifold (P, π_P) and a representation of $\pi_1(B, \pi_B)$ on a group G.

There are indeed examples of no-where tangential Poisson actions which are not fibered Poisson structures:

Example 3.3 *Consider on $\mathbb{C}^2$ the following toric Poisson structure that we will discuss in Chapter 4:*

$$\pi = -2iz_1\bar{z}_1 \frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial \bar{z}_1} - 2iz_2\bar{z}_2 \frac{\partial}{\partial z_2} \wedge \frac{\partial}{\partial \bar{z}_2}.$$

This structure is invariant under the action of $(\mathbb{C}^*)^2$, and thus under the diagonal $\mathbb{C}^*$-action. This implies that is we let R and E denote the rotation vector field (generator of the diagonal $\mathbb{S}^1$-action) and the Euler vector field, respectively, then:

$$\pi_P = \pi - R \wedge E$$

is Poisson and still $\mathbb{C}^*$-invariant.

The original Poisson structure π has four symplectic leaves: the origin, the lines $z_1 = 0$, $z_2 = 0$ and their complement. By construction it is the sum of commuting Poisson structures:

$$\pi = \pi_P + R \wedge E,$$

where the latter is vertical with respect to the diagonal $\mathbb{C}^*$-action. The $\mathbb{C}^*$-orbits/fibers are symplectic submanifolds of the symplectic leaves, which induced Poisson structure given by $R \wedge E$. By defining π_P we obtain a Poisson structure in which the $\mathbb{C}^*$-action becomes no-where tangent to the leaves. To show that this is not a fibered Poisson structure one considers the function:

$$d : \mathbb{C}^2 \to \mathbb{C}, \quad (z_1, z_2) \mapsto z_1 z_2.$$

A short computation shows that its regular level sets are the symplectic 2-dimensional leaves of π_P, and, that, the zero level set $z_1 z_2 = 0$ is the union of the zero dimensional leaves. If π_P were a fibered Poisson structure, its characteristic foliation would be locally trivial. In particular 2-dimensional leaves should accumulate into zero dimensional leaves. But this is not possible because the function d is not proper.

We will finish this section by describing a way of relaxing a bit the condition on the principal action of G being no-where tangential on (P, π_P), so that we still get a Poisson structure with reasonable behavior with respect to the fibration structure:

Proposição 3.1 *Let $pr : P \to B$ be a principal G-bundle, let π_P be a G-invariant Poisson structure on P and let (F, π_F) be a Poisson manifold on which G-acts by tangential Poisson diffeomorphisms. Let π be the induced Poisson structure on the associated bundle M which is the sum of commuting Poisson structures $\pi_0 + \pi_v$, where the latter is vertical.*

Let us further assume that:

(i) for every point p of P the action of G in (P, π_P) is either no-where tangential at p or the orbit is a symplectic submanifold of $\mathcal{O}_p$;

(ii) the orbits of the action of G in F are isotropic submanifolds of the corresponding symplectic leaf.

Then the following properties for π hold:

1. *There is an equality of foliations:*

$$\mathcal{F}_\pi = \mathcal{F}.$$

2. *Each fiber F_b is a Dirac submanifold, but in general not diffeomorphic to (F, π_F) because the Dirac structure is not induced by just the vertical Poisson structure.*

Proof. We will use the same notation that in the proof of Theorem 3.4. In the product trivialization of M given by a slice $V \times F$ at p we must analyze the Hamiltonians coming from the functions on each of the factors. If the action of G is no-where tangent at p then the analysis was already done in the proof of Theorem 3.4. Otherwise, we take the slice at p containing the symplectic orthogonal to $G \cdot p$ inside $\mathcal{O}_p$.

Recall that $V \times \{e\} \times F \subset V \times G \times F$ is a slice to the diagonal G-action. Let l be a function which is the pullback to $V \times F$ of a function on V by the first projection. Because of the slice property of $V \times \{e\} \times F$, the pullback of l by the diagonal G action is a function h which is constant on G-orbits (recall that l is constant in $\{p\} \times F$ and therefore the G-action on F picks always the same value!). Therefore we have:

$$(\pi_P + \pi_F)^\#(dh)(p, y) = \pi_P^\#(dh)(p) \in T_pV,$$

where the latter property uses that h is constant on the symplectic orbit $G \cdot p$ and T_pV contains its symplectic orthogonal. Therefore the Hamiltonians of functions on the base span a subspace $H_p \subset T_p\mathcal{O}$ with trivial intersection with T_yF and mapping isomorphically into $T_b\mathcal{O}_b$.

To prove the equality of foliations we need to show that functions which are the pullback of functions on the fiber have Hamiltonians spanning $T_y\mathcal{O}_y$. As usual, we start with such a function f on F, pull it back first to $\widetilde{f}$ on $V \times F$ and then to a function h on $V \times G \times F$. Once more this function is not necessarily constant on $\{p\} \times G \times \{y\}$. Its Hamiltonian has a double contribution:

$$\pi_P^\#(d_Ph)(p, y) + \pi_F^\#(d_Fh)(y) = \pi_P^\#(d_Ph)(p) + X_f(y).$$

By construction h is constant on V which contains the symplectic orthogonal to the G-orbit. Therefore

$$\pi_P^\#(d_Ph)(p, y) = X_\alpha^P(p),$$

where $\alpha \in \mathfrak{g}$ and X^P_α is the corresponding infinitesimal generator. Therefore when passing to the quotient by the G-action we obtain

$$\pi^{\#}(\widetilde{df})[p,y] = X_f(y) - X^F_\alpha(y). \tag{3-1}$$

Thus we must show that we obtain all Hamiltonian directions in $T_y\mathcal{O}_y$. For Hamiltonians not tangent to $G \cdot y$ at y this is not a problem as (3-1) cannot vanish. Let us assume that $X_f(p)$ is tangent to $G \cdot y$ at y. Because the orbit is isotropic this means that f must be constant on $G \cdot y$. Therefore the function h above becomes constant on $G \cdot p$ and 3-1 becomes

$$\pi^{\#}(\widetilde{df})[p,y] = X_f(y).$$

Hence $\mathcal{F}_\pi = \mathcal{F}$.

To check that the pullback Dirac structure $L_\pi|_{F_b}$ is smooth simply note that because V contains the symplectic orthogonal to the G-orbit, it follows that:

$$L_\pi|_{F_b} = \{(\pi^{\#}_F(\widetilde{df}, df) \,|\, f \in C^\infty(F)\}.$$

Because

$$\pi^{\#}_F(\widetilde{df}) = X_f(y) - X^f_\alpha(y),$$

and because by construction α depends smoothly of $df(p)$, it follows that $L_\pi|_{F_b}$ is smooth. Thus is is a Poisson structure π_b whose characteristic foliation is the same as that of π_F. □

If $G = \mathbb{S}^1$ then any tangential action has isotropic orbits. Another example of tangential action with isotropic orbits is any abelian Hamiltonian action.

Example 3.4 *Consider the coadjoint action of a Lie group G on $(\mathfrak{g}^*, \pi_{can})$. This is a Hamiltonian action and so it is its restriction to a maximal torus [9, Proposition 3.5.6].*

An example of an action whose satisfies requirement (1) in Proposition 3.1 is that of Example 3.3

Below we present another example which also shows that the Poisson Dirac structures on fibers π_b need not be diffeomorphic to π_F:

Example 3.5 *Consider the linear projection*

$$pr : \mathbb{R}^3 \to \mathbb{R}, \quad (b,x,y) \to b.$$

Endow $\mathbb{R}^3$ with the regular Poisson structure

$$\pi_P = b\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}.$$

Let $G = \mathbb{R}^2$ act on $(\mathbb{R}^3, \pi_P)$ by translations on the last two coordinates. This is a Poisson action whose orbits are exactly the symplectic leaves of π_P for $b \neq 0$ and a collection of orbits of dimension zero for $b = 0$. Therefore it is in the hypotheses of Proposition 3.1.

Let $F = \mathbb{R}^4$ endowed with the constant Poisson (symplectic) structure

$$\pi_F = \frac{\partial}{\partial u} \wedge \frac{\partial}{\partial v} + \frac{\partial}{\partial \zeta} \wedge \frac{\partial}{\partial \sigma}.$$

Let $\mathbb{R}^2$ act by translations

$$(r, s) \cdot (u, v, \zeta, \sigma) \mapsto (u + r, v, \zeta + s, \sigma).$$

This is a symplectic action whose orbits are isotropic planes. Therefore by Proposition 3.1 we obtain a Poisson structure π on the associated bundle M, which is a trivial bundle over $\mathbb{R}$ with fiber $\mathbb{R}^4$.

To compute the Poisson tensor π_b we take the linear coordinate functions on the fiber and observe that when pulled back to $\mathbb{R}^3 \times \mathbb{R}^4$ they become:

$$u - x, v, \zeta - y, \sigma.$$

Therefore:

$$\pi_b = \pi_F + b\frac{\partial}{\partial u} \wedge \frac{\partial}{\partial \zeta}.$$

Note that this is a symplectic structure for all b. We may modify our example by taking quotient of the fiber by the integral lattice, so that the new fiber of M is the torus T^4. Now the symplectic structures ω_b have cohomology class which depends linearly on b, and this makes them non symplectomorphic.

It is interesting to change the action on the fiber from the first and third variables, to the first and second. The result is that the orbits are no longer isotropic, but symplectic. The effect on π is that for $b = -1$ the corresponding fiber is not symplectic. It becomes a rank two Poisson structure.

3.3 Cotangent modifications

In this section we will introduce cotangent modifications. This is a specific case of Theorem 3.4 which we want to study in some detail. As we shall see, the cotangent modification aims to enlarge the trivial Poisson fibers of an abelian principal bundle (P, π_P), so that they become symplectic. Moreover, the advantage of having a Lie group as symplectic fiber allows for a partially infinitesimal description of cotangent modifications and related Poisson structures on the same total space. The results in this section will be used in the applications in the coming chapter.

Definition 3.5 *Let $P \to B$ be a locally trivial bundle with fiber F. We have that the* ***cotangent associated bundle*** $m : T_v^*P \to B$ *with fiber T^*F, is the bundle associated to the cotangent lift of the cocycles of P.*

Note that T^*F is a symplectic manifold [10, Chapter 2]. This means that T_v^*P supports a vertical Poisson structure, that is regular and make the fiber into symplectic leaves.

We are interested in enlarging this vertical Poisson structure with another Poisson structure coming from the base. For that we assume that P a principal G-bundle with a Poisson structure π_P so that the G-action is Poisson no-where tangent. Then, by Theorem 3.4 we can construct a fibered Poisson structure on T_v^*P over (B, π_B) with fiber (T^*F, π_{can}).

Now, suppose further that G is an abelian Lie group with Lie algebra $\mathfrak{g}$. Then its tangent bundle is also an abelian Lie group. Upon a choice of inner product on $\mathfrak{g}$ –which we assume from now on– the cotangent bundle T^*G becomes an abelian Lie group with Lie algebra $\mathfrak{g} \oplus \mathfrak{g}^*$. Let V denote the group integrating $\mathfrak{g}^*$, which we will call (vector) subgroup. We have canonical diffeomorphisms

$$T_v^*P \simeq P \times V \simeq P \times \mathfrak{g}^*. \tag{3-2}$$

Definition 3.6 *Let $P \to B$ be a principal G-bundle, where G is an abelian Lie group with a inner product on its Lie algebra. Let π_P be a G-invariant Poisson structure on P so that the G-action is no-where tangent.*

The ***cotangent modification*** *of (P, π_P) is the associated bundle $q : T_v^*P \to B$ endowed with induced fibered Poisson structure:*

$$\pi := \widetilde{pr}_*(\pi_P + \pi_{can}).$$

The following result relates the leaf spaces of (P, π_B), its base (B, π_B) and its cotangent modification:

Lemma 3.7 *Let $pr : P \to B$ be a principal G-bundle endowed with a Poisson structure π_P, so that the G-action is Poisson no-where tangent. Let (B, π_B) the induced Poisson structure on the base and let (T_v^*P, π) be the cotangent modification of (P, π_P).*

We have the sequence of surjective Poisson submersions between fibered Poisson structure

$$(T_v^*P, \pi) \xrightarrow{t} (P, \pi_P) \xrightarrow{pr} (B, \pi_B).$$

so that the composition induces on leaf spaces

$$T_v^*P/\mathcal{F} \to B/\mathcal{F}_{\pi_B}$$

the identity map.

In other words, (T_v^*P, π) is the result of appropriately inflating each leaf of π_B by adding symplectic directions. However, passing from π_B to π_P produces a G-fibration of leaf spaces (and note the absence of a map on leaf spaces from π to π_P.

Proof.

The composition of Poisson surjections uses that T^*G is the product of $G \times V$. Because the vertical Poisson structure coming from π_{can} is symplectic, and because the V-orbits are Lagrangian submanifolds of the T^*G-orbits, taking quotient by the V-action on $(T_v^*P \equiv P \times V, \pi)$ recovers (P, π_P). Because the actions of V and G commute, quotienting (P, π_P) to B produces the same Poisson structure as quotienting T_v^*P to B. For the latter quotient there is an identification of symplectic leaves because

$$q : (T_v^*P, \pi) \to (B, \pi_B)$$

is a fibered Poisson structure with symplectic fibers. □

We would like to characterize cotangent modifications among Poisson structures on T_v^*P. The first requirement is the presence of a T^*G-invariant vertical Poisson structure whose symplectic leaves are the fibers/T^*G-orbits. In this situation the inverse fiberwise symplectic structure is given infinitesimally by a map:

$$\varpi_{\pi_v} : B \to \wedge^2(\mathfrak{g} \oplus \mathfrak{g}^*).$$

Proposition 3.8 *Let π_v be a vertical Poisson structure on $T_v^*P \to B$, where G is an abelian group, such that fibers are symplectic and T^*G acts by Poisson diffeomorphisms.*

Then the following statements are equivalent:

1. $\pi_v = \widetilde{pr}_*(\pi_{can})$;

2. *The following two conditions hold:*

a. $\varpi_{\pi_v} : B \to \wedge^2(\mathfrak{g} \oplus \mathfrak{g}^*)$ *is a constant map;*

b. *Both* $\mathfrak{g}$ *and* $\mathfrak{g}^*$ *are Lagrangian w.r.t.* ϖ_{π_v} *and moreover an orthonormal basis of* $\mathfrak{g}$ *produces a symplectic basis.*

Proof. Let $\pi_v = \widetilde{\mathrm{pr}}_*(\pi_{\mathrm{can}})$ and let $\{\alpha_i\}$ be an (orthogonal) base of $\mathfrak{g}$ and let $\{\xi_i\}$ be its dual base in $\mathfrak{g}^*$. Then we can write:

$$\pi_v = \widetilde{\mathrm{pr}}_*(\pi_{\mathrm{can}}) = \sum X_{\alpha_i} \wedge X_{\xi_i} = \pi_{\mathrm{can}}$$

Hence its infinitesimal counter part $\varpi_{\pi_{\mathrm{can}}}$ is constant and $\mathfrak{g}$ and $\mathfrak{g}^*$ are Lagrangian subspaces. By construction we started with an orthonormal basis which produced a symplectic basis.

For the converse, condition (a) and (b) immediately give:

$$\pi_v = \sum X_{\alpha_i} \wedge X_{\xi_i} = \pi_{\mathrm{can}}.$$

Note that we need the additional (slightly subtle) hypotheses on basis because the group structure on T^*G incorporates the inner product. □

The characterization of cotangent modifications is as follows:

Proposition 3.9 *Let* π *a fibered Poisson structure on* T_v^*P *which is* T^*G*-invariant and has symplectic fibers:*

Then π *is the cotangent modification if and only if:*

1. *The vertical (regular) Poisson structure* π_v *defined by* π *equals* $\widetilde{pr}_*\pi_{can}$*, in which case*
$$\pi = (\pi - \pi_v) + \pi_v$$
is the sum of commuting Poisson structures, where $\pi - \pi_{can}$ *is a Poisson structure for which the* T^*G*-action is Poisson and no-where tangent.*

2. *If, moreover,* $P \subset T_v^*P \equiv V$ *is a Poisson submanifold of* $\pi - \pi_v$*, then* π *is the cotangent modification of the reduction of* (T_v^*P, π) *by* V *(more generally, if a section of* $T_v^*P \to P$ *is a Poisson submanifold, we get a diffeomorphism to the cotangent modification)*

Proof. That the cotangent modification has the above properties is straightforward.

Let us point out that a Poisson structure on a fibration whose fibers are symplectic submanifolds of the symplectic canonically induces a regular vertical structure π_v: it is the result of inverting on each fiber the symplectic form.

Using assumption (1) Proposition 3.7 gives us the explicit formula:

$$\pi_v = \sum X_{\alpha_i} \wedge X_{\xi_i}.$$

Now X_i and $\sum X_{\xi_i}$ are infinitesimal generators of the T^*G-action. By elementary properties of the Schouten bracket the T^*G-invariance of π implies:

$$[\pi, \pi_v] = 0.$$

Hence we obtain the commuting sum of Poisson structures:

$$\pi = (\pi - \pi_v) + \pi_v.$$

Now, if $P \subset T^*_v P$ is a Poisson submanifold of $\pi - \pi_v$, then its characteristic foliation must be tangent to P at the points of P. The V-invariance implies that $\pi - \pi_v$ is the result of propagating $\pi_P := (\pi - \pi_v)|_P$ by the V-action. From this we conclude that π is the cotangent modification of π_P. □

Proposition 3.9 is formalizing Example 3.3 once we remove the origin. There our group G is $\mathbb{S}^1$ and the identification $T^*\mathbb{S}^1 \equiv \mathbb{C}^*$ use the Rotation and Euler vector fields. The product structure in the total space is the usual:

$$\mathbb{C}^2 \backslash \{0\} \equiv \mathbb{S}^3 \times \mathbb{R}^*_+.$$

The Poisson structure π there has indeed symplectic fibers for which $\pi_v = E \wedge R$. However, it is not a cotangent modification, or diffeomorphic to it, because the 2-dimensional symplectic leaves of $\pi - \pi_v$ are not confined inside compact subsets of $\mathbb{C}^2 \backslash \{0\}$.

4
Fibered Poisson structures with finite number of leaves

In this chapter we apply our previous constructions to toric and Bruhat structures (see also Appendices B and C). We will use the analysis of cotangent modifications to show that toric Poisson structures fit into the weaker hypotheses of Proposition 3.1, and, that, upon taking quotient we obtain Poisson structures with a no-where tangent action of a compact torus. We will also note that for Poisson Lie group structures on compact Lie groups, the action of the maximal torus is no-where tangent. With these preliminary results, we present a direct application of Theorem 3.4 to toric Poisson manifolds and coadjoint orbits with their Bruhat Poisson structures. This application provides a way to build new examples of Poisson manifolds with a finite number of symplectic leaves. Finally, we discuss how our construction allows to construct fibered Poisson structures in topologically different fibrations (with the same base and fiber).

The toric Poisson structures on toric varieties and the Bruhat Poisson structures on coadjoint orbits are Poisson structures on compact manifolds which share the following important properties:

1. they have a finite number of symplectic leaves and the partial order given by the inclusion in the closure is described by a combinatorial object (polytope and Weyl group, respectively);

2. they have large abelian symmetry group.

4.1
Toric structures

Let X be a projective toric variety and let $\mathbb{T}$ denote the complex torus acting on X with dense orbit. The properties of **toric Poisson structures** (X, π) were introduced in [11] and briefly discussed in the Appendix B. The next construction –which was partially discussed in Example 3.3– illustrates how toric Poisson structures on $\mathbb{C}^n$ posses an interesting fibering property related to cotangent modifications, which fits into the structures discussed in Proposition 3.9:

Example 4.1 *Consider $\mathbb{C}$ with complex coordinate $z = x + iy$. The infinitesimal generators of the standard action of $\mathbb{C}^*$ on $\mathbb{C}$ are rotation and Euler vector fields:*

$$R := -iz\partial_z + i\bar{z}\partial_{\bar{z}}, \quad E := z\partial_z + \bar{z}\partial_{\bar{z}}.$$

The toric Poisson structure on $\mathbb{C}$ can be written:

$$E \wedge R = -2iz\bar{z}\frac{\partial}{\partial z} \wedge \frac{\partial}{\partial \bar{z}} = (x^2 + y^2)\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial \bar{x}},$$

which shows that it is $\mathbb{C}^*$-invariant.

The Toric Poisson structure on $\mathbb{C}^n$ is the product of toric Poisson structures. In dimension two the domain $\mathcal{U} \subset \mathbb{C}^2$ associated to projective space (toric variety) is $\mathbb{C}^2 \setminus \{0\}$ and the subgroup of $\mathbb{C}^{*2}$ is $\mathbb{C}^*$ diagonally embedded. Thus, as explained in Appendix C and [11] the quotient on $\mathbb{C}^2 \setminus \{0\}$ of

$$\pi_{\mathbb{C}^2} := -2iz_1\bar{z}_1\frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial \bar{z}_1} - 2iz_2\,\frac{\partial}{\partial z_2} \wedge \frac{\partial}{\partial \bar{z}_2}$$

is projective space with its toric Poisson structure:

$$(\mathbb{C}P(1), \pi_B).$$

The projection

$$pr : (\mathbb{C}^2 \setminus \{0\}, \pi_{\mathbb{C}^2}) \to (\mathbb{C}P(1), \pi_B)$$

is a Poisson submersion. The fibers –which are the $\mathbb{C}^*$-orbits– are symplectic manifolds. Then as described in section 3.3 associated to $\pi_{\mathbb{C}^2}$ there is a vertical Poisson structure π_v. Each symplectic structure on (a leaf of) π_v is encoded infinitesimally by a constant symplectic form on the Lie algebra of $\mathbb{C}^*$. If we let α and $i\alpha$ be the vectors in the Lie algebra of $\mathbb{C}^*$ generating the standard rotation and dilation actions, then it follows that

$$\varpi_{\pi_v}^{-1} = \alpha \wedge i\alpha.$$

Thus by Proposition 3.8 the vertical Poisson structure equals the canonical one for a cotangent modification:

$$\pi_{\mathbb{C}^2} = \pi_0 + \pi_{can}.$$

As shown in Example 3.3 this is not a fibered Poisson structure.

If we write $\mathbb{C}^* = \mathbb{S}^1 \times \mathbb{R}^*_+$, then

$$\mathbb{C}^2 \setminus \{0\}/\mathbb{S}^1 \simeq \mathbb{S}^3$$

inherits a Poisson structure π_P. Of course, $(\mathbb{C}^2\backslash\{0\}, \pi_{\mathbb{C}^2})$ cannot be the cotangent modification of $(\mathbb{S}^3, \pi_P)$ (or diffeomorphic to it) as the condition of having Poisson submanifolds in compact regions is not fulfilled (this is what Example 3.3 shows). However, the sequence of Poisson submersions

$$(\mathbb{C}^2\backslash\{0\}, \pi_{\mathbb{C}^2}) \to (\mathbb{S}^3, \pi_P) \to (\mathbb{C}P(1), \pi_B)$$

still meets the thesis of Lemma 3.9

Finally, observe that by construction the residual $\mathbb{S}^1$-action on $(\mathbb{S}^3, \pi_P)$ is no-where tangent. Therefore it is in the hypothesis of our main Theorem for $G = \mathbb{S}^1$. It is also worth noticing that $(\mathbb{C}^2\backslash\{0\}, \pi_{\mathbb{C}^2})$ is in the hypothesis of the weaker Proposition 3.1 for $G = \mathbb{C}^*$.

The previous example generalizes in a suitable sense to all toric Poisson structures:

Proposition 4.1 *Let X be a toric variety obtained as the quotient of $\mathcal{U} \subset \mathbb{C}^n$ by the action of $\mathbb{N} \subset \mathbb{C}^{*n}$. Let $\pi_{\mathbb{U}}$ and π_B be the restriction of the toric Poisson structures on $\mathbb{C}^n$ and the induced toric Poisson structure, respectively. Decompose $\mathbb{N} = NV$, where N is its maximal compact subtorus, $\mathfrak{n}$ is its Lie algebra and $i\mathfrak{n}$ is the Lie algebra of V.*

Then $\pi_{\mathcal{U}}$ has the following properties:

1. *The $\mathcal{N}$-orbits are symplectic manifolds which are given infinitesimally by a constant map:*
$$\varpi_{\pi_v} : X \to \wedge^2(\mathfrak{n} \oplus i\mathfrak{n}).$$
In particular
$$\pi_{\mathcal{U}} = \pi_0 + \pi_v$$
is the sum of commuting Poisson structures.

2. *Within the $\mathbb{N}$-orbits, all N-orbits and V-orbits are Lagrangian submanifolds.*

3. *The sequence of Poisson submersions*
$$(\mathcal{U}, \pi_{\mathcal{U}}) \to (P := \mathcal{U}/V, \pi_P) \to (X, \pi_B)$$
induces the identity form the leaf spaces of $\pi_{\mathcal{U}}$ to the leaf space of π_B.

   The fibrations $(\mathcal{U}/V, \pi_P) \to (X, \pi_B)$ and $(\mathcal{U}, \pi_{\mathcal{U}}) \to (X, \pi_B)$ are in the hypotheses of Theorems 3.4 and Proposition 3.1 for the groups N and $\mathbb{N}$, respectively.

Proof. The relevant observation is that by a density argument (which also uses compactness of X) it is enough to prove the above properties in the dense orbit $\mathbb{C}^{*n}$. There, $\pi_{\mathcal{U}}$ is given infinitesimally by a constant Kahler form on $\mathbb{C}^n$, and this easily implies the three properties above for the complex subgroup $\mathbb{N}$. □

4.2 Bruhat Poisson structures

Let G be a compact, connected, semisimple Lie group and T a maximal torus of G. As explained in Appendix C from the Iwasawa decomposition (see Example C.4) one gets on G structure of Poisson Lie group (G, π) with the following properties which we single out in a Lemma for further reference:

Lemma 4.2

1. *π is (right, say) T-invariant and the action is no-where tangent. In particular*

$$(G, \pi_G) \to (G/T, \pi_B)$$

3.43.4.

2. *All coadjoint orbits G/P (not just the regular one), where P is a parabolic subgroup of G and $T \subset P$, satisfy:*

   a. *The symplectic leaves of the Bruhat-Poisson structure π_B are the Bruhat cells of G/P;*
   b. *π_B is invariant for the left action of T on G/P which is tangential.*

The above properties imply that both a Poisson Lie group and any of its coadjoint orbits fit in the hypothesis of Theorem 3.4 for base/bundle and fiber, respectively.

Example 4.2 *Let $G = SU(2)$ and T be the diagonal maximal torus. The results described in Appendix C say that the leaf space of the Poisson structure is in bijection with the normalizer of T, which has two connected components in correspondence with the Weyl group:*

$$N(T)/T \cong \mathbb{Z}_2.$$

Upon projection each T-family of leaves becomes a single leaf of the regular orbit $(\mathbb{S}^2, \pi_B)$, namely, a point for T and its complement for the other connected component of $N(T)$. On the sphere there is a residual circle action by rotations which is Poisson.

4.3
Fibered Poisson structures with finite symplectic leaves and large symmetry

Here we are apply our principal result to toric varieties and Poisson Lie groups and their coadjoint orbits. As a result, we produce new examples of Poisson structures with a finite number of symplectic leaves and large abelian symmetry.

Theorem 4.3 *Consider the following hypothesis:*

(i) Let (B, π_B) be either a toric variety of complex dimension m or a coadjoint orbit of a compact, semisimple Lie group of rank m, endowed with their toric Poisson and Bruhat-Poisson structure, respectively. Let T_P denote either the compact subgroup N associated to B of the fixed maximal torus of the Lie group;

(ii) Let (F, π_F) be either toric variety of complex dimension n or a coadjoint orbit of a compact, semisimple Lie group of rank n. Let T denote either the compact symmetry subgroup of the toric variety or the fixed maximal torus of the Lie group;

(iii) Let $T_P \to T$ be a group morphism.

Then the above data determines a fibered Poisson structure

$$q : (M, \pi) \to (B, \pi_B)$$

with fiber (F, π_F) which has the following properties:

1. *Its leaf space is in bijection with*
$$B/\mathcal{F}_{\pi_B} \times F/\mathcal{F}_{\pi_F}. \tag{4-1}$$
In particular it is finite and the inclusion on closures defines a partial order in the leaf space which can be described in a combinatorial fashion.

2. *It supports an Poisson action of a compact torus of rank $m + n$.*

Proof. Item (1) above determines a principal T_B-bundle $\mathrm{pr} : P \to B$. By Proposition 4.1 and Lemma 4.2 there is a Poisson structure π_P on P which is T_B-invariant and makes pr a Poisson submersion, and such that the T_B-action is no-where tangent.

Item (2) presents a Poisson manifold (F, π_F) with a tangential T-action, and, together with the morphism in (3), a tangential T_P-action.

Thus we are in the hypothesis of Theorem 3.4. Hence on the associated bundle $q : M \to B$ we obtain a fibered Poisson structure π over (B, π_B) with fiber (F, π_F). We also obtain the bijection 4-1. Since both π_B and π_F have finite number of leafs with a partial order encoded combinatorially, so does π.

Regarding symmetries, on the product Poisson manifold $(P \times B, \pi_P + \pi_B)$ there is a Poisson action of a second torus T^{m+n} of rank $m+n$ which commutes with the diagonal T_P-action. The first factor is the torus T^m coming from the action of the compact torus of the toric Poisson structure on the complex vector space used to construct B; this action descends to P. The second factor is $T^n = T$ the fixed maximal torus acting on the left of the coadjoint orbit. The Poisson T^{m+n}-action clearly commutes with the T_P-action, and thus descends to a Poisson action on (M, π).

□

4.4 Topological types

Due to the difficulty of producing interesting Poisson structures, we would like to know that Theorem 4.3 produces different manifolds, or, at least, different locally trivial fiber bundles.

Firstly, we introduce the following relation among locally trivial fiber bundles $P \to B$ with fiber F, and with a marked fiber

$$i_{b_0} : F \to F_{b_0}.$$

Definition 4.4 *We say that two such marked fiber bundles are* ***equivalent*** *if there exist a bundle isomorphism lifting the identity on the base, and commuting with the markings of the fiber over* b_0*.*

Secondly, we will confine the discussion to the case when the base is $(\mathbb{S}^2, \pi_B)$, where the Poisson structure is either the toric or the Bruhat one, and the fiber is $(G/P, \pi_{BP})$.

This means that $P \to B$ is actually the Hopf fibration

$$\mathbb{S}^3 \to \mathbb{S}^2,$$

and the only remaining piece of data we need to apply Theorem 4.3 is a morphism

$$\mathbb{S}^1 \to T.$$

This morphism can be given infinitesimally, via a so called coweight in the Lie algebra of T. Coweights fits into a full rank lattice $\Lambda_G^\vee$, which contains a second full rank lattice known as the coroot lattice $\Lambda_r^\vee$. The quotient

$$\Lambda_G^\vee/\Lambda_r^\vee$$

is canonically identified with the (finite) fundamental group of G [12, Theorem 7.1].

The next uses the characterization non-equivalent homogeneous bundles over $\mathbb{S}^2$ with fiber a coadjoint orbit

Proposition 4.5 *Let G be a compact, connected, semisimple Lie group. If two coweights of G do not represent the same element in the fundamental group of G, then the Poisson fibered structures*

$$(M,\pi) \to (\mathbb{S}^2, \pi_B), \quad (M',\pi') \to \mathbb{S}^2$$

with fiber $(G/P, \pi_{BP})$ constructed applying Theorem 4.3 are supported in non-equivalent bundles.

Proof. We have the aforementioned isomorphism:

$$\Lambda_G^\vee/\Lambda_r^\vee \simeq \pi_1(\mathrm{Ad}(G)).$$

It follows from [13, Theorem 1.4], that the natural morphism

$$\rho : \pi_1(G) \to \pi_1(\mathrm{Diff}(G/P))$$

if injective. So, if two coweights do not represent the same element on $\pi_1(G)$, then the corresponding (homogeneous) bundles

$$M \to \mathbb{S}^2, \quad M' \to \mathbb{S}^2$$

with fiber G/P are not equivalent. □

We illustrate the previous result with the classically known classification of $\mathbb{S}^2$ bundles over $\mathcal{S}^2$:

Example 4.3 *On $PSU(2) = SO(3)$ one can identify coroot and coweight lattices with $\mathbb{Z}$ and $(1/2)\mathbb{Z}$, so that*

$$\pi_1(SO(3)) = \mathbb{Z}^2.$$

We have two types of non-equivalents fiber bundles over $\mathbb{S}^2$, with fiber $\mathbb{S}^2$, the trivial one and the non-trivial.

We close this thesis with the following open problem: as the previous example shows coweights corresponding to morphisms $\mathbb{S}^1 \to \mathbb{S}^1$ give the same bundle according to their parity. It is natural to investigate if for even coweights, say, all fibered Poisson structures are diffeomorphic, i.e., they are the trivial Poisson structure.

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A
Poisson and Dirac manifolds

This appendix contains a brief introduction to Poisson manifolds including their characteristic or symplectic foliation, their morphisms and a standard construction of Poisson structures on quotients by Poisson actions. We also recall the concept of Dirac manifolds and their morphisms. The main references for this appendix are [14], [4] and [15].

A.1
Poisson structures

A.1.1
Poisson manifolds

Definition A.1 *We say that a pair $(M, \{\cdot,\cdot\})$ is a **Poisson manifold**, where M is a (smooth) manifold, if the bracket $\{\cdot,\cdot\}$ is a $\mathbb{R}$-bilinear, antisymmetric function $\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ that satisfies, for each f_1, f_2, $f_3 \in C^\infty(M)$*

1. *Leibniz identity:*

$$\{f_1, f_2 \cdot f_3\} = \{f_1, f_2\} \cdot f_3 + f_2 \cdot \{f_1, f_3\};$$

2. *Jacobi identity:*

$$\{f_1, \{f_2, f_3\}\} + \{f_2, \{f_3, f_1\}\} + \{f_3, \{f_1, f_2\}\} = 0.$$

*We say that $\{\cdot,\cdot\}$ is a **Poisson structure** or a **Poisson bracket**.*

Example A.1 *Let M be a manifold and define the **zero Poisson structure** by $\{f_1, f_2\} \equiv 0$, for all $f_1, f_2 \in C^\infty(M)$.*

Example A.2 *Let $\mathfrak{g}$ be a Lie algebra of finite dimension. The dual space $\mathfrak{g}^*$ inherits a canonical Poisson structure: for f_1, $f_2 \in C^\infty(\mathfrak{g}^*)$,*

$$\begin{array}{rcl} \{f_1, f_2\} : \mathfrak{g}^* & \longrightarrow & \mathbb{R} \\ \xi & \longmapsto & \xi([df_1(\xi), df_2(\xi)]). \end{array}$$

This is a linear Poisson structure in the sense that the bracket of linear functions is a linear function.

Example A.3 *Let (M, ω) be a symplectic manifold (see [10]). It induces a Poisson structure on M as follows: we define, for all $f_1,\ f_2 \in C^\infty(M)$*

$$\{f_1, f_2\} := \omega(X_{f_1}, X_{f_2}), \tag{A-1}$$

where $X_{f_1}, X_{f_2} \in \mathfrak{X}(M)$ are the Hamiltonians fields associated to f_1, f_2, respectively.

Let M be a smooth manifold. An element $\pi \in \Gamma(\Lambda^2 TM)$ is called a **bivector field** in M. In other words, a bivector field is an application

$$\begin{aligned} \pi : M &\longrightarrow \Lambda^2(TM) \\ x &\longmapsto \pi(x) \in \Lambda^2(T_x M). \end{aligned}$$

Let $(x_1, \cdots, x_n)$ be local coordinates in an open set $U \subset M$, so with respect to these coordinates

$$\pi(x) = \sum_{i<j} \pi_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} = \frac{1}{2} \sum_{i,j} \pi_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}, \tag{A-2}$$

where $\pi_{ij} \in C^\infty(U)$ are functions so that $\pi_{ij} = -\pi_{ji}$, $i, j = 1, \ldots, n$.

A bivector field π determines an application

$$\begin{array}{rcccl} \pi & : & \Omega^1(M) \times \Omega^1(M) & \longrightarrow & C^\infty(\mathbb{R}) \\ & & (\alpha, \beta) & \longmapsto & \pi(\alpha, \beta). \end{array}$$

Let π be a bivector field on M and let $\{\cdot,\cdot\}_\pi : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ be defined by

$$\{f_1, f_2\}_\pi \ := \ \pi(df_1, df_2). \tag{A-3}$$

Then $\{\cdot,\cdot\}_\pi$ is $\mathbb{R}$-bilinear, antisymmetric and satisfies the Leibniz identity. Conversely, we have:

Lemma A.2 *If an application $\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ antisymmetric, $\mathbb{R}$-bilinear is a biderivation, so $\{\cdot,\cdot\}$ defines a bivector field in M, where π is given by the formula (A-3).*

Consider the **Schouten bracket**[1]

$$[\cdot,\cdot] : \mathfrak{X}^2(M) \times \mathfrak{X}^2(M) \longrightarrow \mathfrak{X}^3(M).$$

given by

$$[\pi,\pi](df_1, df_2, df_3) = \{f_1,\{f_2,f_3\}\} + \circlearrowleft .$$

As a consequence of the previous discussion and Lemma $A.2$ we have:

Proposition A.3 *There is a one to one correspondence between Poisson structures on M and bivector fields π on M such that $[\pi,\pi] = 0$.*

Definition A.4 *Let $(M_1,\{\cdot,\cdot\}_1),(M_2,\{\cdot,\cdot\}_2)$ be Poisson manifolds. A **Poisson morphism** is a (smooth) map $\Psi : M_1 \to M_2$ so that, if f_1, $f_2 \in C^\infty(M_2)$, then*

$$\Psi^*\{f_1,f_2\}_2 = \{\Psi^* f_1, \Psi^* f_2\}_1,$$

where $\Psi^* f = f \circ \Psi$, $f \in C^\infty(M_2)$.

Example A.4 *Let $\varphi : \mathfrak{g} \to \mathfrak{h}$ be a morphism of Lie algebras. The dual map $\varphi^* : \mathfrak{h}^* \to \mathfrak{g}^*$ is a Poisson morphism.*

Example A.5 *Let (M,ω,G,μ) be a Hamiltonian G-space. The momentum map $\mu : M \to \mathfrak{g}^*$ is a Poisson morphism with respect the Poisson structure in M described in Example $A.3$ and the linear Poisson structure in $\mathfrak{g}^*$ given in Example $A.2$.*

Definition A.5 *Let $(M,\{\cdot,\cdot\})$ a Poisson manifold and $f_1 \in C^\infty(M)$. The vector field $X_{f_1} \in (M)$ defined by:*

$$X_{f_1}(f_2) := \{f_1,f_2\} = \pi(df_1,df_2),$$

*for all $f_2 \in C^\infty(M)$, is the **Hamiltonian vector field** associated to f_1.*

Jacobi's identity translates into:

$$[X_{f_1}, X_{f_2}] = X_{\{f_1,f_2\}}. \tag{A-4}$$

Let M, N be manifolds and $F : M \to N$ a smooth map. We say that two vector fields $X \in \mathfrak{X}(M)$, $Y \in \mathfrak{X}(M)$ are F-**related** if, for all $x \in M$:

$$Y_{F(x)} = DF(x)(X_x).$$

[1] In full generality the Schouten bracket is a binary operation on multivector fields on M [14]

There is an analogous definition of F-related bivector fields.

The definition of Poisson morphism can be recast using both bivector fields and Hamiltonian vector fields:

Lemma A.6 *If $\Psi : (M_1, \{\cdot,\cdot\}_1) \to (M_2, \{\cdot,\cdot\}_2)$ is a map between Poisson manifolds, then the following statements are equivalent:*

1. *Ψ is a Poisson morphism;*

2. *For all $f \in C^\infty(M)$ the Hamiltonian vector fields of f and $\Psi^* f$ are Ψ-related;*

3. *The bivector fields associated to $\{\cdot,\cdot\}_1$ and $\{\cdot,\cdot\}_2$ are Ψ-related.*

A.1.2 Symplectic Foliation

If $\pi \in \mathfrak{X}^2(M)$ is a bivector field on a manifold M, then we have a map of vector bundles

$$\begin{aligned} \pi^\sharp : T^*M &\longrightarrow TM \\ (\alpha, \beta) &\longmapsto \pi^\sharp(\alpha)(\beta) \coloneqq \pi(\alpha, \beta). \end{aligned}$$

Definition A.7 *Let M be a manifold and $\pi \in \mathfrak{X}^2(M)$ a bivector field. The **rank** of π in $x \in M$ is the rank at x of the linear map $\pi^\sharp : T^*M \to TM$.*

Let (M, π) be a pair where M is a manifold and $\pi \in \mathfrak{X}^2(M)$ is a bivector field. We say that $x_0 \in M$ is a **regular point of** M if exists a neighborhood U of x_0 such that for all $x \in U$ $\text{rank}\pi_x = \text{rank}\pi_{x_0}$. Otherwise, we say that x is a **singular point of** M. The set of regular and singular points of M are denoted M_{reg} and M_{sing}, respectively. It follows that M_{reg} is an open dense subset (with connected components of possibly different rank).

Example A.6 *Let (M, ω) be a symplectic manifold and $\pi \in \mathfrak{X}(M)$ the associated Poisson structure. Then, rank$\pi_x =$ dim$M$, for all $x \in M$.*

Example A.7 *Let $A = (a_{ij})$ be a $n \times n$ antisymmetric matrix and consider the associated Poisson structure π_A in $\mathbb{R}^n$ defined as follows: let $x = (x_1, \ldots, x_n)$ be linear coordinates on $\mathbb{R}^n$. If f_1, $f_2 \in C^\infty(M)$ then*

$$(\pi_A)_x(df_1, df_2) \coloneqq \sum a_{ij} \cdot x_i x_j \frac{\partial f_1}{\partial x_i} \wedge \frac{\partial f_2}{\partial x_j}.$$

Therefore rank$(\pi_A)_x$ is exactly the rank of the matrix than is obtained by removing from A the rows and columns where the coordinates of the point x vanish.

Definition A.8 *Let (M, π) be a Poisson manifold. If π has constant rank then we say that π is a* ***regular Poisson structure****.*

Note that in the case of a regular Poisson structure we obtain a distribution $x \mapsto \mathrm{Im}\pi_x^\sharp$. This is a (smooth) distribution locally generated by vector fields $X_f = \pi^\sharp(df)$, $f \in C^\infty(M)$. The next result describes the properties of this distribution which are a consequence of the formula (A-4):

Theorem A.9 *Let (M, π) a regular Poisson structure. Then the following holds:*

1. *The distribution $\mathrm{Im}\pi^\sharp$ is integrable;*

2. *Each leaf $\mathcal{S}$ of $\mathrm{Im}\pi^\sharp$ is a Poisson submanifold of (M, π) and the induced Poisson structure $\pi_{\mathcal{S}}$ in $\mathcal{S}$ is non-degenerated (i.e., symplectic).*

That is, if (M, π) is a regular Poisson manifold, then M is foliated by symplectic leaves. And, conversely, if $\mathcal{F}$ is a foliation on M and the leaves admit a (smoothly varying) a symplectic form $\omega_{\mathcal{F}}$, then there exists a unique regular Poisson structure $\pi \in \mathfrak{X}^2(M)$ whose symplectic foliation is $(\mathcal{F}, \omega_{\mathcal{F}})$

More generally, any Poisson manifold (M, π) has an associated distribution spanned by Hamiltonian vector fields: $\pi^\sharp(T^*M)$. This distribution is integrated by a foliation in the sense of Definition 2.19[2], and its leaves are Poisson (symplectic) submanifolds of (M, π):

Theorem A.10 (Symplectic foliation) *Let (M, π) be a Poisson manifold. Then the Hamiltonian distribution $\pi^\sharp(T^*M)$ is integrated by a (possibly non-regular) foliation, each of whose leaves is a symplectic submanifold of (M, π).*

Observe that for a Poisson morphism $\Psi : (M, \pi_M) \to (N, \pi_N)$ the equation

$$\Psi_*(X_{\Psi^* f}) = X_f$$

implies that the restriction of Ψ to a symplectic leaf of (M, π_M) is a surjective submersion onto a symplectic leaf of (N, π_N).

Though we shall not be needing more general results on Poisson structures, it is worth mentioning that a great deal of the complexity of the Poisson structure has to do which the behavior of the Poisson structure in the directions transverse to the symplectic leaves. More precisely, this amounts not just to local statements (i.e., behavior of the local Poisson structures transverse to the symplectic leaves constructed upon taking small slices at points), but also to global ones (i.e., behavior of the transverse variation of symplectic forms or, at least, of its spherical classes).

[2]This is a consequence of Weinstein's splitting theorem [16]

A.1.3
Poisson actions

The last definition that we present about Poisson manifolds is the notion of Poisson action. Quotients of Poisson manifolds by Poisson action are the best way to produce new examples of Poisson manifolds.

Definition A.11 *Let (M, π) be a Poisson manifold and G a Lie group. The action $\psi : G \times M \to M$ is a* ***Poisson action*** *if for any $g \in G$, the application $\psi_g : M \to M$ is a Poisson morphism.*

Example A.8 *Consider the torus $\mathbb{T}^4$ together with the bivector field*

$$\pi = \frac{\partial}{\partial \theta_1} \wedge \frac{\partial}{\partial \theta_2} + \frac{\partial}{\partial \theta_3} \wedge \frac{\partial}{\partial \theta_4}$$

and the action of $\mathbb{S}^1$ in $\mathbb{T}^4$ given by:

$$\theta \cdot (\theta_1, \theta_2, \theta_3, \theta_4) = (\theta_1, \theta_2, \theta_3, \theta_4 + \theta).$$

This action is a Poisson action

Let (M, π) be a Poisson manifold together with a Poisson action $\psi : G \times M \to M$. As we know, under certain hypothesis, the quotient space M/G is a smooth manifold. In other words, M is a principal G-bundle with base M/G. Our next result give us necessary conditions for the G-action in M so that the orbit space M/G inherits a Poisson structure.

Theorem A.12 *Let (M, π) be a Poisson manifold endowed with a principal G-bundle structure over M/G and so that the G-action $\psi : G \times M \to M$ is Poisson. Then there exists a unique Poisson structure $\pi_{M/G} \in \mathfrak{X}^2(M/G)$ on M/G for which the projection pr : $M \to M/G$ is a Poisson morphism.*

More precisely, functions on M/G are identified with G-invariant functions on M, and, by hypotheses, this is a Poisson subalgebra.

Example A.9 *Consider the action of $\mathbb{S}^1$ in $\mathbb{T}^4$ given in the Example A.8. Then $M/G = \mathbb{T}^3$ with (periodic) coordinates $\theta_1, \theta_2, \theta_3$ and*

$$\pi_{M/G} = \frac{\partial}{\partial \theta_1} \wedge \frac{\partial}{\partial \theta_2}.$$

Now, we are going to present a special type of Poisson actions. Let G be a Lie group, with lie algebra $\mathfrak{g}$ and (M, π) a Poisson manifold.

Definition A.13 *A Poisson action $\psi : G \times M \to M$ is a* ***Hamiltonian action*** *if there exists an application $\mu : M \to \mathfrak{g}^*$, called* ***moment map****, such that:*

1. *For each $u \in \mathfrak{g}$, the function $\mu^u : M \to \mathbb{R}$, such that, $\mu^u(p) = \mu(p)(u)$, is a Hamiltonian function associated to infinitesimal generator of the action on M:*
$$X_u := \pi^\sharp(d\mu^u);$$

2. *For each $g \in G$ the following diagram is commutative:*

$$\begin{tikzcd} M \arrow[r, "\psi_g"] \arrow[d, "\mu"'] & M \arrow[d, "\mu"] \\ \mathfrak{g}^* \arrow[r, "Ad_g^*"'] & \mathfrak{g}^*. \end{tikzcd}$$

That is the moment map $\mu : M \to \mathfrak{g}^*$ is equivariant with respect to the given G action on M and the coadjoint action, and it is a Poisson morphism (for the linear Poisson structure on $\mathfrak{g}^*$).

A.2 Dirac Structures

Let M be a smooth manifold and consider the **generalized tangent bundle** $\mathbb{T}M := TM \oplus T^*M$, which comes with the natural projections $\mathrm{pr}_T : \mathbb{T}M \to TM$ and $\mathrm{pr}_{T^*} : \mathbb{T}M \to T^*M$. Consider the fiberwise nondegenerate, symmetric and bilinear form:

$$\langle (X, \alpha), (Y, \beta) \rangle := \alpha(Y) + \beta(X)$$

where $x \in M$, X, $Y \in T_xM$ and α, $\beta \in T_x^*M$,. The **Courant bracket** $[\![\cdot, \cdot]\!] : \Gamma(\mathbb{T}M) \times \Gamma(\mathbb{T}M) \to \Gamma(\mathbb{T}M)$ is defined as follows:

$$[\![(X, \alpha), (Y, \beta)]\!] := \Big([X, Y], \mathcal{L}_X\beta - \mathcal{L}_Y\alpha + \frac{1}{2}(d(\alpha(Y) - \beta(X)))\Big).$$

Definition A.14 *A **Dirac structure** on M is a vector subbundle $L \subset \mathbb{T}M$ satisfying*

1. *$L = L^\perp$ in relation to $\langle \cdot, \cdot \rangle$ (i.e., it is a field of Lagrangian subspaces);*

2. *$[\![\Gamma(L), \Gamma(L)]\!] \subset \Gamma(L)$ (i.e., it is involutive for the Courant bracket).*

We call a pair (M, L) as in Definition A.14 a **Dirac manifold** and a subbundle $L \subset \mathbb{T}M$ that satisfies condition (1) in Definition A.14 a **Lagrangian subbundle**.

Remark A.15 *The tensor $\Upsilon \in \Gamma(\Lambda^3 L^*)$ defined by*

$$\Upsilon(a_1, a_2, a_3) := \langle [\![a_1, a_2]\!], a_3 \rangle$$

*for a_1, a_2, $a_3 \in L$, is called the **Courant tensor**. Therefore, condition (2) in Definition A.14 is equivalent to $\Upsilon = 0$.*

Now, we are going to present two important examples of Dirac structures:

Example A.10 *Let M a manifold and $\pi \in \mathfrak{X}^2(M)$ a bivector field. Consider the subbundle given by the graph of $\pi^\sharp$:*

$$L_\pi := \textit{graph}(\pi^\sharp) = \left\{ (\pi^\sharp(\alpha), \alpha) \mid \alpha \in T^*M \right\}.$$

Let $a_i = (\pi^\sharp(df_i), df_i), i = 1,\ 2,\ 3$, so that

$$\Upsilon(a_1, a_2, a_3) = \{f_1, \{f_2, f_3\}\} + \circlearrowleft .$$

Then, by Remark A.15 (M, π) is a Poisson manifold if, and only if, the subbundle L_π is a Dirac structure for M.

For the next example we introduce the following notation: A 2-form $\omega \in \Omega^2(M)$ on a manifold M is called a **presymplectic form** if ω is closed. In this case the pair (M, ω) is called a **presymplectic manifold**. Moreover, we have the induced application

$$\begin{aligned} \omega^\sharp : TM &\longrightarrow T^*M \\ u &\longmapsto i_u\omega \ . \end{aligned}$$

Example A.11 *Like in the previous example, consider for any 2-form $\omega \in \Omega^2(M)$ the subbundle*

$$L_\omega := \textit{Graph}(\omega^\sharp) = \{(X, \omega^\sharp(X)) \mid X \in TM\}.$$

Now, if $a_i = (X_i, \omega^\sharp(X_i))$, $i = 1,\ 2,\ 3$, then we have:

$$\Upsilon(a_1, a_2, a_2) = d\omega(X_1, X_2, X_3).$$

Therefore M is a Dirac manifold if, and only if, ω is presymplectic.

Definition A.16 *Let (M, L) be a Dirac manifold. We say that $f \in C^\infty(M)$ is a **admissible function** if there exists a vector field X such that $(X, df) \in L$. This vector field X is called a **Hamiltonian vector field** for f.*

Note that in Example A.10 every function f is admissible, and, it possess a unique Hamiltonian vector field: the usual Hamiltonian X_f.

Our next step is to define morphisms between Dirac manifolds.

Definition A.17 *Let $(M, L_M), (N, L_N)$ be Dirac manifolds and let $\Phi : M \to N$ be a map. The backward image of L_N and forward image of L_M (by Φ) are:*

$$\{(X, \Phi^*(x)(\alpha)) \,|\, (D\Phi(x)(X), \alpha) \in (L_N)_{\Phi(x)}\}$$

$$\{(D\Phi(x)(X), (\alpha)) \,|\, (X, \Phi^*(x)(\alpha)) \in (L_M)_x\}.$$

The map is called **backward Dirac** *when the backward image of L_N matches L_M. Likewise, Φ is* **forward Dirac** *when the forward image of L_M is L_N.*

If $\Phi : M \to M$ is a diffeomorphism, then Φ is a backward Dirac map if, and only if, Φ is a forward Dirac map. In this case we say that Φ is a **Dirac diffeomorphism.**

Example A.12 *Let $(M, \pi_M), (N, \pi_N)$ be Poisson manifolds. A map $\Phi : M \to N$ is Poisson if, and only if, for any $x \in M$*

$$(\pi_M^\sharp)_{\Phi(x)} \;=\; D\Phi(x) \circ (\pi_N^\sharp)_{\Phi(x)} \circ \Phi^*(x) \tag{A-5}$$

By the Example A.10

$$(L_{\pi_N})_{\Phi(x)} = \{(\pi_N^\sharp(\alpha), \alpha) \,|\, \alpha \in T^*_{\Phi(x)}N\}.$$

So, by formula (A-5) it follows that the forward image of L_{π_M} equals L_{π_N}.

We conclude that a smooth map is a Poisson map if, and only if, it is a forward Dirac map.

Let (M, ω_M), (N, ω_N) be presymplectic manifolds. We say a map $\Phi : M \to N$ is a presymplectic morphism if $\Phi^*\omega_M = \omega_N$.

Example A.13 *Let (M, ω_M), (N, ω_N) be presymplectic manifolds. Following the ideas of the previous example one proves that a map $\Phi : M \to N$ is a presymplectic morphism if, and only if, it is a backward Dirac map.*

Let (N, L_N) be a Dirac manifold and let $\Phi : M \to N$ be a map. The backward image of L_N is a subset of $\mathbb{T}N$ whose intersection with each fiber is a Lagrangian subspace. The requirement for the backward image of L_N to define a Dirac structure on M (so that Ψ becomes a backward Dirac map) is that the backward image be a (smooth) subbundle of $\mathbb{T}M$.

Definition A.18 *Let (N, L_N) be a Dirac manifold. A submanifold $M \subset N$ is called a Poisson-Dirac submanifold of (N, L_N) if the backward image of L_N by the inclusion is smooth.*

The name Poisson-Dirac submanifold is justified by the fact that the Dirac structure on M is actually Poisson (this is a general fact for immersions which are backwards Dirac [15]).

Though we shall not use further results on Dirac structures, it is worth mentioning that any Dirac structure supports a presymplectic foliation; this foliation integrates the distribution $\mathrm{pr}_{TM}(L_M)$ (omitting presymplectic structures, the foliation is the characteristic foliation of an underlying Lie algebroid structure). In particular, the aforementioned result on Poisson-Dirac submanifolds explains why at any point the (local) transverse Dirac structure (defined on a small slice to the presymplectic foliation) is actually Poisson.

B
Toric manifolds

In this appendix we will present the concept of toric varieties. First, we do it from the symplectic viewpoint. Next, we do it from the complex algebraic viewpoint. The latter will be the most useful one for us, as it is the most convenient to introduce the so-called toric Poisson structures. The main references are [11], [17] and [18].

B.1
Symplectic Toric Manifolds

Let G be a Lie group, let $\mathfrak{g}$ denote its Lie algebra and let (M, ω) be a symplectic manifold. An action $\psi : G \times M \to M$ of G in M is a **symplectic action** if, for each $g \in G$, the application $\psi_g : M \to M$ is a symplectomorphism. The notion of a Hamiltonian action for the symplectic case is the same presented in Definition A.13, and, Condition 1 there is translated as

$$d\mu^u = \mathfrak{i}_{X_u}\omega.$$

The quadruple (M, ω, G, μ) is referred to as a **Hamiltonian G-space.**

Remark B.1 *If G is an abelian Lie group then the action of G on itself by conjugation is trivial. Therefore the coadjoint action of G on $\mathfrak{g}^*$ is trivial as well. This happens if G is, for example, the n-dimensional torus $\mathbb{T}^n$. In this case, the moment map associated to the Hamiltonian action is a G-invariant application.*

For now on the Lie group that we will be interested in the compact torus $\mathbb{T}^n$. We identify its Lie algebra with $\mathbb{R}^n$ in such a way that the the kernel of the exponential map is the integral lattice $\mathbb{Z}^n \subset \mathbb{R}^n$ (and thus we identify the dual of the Lie algebra with $\mathbb{R}^n$ as well).

In this case we have the following description for a Hamiltonian action on (M, ω): an action $\psi : \mathbb{T}^n \times M \to M$, with associated moment map $\mu : M \to \mathbb{R}^n$ is Hamiltonian if:

1. for $\{u_1, \cdots, u_n\}$ a basis for $\mathbb{R}^n$, the function $\mu^{u_i} : M \to \mathbb{R}$ defined by $\langle \mu, u_i \rangle$ is a Hamiltonian function associated to the infinitesimal generator X_{u_i};

2. The moment map $\mu : M \to \mathbb{R}^n$ is $\mathbb{T}^n$-invariant, that is, for each $g \in G$, $\mu \circ \psi_g = \mu$.

Definition B.2 *We say that (M, ω) is a* ***symplectic toric manifold*** *if M is a Hamiltonian $\mathbb{T}$ space (M, ω, T, μ) for which the torus action is effective and*

$$\textit{dim} M = 2\textit{dim}\mathbb{T}.$$

The key result for the classification of Toric manifolds is the Convexity theorem of Atiyah [19], and Guillemin-Sternberg [20]: if $(M, \omega, \mathbb{T}^n, \mu)$ is a $\mathbb{T}^n$-Hamiltonian space and M is compact and connected, then the fibers of the momentum map are connected subsets of M and the image of the momentum map is a convex polytope with is the convex hull of the fixed points of the action.

A consequence of this result is that the action must admit at least $n+1$ fixed points and, if the action is effective, one concludes dim$M \geq 2n$.

The image of a moment map is called the **moment polytope**.

Example B.1 *Consider the $\mathbb{S}^1$-action on the symplectic manifold $(\mathbb{S}^2, \omega_{\textit{std}} := d\theta \wedge dh)$ by rotations, with moment map $\mu = h$ and moment polytope $[1, 1]$. Then $(\mathbb{S}^2, \omega_{\textit{std}}, \mathbb{S}^1, h)$ is a symplectic toric manifold.*

Definition B.3 *We say that a polytope $\Delta \subset \mathbb{R}^n$ is a* ***Delzant polytope*** *if Δ satisfies:*

1. *Each vertex $v \in \Delta$ is contained in exactly n edges;*

2. *Each edge is of the form $v + t \cdot u_i$, when $u_i \in \mathbb{Z}^n$;*

3. *The u_i form a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.*

The natural equivalence relation between symplectic toric manifolds is equivariant symplectomorphism. The natural equivalence relation between Delzant polytopes is the orbit relation given by the action of the group of affine transformations of $\mathbb{R}^n$ whose linear part lies in $\mathrm{SL}(n, \mathbb{Z})$.

With these preliminary results, we have the following classification result obtained via symplectic reduction theorem (for more details consult [10]).

Theorem B.4 *We have a $1-1$ correspondence between equivalence class symplectic toric manifolds and equivalence classes of Delzant polytopes.*

B.2
Toric Varieties

Consider $(\mathbb{C}^*)^n$ a (complex) **torus**, this is, the affine variety $(\mathbb{C}^*)^n$ with a structure of a Lie group given by the multiplication coordinate to coordinate of complex numbers.

Definition B.5 *An **action** of $(\mathbb{C}^*)^n$ on a complex variety X is given by an (algebraic) action map:*

$$\psi : (\mathbb{C}^*)^n \times X \to X.$$

Now suppose that X is a irreducible variety that admits a torus action:

Definition B.6 *We say that X is a **toric variety** if the action $\psi : (\mathbb{C}^*)^n \times X \to X$ admits an dense open orbit in X.*

Toric varieties can be affine, projective, smooth, etc.

There is a simple way to construct examples of projective toric varieties. For this consider $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ and the corresponding character (a group homomorphism from $(\mathbb{C}^*)^n$ to $\mathbb{C}^*$):

$$\begin{array}{rcc} \lambda : (\mathbb{C}^*)^n & \longrightarrow & \mathbb{C}^* \\ w = (w_1, \ldots, w_n) & \longmapsto & w^\lambda := (w_1^{\lambda_1} \cdot \ldots \cdot w_n^{\lambda^n}). \end{array}$$

Example B.2 *Consider a collection of infinitesimal characters $(\lambda^{(1)}, \ldots \lambda^{(k)}) \subset \mathbb{Z}^n$. Define the action of $(\mathbb{C}^*)^n$ on $\mathbb{C}P^{k-1}$ as follows:*

$$\begin{array}{rccc} \psi & : & (\mathbb{C}^*)^n \times \mathbb{C}P^{k-1} & \longrightarrow & \mathbb{C}P^{k-1} \\ & & (w, [z_1 : \cdots : z_k]) & \longmapsto & [w^{\lambda^{(1)}} \cdot z_1 : \ldots : w^{\lambda^{(k)}} \cdot z_k], \end{array}$$

Take X_A as the closure of the orbit of this action through $[1 : \ldots : 1]$. Then X_A is a toric variety (we may need to consider a residual action in the presence of stabilizer).

Give an algebra A, their **maximal spectrum** is the set

$$\mathsf{Spec}_{\mathsf{m}}(A) := \{\text{maximal ideals in} A\}.$$

It is a consequence of Hilbert's Nullstellensatz that if X is an affine variety, so $X \simeq \mathsf{Spec}_{\mathsf{m}}\mathbb{C}[X]$. In other words, affine varieties can be identified with finitely generated $\mathbb{C}$-algebras.

Example B.3 *If $A = \mathbb{C}[z_1, \ldots, z_n]$,, then Hilbert's Nullstellensatz implies that*

$$\mathbb{C}^n \simeq Spec_m \mathbb{C}[z_1, \ldots, z_n].$$

Definition B.7 *Let S be a semigroup. We say that $\mathbb{C}[S]$ is a* ***semigroup algebra*** *if $\mathbb{C}[S]$ is a $\mathbb{C}$-algebra generated by elements z^σ, $\sigma \in S$, and whose multiplication is given by:*

$$z^\sigma \cdot z^\delta := z^{\sigma+\delta}.$$

The following result gives a description of toric varieties centered on their function ring (see for example [17]). Let $S \subset \mathbb{Z}^n$ be a finitely generated semigroup.

Proposition B.8 *If $\mathbb{C}[S]$ is a semigroup algebra then the affine variety Spec$_m\mathbb{C}[S]$ is an affine toric variety.*

We have a reciprocal for the above result (see again [17]) that classifies affine toric varieties:

Theorem B.9 *Let $S \subset \mathbb{Z}^n$ be a finitely generate semigroup. Any affine toric variety (with an open dense $(\mathbb{C}^*)^n$-orbit) is of the form Spec$_m\mathbb{C}[S]$.*

Next we introduce the concept of cones and their relation with toric varieties.

Definition B.10 *A* ***convex polyhedral cone*** *in $\mathbb{R}^n$ is the set*

$$C = \left\{ \sum_{i=1}^{r} a_i \cdot v_i \mid a_i \geq 0 \right\},$$

where $v_1, \ldots, v_r \in \mathbb{R}^n$. In this case we say that the v_i's generate C.

If the set of generators belong to $\mathbb{Z}^n$ we say that C is ***rational****.*

The **dual cone** of C is the set

$$C^* := \{f \in (\mathbb{R}^n)^* | f(x) \geq 0, \forall x \in C\}.$$

Example B.4 *The first quadrant of $\mathbb{R}^2$ is a (rational) convex polyhedral cone generated by the canonical basis e_1, e_2 of $\mathbb{R}^2$.*

Example B.5 *In $\mathbb{R}^3$ we have the rational cone generated by the set $\{e_1, e_2, e_1 + e_3, e_2 + e_3\}$.*

Recall that the dual cone C^* are the linear functionals with positive evaluation on C.

Remark B.11 *An important fact [17] is that given a rational cone C then the set*

$$S_C := C^* \cap (\mathbb{Z}^n)^*$$

is a finitely generated semigroup.

The following corollary is a consequence of Proposition B.8 and of Remark B.11:

Corollary B.12 *If $C \subset \mathbb{R}^n$ is a rational cone, then Spec$_m\mathbb{C}[S_C]$ is an affine toric variety.*

To pass from affine to projective toric varieties the naive idea of patching affine charts turns out to work in the algebra geometric context. The efficient way to perform the gluing is via the cones. These idea requires the introduction of faces and fans:

Definition B.13 *A **face** F in a cone C is either the cone itself or the intersection of C with the vanishing hyperplane of a form in C^*.*

Example B.6 *Let C be the cone generated by $\{e_1, e_2, e_1 + e_3, e_2 + e_3\}$. The first element of the dual basis e_1^* belongs to C^*, and it gives rise to the face generated by $\{e_2, e_2 + e_3\}$.*

Let C be a convex polyhedral cone. If $0 \in C$ is a face for C we say that C is **strongly convex**.

Definition B.14 *A **fan** $\mathcal{F}$ is a finite (non-empty) collection of rational strongly convex cones subject to the following conditions:*

1. *For each $C \in \mathcal{F}$, each face of C belongs to $\mathcal{F}$;*

2. *The intersection of any two cones $C_1, C_2 \in \mathcal{F}$ is a face of C_1 and C_2.*

The set $X_{\mathcal{F}}$ resulting by gluing the affine toric varieties $X_C :=$ Spec$_m C[S_C]$ (for all $C \in \mathcal{F}$) is called the **toric variety associated to a fan** $\mathcal{F}$, when we identify X_C with the corresponding Zariski open subset in $X_{C'}$, whenever C is a face of C'.

Example B.7 *Consider the fan $\mathcal{F}$ given by the cones*

$$C = \{0\}\,, \quad C_0 = \mathbb{Z}_0^+(-e_1), \quad C_1 = \mathbb{Z}_0^+(e_1).$$

Note that $\mathbb{C}[S_{C_0}] = \mathbb{C}[z^{-1}]$, $\mathbb{C}[S_{C_1}] = \mathbb{C}[z]$ and $\mathbb{C}[S_C] = \mathbb{C}[z, z^{-1}]$. Using the map $z \to z^{-1}$ to glue the affine variety X_{C_1} in X_{C_0} along of X_C we obtain a toric variety $X_{\mathcal{F}} = \mathbb{C}P^1$.

An important result about $X_{\mathcal{F}}$ is the following:

Proposition B.15 *For a fan $\mathcal{F}$, $X_{\mathcal{F}}$ is a smooth variety if, and only if, every cone C on $\mathcal{F}$ is smooth.*

$X_{\mathcal{F}}$ is a normal possibly non-smooth algebraic variety. In fact, fans classify normal toric varieties [21]:

Theorem B.16 *Any normal toric variety is equivalent to a $X_{\mathcal{F}}$, where $\mathcal{F}$ is a fan in $\mathbb{R}^n$.*

Finally, let us discuss projective toric varieties. Suppose that X is a toric variety, with torus $(\mathbb{C}^*)^n$. We say that X is a **equivariantly projective toric variety** if there exist a embedding $X \hookrightarrow \mathbb{P}^k$ linearizing the action, i.e., sending the torus to a subtorus of the projective linear group. Note that the variety X_A given in the Example (B.2) is a example of equivariantly projective toric variety.

The way to associate a fan to X_A is as follows: more generally, start with V be finite dimensional $(\mathbb{C}^*)^n$-module and consider the corresponding decomposition according to weights (i.e., diagonalize simultaneously all the operators corresponding to elements of our abelian group): for $\lambda \in \mathbb{Z}^n$, the **weight space** associated to λ is the vector subspace

$$V_\lambda = \left\{ v \in V \mid w \cdot v = w^\lambda \cdot v, \ \forall w \in (\mathbb{C}^*)^n \right\}.$$

The module V decomposes in the following way:

$$V \simeq \bigoplus_{\lambda \in \mathbb{Z}^n} V_\lambda.$$

Given $v \in V$, consider the set $A_v := \{\lambda \in \mathbb{Z}^n \mid v_\lambda \neq 0\}$. The **weight polytope** P_v is defined to be the convex hull in $\mathbb{R}^n$ of the set A_v. We have then the following result (see for example [17]).

Theorem B.17 *Let X_A be the toric variety described in the Example (B.2). There is a bijection between faces of the the polytopes P_v and the orbits of the action of $(\mathbb{C}^*)^n$ in X_A.*

Now, let $P \subset \mathbb{R}^n$ a polytope and $f \in (\mathbb{R}^n)^*$.

Definition B.18 *Let F a face of P. The* ***cone associated to*** *F is the closure of the subset*

$$C_{F,P} = \{f \in \mathbb{R}^n)^* | \mathit{supp}_P f = F\}$$

where $\mathit{supp}_P f$ is the set of points in P where f achieves its minimum.

Then, the **fan of the polytope** P is the collection $\mathcal{F}_P$ of the cones $C_{F,P}$ for all faces of P.

Let V be a vector space and suppose that $(\mathbb{C}^*)^n$ act linearly on V and on $\mathbb{P}(V)$. Let $v \neq 0$ on V and let $\bar{\mathcal{O}}_v$ be the closure on $\mathbb{P}(V)$ of the $(\mathbb{C}^*)^n$-orbit through $[v]$. Then,the toric variety is equivalent to X_{A_v}. Then, we have the following result

Proposition B.19 *The fan of the toric variety $\bar{\mathcal{O}}_v$ equals the fan of the weight polytope P_v.*

B.3
Equivalence between symplectic and algebraic points of view

Using the idea presented in [17] we show a equivalence between symplectic toric manifolds and toric varieties.

Consider Δ a Delzant polytope, whose vertex belong to $\mathbb{Z}^n$. Then exists a $\mathbb{T}^n$-Hamiltonian space $(M_\Delta, \mu_\Delta, \mathbb{T}^n, \omega_\Delta)$, such that

$$\mu_\Delta(M_\Delta) = \Delta.$$

Let $A = \mathbb{Z}^n \cap \Delta = \left\{\lambda^{(1)}, \ldots, \lambda^{(k)}\right\}$. By Example (B.2) we have that X_A is an equivariantly projective variety (for the torus $(\mathbb{C}^*)^n$). Consider the embedding $i : X_A \hookrightarrow \mathbb{P}^{k-1}$. This induces a symplectic structure $\omega_A := i^*(-2\omega_{FS})$.

We have that $(\mathbb{P}^{k-1}, -2\omega_{FS})$, where ω_{FS} is the Fubini-Study's form, is a $\mathbb{T}^n$-Hamiltonian space, with moment map $\mu : \mathbb{P}^{k-1} \to \mathbb{R}^n$ given by:

$$\mu[z_1 : \ldots : z_n] = \frac{\displaystyle\sum_{j=1}^{k} \lambda^{(j)} \cdot |z_j|^2}{\displaystyle\sum_{j=1}^{k} |z_j|^2}.$$

So, the induced action of $\mathbb{T}^n$ in X_A is a Hamiltonian action with moment map $\mu_A := \mu\mid_{X_A}$, so that $\mu(X_A) = \Delta$.

B.4
Toric Poisson structures

Definition B.20 *Let X be a toric manifold. Then X admits an open dense orbit by an action of a torus $\mathbb{T}$. A **toric Poisson structure** π on X is a Poisson structure which is invariant under the $\mathbb{T}$-action and whose symplectic leaves are the $\mathbb{T}$-orbits.*

Consider the $\mathbb{T}^n$-action on $\mathbb{C}^n$ by left multiplication on the coordinates of $\mathbb{C}^n$ and the bivector field on $\mathbb{C}^n$

$$\pi := \sum_{j=1}^{n} -2iz_j\bar{z}_j \frac{\partial}{\partial z_j} \wedge \frac{\partial}{\partial \bar{z}_j}.$$

Then, in [11] we have the following result:

Lemma B.21 *The bivector field $\pi_{\mathbb{C}}$ is a Poisson structure in $\mathbb{C}^n$ that is invariant by the $\mathbb{T}^n$-action. Moreover, the symplectic leaves of $(\mathbb{C}^n, \pi_{\mathbb{C}})$ are exactly the orbits of the $\mathbb{T}^n$-action on $\mathbb{C}^n$.*

Let Δ be a Delzant polytope and $\phi : \mathbb{C}^n \to \mathfrak{t}$ the extension of the application which associates to e_l the element u_l on the dual polytope Λ. Let $\mathfrak{n} := (\text{ker}\phi : \mathbb{C}^n \to \mathfrak{t})$ and $\mathcal{U}$ the union of the orbits of the $\mathbb{T}^n$-action on $\mathbb{C}^n$. If $\mathbb{N}$ is a subgroup of $\mathbb{T}^n$ such that $\mathfrak{n}$ is their Lie algebra then $\mathbb{N}$ acts freely on $\mathcal{U}$. In fact, the combinatorial approach above selects the collection of $\mathbb{T}^n$-orbits of $\mathbb{C}^n$ where the action of $\mathbb{N}$ is free.

Consider the quotient

$$X = \mathcal{U}/\mathbb{N}$$

and the induced Poisson structure π on X. Then, we have the following result [11]:

Theorem B.22 *The quotient X is the toric variety determined by Δ, and the induced Poisson structure is toric. In particular the symplectic leaves of (X, π) are in bijection with the set of faces of Δ.*

C
Poisson structure on Lie groups

In this appendix we are interested to introduce certain Poisson structures on compact Lie groups and their (co)adjoint orbits. For this we discuss the notion of Poisson Lie groups. The results of the appendix can be found in [14], [22] and [23].

C.1
Poisson Lie groups and Lie bialgebras

Definition C.1 *Let G be a Lie group endowed with a Poisson structure $\pi \in \mathfrak{X}^2(G)$. We say that G is a **Poisson Lie group** if the multiplication on G*

$$\begin{array}{rcccc} m & : & G \times G & \longrightarrow & G \\ & & (g,h) & \longmapsto & g \cdot h, \end{array}$$

*is a Poisson morphism. In this case we say that $\pi \in \mathfrak{X}^2(G)$ is a **Lie Poisson structure** or a **Lie Poisson bivector** on G.*

Let $\pi \in \mathfrak{X}^2(G)$ be a Poisson structure on G. That π be a Lie-Poisson structure can be recast by means of the left and right translation $\mathsf{L}_g : G \to G$, $\mathsf{R}_g : G \to G$. Indeed, π is a Lie Poisson structure on G if, and only if, $\pi \in \mathfrak{X}^2(G)$ is **multiplicative**: for each g, $h \in G$

$$\pi(g \cdot h) \quad = \quad (\mathsf{L}_g)_* \pi(h) + (\mathsf{R}_h)_* \pi(g). \tag{C-1}$$

Example C.1 *Any Lie group endowed with the zero Poisson structure is a Poisson Lie group.*

Example C.2 *Let (G, π_G), (H, π_H) be two Poisson Lie groups and denote by $m_G : G \times G \to G$, $m_H : H \times H \to H$ the multiplications on G and H, respectively. Then, the Cartesian product $(G \times H, \pi_G \oplus \pi_H)$, together with the multiplication*

$$\begin{array}{rcccc} m & : & (G \times H) \times (G \times H) & \longrightarrow & G \times H \\ & & ((g_1,h_1),(g_2,h_2)) & \longmapsto & (m_G(g_1,g_2), m_H(h_1,h_2)), \end{array}$$

is a Poisson Lie group.

Example C.3 *Let G be an abelian Lie group endowed with a Poisson structure $\pi \in \mathfrak{X}^2(G)$, and let $\mathfrak{g}$ be the Lie algebra of G. Then π is multiplicative if, and only* *if, the application*

$$\begin{array}{rcl} \pi_R : G & \longrightarrow & \mathfrak{g} \wedge \mathfrak{g} \\ g & \longmapsto & (R_{g^{-1}})_* \pi(g) \end{array}$$

is a morphism of (abelian) Lie groups.

The next result aims to present the infinitesimal object associated to a connected Poisson Lie group. Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let $\pi \in \mathfrak{X}^2(G)$ be a bivector field. By [14, Theorem $5.1.3$] we have that $\pi \in \mathfrak{X}^2(G)$ is multiplicative if, and only if, the linearization of π at the identity $D\pi(e) : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ is a 1-cocycle for the adjoint representation.

The map dual to the linearization of π at the identity defines an antisymmetric, $\mathbb{R}$-bilinear bracket on $\mathfrak{g}^*$:

$$\begin{array}{rcccl} [\cdot,\cdot]_\pi & : & \mathfrak{g}^* \wedge \mathfrak{g}^* & \longrightarrow & \mathfrak{g}^* \\ & & (\delta, \beta) & \longmapsto & [\delta, \beta]_\pi := D\pi(e)(\delta, \beta). \end{array}$$

Theorem C.2 *Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Let $\pi \in \mathfrak{X}^2(G)$ be a Lie Poisson structure on G. Then the dual map to the linearization of π at the identity give us a Lie algebra structure on $\mathfrak{g}^*$:*

$$[\delta, \beta]_\pi(X) \ := \ D\pi(e)(X)(\delta, \beta), \tag{C-2}$$

when $X \in \mathfrak{g}$ e δ, $\beta \in \mathfrak{g}^*$.

The above discussion motivates the following definition:

Definition C.3 *Let $\mathfrak{g}$ be a Lie algebra. We say that the pair $(\mathfrak{g}, \mathfrak{g}^*)$ is a **Lie bialgebra** if, there exist a Lie algebra structure on $\mathfrak{g}^*$ such that the application $\mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ dual to the bracket $[\cdot,\cdot]_{\mathfrak{g}^*} : \mathfrak{g}^* \wedge \mathfrak{g}^* \to \mathfrak{g}^*$, is a 1-cocycle relative to the adjoint representation of G on $\mathfrak{g}$.*

Let $(\mathfrak{g}, \mathfrak{g}^*)$ be a Lie bialgebra. Let G e G^* be the connected and simply connected Lie groups with Lie algebras $\mathfrak{g}$, $\mathfrak{g}^*$, respectively. The group G^* is called the **dual Lie group** of G.

Theorem C.4 *Let (G, π) be a Poisson Lie group, with Lie algebra $\mathfrak{g}$.*

Then the pair $(\mathfrak{g}, \mathfrak{g}^*)$ admits a Lie bialgebra structure. Conversely, if $\tilde{G}$ is the connected and simply connected Lie group integrating $\mathfrak{g}$, then the Lie bialgebra structure on $(\mathfrak{g}, \mathfrak{g}^*)$ defines a multiplicative bivector field $\pi \in \mathfrak{X}^2(\tilde{G})$ such that the pair $(\tilde{G}, \pi)$ is a Poisson Lie group with Lie bialgebra the given bialgebra structure (and it also induces a Poisson Lie group structure on the dual Lie group of $\tilde{G}$).

The next result presents a useful characterization of Lie bialgebras:

Theorem C.5 *Let* $(\mathfrak{g}, \mathfrak{g}^*)$ *be a Lie bialgebra and denote by* $[\cdot,\cdot]_1$ *e* $[\cdot,\cdot]_2$ *the Lie brackets on* $\mathfrak{g}$ *and* $\mathfrak{g}^*$*, respectively. Let* $\langle\cdot,\cdot\rangle$ *be the canonical non-degenerate, bilinear, symmetric pairing on* $\mathfrak{g}\oplus\mathfrak{g}^*$*:*

$$\langle v_1+\beta_1, v_2+\beta_2\rangle \;=\; \beta_1(v_1)+\beta_2(v_1) \tag{C-3}$$

and define the following bracket on $\mathfrak{g}\oplus\mathfrak{g}^*$*:*

$$\begin{aligned}[v_1+\beta_1, v_2+\beta_2] \;:=\; & [v_1,v_2]_1 - ad^*_{\beta_2}(v_1) + ad^*_{\beta_1}(v_2) + [\beta_1,\beta_2]_2 \\ & -ad^*_{v_2}(\beta_1) + ad^*_{v_1}(\beta_2),\end{aligned}$$

Then this bracket defines a Lie algebra structure on $\mathfrak{g}\oplus\mathfrak{g}^*$ *such that the canonical symmetric pairing is an* ad*-invariant.*

Conversely, if a Lie algebra $\mathfrak{g}_1\oplus\mathfrak{g}_2$ *admits an* ad*-invariant non-degenerate, symmetric, bilinear pairing under which* $\mathfrak{g}_1$ *and* $\mathfrak{g}_2$ *are isotropic subspaces, then the pair* $(\mathfrak{g}_1,\mathfrak{g}_2)$ *admits a canonical Lie bialgebra structure.*

The previous results motivates the following definition:

Definition C.6 *A* ***Manin triple*** *is a triple of Lie algebras* $(\mathfrak{g},\mathfrak{g}_1,\mathfrak{g}_2,\langle\cdot,\cdot\rangle)$ *together with an* ad*-invariant non-degenerate, symmetric, bilinear pairing on* $\mathfrak{g}$ *such that:*

1. $\mathfrak{g}_1$, $\mathfrak{g}_2$ *are Lie subalgebras of* $\mathfrak{g}$*;*
2. $\mathfrak{g}=\mathfrak{g}_1\oplus\mathfrak{g}_2$ *as vector spaces;*
3. *if is a inner product on* $\mathfrak{g}$, *then* $\mathfrak{g}_1,\mathfrak{g}_2$ *are isotropic relative to the pairing, i.e., the splitting above is Lagrangian relative to the pairing.*

That is, the Theorem C.5 ensures the existence of a $1-1$ correspondence between Lie bialgebras and Manin triples. An example will be introduce latter.

C.2
Double Lie groups and double Lie algebras

Now we go back from infinitesimal to global objects (forgetting for the moment about Poisson structures):

Definition C.7 *Let G be a Lie group and let G_1, G_2 be closed Lie subgroups of G. We say that (G, G_1, G_2) forms a **double Lie group** if the product*

$$\begin{array}{rcccc} \mathfrak{m} & : & G_1 \times G_2 & \longrightarrow & G \\ & & (g_1, g_2) & \longmapsto & g_1 \cdot g_2, \end{array}$$

is a diffeomorphism. In this case, we will denote $G := G_1 \bowtie G_2$.

Definition C.8 *We say that (G, G_1, G_2) forms a **(local) double Lie group** if, exist Lie subgroups $G_1^{'}$, $G_2^{'}$ of G, locally isomorphic to G_1, G_2, respectively, such that the product*

$$\begin{array}{rcccc} \mathfrak{m} & : & G_1^{'} \times G_2^{'} & \longrightarrow & G \\ & & (g_1^{'}, g_2^{'}) & \longmapsto & g_1^{'} \cdot g_2^{'}, \end{array}$$

is a local diffeomorphism around the identity.

We have the obvious associated infinitesimal object:

Definition C.9 *Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{g}_1, \mathfrak{g}_2$ Lie subalgebras of $\mathfrak{g}$. We say that $(\mathfrak{g}, \mathfrak{g}_1, \mathfrak{g}_2)$ forms a **double Lie algebra** if $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$.*

The next is an immediate consequence of the Lie's third Theorem (see for example [24]):

Corollary C.10 *Let (G, G_1, G_2) be a (local) double Lie group, whose associated Lie algebras to G, G_1 and G_2 are $\mathfrak{g}$, $\mathfrak{g}_1$ and $\mathfrak{g}_2$, respectively. Then $(\mathfrak{g}, \mathfrak{g}_1, \mathfrak{g}_2)$ forms a double Lie algebra. Conversely if $(\mathfrak{g}, \mathfrak{g}_1, \mathfrak{g}_2)$ forms a double Lie, then (G, G_1, G_2) is local double Lie group.*

Under appropriate additional hypotheses local objects can be removed from the picture:

Proposition C.11 *Let $(\mathfrak{g}, \mathfrak{g}_1, \mathfrak{g}_2)$ be a double Lie algebra and let G be the connected and simply connected Lie group which integrates $\mathfrak{g}$. If G_1 and G_2 are connected Lie groups that integrate $\mathfrak{g}_1$ and $\mathfrak{g}_2$, respectively, G_1 is compact and G_2 is connected, then (G, G_1, G_2) forms a double Lie group.*

The following result is crucial to the construction of the Poisson-Lie groups we will be interested in:

Theorem C.12 (Iwasawa decomposition) *Let G be a complex semi-simple Lie group and let $\mathfrak{g}^{\mathbb{R}}$ denote its (complex) Lie algebra $\mathfrak{g}$ viewed as a real Lie algebra. Let $\mathfrak{k}$ be a compact real form of $\mathfrak{g}$, $\mathfrak{t}$ a maximal abelian subalgebra of $\mathfrak{k}$. Fix a root partial order for the adjoint action of the (complex) Cartan subalgebra $\mathfrak{h} := \mathfrak{t} + i \cdot \mathfrak{t}$ and let $\mathfrak{n}_+$ be the sum of eigenspaces associates set of positive real roots of $(\mathfrak{g}, \mathfrak{h})$.*

Then $\mathfrak{g}^{\mathbb{R}}$ can be decomposed as direct sum of subalgebras

$$\mathfrak{g}^{\mathbb{R}} = \mathfrak{k} \oplus i \cdot \mathfrak{t} \oplus \mathfrak{n}_+,$$

and $G = G^{\mathbb{R}}$ can be factorized as product of (real) subgroups

$$\begin{array}{ccc} K \times A \times N & \longrightarrow & G^{\mathbb{R}} \\ (k, t, n) & \longmapsto & k \cdot t \cdot n, \end{array}$$

where K is the compact real form integrating $\mathfrak{k}$, $A = \exp(i\mathfrak{t})$ is an abelian subgroup diffeomorphic to a vector space and N is the connected subgroup of $G^{\mathbb{R}}$ integrating $\mathfrak{n}^+$.

Example C.4 *With the notation of Theorem C.12, consider $\mathfrak{b} \cdot \mathfrak{t} \oplus \mathfrak{n}_+$ which is a solvable subalgebra of $\mathfrak{g}^{\mathbb{R}} = \mathfrak{t} \oplus \mathfrak{b}$. Let $B = AN$ be the connected Lie subgroup of $G^{\mathbb{R}}$ with Lie algebra $\mathfrak{b}$. Then the application*

$$\begin{array}{ccc} K \times B & \longrightarrow & G^{\mathbb{R}} \\ (k, b) & \longmapsto & k \cdot b, \end{array}$$

is a diffeomorphism, that is, $(G^{\mathbb{R}}, K, B)$ is a double Lie group. Consequently $(\mathfrak{g}^{\mathbb{R}}, \mathfrak{k}, \mathfrak{b})$ is a double Lie algebra.

Here is a more concrete example:

Example C.5 *Take $G = SL(n\mathbb{C})$ and choose as maximal compact subgroup $K = SU(n)$ and as maximal torus in K the diagonal special unitary matrices. Under the standard choice of positive roots (differences of entries in the diagonal matrices in $\mathfrak{h}$), the sum of the eigenspaces corresponding to positive roots are the strictly upper triangular matrices in $SL(n\mathbb{C})$:*

$$\mathfrak{n} = \left\{ \begin{bmatrix} 0 & a_{12} & \cdots & a_{1n} \\ 0 & 0 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} |a_{ij} \in \mathbb{C} \right\},$$

$$i\mathfrak{t} = \left\{ A = diag(a_{11}, \cdots, a_{nn}) \,|\, a_{ii} \in \mathbb{R},\ \sum_{i=1}^{n} a_{ii} = 0 \right\},$$

$$\mathfrak{b} = \left\{ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} \mid a_{ij} \in \mathbb{C},\ \textit{if } i \neq j,\ a_{ii} \in \mathbb{R},\ \sum_{i=1}^{n} a_{ii} = 0 \right\}.$$

At the Lie group level we have that $SL(n,\mathbb{C}) = SU(n)B$ is a double Lie group where

$$B = \left\{ A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} \mid a_{ij} \in \mathbb{C},\ \textit{if } i \neq j,\ a_{ii} \in \mathbb{R}^+,\ \textit{det}(A) = 1 \right\}.$$

Back to Poisson geometry, the Iwasawa decomposition induces a canonical Manin triple structure on $(\mathfrak{g}^{\mathbb{R}}, \mathfrak{k}, \mathfrak{b})$.

For this consider $\langle \cdot, \cdot \rangle$ the imaginary part of the Killing form on $\mathfrak{g}$. This is an ad-invariant non-degenerate, symmetric, bilinear paring such that $\mathfrak{k}$ and $\mathfrak{b}$ are isotropic with respect to it. Therefore $(\mathfrak{g}^{\mathbb{R}}, \mathfrak{k}, \mathfrak{b}, \langle \cdot, \cdot \rangle)$ is a Manin triple. Then this induces on K and B Poisson Lie structures.

C.3
Actions on Poisson Lie groups and coadjoint orbits

Let (M, π) be a Poisson manifold. As we know, π induces a linear application

$$\pi^\sharp : T^*M \longrightarrow TM,$$

Now, consider a Poisson Lie group (G, π_G) with dual Poisson Lie group G^* and Lie algebra $\mathfrak{g}$. For each $\alpha \in \mathfrak{g}^*$ consider α_L, α_R the corresponding left and right invariant 1-forms on G, respectively. Consider the applications

$$\begin{array}{rcl} \lambda, \rho : \mathfrak{g}^* & \longrightarrow & \mathfrak{X}(G) \\ \alpha & \longmapsto & \lambda(\alpha) := \pi^\sharp(\alpha_L), \quad \rho(\alpha) := -\pi^\sharp(\alpha_R). \end{array}$$

By [14, Proposition $5.4.8$], we have that λ and ρ are Lie algebra morphisms and antihomomorphism, respectively.

Definition C.13 *Consider $\lambda(\alpha)$, $\rho(\alpha) \in \mathfrak{X}(G)$. Then, the integrations of λ, ρ induce left and right actions of G^* on G (assuming G^* to be simply connected). We will call these the* ***dressing actions*** *or* ***dressing transformations****.*

The importance of dressing actions is given by the following result on [22]:

Proposition C.14 *If (G, π) is a Poisson Lie group with (simply connected) dual G^*, then the dressing actions of G^* on G are Poisson actions.*

Example C.6 *Let G be a connected complex semi-simple Lie group with Iwasawa decomposition $G^{\mathbb{R}} = KAN$, and let π_{PL} be the Poisson Lie group structure on K. Consider the double Lie group factorization $G^{\mathbb{R}} = KB$ and the corresponding first projection: $pr_1 : G^{\mathbb{R}} \to K$.*

The right dressing action of $B = K^*$ on K is given by:

$$\begin{array}{ccc} K \times B & \longrightarrow & K \\ (k, b) & \longmapsto & pr_1(b^{-1} \cdot k), \end{array}$$

Here is an important result for the purposes of this thesis [23]:

Theorem C.15 *Let G be a connected complex semi-simple Lie group with Iwasawa decomposition $G^{\mathbb{R}} = KAN$, where the Lie algebra of A is $i\mathfrak{t}$ with $\mathfrak{t}$ the Lie algebra of a maximal torus $T \subset K$. Then the following holds:*

1. *The group K supports a Poisson Lie group structure π_{PL};*
2. *The symplectic leaves of π_{LP} are the orbits of either the left or right dressing actions;*
3. *The normalizer of T is transverse to the symplectic leaves and intersects each leaf in just one point. In other words, the composition*

   $$N(T) \hookrightarrow K \to K/\mathcal{F}_{\pi_{PL}}$$

   is a homeomorphism;
4. *The left and right multiplication of T on (K, π_{PL}) is a Poisson action.*

Example C.7 *The special unitary groups are Poisson Lie groups. More concretely, $SU(2)$, which is diffeomorphic to the 3-sphere, supports a Poisson Lie group structure whose symplectic leaves are parametrized by the normalizer of the diagonal torus, which is diffeomorphic to two copies of the circle.*

Our final discussion concerns induced Poisson structures on (co)adjoint orbits. To start with, we record the properties of the Poisson structure induced on the regular orbit [23]:

Theorem C.16 *The Poisson Lie group structure on K induces a Poisson structure on π on K/T with the following properties:*

1. *The projection map is a Poisson surjective submersion:*

$$(K, \pi_{PL}) \to (K/T, \pi);$$

2. *The left T action on $(K/T, \pi)$ is Poisson, it is tangent to the symplectic leaves, and moreover the T-orbits are isotropic submanifolds of the corresponding symplectic leaf;*

3. *The projection induces a surjection between leaf spaces:*

$$N(T) \to W = N(T)/T.$$

In particular the symplectic leaves of π are parametrized by the Weyl group.

In fact, more can be said about the symplectic leaves of the Poisson structure on K/T, and, the discussion extends to all (co)adjoint orbits:

Recall that the complex Lie group has a Bruhat decomposition

$$G = \bigsqcup_{w \in W} \mathcal{B} w \mathcal{B},$$

where $\mathcal{B}$ is for example the Borel subgroup

$$\mathcal{B} = TAN.$$

(For a proof of this result see for example [24, Theorem 1.4]).

As each coadjoint orbit on $\mathcal{O}$ in G is of the form

$$\mathcal{O} \simeq G/P \simeq K/P \cap K$$

when P is a parabolic subgroup of G (a subgroup containing a Borel subgroup), the Bruhat decomposition of G induces a Bruhat decomposition of any coadjoint orbit $K/P \cap K$. Each stratum –called a Bruhat cell– is diffeomorphic to a complex vector space and they are parametrised by the residual Weyl group $W_P := W \cap P$.

Here is Lu and Weinstein fundamental result on the induced Bruhat-Poisson structures on compact coadjoint orbits and their fundamental properties [22]:

Theorem C.17 *Let K be a compact semi-simple Lie group and let $\mathcal{O} \simeq K/P \cap K$ be a coadjoint orbit. Then the following holds:*

1. *$\mathcal{O}$ admits the so-called Bruhat-Poisson structure so that the projection*

$$(K, \pi_{PL}) \to (\mathcal{O}, \pi)$$

is a Poisson map;

2. *The symplectic leaves of $(\mathcal{O}, \pi)$ are the orbits of the right dressing action of K^* and coincide with the Bruhat cells. In particular the leaf space is parametrised by the residual Weyl group W_P;*

3. *The left T action on $\mathcal{O}$ is Poisson, tangent to the symplectic leaves, and with isotropic orbits.*

All in all, we want to emphasize the following properties of the Bruhat-Poisson structures:

1. They have a finite number of leaves;

2. The have a Poisson action of T with isotropic orbits.