Pontifícia Universidade Católica
do Rio de Janeiro

Arthur Galego Mendes

Essays on Monetary and Fiscal Policy

Tese de Doutorado

Thesis presented to the Programa de Pós–graduação em Economia of PUC-Rio in partial fulfillment of the requirements for the degree of Doutor em Economia.

Advisor: Prof. Tiago Couto Berriel

Rio de Janeiro
September 2018

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Bibliographic data

Arthur Galego Mendes

Essays on Monetary and Fiscal Policy / Arthur Galego Mendes; advisor: Tiago Couto Berriel. – Rio de janeiro: PUC-Rio, Departamento de Economia, 2018.

v., 179 f: il. color. ; 30 cm

Tese (doutorado) - Pontifícia Universidade Católica do Rio de Janeiro, Departamento de Economia.

Inclui bibliografia

1. Economia – Teses. 2. – Teses. 3. Armadilha da Liquidez;. 4. Balanço do Banco Central;. 5. Limite Inferior de Juros;. 6. Regras Fiscais;. 7. Comodities;. I. Berriel, Tiago. II. Pontifícia Universidade Católica do Rio de Janeiro. Departamento de Economia. III. Título.

CDD: 620.11

To my parents, for their support
and encouragement.

Acknowledgments

I would like to thank my advisor Professor Tiago Couto Berriel, the dean of the Graduate Program Professor Gustavo Gonzaga, the members of the committee, and my mother and father. This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance Code 001. I also would like to thanks Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) for financing support.

Abstract

Arthur Galego Mendes; Berriel, Tiago (Advisor). **Essays on Monetary and Fiscal Policy**. Rio de Janeiro, 2018. 179p. Tese de doutorado – Departamento de Economia, Pontifícia Universidade Católica do Rio de Janeiro.

This thesis is composed of 3 chapters. In the first chapter, It's shown that when a central bank is not fully financially backed by the treasury and faces a solvency constraint, an increase in the size or a change in the composition of its balance sheet (quantitative easing - QE) can serve as a commitment device in a liquidity trap scenario. In particular, when the short-term interest rate is at the zero lower bound, open market operations by the central bank that involve purchases of long-term bonds can help mitigate deflation and recession under a discretionary policy equilibrium. Using a simple endowment-economy model, it's shown that a change in the central bank balance sheet, which increases its size and duration, provides an incentive to the central bank to keep interest rates low in the future to avoid losses and satisfy its solvency constraints, approximating its full commitment policy. In the second chapter, the validity of the novel mechanism developed in chapter 1 is tested by incorporating a financially-independent central bank into a medium-scale DSGE model based on Smets and Wouters (2007), and calibrating it to replicate key features of the expansion of size and composition of the Federal Reserve's balance sheet in the post-2008 period. I find that the programs QE 2 and 3 generated positive effects on the dynamics of inflation, but mild effects on the output gap. The third chapter of the thesis evaluates the welfare consequences of simple fiscal rules in a model of a small commodity-exporting country with a share of financially constrained households, where fiscal policy takes the form of transfers. The main finding is that balanced budget rules for commodity revenues often outperform more sophisticated fiscal rules where commodity revenues are saved in a Sovereign Wealth Fund. Because commodity price shocks are typically highly persistent, the households' current income is close to their permanent income, so commodity price shocks don't need smoothing, making simple balanced budget rules close to optimal.

Keywords

Liquidity Trap; Central Bank Balance Sheet; Zero Lower Bound; Fiscal Rules; Commodities;

Resumo

Arthur Galego Mendes; Berriel, Tiago. **Ensaios sobre Políticas Monetárias e Fiscais**. Rio de Janeiro, 2018. 179p. Tese de Doutorado – Departamento de Economia, Pontifícia Universidade Católica do Rio de Janeiro.

Esta tese é composta por 3 capítulos. No primeiro capítulo mostro que quando um banco central não é totalmente apoiado financeiramente pelo tesouro e enfrenta uma restrição de solvência, um aumento no tamanho ou uma mudança na composição de seu balanço pode servir como um mecanismo de compromisso em um cenário de armadilha de liquidez. Em particular, quando a taxa de juros de curto prazo está em zero, operações de mercado aberto do banco central que envolvam compras de títulos de longo prazo podem ajudar a mitigar a deflação e recessão sob um equilíbrio de política discricionária. Usando um modelo simples com produto exógeno, mostramos que uma mudança no balanço do banco central, que aumenta seu tamanho e duração, incentiva o banco central a manter as taxas de juros baixas no futuro, a fim de evitar perdas e satisfazer a restrição de solvência, aproximando-se de sua política ótima de commitment. No segundo capítulo da tese, eu testo a validade do novo mecanismo desenvolvido no capítulo 1, incorporando um banco central financeiramente independente em um modelo DSGE de média escala baseado em Smets e Wouters (2007), e calibrando-o para replicar principais características da expansão do tamanho e composição do balanço do Federal Reserve no período pós-2008. Eu observo que os programas QE 2 e 3 geraram efeitos positivos na dinâmica da inflação, mas impacto modesto no hiato do produto. O terceiro capítulo da tese avalia as consequências em termos de bem-estar de regras fiscais simples em um modelo de um pequeno país exportador de commodities com uma parcela da população sem acesso ao mercado financeiro, onde a política fiscal assume a forma de transferências. Uma constatação principal é que as regras orçamentárias equilibradas para as receitas de commodities geralmente superam as regras fiscais mais sofisticadas, em que as receitas de commodities são salvas em um Fundo de Riqueza Soberana. Como os choques nos preços das commodities são tipicamente altamente persistentes, a renda atual das famílias está próxima de sua renda permanente, tornando as regras orçamentárias equilibradas próximas do ideal.

Palavras-chave

Armadilha da Liquidez; Balanço do Banco Central; Limite Inferior de Juros; Regras Fiscais; Comodities;

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1
Quantitative Easing as a Commitment Device in a Liquidity Trap

1.1
Introduction

Since the financial crisis of 2008, many central banks have been forced to change their main policy tool away from the short-term interest rates. As the policy rates reached the zero lower bound (ZLB), they lost their suitability as instruments to stimulate the economy. In a sluggish recovery, there has been a search for alternative expansionary monetary policies. The expansion of central bank's balance sheets has been the most common choice. In the United States, the Federal Reserve (Fed) purchased a total of \$1.75 trillion in agency debt, mortgage-backed securities (MBS) and Treasuries in the ”QE1”, followed by a second Treasury-only program of \$600 billion in the fall of 2010. In September 2011, the Fed introduced QE3, increasing the amount of long-term bonds in its balance sheet. Other countries also followed similar strategies. In March 2009, the Bank of England (BoE) announced it would purchase a total of £75 billion of U.K. gilts, which, after subsequent increases, was expanded to £375 billion in July 2012. On 4 April 2013, the Bank of Japan (BoJ) announced a plan to purchase ¥7.5 trillion of bonds a month and double its monetary base. More recently, on January 2, 2015, the European Central Bank (ECB) announced monthly asset purchases of 60 billion euros, to be carried out until at least September 2016.

The stimulative role of QE has since been the focus of intensive debate. Empirically, many studies have demonstrated the effects of these programs on asset prices and interest rates.[1] However, the precise theoretical channel through which these programs affect real variables is unclear and is still under the scrutiny of the academic debate. Most recent mechanisms rely on segmented markets or other sources of financial frictions in order to generate real effects.[2] In this paper, we provide an alternative mechanism in which

[1] See Gagnon et al. (2011), Hamilton and Wu (2012), Krishnamurthy and Vissing-Jorgensen (2011), and Williams (2011) and references therein.

[2] Among others, we refer to Gertler and Kiyotaki (2010), Gertler and Karadi (2013), Vayanos and Vila (2009), and Curdia and Woodford (2011)

changes in central bank balance sheet have real effects. Specifically, when the central bank is restricted from incurring in huge financial losses, these programs act as a credible restriction on future monetary policy actions.

In addition, we show that central banks that face solvency constraints can use their balance sheets to mitigate the credibility issues that arise in optimal policy in a liquidity trap. In other words, a central bank that is restricted in the losses it can have is subject to a possible commitment mechanism: if its balance sheet is large or shows long enough duration, possible adverse asset price movements coming from interest rate hikes are going to be avoided, restricting upward shifts in the policy rates and leading to a credible higher inflation path in the future. This commitment mechanism allows a discretionary central bank to approximate optimal commitment policies and provides a theoretical justification for the recent adoption of QE programs by several central banks as their short-term interest rates have reached the ZLB.

Identifying channels through which large purchase programs, such as QEs, have real effects is no trivial task. It has been well known since Wallace (1981) that changes in the size or the composition of the central bank's balance sheet has no effect on equilibrium allocations within the framework of general equilibrium models: in a representative agent-based model, a mere shuffling of assets between the central bank and the private sector should not change equilibrium asset prices in the economy. Instead, macroeconomic theory prescribes a rather different policy in the liquidity trap scenario. As first noted by Krugman (1998), optimal monetary policy at the ZLB entails a commitment to keep short-term interest rates low for a long period in the future. This policy generates a higher level of expected real income and inflation in the future and provides the economy with the necessary incentives for greater real expenditure and larger price increases in the present. The problem also emphasized in Krugman (1998), is how to make low interest rates in the future credible: the central bank may renege ex post on its promises to pursue its goal of price stability. In fact, why would the central bank generate undesired inflation simply because of a binding constraint in the past?

Addressing this credibility problem, Woodford (2012) suggests the use of explicit statements by central banks about the outlook for future policy in addition to their announcements about the immediate policy actions that are in course. This type of policy, or *forward guidance*, is intended to facilitate the implementation of the optimal policy, as it makes it unambiguously clear that the central bank intends to maintain the benchmark rate at its lower bound for extended periods. Despite all the discussion of its effectiveness in practice, these announcements only constitute a commitment device if associated with costs of reneging (either moral or pecuniary).

Nakata (2018) finds that a central bank has the incentive to maintain the original announced path of low nominal interest rates, in order to build reputation, if contractionary shocks hit the economy frequently. If the central bank reneges on the promise of low policy rates, it will lose reputation and the private sector will not believe such promises in future recessions. However, it is possible that most central bankers see it differently, and fear that even a temporary inflation overshoot could undermine the central bank's reputation of pursuing price stability as their primary objective.

Instead of relying on hidden reneging costs, we design a mechanism through which the credibility problem in a liquidity trap scenario can be mitigated if central banks face solvency constraints. More specifically, this mechanism allows this type of central bank to commit to lower future interest rate through a large-scale purchase of long-term securities that creates an incentive not to raise interest rates in the future and thus, avoid losses on its balance sheet.

This result relies on two basic assumptions: (*i*) central banks are not financially backed by the treasury in all possible states of nature, and (*ii*) central banks cannot become insolvent. The first observation limits cross-transfers between these authorities and imposes a budget constraint to the central bank. The second implies that central banks with limited fiscal backing cannot run unlimited losses.[3] We view these assumptions as a consequence of a self-imposed behavior motivated by the political embarrassment caused by large financial bail-outs. Together they provide an additional restriction to monetary policymakers: they cannot undertake actions that lead to excessive losses in their balance sheets. Accordingly, a current large-scale purchase of long-term securities can credibly lock the central bank into low interest rates in the future because interest rate hikes may threaten the central bank's solvency.[4]

Eggertsson (2006) was one of the first works to analyze deflation as a credibility problem, and to formally think about a time-consistent implementation of the commitment solution in a liquidity trap. He proposes that a government can credibly commit to "being irresponsible" by increasing deficit during a liquidity trap. Inflation expectations would increase because higher nominal debt gives the government an incentive to inflate the real value of the

[3]This is directly related to the literature that assumes balance sheet concerns on the part of the central bank, such as Sims (2004), Berriel and Bhattarai (2009), and Jeanne and Svensson (2007).

[4]For further reference on how interest rates affect central bank's balance sheets, see Hall and Reis (2015).

debt away, instead of raising revenues through an increase in distortionary tax rates.

This work is also closely related to Jeanne and Svensson (2007) (JE07 hereafter). They showed that if central banks in small open economies have capital concerns, then it is possible to create a commitment mechanism that allows independent central banks to achieve a higher future price level through a present currency depreciation. This paper differs from JE07 in two important aspects. First, the commitment mechanism we designed does not rely on the small open economy assumption and hence is more suitable for the U.S. economy. Second, in JE07 capital concerns is modeled as ad-hoc preferences against low levels of capital that are difficult to assess and interpret in practice. Instead, we rely on the more realistic assumption that central banks will not undertake any actions that may undermine their capacity or independence to carry out monetary policy in the future. This is in line with Del Negro and Sims (2005), where low levels of capital can prevent a central bank from avoiding self-fulfilling hyper-inflationary equilibria, and Buiter (2008), where the scale of the recourse to seigniorage required to safeguard central bank solvency may undermine price stability. Bhattarai et al. (2015) focus on the implications of joint monetary and fiscal policy to a similar problem, while here we focus on the implications of limited losses of the central bank.

The rest of the chapter is organized as follows. Section (1.2) describes a simple endowment economy model with a financially independent central bank that conducts monetary policy under discretion and commitment and is allowed to buy short and long-term government bonds. In section (1.3) its shown analytically how an increase in the size and composition of the central bank's balance sheet can serve as a commitment device to low interest rates in the future during a liquidity trap scenario. Section (1.4) discusses the results of the previous section and simulates the impact of QE on the Fed's balance sheet and the dynamics of inflation in the exogenous-income model. Section (1.5) concludes.

1.2
A Simple Endowment Economy Model

1.2.1
The Model Overview

In this section, we consider a one-good, representative agent economy. The household consumes and saves by buying riskless government bonds of different maturities. In this simple economy, we abstract from production and assume that consumption each period is restricted to an exogenous endowment process. The central bank is not fully financed by the Treasury and conducts monetary policy through a price-level targeting regime in which the policy rate is set to minimize a quadratic loss function of the price level. We introduce money in this economy by imposing a cash-in-advance constraint: at the beginning of each period, individuals trade cash for bonds, with net nominal interest rate i_t. Their consumption during the period is constrained by the cash with which they generate from this trading. We show in section 3 that the economy falls into a liquidity trap in period 1, with price level below the target, as a result of an unanticipated fall in the expected endowment growth. The same scenario might arise in period 2, conditional on the realization of the endowment process.

1.2.2 The Household

The household's utility function is assumed to take the form,

$$U_t = \mathbb{E}_t \sum_{i=0}^{\infty} \beta^i \frac{C_{t+i}^{1-\sigma}}{1-\sigma}$$

where C_t is consumption in period t, $\mathbb{E}_t$ is the expectation operator conditional to available information in period t, β is the discount factor, and σ the coefficient of risk aversion. The household seeks to maximize her utility subject to the following budget constraint,

$$P_t Y_t + (1+i_{t-1}) B_{t-1}^{hh} + \left(\frac{1 + (1-\delta_b) Q_t}{Q_{t-1}} \right) B_{t-1}^{l,hh} + (1+i_{t-1}^m) M_{t-1} = P_t C_t + Z_t + B_t^{hh} + B_t^{l,hh} + M_t \tag{1-1}$$

where Y_t is a stochastic endowment process, M_t, B_t^h and $B_t^{l,h}$ are respectively the total of money, short and long-term claims on the government debt held by the household. The short-term bond costs 1 dollar in period t and pays nominal interest rate $1 + i_t$ in period $t + 1$. We allow the central bank to pay nominal net interest rate, i_t^m, on the its monetary liabilities, M_t. The long-term bond costs Q_t dollars in period t and pays a one-dollar coupon in period $t+1$. In $t+2$, the bond will pay a fraction $(1-\delta_b)$ of the coupon, $(1-\delta_b)^2$ in $t+3$, and so on. Lower values of δ_b correspond to portfolios with longer maturities. We add up bonds in terms of the amount of output they will pay in the current period so each period the bonds inherited from previous periods shrink by the factor $1-\delta_b$, and $1/\delta_b$ is the average maturity of the long-term bond. Each period the government collects real lump-sum tax, Z_t.

1.2.3
The Endowment Process

As mentioned before, there is no production and each period consumption is restricted to the exogenous income process, Y_t. Its assumed that, from indefinitely long before period 1, the agent has been receiving a certain income $y^* e^{\bar{y}}$. In period 1, however, the agent is informed that from period 2 onwards income will follow the process described by expression (1-2), and that the resolution of uncertainty on this process will become available information to the agent only in period 2,

$$(Y_1, Y_2, Y_3, ..., Y_{N-1}, Y_N, Y_{N+1}, ...) =$$
$$= \begin{cases} (y^* e^{\bar{y}}, y^*, y^*, ..., y^*, y^*, y^*, ...) & \text{with probability } 1-\mu \\ (y^* e^{\bar{y}}, y^*, y^* e^{\underline{y}}, ..., y^* e^{\underline{y}}, y^*, y^*, ...) & \text{with probability } \mu \end{cases} \tag{1-2}$$

where y^* is the income of the upcoming steady state, $\bar{y} > 0$ and $\underline{y} < 0$. In the section (1.3) we show that in period 1 the unexpected fall in income growth pushes the economy into a liquidity trap as a result of the interaction between the agent's excess savings and the ZLB. This liquid trap scenario continues in period 2, with probability μ, in the low-income realization of process (1-2), or reverts with probability $1-\mu$ in the high-income realization of (1-2).

Figure (1.1) depicts the endowment process. Note that after N periods, the income returns to steady-state level, y^*, independently of the realization of process (1-2), so we have a well defined non-stochastic steady state.

1.2.4
The Public Sector

1.2.4.1
The Central Bank

Increasing literature points to the fact that central banks are not fully financed by the treasury in all contingencies. This is more evident in cases where the central bank faces risks of unusually large losses in its balance sheet. Following these concerns, we introduce a central bank that is not fully financially backed by the treasury. Since the central bank cannot rely on the treasury for all its financial needs, it is subject to a period-by-period budget constraint,

$$B_t^{cb} + B_t^{cb,l} + D_t = (1+i_{t-1})B_{t-1}^{cb} + \left(\frac{1+(1-\delta_b)Q_t}{Q_{t-1}}\right) B_{t-1}^{cb,l} + M_t - (1+i_{t-1}^m)M_{t-1}$$

where B_{t-1}^{cb} and $B_{t-1}^{l,cb}$ denote the dollar-denominated stock of short and long-term government bonds held by the central bank in period t, respectively. The variable M_t is the outstanding central bank monetary liabilities, and D_t denotes dividends paid to the treasury. All measured in nominal US dollars.

One important assumption is that, while considering its budget constraint, all assets of the central bank are marked-to-market. This is a trivially appropriate assumption for modeling the ECB or the Bank of England, which are obliged by law to report this type of pricing. However, if one considers the Fed, this assumption is debatable. In principle, the Fed *can* and actually has adopted historical prices in calculating gains or losses in his balance sheet. We argue that if one is worried about the political implications of a possible recapitalization or decrease in revenues from the Fed to the U.S. Treasury, a large gap between historical and market valuations would be an embarrassment for the Fed. So, even without reporting it, the Fed would care about marked-to-market gains and losses in its balance sheet.

This paper adopts the view that a central bank endorses the mark-to-market accounting, assessing the value of its portfolio at market prices under any circumstance. In this case, the central bank's net income is defined as the sum of net interest income with the capital gain. The first term of expression (2-2) denotes the nominal net interest income, NII_t: the net return on the short and long-term bonds, less the interest paid on the central bank's monetary liabilities. Note that the net return of the long-term bond is the one-dollar coupon less the depreciation of the bond. The second term of the expression denotes capital gains or losses: the dollar value of the holdings of long-term bonds multiplied by the percentage change in the market value of the bond.

$$NI_t = \underbrace{i_{t-1}B_{t-1}^{cb} + \left(\frac{1-\delta_b Q_{t-1}}{Q_{t-1}}\right) B_{t-1}^{cb,l} - i_{t-1}^m M_{t-1}}_{\text{Nominal Net Interest Income } (\equiv NII_t)} + \underbrace{\left(\frac{Q_t^b}{Q_{t-1}^b} - 1\right) B_{t-1}^{cb,l}}_{\text{Nominal Capital Gains or Losses}} \tag{1-3}$$

Net income is an important concept because it underlies the size of remittances foreseen in the contract between the fiscal and the monetary authorities. Usually, central banks transfer a share of its net income to the treasury in terms of seigniorage revenues. We model this agreement by assuming the following dividend rule,

$$D_t = NI_t$$

The rule D_t is key in this paper. It is important to note that these transfers paid by the central bank to the treasury could be negative. Such transfer payment from the treasury to the central bank can be viewed as the mechanism through which the treasury can inject capital into the central bank, i.e., transfer resources to the central bank in order to recapitalize it.

In normal times, the assets and liabilities of a central bank are nearly riskless and net income is usually positive. When the central bank holds other types of assets, especially private debt, and assets subject to nominal losses, net income could be negative with significant probability. Negative net income requires fiscal backing to the central bank. The act of capitalizing the central bank would have to be approved by fiscal authorities, subject to the underlying political process.

Even if feasible in economic terms, a fiscal bailout of the central bank is not necessarily politically implementable. In many occasions, the taxpayer is simply not willing to abdicate on real resources (and, thus, consumption), in order to support the central bank's balance sheet. An interesting example is the ECB, where it is not clear how losses would be split among different fiscal authorities. We include these considerations in the model by assuming the following dividend rule between the central bank and the treasury,

$$D_t = \begin{cases} NI_t & \text{if } NI_t \geq -\xi \\ 0 & \text{otherwise} \end{cases}$$

where $\xi \geq 0$. Positive net income is transferred to the treasury as dividends on seignorage. We allow for a limited degree of fiscal backing. The treasury is allowed to cover central bank's losses if it sits below a predefined threshold. If $NI_t < -\xi$, recapitalization is blocked by fiscal authorities.

The central bank's net worth, NW_t, is defined as the excess of the value of the bond portfolio marked to market over the size of the monetary liabilities, $NW_t = B_t^{cb} + B_t^{cb,l} - M_t$. Note that we can rewrite the central bank's balance sheet recursively as

$$NW_t = NW_{t-1} + NI_t - D_t \tag{1-4}$$

Central bank insolvency is an issue of considerable controversy since the vast majority of its liabilities is irredeemable. As pointed out by Sims

(2004), while a central bank can always pay all its home-currency denominated expenses (financial or operational) through the issuance of base money it may not be optimal or even acceptable: it may generate inadmissible high rates of inflation. In addition, there are limits to the amount of real resources the central bank can appropriate by increasing the issuance of nominal base money.[5] Hence, despite the central bank's special ability to issue not just non-interest-bearing but also irredeemable liabilities, central bank's solvency is questioned if its capital falls below some specified level. According to Hall and Reis (2015), a central bank is independent as long as it adheres to its dividend rule and the rule does not imply explosive growth of reserves. These authors take interest rates and inflation as given and study the implications of the exit from the Great Recession to the financial stability of the Fed. We take the opposite approach: *we assume that central banks remain solvent and study the implications for optimal monetary policy.* We rule out insolvency in the model by imposing a lower bound on the central bank's net worth[6],

$$NW_t \geq -\phi \tag{1-5}$$

where the parameter ϕ can be interpreted as a physical limit imposed by fiscal authorities or a self-imposed restriction in light of the uncertainties of a bail-out. We take a similar approach to Hall and Reis (2015) and assume that ϕ represents the present value of seignorage revenues, so that the central bank does not need to receive a positive transfer from the fiscal authority in present value. Expression (1-5) implies that policymakers are prohibited to undertake policies schemes that lead the central bank to insolvency or that severely compromise the financial status of the bank.

This solvency constraint is related to the literature that assumes balance-sheet concerns on the part of the central bank. Isard (1994) presented a model of currency crises in which the central bank cares about the value of its foreign exchange reserves. More recently, Jeanne and Svensson (2007) assumed that the central bank has an objective function with a fixed loss suffered if the capital of the central bank falls below a critical level. Berriel and Bhattarai (2009) modeled balance sheet concerns by including a target for real capital in the central bank's loss function. These works assume ad-hoc preferences of the central bank against negative or even low levels of capital. Note that the solvency constraint (1-5) simply prevents the central banker from taking certain policy actions in certain situations, and says nothing about central bankers' preferences about capital adequacy. This is in line with Del Negro

[5]See Buiter (2008).

[6]Note that we are imposing this solvency restriction in nominal terms. This simplifying assumption is dropped in the quantitative model when we take this restriction in real terms.

and Sims (2015), where low levels of capital may prevent a central bank from avoiding self-fulfilling hyperinflationary equilibria.

It remains to specify the objective of monetary policy and how the central bank manages different instruments to achieve its goals. Its assumed that the central bank has an objective function corresponding to a price-level targeting regime. The central bank's intertemporal loss function can be written as

$$L_t = \mathbb{E}_t \sum_{i=0}^{\infty} \beta^i (log(P_{t+i}) - 1)^2$$

where 1 is the target on the price level. The price level target is a simplification that allows for a simple analytical solution of the model. The quantitative model of chapter (2), substitutes this ad-hoc assumption by a standard loss function in terms of inflation and the output gap.

The monetary authority has three instruments to achieve its goal of price stability: the policy rate (i_t), interest paid on reserves (IOR) (i_t^m), and quantitative easing ($B_t^{cb,l}$). We assume that in each period, the central bank sets the IOR equal to the policy rate,

$$i_t = i_t^m \tag{1-6}$$

This is a convenient assumption since it implies zero net interest income at all periods, allowing for a simple analytical solution of the model. Moreover, (1-6) captures an important feature of the new way that many central banks have been operating around the world since the financial crisis. For example, in the US, the Emergency Economic Stabilization Act of 2008 allowed the Fed to begin paying interest on excess reserve balances ("IOER") as well as required reserves.

In the model, the quantitative easing policy is of the simplest form possible: the central bank implements a constant target for holdings of long-term government bonds in its balance sheet,

$$B_t^{cb,l} = B_*^l \tag{1-7}$$

where B_*^l is not under the control of policymakers but rather chosen to match the size of Fed's balance sheet observed in the data. Note that the purpose of this work is not to investigate the optimality of QE but rather to take it as given and assess its implications to conventional monetary policy when the central bank is fiscally constrained.

We assume that the central bank implements adjustments to the policy rate through conventional open-market operations with short-term bonds. In this case, holdings of short-term bonds, B_t^{cb}, is determined in the balance sheet so that the central bank supplies the desired liquitity required by households,

$$B_t^{cb} = NW_t + M_t - B_t^{cb,l} \tag{1-8}$$

Monetary policy is also subject to the zero lower bound (ZLB) on the short-term nominal interest rate,

$$i_t \geq 0 \tag{1-9}$$

1.2.4.2
The treasury and fiscal policy

For simplicity we abstract from government expenditure and assume that the fiscal authority receives dividends from the central bank, D_t, and collects lump-sum taxes, Z_t. The treasury's budget constraint can be written as,

$$\begin{aligned} D_t + Z_t + B_t^{hh} + B_t^{cb} + B_t^{cb,l} + B_t^{hh,l} = (1 + i_{t-1}) \left(B_{t-1}^{cb} + B_{t-1}^{hh} \right) + \\ + \left(\frac{1 + (1 - \delta_b) Q_{t-1}}{Q_{t-1}} \right) \left(B_{t-1}^{l,hh} + B_{t-1}^{l,cb} \right) \end{aligned} \tag{1-10}$$

We specify fiscal policy in terms of a rule that determines the evolution of lump-sum taxes responding to contemporaneous level of total real government debt,

$$\frac{Z_t}{P_t} = exp \left\{ \phi_z \left(\frac{B_{t-1}^{hh} + B_{t-1} + B_{t-1}^{hh,s} + B_{t-1}^{s}}{P_t} \right) \right\} \tag{1-11}$$

We choose the parameter ϕ_z so that fiscal policy is “Ricardian": lump-sum taxes adjust sufficiently fast to ensure that the trajectory of government debt is non-explosive, regardless the path of the price level.[7]

[7]The terminology “Ricardian" fiscal policy is borrowed from Woodford (2001). “Passive" fiscal policy has equivalent interpretation, as in Leeper (1991). Note that model is not overdetermined because the treasury's budget constraint is a mirror of the household's budget constrain, so that B_t^{hh} will always adjust to close the treasury's budget regardless of Z_t .

1.2.5 Equilibrium

Consider a linear rational expectations model formed by the system of equations (1-1)-(1-11) linearized around the zero-inflation steady state. If the fiscal authority is not allowed to back the central bank in case of insolvency, i.e. $\xi = \phi$, we can reduce the model to a 3-equations linear system

$$\hat{y}_t = \hat{y}_{t+1|t} - \sigma^{-1}\left(i_t - (\hat{p}_{t+1|t} - \hat{p}_t) - \rho\right) \tag{1-12}$$

$$\hat{q}_t = \beta(1-\delta_b)\hat{q}_{t+1|t} - (i_t - \rho) \tag{1-13}$$

$$\tilde{ni}_t = b^l_* \left(\hat{q}_t - (1+\rho-\delta_b)\hat{q}_{t-1}\right) \tag{1-14}$$

together with the non-linear constraints

$$\tilde{ni}_t \geq -(\phi + ni_*) \tag{1-15}$$

$$i_t \geq 0 \tag{1-16}$$

and the log-linearized endowment process,

$$(\hat{y}_1, \hat{y}_2, \hat{y}_3, ..., \hat{y}_{N-1}, \hat{y}_N, \hat{y}_{N+1}, ...) = \begin{cases} (\bar{y}, 0, 0, ...) & \text{with prob } 1-\mu \\ (\bar{y}, 0, \underline{y}, ..., \underline{y}, 0, 0, ...) & \text{with prob } \mu \end{cases} \tag{1-17}$$

where stared variables denote steady state levels, hatted variables denote percent deviation from steady state $\left(\hat{x} = \frac{x_t - x_*}{x_*}\right)$, tilded variables denote deviations from steady state as a share of steady state GDP $\left(\tilde{x} = \frac{x_t - x_*}{y_*}\right)$ and $\theta_b \equiv \beta(1-\delta_b)$. A list of all linearized equations and a proof for the proposition are provided in the technical appendix (5.1).

Notation *Let s_i denote the nodes of the exogenous income process* (1-17)*, for $i \in \{l, h\}$. Where "h" indicates the realization of the high-income state and "l" represents the low-income state.*

Definition 1 *We define a discretion equilibrium as a sequence for prices $\{\hat{p}_t, i_t, \hat{q}_t\}$ and quantities $\{\tilde{ni}_t, \hat{y}_t\}$ as functions of the stochastic variable $\{s_i\}$ and the endogenous state $\{\hat{q}_{t-1}\}$ such that the central bank's intertemporal loss function, L_t, is minimized every period subject to* (1-12)-(1-17) *when the central bank cannot commit to future policies.*

Definition 2 *We define a commitment equilibrium as a sequence for prices*

$\{\hat{p}_t, i_t, \hat{q}_t\}$ and quantities $\{\tilde{ni}_t, \hat{y}_t\}$ as functions of the stochastic process $\{s_i\}$ and the endogenous state $\{\hat{q}_{t-1}\}$ such that the central bank's intertemporal loss function, L_t, is minimized once and for all in period 1 subject to (1-12)-(1-17) *when the central bank can commit to future policies.*

1.3
Fiscally Constrained Central Bank and Quantitative Easing

In this section, we assume that $\phi = \xi < \infty$ so that the solvency constraint is a relevant restriction to equilibrium. We show that a long-term bond purchase program (or QE) can help mitigating deflation in a discretionary equilibrium. This is because a change in the size and composition of the balance sheet provides the central bank with the incentive to keep low interest rates in period 2 and avoid large financial losses.

More specifically, we show that for any given loss limit, ϕ, there is a positive level of long-term bond holdings, $b_*^{cb,l}$, such that, if the zero lower bound is binding in period 1 then the solvency constraint is binding, at least, in the high-endowment state of period 2. The solvency constraint prevents an interest rate rike and alleviates deflationary pressures in period 1.

Definition 3 *Let $\tilde{\phi}_b \equiv \frac{\phi}{(1+\rho-\delta_b)b_*^l}$. The fraction $\tilde{\phi}_b$ can be interpreted as an inverse measure of risk. It denotes the largest percentage fall in the market value of the long-term bond within one period, conditional on the central bank remaining solvent in that period.*

Note that both an increase in the size or duration of the balance sheet will decrease $\tilde{\phi}_b$. A lower $\tilde{\phi}_b$ means that the central bank balance sheet is less resilient to the volatility of long-term bond prices.

Next, we make three assumptions about the parameter space. Assumptions (i) and (ii) ensure that the endowment process (1-17) pushes the economy against the ZLB in period 1 and in the low-income state of period 2. Condition (iii) guarantees that the solvency constraint is tight enough to restrict an interest rate hike in the high-income state of period 2, but not so tight that the central bank is insolvent regardless the choice of the policy rate.

A1 Assume (i) $\underline{y} < -\rho\sigma^{-1}$, (ii) $\bar{y} \geq \rho\sigma^{-1}(2+\beta) + (1+\beta)\mu\underline{y}$ *and* (iii) $\rho \leq \tilde{\phi}_b \leq \rho(1+\mu\theta_p)$

1.3.1
Solving the Model under Discretionary Monetary Policy

In this endowment economy, it is intuitive to think in terms of an equilibrium real interest rate, which will be in effect no matter the behavior of nominal prices. In normal times, when expected income growth is non-negative, the equilibrium real interest rate is positive and policymakers can readily implement the policy rate that is consistent with the price-level target if this policy does not threaten the solvency of the central bank.

Under the specific assumptions of this model, the equilibrium interest rates will be positive in the high-income state of period 2 and from period 3 onwards,

$$r_t^n(s_i) \equiv i_t - (\hat{p}^i_{t+1|t} - \hat{p}^i_t) = \rho > 0 \qquad \text{for all } 3 \leq t < N, \text{ and } t = 2 \text{ if } s_i = s_h,$$

where $r_t^n(s_i)$ denotes the natural interest rate in period t when state i occurred. If the solvency does not bind, one can immediately guess at the solution: the central bank sets the nominal interest rate equal to ρ and the price level immediately converges to the target. Condition $\tilde{\phi}_b \geq \rho$ assures that the loss limit is large enough to allow the central bank to pursue this policy scheme from period 3 onwards, independent of the realization of the income process.

Assume A1, $\phi = \xi$ and $N \to \infty$. The equilibrium under dicretionary monetary policy is characterized by $\{\hat{p}_t(s_i), i_t(s_i), \hat{q}_t(s_i)\} = \{0, \rho, 0\}$ for $t \geq 3$ and $s^i = \{s^l, s^h\}$. A proof is provided in the technical appendix (5.2).

Low-Income State of the Second Period. Condition (i) brings about a large fall in expected income growth that pushes the natural rate of interest into negative territory, and the central bank faces a credibility problem, as in Krugman (1998),

$$r_2^n(s^l) = i_2(s^l) - (\underbrace{\hat{p}_{3|2}(s^l)}_{=0} - \hat{p}_2(s^l)) = \rho + \sigma\underline{y} < 0$$

Because of the ZLB, the only way the economy can achieve negative real interest rates is by generating inflation expectations. Because the central bank cannot commit to a higher target in period 3, the price level will have to fall below the target in order to clear the goods market in the low-income state of period 2. Note that, as in Wallace (1981), the price level will fall regardless of the current money supply because any excess money will simply be kept rather than spent. This happens because once the nominal rate reaches zero, money and bonds become perfect substitutes and no matter how much liquidity the central bank injects in the economy, it can no longer affect asset prices. Lemma (1.3.1) summarizes the equilibrium outcome in this state,

Assume A1, $\phi = \xi$ and $N \to \infty$. The equilibrium under discretion in the low-income state of period 2 is characterized by

$$\{\hat{p}_2(\hat{q}_1, s_l), i_2(\hat{q}_1, s_l), \hat{q}_2(\hat{q}_1, s_l)\} = \{\rho + \sigma \underline{y}, 0, \rho\} \tag{1-18}$$

A proof is provided in the technical appendix (5.2).

High-Income State of the Second Period. In the high-income state of period 2, the central bank faces a positive natural real interest rate, ρ, and hence can achieve the target by setting the policy rate equal to ρ,

$$r_2^n(\hat{q}_1, s_h) = i_2(\hat{q}_1, s_h) - (\underbrace{\hat{p}_{3|2}(\hat{q}_1, s_h)}_{=0} - \hat{p}_2(\hat{q}_1, s_h)) = \rho > 0$$

The key difference in this state is that the solvency constraint might bind and prevent the central bank from choosing the optimal discretionary policy. The price of the long-term bond in the previous period, $\hat{q}_1$, plays an important role in determining the equilibrium in this state. Recall that $\tilde{\phi}_b$ is the largest fall in the price of long-term bonds that the central bank can absorb while still solvent. Hence, if $\hat{q}_1 > \tilde{\phi}_b$, the central bank will be forced deviate from its optimal (unconstrained) discretionary policy towards a more expansionary interest rate policy. Lemma (1.3.1) summarizes,

Assume A1, $\phi = \xi$ and $N \to \infty$. The equilibrium under discretionary monetary policy in the high-income state of the second period is characterized by

$$i_2(\hat{q}_1, s_h) = \begin{cases} \rho & \text{if} \quad \hat{q}_1 \leq \tilde{\phi}_b \\ \rho - (1 + \rho - \delta_b)(\hat{q}_1 - \tilde{\phi}_b) & \text{if} \quad \tilde{\phi}_b < \hat{q}_1 \leq \tilde{\phi}_b + \frac{\rho}{1+\rho-\delta_b} \\ 0 & \text{if} \quad \tilde{\phi}_b + \frac{\rho}{1+\rho-\delta_b} < \hat{q}_1 \end{cases}$$

$$\hat{p}_2(q_1, s_h) = \hat{q}_2(q_1, s^h) = \rho - i_2(q_1, s^h) \tag{1-19}$$

$$\tilde{nw}_2(q_1, s_h) = \begin{cases} -\left(\frac{\phi}{\tilde{\phi}_b}\right)\hat{q}_1 & \text{if} \quad \hat{q}_1 \leq \tilde{\phi}_b \\ -\phi & \text{if} \quad \tilde{\phi}_b < \hat{q}_1 \leq \tilde{\phi}_b + \frac{\rho}{1+\rho-\delta_b} \end{cases}$$

A proof is provided in the technical appendix (5.2).

When the $\hat{q}_1 > \tilde{\phi}_b$, the solency constraint binds and prevents the central bank from raising interest rates, resulting in an undesired high price level, $\hat{p}_2(q_1, s_h) > 0$. Moreover, in the range between $\tilde{\phi}_b$ and $\tilde{\phi}_b + \frac{\rho}{1+\rho-\delta_b}$, the equilibrium price level increases with $\hat{q}_1$ at the rate $1 + \rho - \delta_b$.

First Period. In the first period the central bank chooses the policy rate i_1 to minimize the intertemporal loss function, L_1, taking into account that this decision affects $\hat{q}_1$, and hence expectations about next periods's price level. The private sector condition its expectations to $\hat{q}_1$, using expressions (1-18) and (1-19) and the probability distribution of the endowment process,

$$\hat{p}_{2|1}(q_1) = \mu\hat{p}_2(\hat{q}_1, s_l) + (1-\mu)\hat{p}_2(\hat{q}_1, s_h) \tag{1-20}$$
$$\hat{q}_{2|1}(\hat{q}_1) = \mu\hat{q}_2(\hat{q}_1, s_l) + (1-\mu)\hat{q}_2(q_1, s_h) \tag{1-21}$$

The problem faced by the central bank is

$$\begin{aligned}
\underset{\{i_1 \geq 0\}}{min} \quad & \frac{1}{2}\left[\hat{p}_1^2 + \beta\left(\hat{p}_{2|1}(\hat{q}_1)\right)^2\right] \\
\text{s.t.} \quad & r_1^n = i_1 - \left(\hat{p}_{2|1}(\hat{q}_1) - \hat{p}_1\right) = \rho - \sigma\bar{y} \\
& \hat{q}_1 = \theta_q\hat{q}_{2|1}(\hat{q}_1) - (i_1 - \rho) \\
& \tilde{ni}_1 = b_*^l\left(\hat{q}_1 - (1+\rho-\delta_b)\hat{q}_0\right) \geq -\phi \\
& \text{(1-18) - (1-21) given } \hat{q}_0 = 0
\end{aligned}$$

Proposição 1.1 *Assume A1, $\phi = \xi$ and $N \to \infty$. The equilibrium under discretionary monetary policy in the first period is characterized by*

$$\begin{aligned}
i_1 &= 0 \\
\hat{q}_1 &= \rho\left(\frac{(1+\mu\theta_q)}{1-(1-\mu)(1+\rho-\delta_b)\theta_q}\right) - \theta_q\Xi\tilde{\phi}_b \\
\hat{p}_1 &= \underbrace{\rho - \sigma\bar{y} + \mu(\rho + \sigma\underline{y})}_{\textit{unconstrained discretion}} + \underbrace{(1-\mu)\Xi\left(\rho(1+\mu\theta_q) - \tilde{\phi}_b\right)}_{\textit{QE effect}}
\end{aligned} \tag{1-22}$$

where $\Xi \equiv \left(\frac{(1+\rho-\delta_b)}{1-(1-\mu)(1+\rho-\delta_b)\theta_q}\right)$. A proof is provided in the technical appendix (5.2).

As highlighted by expression (1-22), the equilibrium price level in period 1 is the sum of the baseline price level (that would prevail in the absence of the solvency constraint) with the QE effect. The QE effect is always non-negative and depends on the size and duration of the central bank's balance sheet.[8] When the central bank expands its balance sheet through an increase in b^l_*, the parameter $\tilde{\phi}_b$ shrinks, tightening the solvency constraint in the high-income state of period 2, and boosting the price level in period 1 when the economy is stuck at the ZLB. Moreover, the marginal effect of QE depends on the average duration of the balance sheet: Ξ increases with $1/\delta_b$, and $\hat{p}_1$ collapses to the baseline price level as $1/\delta_b \to 1$.

These results provide a theoretical support for the use of non-conventional monetary policies, such as QE, by central banks to achieve price stability when the short-term policy rate is up against the ZLB.

1.4 Discussion

In this section, we use the simple model developed in sections 1.2 - 1.3 to analyze the effects of unconventional monetary policy on a fiscally constrained central bank since the financial crisis in 2008. We calibrate the model to resemble basic features of the US economy and the Fed's balance sheet in the period 2009-2013 when the bank implemented three rounds of large-scale asset pruchase programs: QE1, QE2, and QE3. We then use the calibrated model to evaluate the consequences of these programs to the financial stability of the Fed, and the dynamics of inflation.

Calibration. We follow Hall and Reis (2015) and use data in the annual report of the Federal Reserve on the value and maturity of the U.S. Treasury securities it holds, to calculate the value-weighted average maturity of the Fed's financial assets. Between 2009 and 2013, the average maturity of the Fed's portfolio was 7.8 years. We measure the holdings of long-term bonds, $b^{cb,l}_*$, as the total U.S. Treasury and agency securities held by the Fed as a share of GDP. We assume that the Fed holds long-term bonds in an amount equal to 15 percent of annual GDP throughout the period in consideration. The loss limit, ϕ, is set to 2 percent of annual GDP, which is the present value of future seignorage revenues used in Hall and Reis (2015).

Hall and Reis (2015) estimated the expected duration of the crisis from 2009 to be 5 years (exit rate of 20 percent per year). To replicate the crisis expected duration in the 3-period structure of the model, we choose $\beta = 0.94$ and $\mu = 0.65$. Thus, steady-state annual real interest rate is 2 percent if we interpret each period of the model as being 3 years long. In this case, in period 1 the economy faces a 3-year crisis with low income and zero interest rates. In period 2, either the liquidity trap persists for another 3 years with

[8]The non-negativity of the QE effect depends on condition (*iii*) of A1.

65% probability or the economy recovers with 35% chance. The parameters $\bar{y}$ and $\underline{y}$ are chosen so that the expected cumulated forgone income is 25% of annual GDP, which is the accumulated gap between potential and actual GDP observed in the data during the period 2009-2013 (see figure 1.4 for details).[9] Table 1.1 in section (1.6) summarizes the model's calibration and steady state.

The Effects of Quantitative Easing. The left-hand side of figure 1.2 plots the state-contingent equilibrium paths for the price level, interest rate, long-term bond price and the Fed's net worth. The blue dashed line shows the evolution of these variables in the high-income state of the income process and the red line represents the low-income state. For the sake of comparision with well known results in the literature, we plot on the right-hand side of figure 1.2 the equilibrium paths of the nominal interest rate and the price level under the baseline commitment (upper panel) and discretion (lower panel), both without QE ($b^l_* = 0$).To simplify the comparison, we asume that under commitment the Fed has the ability to commit to a price level target only in period 2.[10]

When the economy enters the crisis state in period 1, the central bank lowers the short-term interest rate to zero to counteract the deflationary pressures coming from the household's attempt to smooth consumption. As a result of lower, current, and expected interest rates, the bond price increases 10% and the Fed makes a large profit from its holdings of long-term bonds, 1.5% of GDP ($225 billion). Complying with the dividend rule, the Fed remits the current net income to the Treasury. In the high-income state of the second period, economic conditions improve and the Fed raises interest rates to 2% a year, the long-run level. The bond price plummets from the 10% peak in the first period and the bank suffers a large loss of 1.5% of GDP (symmetrical to the gain in period 1). However, the loss *is not large enough* to put the financial stability of the Fed at risk. The present value of seignorage is estimated to be 2% of GDP and hence the Fed remains solvent. This result is in line with Hall and Reis (2015).[11]

Because the solvency constraint is not active, the private sector expects a contractionary monetary policy when the crisis is over, resulting in a 1.2%

[9]The accumulated gap between potential and actual during the crisis in the model is given by $\bar{y} + \mu(\bar{y} - \underline{y})$

[10]As aforementioned, without QE, assets held by the Central Bank are riskless and the solvency constraint can be disregarded.

[11]Although this model does not account for many important features of the crisis and the operating system of the Fed, our results seem to capture well the dynamics of the Fed's balance sheet estimated by Hall and Reis (2015), using a much more detailed approach. Hall and Reis (2015) find that the Fed would earn around $ 110 billion in profits at the outset of the crisis, and loose $220 billion when the economy shifts from the crisis state to normal times. They also conclude that, although theoretically plausible, it is unlikely that the Fed faces a real risk of insolvency.

fall in the price level in the first period. As illustrated in the upper panel on the RHS of figure 1.2, the optimal policy involves a commitment to a higher price-level target and lower nominal interest rate in the high-income state of the second period. The lower panel shows that the conventional discretionary equilibrium without QE is identical to the outcome with QE, and hence QE had no effect whatsoever.[12]

Optimal Quantitative Easing. Is there a specific size (or duration) of the Fed's balance sheet that, at least theoretically, policymakers could implement the commitment equilibrium in a discretionary and time-consistent setup? Figure 1.3 shows the state-contingent path of key variables when the Fed holds long-term bonds in an amount equal to 21 percent of annual GDP. The equilibrium paths of short-term interest rates and price level replicate exactly the optimal commitment equilibrium. The solvency constraint prevents the central bank from raising the interest rate in the high-income state of period 2. As a result, the price-level overshoots the target, providing the desired inflation expectations to lower real interest rates in period 1, when monetary policy is stuck at the ZLB, mitigating deflation in that period.

1.5 Conclusion

This chapter provides a theory of Quantitative Easing. It's shown that a central bank that is financially independent from the treasury can use a large-scale purchase of long-term bonds as a commitment device to keep interest rates low in the future. This is because such an open market operation provides an incentive to the central bank to keep interest rates low in future to avoid losses in its balance sheet.

A preliminary conclusion of this chapter is that the unconventional policies that resulted in the Fed borrowing 2.25 trillions of dollars (15% of GDP) from commercial banks to buy risky long-term U.S. Treasury securities did not threaten the solvency of the bank. However, a larger version of QE, in which the Fed purchases long-term bonds in an amount of equal to 21 % of annual GDP, would have had supported the optimal commitment outcome in a discretionary setup.

[12]Eggertsson and Woodford (2003) and Billi (2013) argue that discretionary monetary policy under a price level targeting can approximate the optimal commitment solution. If the price-level target is not reached because of the ZLB, the central bank increases its target for the next period. This, in turn, increases inflation expectations further in the liquidity trap, which reduces the real interest rate, stimulating the economy. However, note that the discretionary policy creates some inflation in the high-income state of the second period, but much less than what is desirable under the fully optimal commitment.

Although the Fed remains solvent when it raises interest rates on the exit of the crisis, it suffers a very large capital loss. It raises the questions of whether this result is robust to a richer environment with endogenous production, sticky prices and wages, capital accumulation and a more realistic structure of exogenous shocks. Moreover, we would like to see if these results go through in a model with inflation targeting (instead of price-level targeting), variables related to the balance sheet expressed in real terms and more realistic assumptions about the financial arrangement between the Fed and the Treasury. Chapter 2 addresses these concerns using a standard medium-scale DSGE model based on Smets and Wouters (2007).

1.6 Figures and Tables

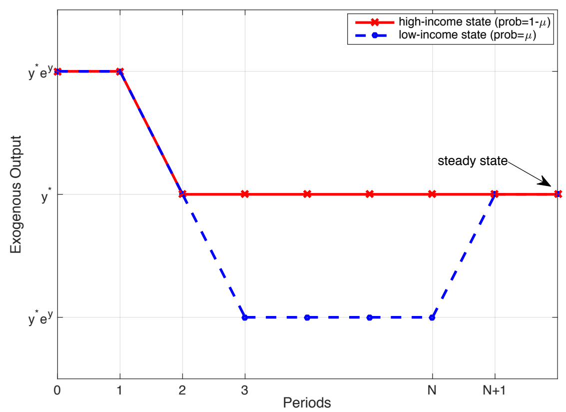
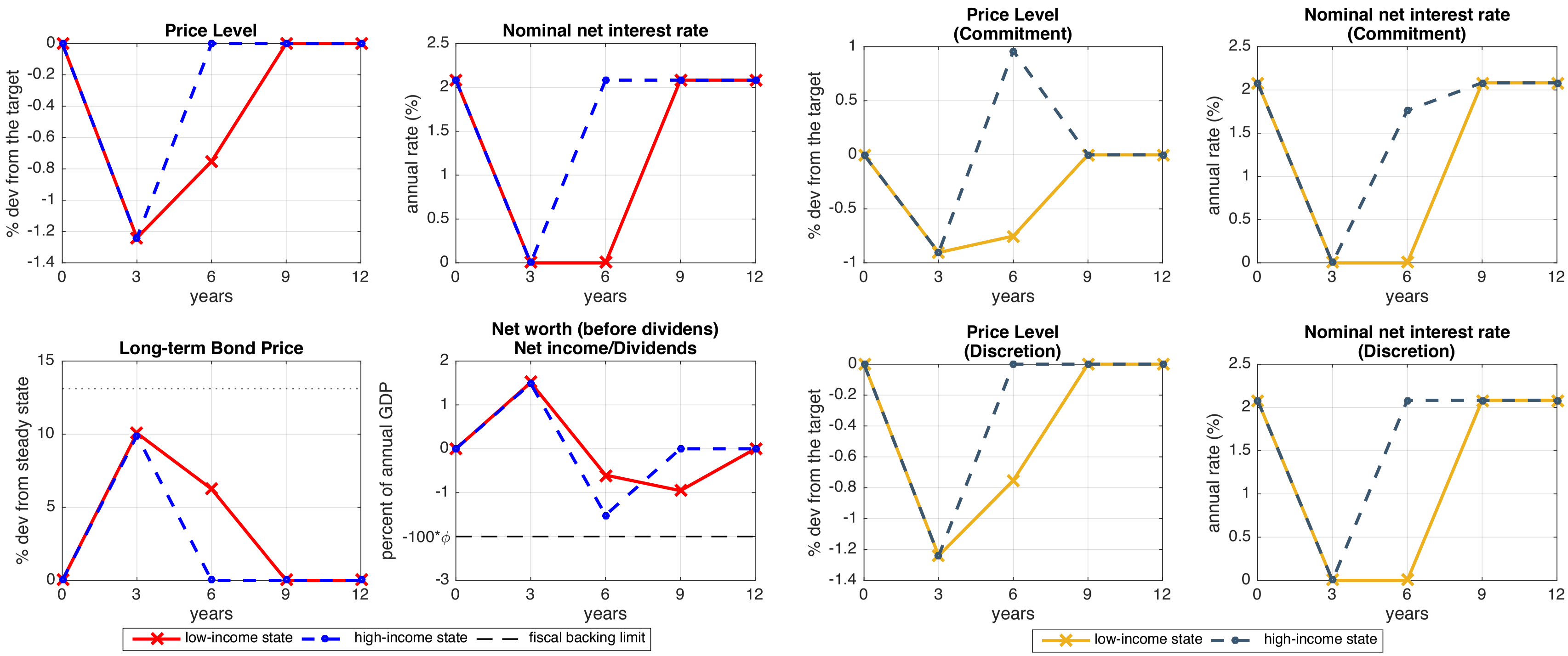


Figure 1.1: **A Simple Endowment-Economy Model:** The Endowment Process. The red line denotes the endowment process conditional to the realization of the high-income scenario, and the blue dashed line conditional to the realization of the low-income scenario.



(a) Discretion & Baseline Calibration

(b) Discretion and Commitment & no QE

Figure 1.2: **Inflation Dynamics and the Financial Stability of the Fed in a Liquidity Trap.** Benchmark calibration (panel a): state-contingent path of the short-term nominal interest rate, price-level, price of long-term bonds and the Fed's net worth under discretion. Alternative calibration without QE ($b^l_* = 0$) (panel b): optimal commitment (upper panel) and discretion (lower panel).

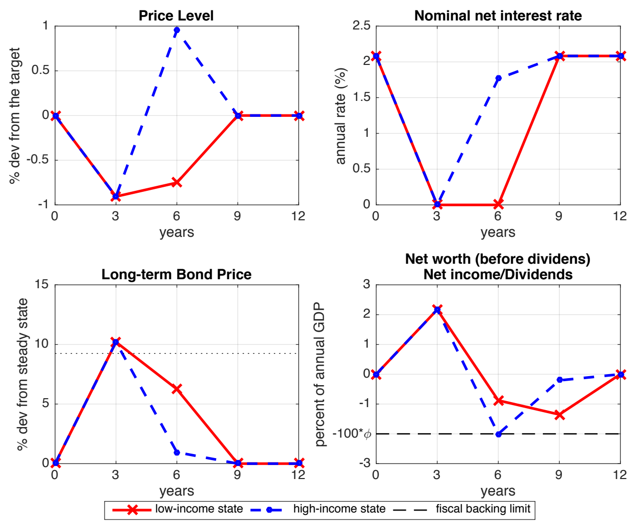


Figure 1.3: **Optimal Quantitative Easing.** State-contingent path of the short-term nominal interest rate and the price-level under discretion when the Fed holds 21 percent of annual GDP in long-term bonds.

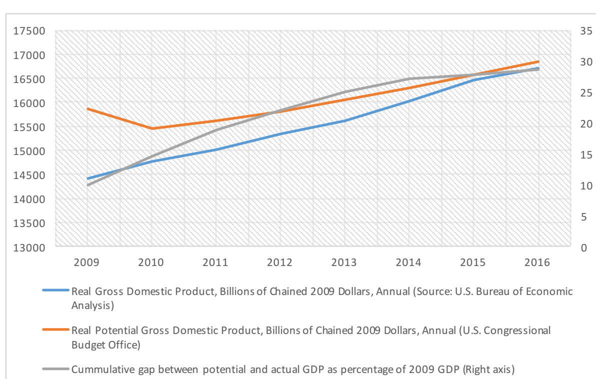


Figure 1.4: **Benchmark Calibration of the Endowment-Economy Model:** Cummulative Gap between Potential and Actual GDP as a percentage of annual 2009 GDP from 2009 to 2016.

Table 1.1: Endowment Economy Model: Summary of Baseline Calibration and Steady State

	Value	**Symbol**	**Target**	**Source**
Panel A: Parameter Calibration				
Intertemporal discount factor	0.95	β	3-year period & 2% annual real interest rate	
Rate of coupon decay	0.04	δ	7.8 years of Fed's average portfolio maturity	HR2015
Crisis exit probability	0.35	$1-\mu$	5 years of expected crisis duration	HR2015
Pre crisis % income above trend	.035	$\bar{y}$	cumulative loss of output (25% of 2009 GDP)	BEA/CBO
Crisis % income below trend	-.035	$\underline{y}$	cumulative loss of output (25% of 2009 GDP)	BEA/CBO
Loss limit	.007	ϕ	PV of seignorage (2 percent of annual GDP)	HR2015
Coefficient of risk aversion	2	σ		
Panel B: Zero-Inflation Steady State				
Fed's holdings of long-term bonds	0.05	b^l_*	15 pecent of annual GDP	HR2015
Fed's liabilities	0.05	m_*	15 pecent of annual GDP	HR2015
Federal Funds Rate	0.06	i_*	2 percent annual real interest rate	
Interest on Reserves	0.06	i^m_*	2 percent annual real interest rate	
Bond price	5.7	q_*		
Fed's holdings of short-term bonds	0	b_*		
Fed's net worth	0	nw_*		
Fed's net income	0	ni_*		
Remmitances to the Treasury	0	d_*		

2
The Effects of Quantitative Easing on the Balance Sheet of the Federal Reserve and the U.S. Economy

2.1
Introduction

Chapter 1 concluded that the unconventional monetary policy undertook by the Fed between 2009 and 2013 that resulted in the borrowing of 2.25 trillions of dollars from commercial banks to buy risky long-term U.S. Treasury securities, did not threaten the solvency of the Fed. This chapter revisits the question using a much more appropriate framework to assess the impact of the QEs on the Fed's balance sheet. Repeating the exercise is important because the mechanism through which QE affects inflation developed in chapter 1 relies on the Fed facing a positive probability of becoming insolvent under the optimal discretionary monetary policy.

The model used in this chapter is based on the New Keynesian DSGE model developed in Smets and Wouters (2007) (SW07 hereafter). The choice of the SW07 model is appropriate since it is widely disseminated as a standard framework for quantitative policy analysis for the U.S. economy. We introduce three main innovations in the baseline model in order to analyze the impact of QE on the balance sheet of the central bank and its consequences to inflation, employment, and output. First, we impose a ZLB. Second, we substitute the usual Taylor rule with the assumption that the policy rate is set optimally under commitment and discretion. Third, we add variables and equations that are related to the balance sheet of the central bank, and introduce two types of solvency constraints.

Section (2.2) describes (*i*) the key features of the baseline SW07 DSGE model, (*ii*) the new feaures added to the model, (*iii*) the solution method with monetary policy under discretion and commitment, (*iv*) the method developed to make forecasts taking into consideration the non-linear restrictions of the model, and (*v*) the calibration strategy to match key features of the U.S. economy and the Fed's balance sheet during the ZLB period. Section (2.3) briefly discusses the empirical performance of the model. Section (2.4) uses the quantitative model to answer two questions. First, assuming that the Fed is financially unconstrained, i.e., monetary policy is conducted regardless the impact on the Fed's balance sheet. What is the likelihood that the Fed will become insolvent on the exit of the crisis and consequent normalization of monetary policy? Second, imposing the two solvency constraints in the model, what is the magnitude and duration of the gap between the unconstrained and constrained paths for the policy rate during the normanlization of monetary policy? Also, what are the consequences to the dynamics of inflation and the output gap in the U.S. economy? Section (2.5) concludes.

2.2 The Quantitative Model

This section describes the quantitative model. First, it presents a discussion of the main features of the SW07 DSGE model. Second, it describes the new feaures added to the baseline SW07, the solution method with monetary policy under discretion and commitment, the method developed to make forecasts taking into consideration the non-linear restrictions of the model, and the calibration. Finally, is provided a discusion of the model's ability to replicate the behavior of the U.S. economy and the Fed's balance sheet during the ZLB period.

2.2.1 Main Features of the Smets and Wouters 2007 DSGE Model

SW07 use US data on real wages, hours worked, real GDP, consumption, investment, prices and the short-term nominal interest rate to estimate a medium-scale DSGE for the US economy, covering the period 1966:Q1 - 2004:Q4. The dataset allows the introduction of seven types of structural shocks: productivity, risk-premium, investment opportunities, exogenous spending, monetary policy, price and wage markup shocks. A large set of frictions is introduced so the model-based response of the observed variables to shocks captures some key properties of VAR-estimated IRFs. We describe the main features of the SW model, focusing on the role played by each shock and friction. The reader is referred to the original article for a full description and derivation of the model.

External habit, sticky wages, wage indexation and wage markup shocks. Households maximize a non-separable (GHH) utility function of consumption and labor over an infinite life horizon. Consumption enters the utility function relative to a time-varying external habit variable and labor is differentiated by a union, so there is monopoly power over wages and allows for the introduction of sticky nominal wages as in Calvo (1983). Due to nominal wage stickiness and partial indexation of wages to inflation, real wages adjust only gradually to the desired wage markup. Also, due to time-varying demand elasticity (as in Kimball (1995)), the real wage is a function of expected and past real wages and the exogenous wage markup.

Capital adjustment costs, variable capital utilization and investment-specific technology shocks. Households rent capital services to firms and decide how much capital to accumulate. Capital accumulation is subject to adjustment costs, and capital utilization is variable. The relative efficiency of investment expenditures are subject to investment-specific technology shocks.

Sticky prices, price indexation, TFP and price markup shocks. Firms produce differentiated goods by hiring labor and capital services, set prices as in Calvo (1983) and are subject to shocks to total factor productivity (TFP). Partial indexation of prices and time-varying demand elasticity for differentiated goods are allowed so that current inflation depends on expected future marginal costs, past inflation rate and also on price markup shocks.

Exogenous expenditure, monetary policy, financial frictions and risk-premium shocks. An exogenous expenditure shock is introduced in the model to capture net export revenues or government expenditure shocks. Households can use government bonds to smooth consumption over time. The central bank follows a generalized Taylor rule by gradually adjusting the short-term nominal interest rate of these bonds in response to inflation and output gap (deviation of actual output from the counterfactual flexible-price economy). To capture the degree of interest rate smoothing observed in the US data the Taylor rule is allowed to respond to lagged values according to the autoregressive coefficient ρ_r. Finally, a risk premium shock represents a wedge between the interest rate controlled by the central bank and the return on bonds.

All shocks are assumed to follow an AR(1) process with an IID-Normal error term with zero mean, estimated persistence and standard deviation. Price and wage markup shocks are allowed to incorporate a moving average error term.

2.2.2 New Features: The Central Bank and The Treasury

The central bank side of the economy is similar to the one developed in the section (1.2) of chapter 1. The central bank issues nominal liabilities, buys short and long-term bonds, and make payments to the treasury on a regular basis. The dynamics of the central bank net worth is described by following equation,

$$nw_t = \frac{nw_{t-1}}{\gamma \pi_t} + ni_t - d_t \tag{2-1}$$

where lower case variables represent detrended real variables. We detrend variables with γ, the steady-state growth rate of the economy, and deflate nominal variables with P_t, the consumer price index. Variables nw_t, ni_t and d_t denote net worth, net income and remittances to the treasury in period t, respectively. The variable π_t denotes the inflation rate between periods t and $t-1$.

As in the model of section (1.2), we can disaggregate the central bank net income into net interest income and capital gains and losses,

$$ni_t = \underbrace{i_{t-1}\frac{b^{cb}_{t-1}}{\gamma \pi_t} + \left(\frac{1-\delta_b Q^b_t}{Q^b_{t-1}}\right)\frac{b^{l,cb}_{t-1}}{\gamma \pi_t} - i^m_{t-1}\frac{m_{t-1}}{\gamma \pi_t}}_{\text{Net Interest Income } (\equiv\, nii_t)} + \underbrace{\left(\frac{Q^b_t}{Q^b_{t-1}} - 1\right)\frac{b^{l,cb}_{t-1}}{\gamma \pi_t}}_{\text{Capital Gain}} \tag{2-2}$$

where b^{cb}_{t-1} and $b^{l,cb}_{t-1}$ denote real value of short and long-term government bonds held by the central bank in period t, respectively. Variable m_t represents the central bank's outstanding monetary liabilities (or reserves), i^m_t is the interest rate payed on reserves (IOR), and i_t is the interest rate payed on short-term government bonds. Q^b_t is the price of long-term government bonds.

In most developed countries, the institutional arrangement between the monetary and fiscal authorities normally determines that a share of the central bank's net income must be remitted to the Treasury. However, the practiced concept of net income differ across countries depending on the type of accounting framework that the central bank adopts. If assets are "marked to maket", net income reflects gains and losses from the variation of the maket price of long-term bonds. However, if a central bank adopts historical pricing in calculating the value of its portfolio, the concept of net income will not incorporate gains and losses from price changes. We incorporate these considerations in the model by considering two types of dividends rules, one based on the net income and another based on the net *interest* income.[1] Moreover, as in the model of section (1.2), we assume that the Treasury does not recapitalize the central bank in case of negative income,

[1] This is appropriate in the context of this model because delta bonds are perpetuities that never mature and historical pricing assumes that bonds are worth the nominal principal returned at maturity.

$$d_t = max(0, (1-\zeta)\Theta_t) \quad \text{where } \Theta_t \in \{nii_t, ni_t\} \tag{2-3}$$

where d_t denotes dividends, and ζ is the share of net income (or net interest income) retained at the central bank to build paid-in capital.

Hall and Reis (2015) argue that many central banks have a mechanism that allows them to recover from the issuance of reserves required to make up for negative income. We do that by adding an exclusion clause that authorizes the central bank to refuse to hand over its income to the treasury for a certain period of time. We introduce these considerations in the model by creating a new “deferred assets account". That account gets credited when the Fed’s income is negative, and represents a claim on future central bank income, which would have been returned to the treasury according to the dividend rule, but instead is retained at the central bank in order to rebuild the bank’s net worth. The deferred assets account is described by the following equation,

$$z_t = \frac{z_{t-1}}{\gamma\pi_t} + (d_t - (1-\zeta)\Theta_t) \tag{2-4}$$

an the dividend rule gets replaced by,

$$d_t = \begin{cases} 0 & \text{if } \Theta_t < 0 \text{ or } z_t > 0 \\ (1-\zeta)\Theta_t & \text{otherwise} \end{cases} \tag{2-5}$$

In 2008 the Board of Governors of the Federal Reserve System received authorization to pay interest on balances held by comercial banks at Reserve Banks (IOER). During the monetary policy normalization, the Fed moves the FFR into the target range set by the FOMC primarily by adjusting the IOER. In the context of this model, it is equivalent to setting the interest rate payed on reserves equal to the policy rate,

$$i_t^m = i_t \tag{2-6}$$

This assumption turns out to be highly convenient in terms of the tractability of the model. It allows us to introduce reserves in the standard SW07 DSGE model without having to make any specific assumptions about the household’s demand for liquidity.[2] Moreover, it allows the central bank to tight monetary conditions without the need to sell assets on its balance sheet.

[2]The only assumption underlying $i_t = i_t^m$ is that the money supply must be large enough to satiate the private sector demand for liquidity, which has been a trivial assumption since the start of balance sheet expansion in 2008.

Also, note that, due to the maturity mismatch between the Fed's assets and liabilities, condition (2-6) does not imply that net interest income will always be equal zero.

We assume that the central bank conducts purchases of short and long-term bonds following simple autoregressive rules (in terms of detrended real market value),

$$b_t^{l,cb} = (\gamma\pi_*)^{\rho_{cb}} \, (y_* b_*^{l,cb})^{1-\rho_{cb}} \left(\frac{b_{t-1}^{l,cb}}{\gamma\pi_*}\right)^{\rho_{cb}} exp(\epsilon_t^{l,cb}) \tag{2-7}$$

$$b_t^{cb} = (\gamma\pi_*)^{\rho_{cb}} \, (y_* b_*^{cb})^{1-\rho_{cb}} \left(\frac{b_{t-1}^{cb}}{\gamma\pi_*}\right)^{\rho_{cb}} exp(\epsilon_t^{cb}) \tag{2-8}$$

where π_* and y_* denote the steady states of inflation and (detrended) output, respectively, $\rho_{cb} \in (0,1)$ and $(\epsilon_t^{l,cb}, \, \epsilon_t^{cb})$ are i.i.d exogenous shocks. The steady state level of the central bank holdings of short and long-term bonds, b_*^{cb} and $b_*^{l,cb}$, are chosen to match the average size of Fed's balance sheet observed in the ZLB period.[3] In the experiments of the next sections, we feed into the model a sequence of shocks $\epsilon_t^{l,cb}$ and ϵ_t^{cb} so that the model replicates the historical time series of assets held by the Fed. The level of reserves (liabilities) required to back the central bank's purchases of short and long-term bonds is given by,

$$m_t = b_t^{cb,l} + b_t^{cb} - nw_t \tag{2-9}$$

Finally, as in section (1.2.4.2), we abstract from government expenditures and assume that the treasury receives dividends from the central bank, d_t, and collects lump-sum taxes, τ_t. The budget constraint of the treasury can be written as,

$$\begin{aligned} d_t + \tau_t + b_t^{cb} + b_t^{l,cb} + b_t^{hh} + b_t^{l,hh} = \\ (1 + i_{t-1}) \left(\frac{b_{t-1}^{cb} + b_{t-1}^{hh}}{\gamma\pi_t}\right) + \left(\frac{1 + (1-\delta_b)Q_t}{Q_{t-1}}\right) \left(\frac{b_{t-1}^{l,cb} + b_{t-1}^{l,hh}}{\gamma\pi_t}\right) \end{aligned} \tag{2-10}$$

The treasury adjusts the real primary fiscal surplus in response to the lagged real value of the total government debt, as in Leeper (1991),

[3] Note that the steady state level of the central bank holdings of short and long-term bonds are expressed as a share of detrended output.)

$$d_t + \tau_t = exp\left\{\phi_z\left(\frac{b^{hh}_{t-1} + b_{t-1} + b^{hh,l}_{t-1} + b^{l}_{t-1}}{\gamma\pi_t}\right)\right\}$$

where we choose the parameter ϕ_z so that fiscal policy is passive.

The model is log-linearized around its steady-state balanced growth path and cast into a system of linear rational-expectations equations. For later reference, it will be useful to characterize the model in matrix form as,

$$\begin{bmatrix} H_{XX} & 0 \\ H_{xX} & H_{xx} \end{bmatrix} \begin{bmatrix} X_{t+1} \\ \mathbb{E}_t x_{t+1} \end{bmatrix} = \begin{bmatrix} A_{XX} & A_{Xx} \\ A_{xX} & A_{xx} \end{bmatrix} \begin{bmatrix} X_t \\ x_t \end{bmatrix} + \begin{bmatrix} B_X \\ B_x \end{bmatrix} i_t + \begin{bmatrix} C_X \\ C_x \end{bmatrix} \epsilon_t \quad \text{(2-11)}$$

where X_t is a vector of endogenous predetermined variables, x_t a vector of non-predetermined variables, i_t is the short-term nominal interest rate and ϵ_t is a vector that collects the exogenous shocks. The forward-looking aspect of private agent's behavior is summarized by the lower block of (2-11). The upper block of (2-11) is inherited from the past and often describes the dynamic behavior of stock variables.

Solvency Constraints. Let ϕ denote the present value of future seignorage revenues. We assume that in each period: (i) the central bank's net worth cannot fall below $-\phi$, and (ii) the balance in the deferred assets account cannot exceed ϕ. The first restriction is similar to section (1.2), and means that policymakers cannot undertake policy actions that lead to insolvency. According to Hall and Reis (2015), the balance in the deferred assets account is a useful metric for judging the bank's financial stability. A large value of ϕ will prevent a permanent increase of reserves following a negative income shock by cutting subsequent dividends and using the proceeds to pay off the initial expansion of reserves. However, a balance above ϕ means that the central bank will need to receive a positive transfer from the treasury in present value. We assume that central bankers will dismiss any policy outlook that leads to this outcome in order to preserve the independence of the bank. Formaly, the solvency constraints are given by the following expressions,

$$nw_t \geq -\phi \quad \text{and} \quad z_t \leq \phi \quad \text{(2-12)}$$

The two restrictions are nearly equivalent when the dividend rule is based on the central bank's net income. However, under the interest-income dividend rule, the lower bound on the central bank's net worth can be binding while the balance on the deferred account is low. This is because the central bank can be reporting positive net interest income, and hence paying positive dividends to the treasury, while it is suffering large capital losses due to the depreciation of the market value of the bond portfolio.

Optimal Monetary Policy. An advantage of having a structural model based on optimizing behavior is that it provides a natural objective for the monetary policy: the maximization of the expected utility of the representative household. Following Woodford (2003, chap. 6), we can express a second-order Taylor series approximation to this objective as a quadratic function of price inflation, the output gap, and the nominal interest rate.[4] We follow this literature and consider the loss function,

$$L_t \equiv \frac{1}{2}\left[(\pi_t - \pi^*)^2 + \lambda_x \tilde{x}_t^2 + \lambda_i (i_t - i^*)^2\right] = x_t' W x_t$$

where π^* is the inflation target, $\tilde{x}_t$ is the output gap, i^* the steady-state nominal net interest rate, W is a positive definite matrix that collects these variables in the vector of forward looking variables, x_t, and λ_x and λ_i are the weights that the central bank attributes to the stabilization of the output gap and interest rate smoothing relative to inflation. We define the intertemporal loss funtion in period t as the expected discounted sum of all the period losses from period t onwards.

Discretion. Here we consider an equilibrium that occurs when policy is conducted under discretion so that the central bank is unable to commit to any future actions. The central bank problem is to choose a sequence $\{i_t\}_{t\geq 0}$ as function of the exogenous process $\{\epsilon_t\}_{t\geq 0}$ and the endogenous state $\{X_t\}_{t\geq 0}$ so as to minimize period-by-period the intertemporal loss function, subject to (2-11), given a initial condition X_0. The solution of this problem satisfies the following bellman equation,

$$v_t\left(X_t, \epsilon_t\right) = \min_{i_t \geq 0} \left\{\frac{1}{2} x_t' W x_t + \beta \mathbb{E}_t v_{t+1}\left(X_{t+1}, \epsilon_{t+1}\right)\right\} \quad \text{(2-13)}$$
$$\text{s.t. (2-11) and (2-12) given } X_0$$

Standard methods to find the solution to this problem do not apply in this

[4] Benigno and Woodford (2003) show that we can approximate the policy problem of maximizing the representative household utility by the simpler problem of minimizing a quadratic loss function of inflation and output gap. This approximation assumes that central banks do not care about the path of nonimal interest rate that is required to implement a specific path of inflation and output gap. However, there are substantianl evidence that central banks also seek to reduce the volatility of nominal interest rates (Goodfriend (1991)). Giannoni and Woodford (2003) show that transactions frictions can generate microeconomic-founded justification for interest rate smoothing.

case because of the large number of endogenous state variables in the system and because of the non-linearity introduced by the ZLB and the solvency constraints. To deal with the non-linearities, we follow Guerrieri and Iacoviello (2015) and solve the model in a piecewise fashion. To deal with the large number of endogenous state variables, we use the optimal linear regulator of a version of the dynamic Stackelberg problem in Ljungqvist and Sargent (2004). Section (6.5) in the technical appendix provides a thorough description of the solution method.

Commitment. Here we consider an equilibrium that occurs when policy is conducted under commitment so that the government is able to commit to future actions. Consider minimizing the intertemporal loss function, under commitment once-and-for-all in period $t = 0$, subject to (2-12), (2-11) for $t \geq 0$ and X_0 given. The method to find the optimal policy under commitment consists in setting-up the Lagrangian funtion, deriving the first-order conditions, combining these with the model's dynamic equations, and solving the resulting linear rational expectation system using the piecewise linear solution. Section (6.6) in the technical appendix provides a thorough description of the solution method.

2.2.3
Generating Forecasts at the Zero Lower Bound.

The model described in the previous section is used to make forecasts of the central bank balance sheet and other key macroeconomic variables in the ZLB period. To generate the forecasts, it is needed to estimate the time series of non-observable variables contained in the vector X_t, and the structural shocks, ϵ_t. The standard procedure used in the literature for this estimation is to apply a Kalman filter. One caveat of using this method is that the model with optimal monetary policy is absent from monetary-policy shocks. Hence, to implement the Kalman filter we must exclude from the analysis one of the US data series used to estimate the original model in SW07.

To avoid the loss of valuable information, we opted to use an alternative method. As the Kalman filter, it works recursively and requires only the last best guess, rather than the entire history, of the model's state to calculate a new state. It's assumed that the model is at the steady state level immediatelly prior to 1985:Q1, the first observation of the sample. Given $X_{1984:Q4}$, we choose $\epsilon_{1985:Q1}$ to minimize the model's sum of squared prediction errors (the distance between the model's measurement equations and the observed variables). Calculated the vector of shocks, we can use the model to update the endogenous state, $X_{1985:Q1}$, and repeat the process recursively until 2015:Q4.[5] Thi method is based on Guerrieri and Iacoviello (2017) and the reader is referred to the article for futher details. Section (2.3) provides a discussion of the estimation results.

2.2.4 Calibration

The model used to estimate the time series of the non-observed variables, and to make forecasts about the future behavior of the U.S. economy and the Fed's balance sheet, is calibrated separately for the pre-ZLB period (1985:Q1 to 2008:Q3) and the ZLB period (2008:Q4 to 2015:Q4). This division intends to address concerns about the new levels of the natural rate of interest, inflation and output growth trend, as well as the vast expansion of the size and duration of the Fed's balance sheet since 2008. Table 2.1 in the section (2.6) summarizes the calibration.

In the pre-ZLB period, all structural parameters, frictions and shock processes are calibrated equal to the mean of the posterior distribution of the parameters obtained by bayesian methods in SW07. In the ZLB period, all paramenters of the pre-ZLB calibration are preserved, except from the intertemporal discount factor, β, the long-run inflation rate, π_*, and the long-run growth trend, γ. These parameters are changed so that the steady-state inflation, GDP growth and nominal interest rate implied by the model are in line with the post-2008 scenario of weak aggregate demand and low growth that has contributed to a general revision of the long-term nominal interest rates in the United States.[6] To recalibrate these parameters we use data based on the FOMC member's individual projections of the nominal interest rate, inflation and output growth under appropriate monetary policy, disclosed by the Summary of Economic Projetions (SEP) since 2012:Q1. The mean of the FOMC projections suggest that the new long-run level of the FFR is 4 percent annual, while inflation and GDP growth are 2 percent. Panel BI of table 2.1

[5] The minimization is carried out with an OLS algorithm up to 2008:Q3, the last period before the ZLB binds. In the ZLB period, 2008:Q4 to 2015:Q4, the relationship between the measurement equations and the structural shocks is non-linear because the duration of the ZLB is conditional on the realization of the shock (See Guerrieri and Iacoviello (2015)). In this case, a numerical algorithm is employed to find the vector of shocks that minimize the model's sum of squared prediction error.

[6] Several factors have contributed to generate a long-run downward trend in the equilibrium real interest rate in developed countries and particularly in the United States. Shifts in demographics, a slowdown in trend productivity growth, increase in inequality, the scarcity of safe assets, deleveraging shocks and a reduction in demand for capital goods are the most likely explanations for the decline in the interest rates.See Summers (2013), Taylor (2014), Krugman (2013) and Eggertsson and Mehrotra (2014), Carvalho et al. (2016), Caballero et al. (2016) and Holston et al. (2016).

describes the calibration of the steady state. The parameter π_* is set to 1.005 so that the model's new annual inflation steady state is 2 percent (down from 3 percent in SW07); $\beta = 0.999$ and $\gamma = 1.003$ so that the steady state of the nominal interest rate is 4 percent annual (down from 6% in SW07) and the trend annual growth rate of the economy is 1.2% (down from 1.7% in SW07).

Federal Reserve Bank Balance Sheet. The Fed's balance sheet is calibrated using seven key quarterly time series provided by the Board of Governors of the Federal Reserve System: monetary base (Federal Reserve Notes plus deposits held by depository institutions), total capital, assets (U.S. Treasury securities plus mortgage-backed securities), interest income from Treasury securities, interest paid on reserves, dividends on capital stock and earnings remittances to the Treasury (interest on Federal Reserve Notes). We calibrate the steady steady of the model to match the average of each series over the two subperiods, expressed as a share of the quarterly U.S. GDP (QGDP). A complete description of the data is provided in the Data Appendix (6.1).

Panel A of table 2.1 summarizes the calibration of the steady state of the central bank's balance sheet and compares it with the averages observed in the data. In the pre-ZLB period, the Fed's bond portfolio was worth on average 22 percent of the QGDP. Most of these assets were backed by liabilities and the Fed's net worth averaged only 0.8 percent of the QGDP during the period. The Fed held only riskless short-term U.S. Treasury bonds and paid no interest on reserves. These operations provided the Fed with an average income stream equal to 0.3 percent of the QGDP, that was almost entirely transfered as dividends to the U.S. Treasury. Column (4) shows that the model and the data line up very well during the pre-ZLB period.

Column (5) of table 2.1 displays the results for the ZLB period. The main difference between the two periods is the dramatic expansion of the size and duration of the balance sheet. The size of the Fed's assets portfolio averaged 77 percent of the QGDP and the average duration raised to 7.8 years. Since the purchase of these assets was funded essentially by newly created bank reserves, the Fed's liabilities also increased in the period and represented on average 75 percent of the QGDP. The Fed's income and expenses increased proportionately less than the assets and liabilities because the yield curve shifted downwards in the period. Roughly speaking, interest income, dividends, surplus to capital stock and net worth nearly doubled relative to the pre-ZLB era.

As in the endowment-economy model of section (1.2), we follow Hall and Reis (2015) and set the loss limit, ϕ, equal to the estimated present value of the Fed's seignorage revenues, 8 percent of the QGDP. The depreciation rate of the long-term bond is set to $\delta_b = 0.03$, so that duration equals 7.8 years. Since we assume that the Fed holds only long-term bonds in the ZLB period, the duration of the Fed's assets is also 7.8 years, which is the estimated value-weighted average maturity of the Fed's financial assets between 2009 and 2013 (see Hall and Reis (2015)).[7]

There are other two parameters of our choice to calibrate the central bank's balance sheet: the steady state level of liabilities, m_*, and the share of net income kept at the Fed, ζ. We set $m_* = 0.75$ to match the data. Since there is no evidence that the arrangement between the Fed and the Treasury has changed since 2008, we opted to keep $\zeta = 0.035$, as in the pre ZLB period. Since the observed Fed expenses with interest payments on excess reserves is very small, 0.03 percent of the QGDP, for simplicity we assume $i^m_* = 0$ so that the interest cost is zero in steady state.

Although the implied model's steady state moderately overestimates the Fed's income and net worth observed in the data, it replicates fairly well the main features of the Fed's balance sheet. Note that the model's exaggeration of the Fed's income is due to the fact that we set the steady-state real interest rate at 2% per year, which is too high compared to the actual FFR in the ZLB period.

Loss Function and the Federal Funds Rate. We follow Giannoni and Woodford (2003) and set the relative weight of output gap to inflation to 1%, $\lambda_x = 0.01$. A positive weight assigned to the stabilization of the nominal interest rate is a necessary condition for stability in the models under commitment and discretion. We choose, $\lambda_i = 0.03$, so that the model-based first-order autocorrelation of the nominal interest rate is in line with the actual FFR autocorrelation estimated over the pre-ZLB sample 1985:Q1 - 2008:Q3.

One caveat of borrowing all parameters governing the shock processes from a model estimated with a Taylor rule, is that the nominal interest rate can display excessive volatility under discretionary optimal monetary policy. We observed that the persistence of the impulse response of the policy rate to a wage mark-up shock is unrealistically high under discretion. As a result of that, the variance of the policy rate largely overestimates the realized variance of the FFR. To restore the good empirical properties of the policy rate implied by the model, we reduce the persistence of the wage mark up shock. Table (2.2) shows the actual and model-based autocorrelations and stadandard deviations of the FFR under different calibrations and assumptions of monetary policy. The last column shows that, when we don't adjust the persistence of the wage

[7]Hall and Reis (2015) use data in the annual report of the Federal Reserve on the value and maturity of the Treasury securities it holds, to calculate the value-weighted average maturity of the Fed's financial assets

mark-up shock ($\rho_w = 0.96$), the quarterly standard deviation of the FFR is almost two times larger in the model under discretion than in the data: 13.7% and 7%, respectively. The first column of the table shows that when we set $\rho_w = 0.9$, the standard deviation of the model under discretion reduces to 8.6%, which is slightly higher than the data and, as expected, significantly higher than the model under commitment, 3.5%.

2.3 Quantitative Model Performance

In this section, we briefly discuss the historical contribution of each structural shock to explain the observed variables over the sample period, and the ability of the model to make reasonable predictions about the federal funds rate and other variables related to the Fed's balance sheet, particularly during the ZLB period.

Shocks and the Great Recession. Figure 2.1 depicts the estimated structural shocks over the entire sample period in the discretion and commitment models. One can see that the risk-premium shock is the most important driver of the great recession. In both models, a very large risk-premium shock hits the U.S. economy in the last quarter of 2007. Following the initial disturbance in the financial sector, a long sequence of adverse investment and expenditure shocks deepened and lengthened the crisis.[8] Figures 2.2 and 2.3 compare the smoothed observed variables with the data. Both models fit well the U.S. time series, especially during the ZLB period.

Federal Funds Rate. We compare the model-based forecasts of the FFR during the ZLB period with the FOMC member's individual projections of the FFR disclosed by the SEP since 2012:Q1. Figures (2.4) - (2.7) depict the forecasts of the FFR using the models under commitment (red line), discretion (blue line) and a Taylor rule (green dashed line) with information available in each quarter of the period 2012:Q1 - 2015:Q4. We compare the paths of future interest rate implied by each model with the mean (red circle), median (green cross) and mode (red cross) of the FOMC's members views on the appropriate level of the FFR at selected points in the future (the SEP dots). Black dots display the projections of individual FOMC's members (19 in total). As in Krugman (1998) and Eggertsson and Woodford (2003), the nominal interest rate is kept at the ZLB for a longer length of time under commitment relative to discretion. Figure (2.8) shows that, while inflation remains persistently below the target when the policy is conducted under discretion or according

[8]Note that the model under commitment requires much larger risk-premium shocks to replicate the financial crisis in the last quarter of 2007 than the model under discretion. This is due to the more accommodative monetary policy in the commitment setup.

to a Taylor rule, the model under commitment predicts a quick rebound and overshoot of the 2 percent target. Overshooting the target lowers the real interest rates and stimulates the economy during the ZLB.

Fed's Balance Sheet. Figures 2.9 and 2.10 depict actual and model-based Fed's income and expenses under discretion and commitment, respectively. The purple and dashed green lines show the path of the smoothed variables from 2005:Q1 to 2015:Q4, when the dividend rule is based on the Fed's net income and net interest income, respectively.[9] One can see that both models track well the upward trend in the Fed's net income during the ZLB period. Because the Fed adopts historical prices in calculating gains or losses in its balance sheet, reported capital gains are zero over the sample period. Note, however, that market evaluations are the main source of volatility in the model-based predictions of income. In a single period, gains and losses from changes in the market price of long-term bonds can be as large as 1.5 percent of quarterly GDP when monetary policy is conducted under discretion. Consequently, the model can replicate satisfactorily the observed stream of dividends when the dividend rule is based on the Fed's net interest income but overestimates the volatility of remittances when the dividend rule is based on the Fed's net income. Also, despite that the Fed has started paying interest on excess reserves since 2008, the total interest cost is relatively small, peaking 0.04 percent of quarterly GDP in 2013-Q4.

We feed the model with a sequence of asset-purchase shocks (see equation (2-7)) such that total assets held by the central bank in the model replicates the Fed's holdings of U.S. Treasury bonds and mortgage-backed securities observed in the data. We then compare the resulting size of liabilities and net worth that the model generates with those in the data. Figure (2.11) shows that both models, under discretion and commitment, can capture well the implied increase in the Fed's liabilities as well as the upward trend in net worth during the ZLB period.

Because the Fed holds bonds of different maturities, accurate projections of the Fed's future income and capital losses require a model that captures well not only the dynamics of the FFR but also the behavior of the entire yield curve. Because we make the simplifying assumption that in the ZLB period the Fed holds only bonds with 7.8 years of duration, the observed average duration, it is important that the model is capable of making good predictions of this specific segment of the yield curve. Figure (2.12) displays the actual and model-based yields on U.S. Treasury bonds of 1, 2, 5, 10, 20 and 30 years duration. Note the model performs very well with low-yield bonds but the fit

[9]For simplicity, we assume that the Fed is fully backed by the Tresury in this execise.

deteriorates as the duration increases. However, looking at the yields on 5 and 10-year bonds, the models seems to perform satisfactorily. The model with commitment tracks well the yield trends while the discretion overestimates that by about 25 basis points on average. Noteworthy, both models fail to predict the high volatility of yields in the ZLB period.[10] Section (6.7) in the technical appendix provides the method used to compute the yield to maturity of the delta bonds.

Finally, figure (2.13) shows the smoothed price of long-term bonds under commitment and discretion. When the crisis hit the U.S economy in 2008:Q4, private agents revise their expectation of interest rates downwards and the price of long-term bonds increase substantially: a 14% rise under commitment and 11% under discretion. After the peak, prices converge slowly to their new, and higher, long-run level. However, note that the volatility of convergence is much higher when monetary policy is conducted under discretion, posing a bigger risk to the stability of the Fed.

2.4 Quantitative Model Results

In this section, we use the quantitative model to test the resilience of the Fed's balance sheet to shifts in the FFR during the normalization of monetary policy following the ZLB period, assuming that the Fed conducts monetary policy with no regard to the solvency constraints. We then analyze how the solvency constraint forces the Fed to deviate from the baseline optimal policy in order to remain financially sound, and the spillovers on the equilibrium dynamics of inflation and the output gap.

2.4.1 The Fed's Financial Stability.

We assess the likelihood that the Fed violates at least one of the solvency constraints specified in expression (2-12) when conducting monetary policy under commitment and discretion during the monetary policy normalization following the ZLB period. We perform Monte-Carlo forecasts with information available in every quarter of the period 2008:Q4 - 2015:Q4, and project the Fed's balance sheet ten years into the future to assess its resilience to the seven types of structural shocks.[11] In order to provide a comprehensive but

[10]Given the asymmetric arrangement between the Fed and the Treasury, high yield volatility is counterproductive for the stability of the bank (high earnings are remitted while high losses are internalized). Hence, if anything, the models underestimate the real risk that monetary policy poses to the Fed.

[11]We estimate the endogenous state of the economy, $X_{2010:Q3}$, and the contemporaneous shock, $\epsilon_{2010:Q4}$, using the method described in section (2.2.3). Then, we draw 1000 trajec-

concise discussion of the main mechanisms driving the dynamics of the Fed's balance sheet, we focus our attention on the announcement date of QE 2 in 2010:Q4. We present a summary of the results for every quarter of in the period 2008:Q4 - 2015:Q4 in the table (2.4).

2.4.1.1 Net-Income Based Dividend Rule.

We start the analysis considering the dividend rule based on the Fed's net income (see equation (2-5)). Panels (a) and (b) of figure 2.14 display the paths of key variables simulated with the model under discretion and commitment, respectively. The solid black line shows the smoothed variables until 2010:Q4, shaded gray areas, and the dashed black line represent the percentiles and the median of the forecast distribution, respectively. The red line corresponds to projections in the absence of further shocks to the economy after 2010:Q4.

As expected, monetary policy under discretion is more contractionary than policy under commitment. Looking at the median of the forecast distribution, the Fed keeps the FFR at the ZLB for a longer length of time when operating under commitment. The lift-off starts in 2013:Q1 under discretion and only 6 quarters later under commitment. Another interesting distinction between the two models is that, while the FFR converges monotonically to the target under discretion, the optimal path of the FFR under commitment first overshoots the target before converging to it.

Persistence and volatility of the FFR. As in the exogenous-income model, contractionary monetary policy per se does not generate losses to the Fed. The impact of monetary policy on the Fed's balance sheet depends on the market's ability to accurately anticipate the future path of nominal interest rates and incorporate that information in the price of the long-term bond. When the FFR is highly volatile, forecast errors are large and the Fed makes large capital gains and losses due to revaluations of the bond's price, as market participants learn their prediction errors.[12] [13]

Columns (1) and (2) of table (2.3) report the persistence and forecast error standard deviation from the simulated data of the FFR, long-term bond prices, capital gains and net worth under commitment and discretion, respec-

tories of the structural shocks from the posterior distribution, solve the model using the piecewise method (described in sections (6.4), (6.5), and (6.6) in the technical appendix) and compute the paths of the endogenous variables implied by the model for each realization of the shocks.

[12]Recall that the equilibrium price of the long-term bond is given by the expected present value of the flow of payments (or coupons) provided by bond.

[13]Note that capital losses are only slightly negative at the median of the FFR distribution, reflecting just the depreciation of the delta bonds and not losses due to unexpected revaluations of the bond price.

tively.[14] Because the optimal commitment allows the Fed to smooth the policy-rate response to strutural shocks over time, the implied persistence of the FFR is significantly higher relative to discretion. The estimated autoregressive coefficient of the FFR in the model under commitment is 0.96, implying that an unexpected movement in the FFR has a half life of 16 quarters. In the discretionary setup, the absence of the ability to commit to future policy causes shocks to the FFR to be relatively short-lived, with a half life of only 2.3 periods (autoregressive coefficient equal to 0.74).

When monetary policy is conducted under discretion, the implied FFR is *more* volatile than under commitment. The average standard deviation of the FFR's forecast errors under discretion is 61 basis points, roughly twice as large as the standard deviation implied by the monetary policy under commitment. Figure 2.14 shows that, under commitment, the lift-off is clustered around 2014:Q3 and there is little dispersion of the FFR in the post-ZLB period. However, under discretion, the duration of the ZLB ranges from 4 to 13 quarters and the distribution of the FFR after the lift-off remains notably volatile. The high volatility of the FFR is passed on to the price of the long-term bond and to the Fed's balance sheet, causing the standard deviation of the Fed's net worth to be more than four times higher under discretion than under commitment, 3.32 and 0.79 percent of quarterly GDP respectively.

The consequences of the high volatility of the FFR are highlighted by both measures of financial strength defined by expression (2-12). Panel C of table (2.3) displays the bottom values of the Fed's net worth and the peak values of the balance on the deferred assets account within the 30th, 20th and 10th percentile of the forecast distribution. Column 2 shows that the Fed is in generally sound financial condition under commitment since even in the worst case scenario it does not violate either solvency constraints. The situation is rather different under discretion. Within the 10th percentile, the Fed's balances on the deferred account reaches 8.04 percent of quarterly GDP, which is above the estimated present value of future seignorage revenues, ϕ, violating the second condition of (2-12). Also in the 10th percentile, the Fed's net worth hits -7.42 of quarterly GDP, coming very close to being technically insolvent and raising serious concerns about the financial stability of the Fed.

Pane A of table (2.4) reports the quarters in which the Fed violates each solvency constraint within the 5^{th}, 10^{th} and 20^{th} percentiles of the forecast

[14] We use the simulated data to compute the n-ahead forecast error standard deviation, $\sigma_n \equiv \sqrt{\sum_{i=1}^{m}(x_{i,t+n} - \mathbb{E}_t x_{t+n})^2}$, where $x_{i,t+n}$ is the i^{th} simulated value of variable x in period $t+n$ and m is the number of simulations. The average forecast error standard deviation is the average of σ_n over $n \in \{1, 2, ..., 50\}$. We estimate the persistence of each simulated time path of variable x, $\{x_{i,t}\}_{t=1}^{50}$, from the regression $x_{i,t} = \alpha_i + \rho_i x_{i,t-1} + \epsilon_t^i$. The average persistence of variable x is the average of ρ_i over $i \in 1, ..., m$.

distribution, during 2008:Q4 - 2015:Q4 (between the announcement of QE 2 and the tapering). The results show that the Fed faced a small but positive risk of becoming insolvent. The probability of hitting the constraint on the Fed's net worth is fairly similar to the probability of hitting the upper bound on the deferred account. In summary, while it is highly unlikely that a solvency constraint is violated in 5 of the 13 quarters of the period, there is at least a 5 percent chance that one of the constraints is violated in the other 8 quarters.

2.4.1.2
Net Interest Income-Based Dividend Rule.

The Fed's financial stability is more vulnerable to interest-rate volatility when the dividend rule is based on the interest income. Columns (3) and (4) of table (2.3) show that in 2010:Q4, the Fed violates the solvency constraint even within the 30th percentile of the forecast distribution, under commitment and discretion. Under discretion, the Fed's net worth sinks to -10.9 percent of the quarterly GDP within the 30th percentile of the forecast distribution, and -13.6 within the 10th percentile. Moreover, net worth is much more volatile than the previous case: forecast error standard deviation equals 5.4 percent of the quarterly GDP under discretion and 1.57 under commitment.

Absence of an insurance from the Treasury. When dividends are based on the Fed's net income, remittances to the treasury is procyclical: increases when bond prices are high and is cut back when prices fall. This arrangement benefits the Fed as it shares the risk of holding long-term bonds with the Treasury. This advantage vanishes when the dividend rule does not account for capital gains and losses. Under the net interest income-based dividend rule, the Fed is forced to hand over to the Treasury a share $1-\zeta$ of the net interest income even if a decline in the price of long-term bonds brings the Fed's net income to negative ground.

Figure (2.14) shows that interest income becomes a very important source of income as the U.S economy recovers from the crisis and the Fed starts normalizing monetary policy. The growth of net interest income since 2010:Q4 is remarkable, in both models it roughly tripled by 2013:Q4, providing not only a substantial but also stable income flow to the Fed.[15] On the other hand, the Fed suffered large capital losses during the same period: the bond portfolio depreciated on average 0.5% of quarterly GDP per quarter.

[15]Interest income grows despite the fact that the Fed also pays interest on reserves because often the entire yield curve steepens when the Fed raises the FFR (the difference between yields on short-term bonds and yields on long-term bonds increases). Hence, the increment in interest income from the portfolio of long-term bonds outweighs the increase in interest payments on reserves and the Fed's overall net interest income expands

Due to the large imbalances between the interest and capital accounts, the overall stability of the Fed depends crucially on the type of dividend rule in place. Figures (2.14) and (2.15) show that while the average payout to the Treasury was 0.7 percent of the quarterly GDP per quarter with the dividend rule based on the interest income, the Fed sent on average *zero resources* to the Treasury under the net-income dividend rule. As a result of large capital losses and large payouts to the Treasury during an extended period of time, the Fed's net worth deteriorates severely, driving the Fed technically insolvent in more the 50% of the simulated paths, as illustrated in figure (2.15). The dashed black line shows that the median of the forecast distribution hits the solvency constraint in 2014:Q1.

Panel B of table (2.4) indicates the quarters in which the Fed violates each solvency constraint during the period 2010:Q4 - 2013:Q4, when the dividend rule is based on the Fed's net interest income. One can see that when conducting monetary policy under discretion, there is at least a 20% probability that the Fed's net worth will fall below -8% of the quarterly GDP *and become technically insolvent*. Moreover, when considering the forecasts in the absence of further shocks to the economy, the Fed violates the net worth constraint in 9 of the 13 quarters. On the other hand, since the payout to the Treasury is almost always positive when the dividend rule is based on the net interest income, the balance on the deferred account is essentially zero throughout the period and the Fed never violates this constraint.

2.4.2
The Role of the Solvency Constraints.

The previous section concluded that the implementation of QE programs 2 and 3 posed a threat to the Fed's financial soundness, particularly in the case of discretionary policy coupled with a dividend rule based on the Fed's interest income. In this section, we analyze the consequences of imposing the lower bound on the Fed's net worth to the equilibrium dynamics of the FFR, inflation and the output gap.

Figure (2.16) displays forecasts of the FFR, inflation and the Fed's net worth when the dividend rule is based on the Fed's net interest income. The forecasts are carried out with available information at (*i*) the end of QE 2 in 2011:Q2, (*ii*) the announcement date of QE 3 in 2012:Q3, and (*iii*) Ben Bernake's tapering announcement date in 2013:Q4; and assuming that no further shocks will hit the economy after each forecast initial date (i) - (iii).[16] The black dashed line shows the predicted evolution of these variables when the monetary authority acts under discretion and is subject to the solvency constraint (2-12). For comparison, we include the predictions from the baseline discretion (blue line) and commitment (red line) models, in which the solvency constraint is lifted ($\phi \to \infty$).

The top row of the figure (2.16) illustrates the results for the forecast from 2010:Q4. The baseline discretion model predicts that the FFR will lift off from zero seven quarters later in 2013:Q1, and quickly converge to the 4% targeted rate. This course of action yields insufficient demand and inflation running below the target for several periods. A side effect of this policy plan is the impact on the Fed's balance sheet. The right-hand side plot shows that the rapid normalization of the FFR drives the Fed's net worth below -8% of the quarterly GDP. The black dashed line shows that to satisfy the solvency constraint, Fed officials must deviate from the baseline policy plan. The liftoff takes place in the same quarter of the baseline discretion but the convergence to the target is slightly slower. In 2014:Q1, the net worth hits - 8 percent of the quarterly GDP and the Fed is forced to cease the tightening cycle for 5 periods. During this interval, a share of the bond portfolio depreciates, making room for further increments of the FFR while preserving the solvency of the Fed.

The impact on inflation is substantial: the average annual inflation rate in the first two years of the forecast increased 0.35 percentage points over the baseline discretion. Figure (2.17) shows that the impact on the output gap is positive but much weaker than the observed impact on inflation: the accumulated annual output gap within the first two years of forecast increased by only 0.2 percent of quarterly GDP relative to the baseline discretion. This unusual implication of the constrained discretion model can be used to rationalize the common criticism in the literature that DSGE models fail to explain the stabilization of inflation at positive rates in the presence of long-lasting negative output gaps, and others that find a large divergence between the inflation predicted by the baseline SW07 model and actual inflation.[17] Finally, the middle and bottom rows of the figure (2.16) show the results of 2011:Q2 carries over to the announcement date of QE 3 and the tapering.

[16]We opted to exclude the QE 1 period from the analysis since the Fed had only moderately extended the maturity of its assets prior to the implementation of QE 2. We also excluded the announcement date of QE 2 because the net worth constraint is not violated when considering the forecasts absent from future shocks (see figure (2.14) and table (2.4))

[17]See Hall (2011), King and Watson (2012) and Ball and Mazumder (2011).

2.5 Conclusion

The plausibility of our theory of Quantitative Easing developed in Chapter 1 hinges on the assumption that actual QE programs threaten the financial stability of central banks at the exit of the ZLB. We test this hypothesis in a simple stylized model calibrated to the U.S. economy and the Fed's balance sheet and find that although the Fed suffers large capital losses at the exit of the ZLB, it is not subject to insolvency. We then use a workhorse medium-scale DSGE based on SW07 and find that the Fed is at risk of insolvency, particularly if monetary policy is conducted under discretion and remittances to the Treasury are based on the Fed's net interest income.

Finally, we use the DSGE model based on SW07 to analyze the consequences of the programs QE 2 and QE 3 to the equilibrium dynamics of the FFR, inflation, and the output gap, assuming that the Fed is subject to solvency constraints. We find that the Fed is forced to deviate from the baseline optimal path of the FFR, creating significant additional inflation but mild impacts on the output gap.

2.6 Tables

Table 2.1: Quantitative Model: Summary of Baseline Calibration and Steady State: pre ZLB period (1985:Q1 to 2008:Q3) and ZLB period (2008:Q4 to 2015:Q4)

	Value (pre ZLB) (1)	**(ZLB) (2)**	**Target (3)**	**Model/Data (pre ZLB) (4)**	**(ZLB) (5)**	**Source (6)**
Panel A: Fed Balance Sheet						
AI. Steady State				*as % of Quarterly GDP*		
Liabilities (reserves) (m_*)	0.21	0.75	Avg Fed liabilities	21/21	75/75	FRB/FRED*
Short-term bonds (b_*)	0.22	0	Avg Fed assets	22/22	0/-	FRB/FRED
Long-term bonds (b^l_*)	0	0.75	Avg Fed assets	0/0	78/77	FRB/FRED
Net worth (nw_*)	0.01	0.03	Avg Fed net worth	1/0.8	3/1.3	FRB/FRED
Interest income	.003	.007	Avg Fed int income	0.3/0.3	0.7/0.5	FRB/FRED
Interest cost	0	0	Avg Fed interest cost	0/0	0/0.03	FRB/FRED
Net income (ni_*)	.003	.007	Avg Fed net int inc#	0.3/0.3	0.7/0.4	FRB/FRED
Surplus to capital stock	10^{-4}	10^{-4}	Avg Fed surplus/capital	0.01/0.01	.026/.018	FRB/FRED
Remittances to Treasury (d_*)	.003	.007	Avg Fed remittances	0.3/0.3	0.7/0.4	FRB/FRED
IOER (i^m_*)	0	0	IOER	0/0	0/0.25	FRB/FRED
AII. Parameters						
Rate of coupon decay (δ_b)	0.03	0.03	Avg Fed port/duration	0/<1 yrs	7.8/7.8 yrs	HR 2015
Loss limit (ϕ)	0.08	0.08	PV of seignorage rev	8% of quarterly GDP		HR 2015
Share on paid-in capital (ζ)	.035	.035	Avg Fed net worth/surplus to capital			FRB/FRED
Panel B: Changes relative to SW07						
BI. Steady State (annual %)				*Annual % rate*		
Growth rate trend (γ)	1.004	1.003	SEP FFR/GDP growth	1.7/-	1.2/2	SEP
Inflation rate (π_*)	1.007	1.005	SEP FFR/PCE growth	3/-	2/2	SEP
FFR (i_*)	0.015	0.01	SEP FFR	6/6	4/4	SEP
BII. Parameters				*Annual % rate*		
Intertemporal discount (β)	.998	.999	Federal Funds Rate	6/6	4/4	FRB/SEP**
Weight on stabilization (λ_i)	0.03	0.03	FFR autocorrelation	.86 (disc) .96 (commit)/0.97		See Table (2.2)
Persist. wage mk shocks (ρ_w)	0.89	0.89	FFR standard deviation	8.6 (disc) 3.5 (commit)/7		See Table (2.2)
Weight on output gap (λ_x)	0.01	0.01	Woodford (2003)			

*Board of Governors of the Federal Reserve System (US), retrieved from FRED, Federal Reserve Bank of St. Louis; https://fred.stlouisfed.org/series/TREAST, December 14, 2017.

**Summary of Economic Projections released with the FOMC minutes; https://www.federalreserve.gov/monetarypolicy.htm

Table 2.2: Mordel-Based and Actual Quarterly First-Order Autocorrelation and Standard Deviation of the Federal Funds Rate: Pre-ZLB Sample Period 1985:Q1-2008:Q3.

	Standard Deviation (%) / First-Order Autocorrelation		
	$\rho_w = 0.9$ & $\lambda_i = 0.03$ (*baseline calibration*)	$\rho_w = 0.9$ & $\lambda_i = 0$ (*no FFR stabilization*)	$\rho_w = 0.96$ & $\lambda_i = 0.03$ (*SW07* ρ_w)
Commitment	3.56/0.96	no equilibrium	3.59/0.96
Discretion	8.63/0.86	no equilibrium	13.7/0.91
Taylor Rule	5.17/0.87		5.96/0.90*
Data**	**7.03/0.98**		

* This specification corresponds to the mean of the posterior distribtion in SW07 (sample 1966:Q1 - 2004:Q4).
** Estimates of std. deviation and autocorrelation based on AR(1) reg on FFR time series 1985:Q1-2008:Q4.

Table 2.3: Statistical Summary of Forecasts of the FFR and the Fed's Balance Sheet at the Announcement Date of QE 2 in 2010:Q4.

Dividend Rule Based on:	*Net Income*		*Net Interest Income*	
	Discretion	*Commitment*	*Discretion*	*Commitent*
	(1)	*(2)*	*(3)*	*(4)*
Panel A: Forecast error std. deviation.				
Federal funds rate *(basis points)*	61	28	"	"
Price of long-term bond *(real U.S. dollars)*	1.69	0.52	"	"
Capital gains and losses *(as % of QGDP)*	1.44	0.65	"	"
Net worth *(as % of QGDP)*	3.32	0.79	5.44	1.57
Panel B: Persistence of the forecasts				
Federal funds rate	0.74	0.96	"	"
Price of long-term bond	0.83	0.93	"	"
Capital gains and losses	0.01	0.20	"	"
Net worth	0.78	0.93	0.84	0.93
Panel C: Peak/Bottom of the forecasts.				
within the 30th percentile:				
Deferred account	5.01	3.54	-	-
Net worth	-4.29	-2.83	-10.93	-9.25
within the 20th percentile:				
Deferred account *(as % of QGDP)*	5.79	3.77	-	-
Net worth *(as % of QGDP)*	-5.10	-3.07	-11.61	-9.58
within the 10th percentile:				
Deferred account	8.04	4.38	-	-
Net worth	-7.42	-3.71	-13.65	-10.32

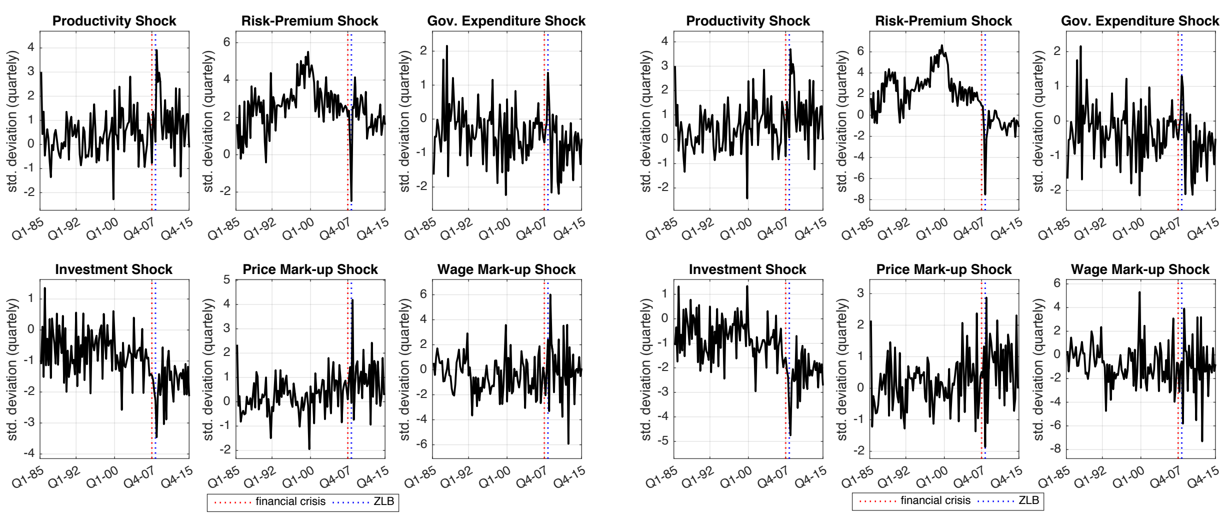
* See footnote 27 for the details of the calculation of average persistence and forecast error standard deviation.

Table 2.4: Fed's Financial Stability under Discretionary Monetary Policy: Summary of the Monte Carlo Forecast Distribution from the Announcement Date of QE 2 in 2010:Q4 to the Annoucement Date of Tapering in 2013:Q4. Affirmative (X) and negative (.)

	2010	*2011*				*2012*				*2013*			
	Q4	*Q1*	*Q2*	*Q3*	*Q4*	*Q1*	*Q2*	*Q3*	*Q4*	*Q1*	*Q2*	*Q3*	*Q4*
Panel A: net-income based dividend rule													
Fed's net worth below -8% of QGDP													
within the 5^{th} percentile	X	X	X	X	.	X	X	.	X	.	.	X	.
within the 10^{th} percentile	.	.	.	.	.	.	.	.	.	.	.	.	.
within the 20^{th} percentile	.	.	.	.	.	.	.	.	.	.	.	.	.
forecast (absence of further shocks)	.	.	.	.	.	.	.	.	.	.	.	.	.
Deferred Account above 8% of QGDP													
within the 5^{th} percentile	X	X	X	X	.	X	X	.	X	.	.	.	X
within the 10^{th} percentile	.	.	X	.	.	.	.	.	.	.	.	.	.
within the 20^{th} percentile	.	.	.	.	.	.	.	.	.	.	.	.	.
forecast (absence of further shocks)	.	.	.	.	.	.	.	.	.	.	.	.	.
Panel B: interest-income based div. rule													
Fed's net worth below -8% of QGDP													
within the 5^{th} percentile	X	X	X	X	X	X	X	X	X	X	X	X	X
within the 10^{th} percentile	X	X	X	X	X	X	X	X	X	X	X	X	X
within the 20^{th} percentile	X	X	X	X	X	X	X	X	X	X	X	X	X
forecast (absence of further shocks)	.	X	X	.	.	X	X	X	X	.	X	X	X
Deferred Account above 8% of QGDP													
within the 5^{th} percentile	.	.	.	.	.	.	.	.	.	.	.	.	.
within the 10^{th} percentile	.	.	.	.	.	.	.	.	.	.	.	.	.
within the 20^{th} percentile	.	.	.	.	.	.	.	.	.	.	.	.	.
forecast (absence of further shocks)	.	.	.	.	.	.	.	.	.	.	.	.	.

Number of simulations = 400.

2.7 Figures



(a) Discretion **(b) Commitment**

Figure 2.1: **Estimated Structural Shocks from Discretion and Commitment Models (benchmark calibration).** *Notes:* estimation based on the adapted OLS filter for the pre-ZLB sample 1985:Q1 - 2008:Q3 and based on the filter developed by Guerrieri & Iacoviello (2014) for the post-ZLB sample 2008:Q4 - 2015:Q4.

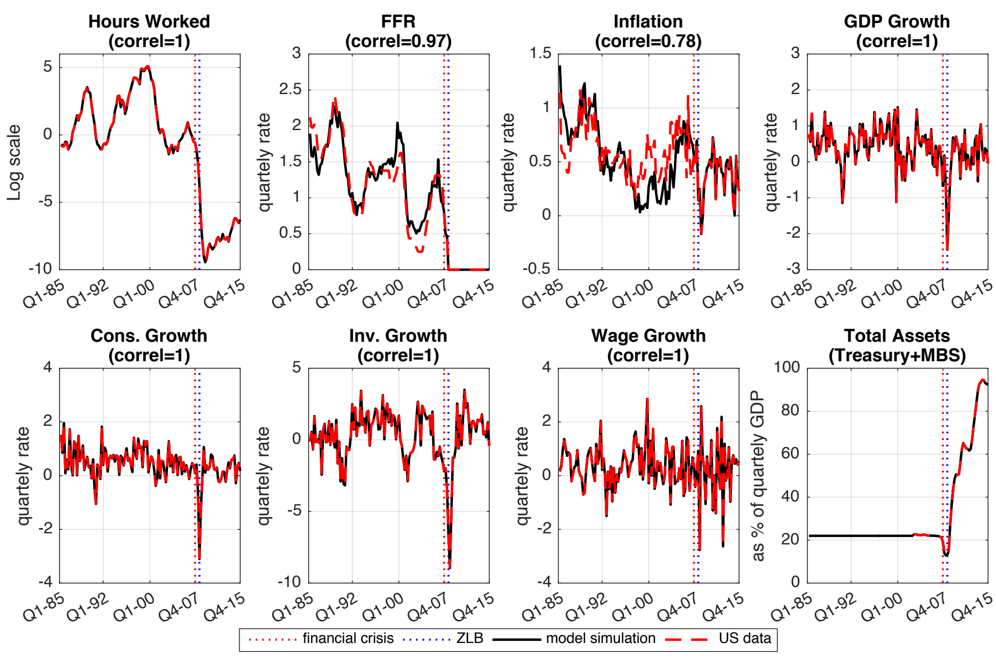


Figure 2.2: **Actual and Model-Based Observed Variables from the Model under Discretion (baseline calibration).** *Notes:* estimation based on the adapted OLS filter for the pre-ZLB sample 1985:Q1 - 2008:Q3 and based on the filter developed by Guerrieri & Iacoviello (2014) for the post-ZLB the subsample 2008:Q4 - 2015:Q4.

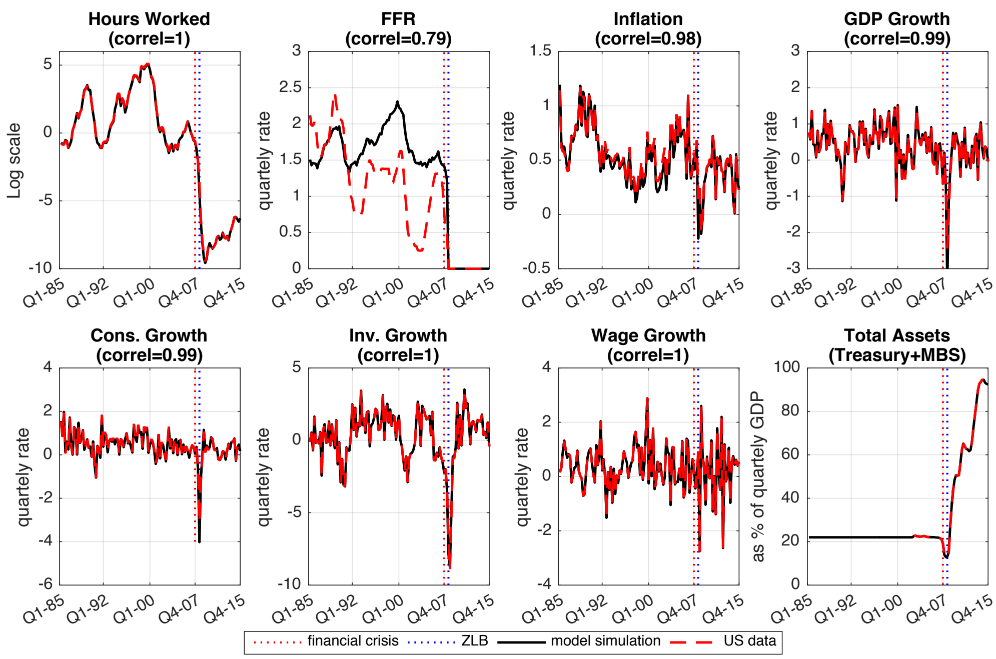


Figure 2.3: **Actual and Model-Based Observed Variables from the Model under Commitment (baseline calibration).** *Notes:* estimation based on the adapted OLS filter for the pre-ZLB sample 1985:Q1 - 2008:Q3 and based on the filter developed by Guerrieri & Iacoviello (2014) for the post-ZLB the subsample 2008:Q4 - 2015:Q4.

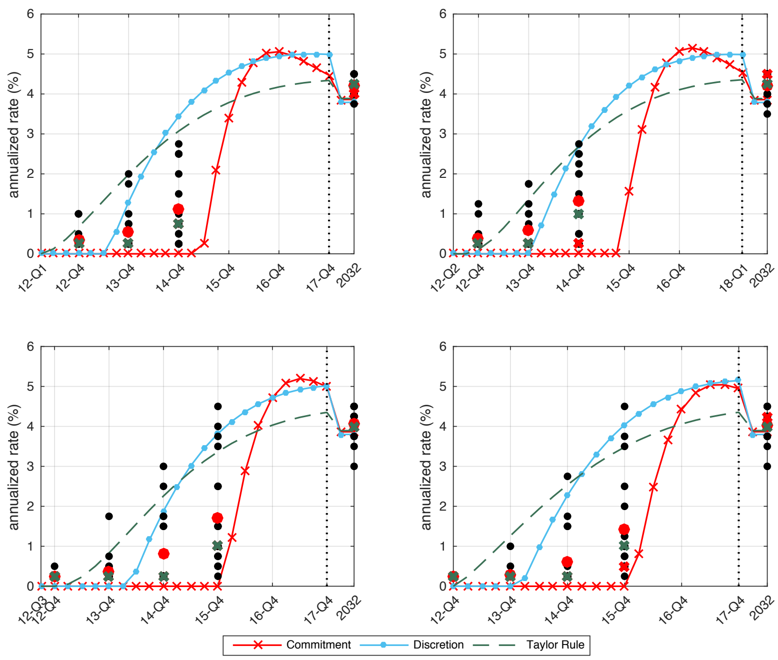


Figure 2.4: **Forecast of the Federal Funds Rate, 2012:** Baseline Calibration of Discretion, Commitment, Taylor Rule and the Summary of Economic Projections. *Notes:* the red circle, red cross and green cross display, respectively, the mean, mode and median of the FOMC member's individual projections of the Federal Funds Rates under appropriate monetary policy, disclosed by the SEP. Black dots display the projections of individual FOMC's members (19 total).

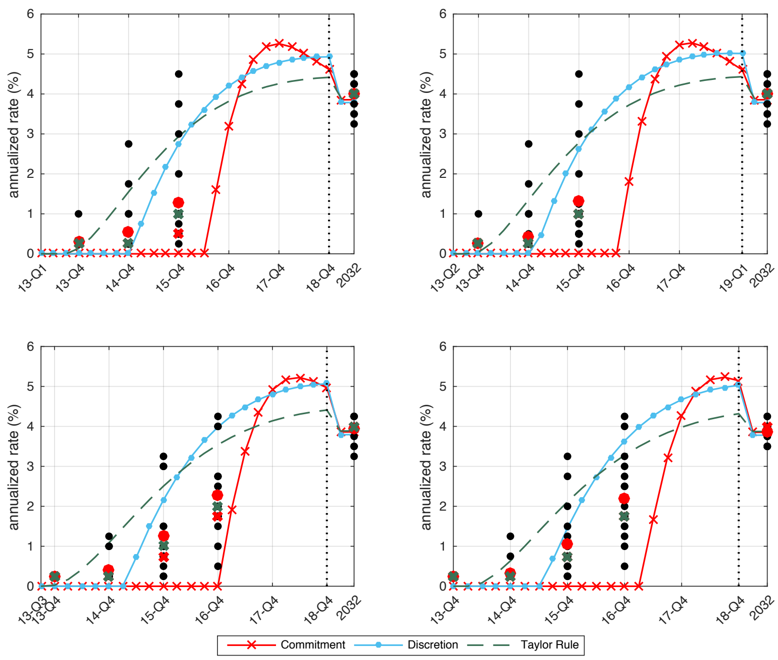


Figure 2.5: **Forecast of the Federal Funds Rate, 2013:** Baseline Calibration of Discretion, Commitment, Taylor Rule and the Summary of Economic Projections. *Notes:* the red circle, red cross and green cross display, respectively, the mean, mode and median of the FOMC member's individual projections of the Federal Funds Rates under appropriate monetary policy, disclosed by the SEP. Black dots display the projections of individual FOMC's members (19 total).

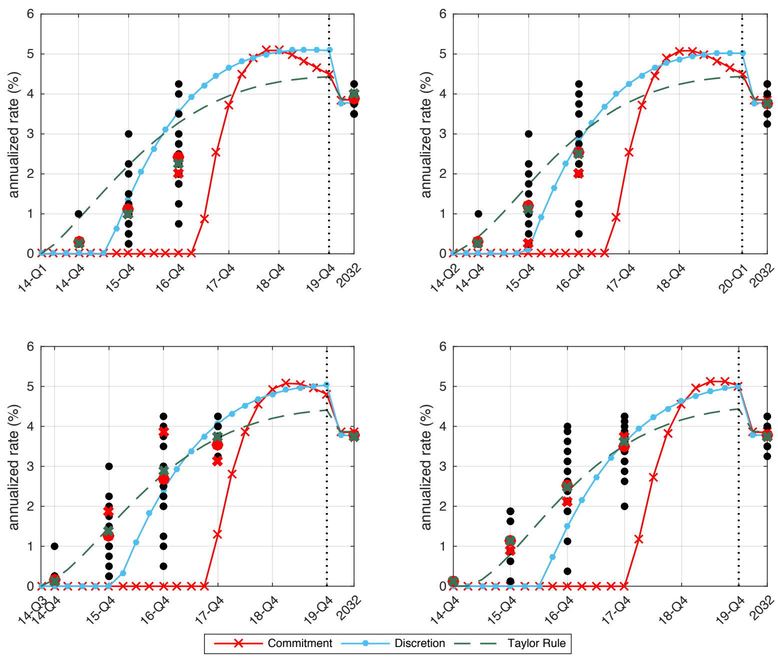


Figure 2.6: **Forecast of the Federal Funds Rate, 2014:** Baseline Calibration of Discretion, Commitment, Taylor Rule and the Summary of Economic Projections. *Notes:* the red circle, red cross and green cross display, respectively, the mean, mode and median of the FOMC member's individual projections of the Federal Funds Rates under appropriate monetary policy, disclosed by the SEP. Black dots display the projections of individual FOMC's members (19 total).

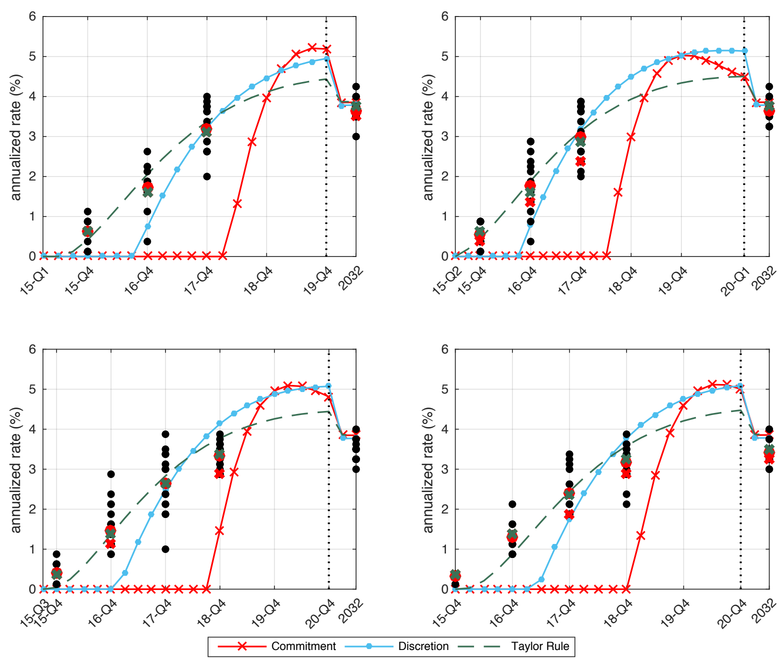


Figure 2.7: **Forecast of the Federal Funds Rate, 2015:** Baseline Calibration of Discretion, Commitment, Taylor Rule and the Summary of Economic Projections. *Notes:* the red circle, red cross and green cross display, respectively, the mean, mode and median of the FOMC member's individual projections of the Federal Funds Rates under appropriate monetary policy, disclosed by the SEP. Black dots display the projections of individual FOMC's members (19 total).

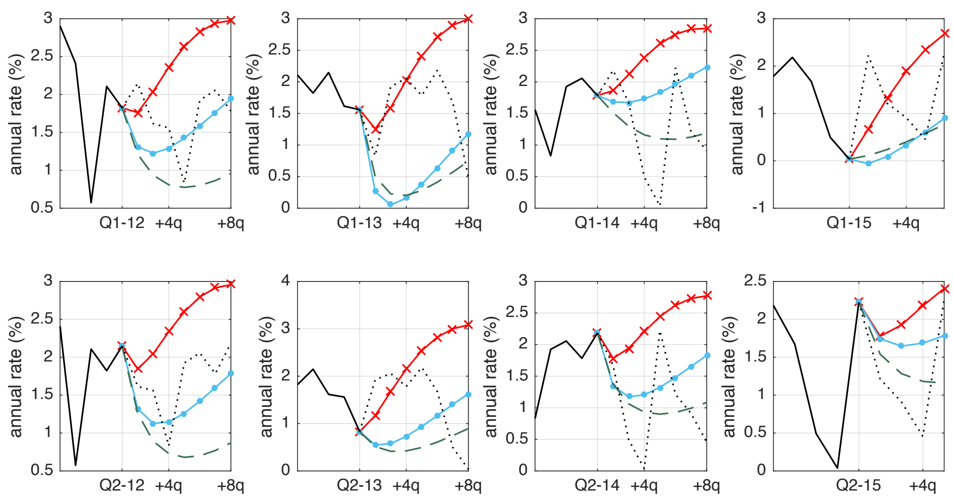


Figure 2.8: **Forecast of the Inflation Rate (PCE) at the Mean of the Distribution, 2012 - 2015:** Baseline Calibration of Discretion, Commitment, Taylor Rule and actual US data (black line). First row: projection with information available at the first quarters of years 2012 - 2015. Second row: projection with information available at the second quarters of the years 2012-2015.

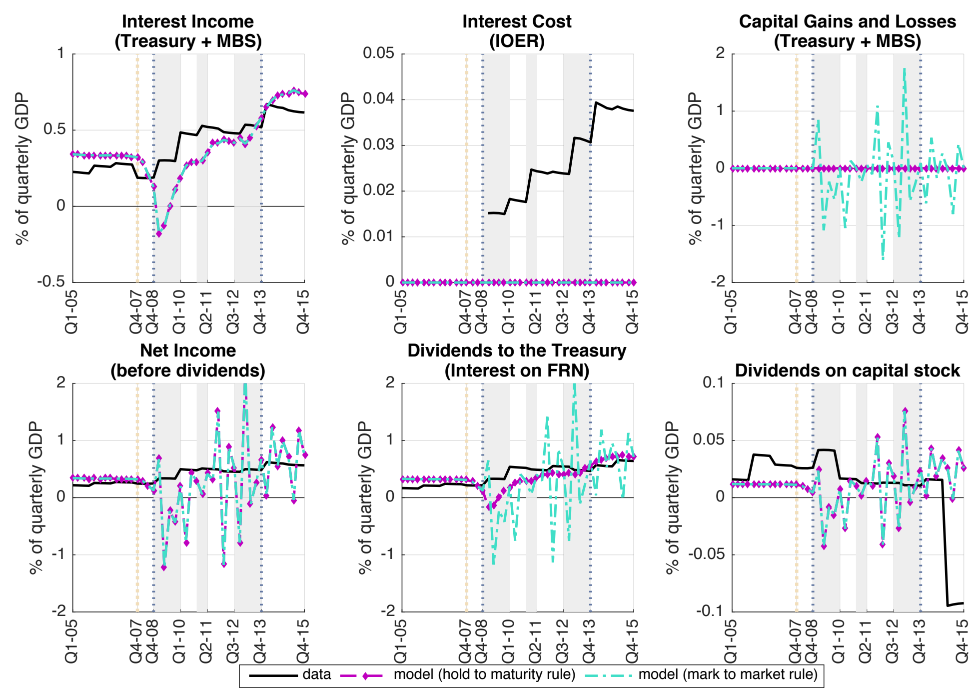


Figure 2.9: **Actual and Model-Based Fed's Income and Expenses: Smoothed Variables from the Discretion Model (benchmark calibration & full fiscal backing case).** *Notes:* estimation based on the adapted OLS filter for the pre-ZLB sample 1984:Q1 - 2008:Q1 and based on the filter developed by Guerrieri & Iacoviello (2014) for the post-ZLB sample 2008:Q2 - 2015:Q4. *Data source:* Board of Governors of the Federal Reserve System (US).

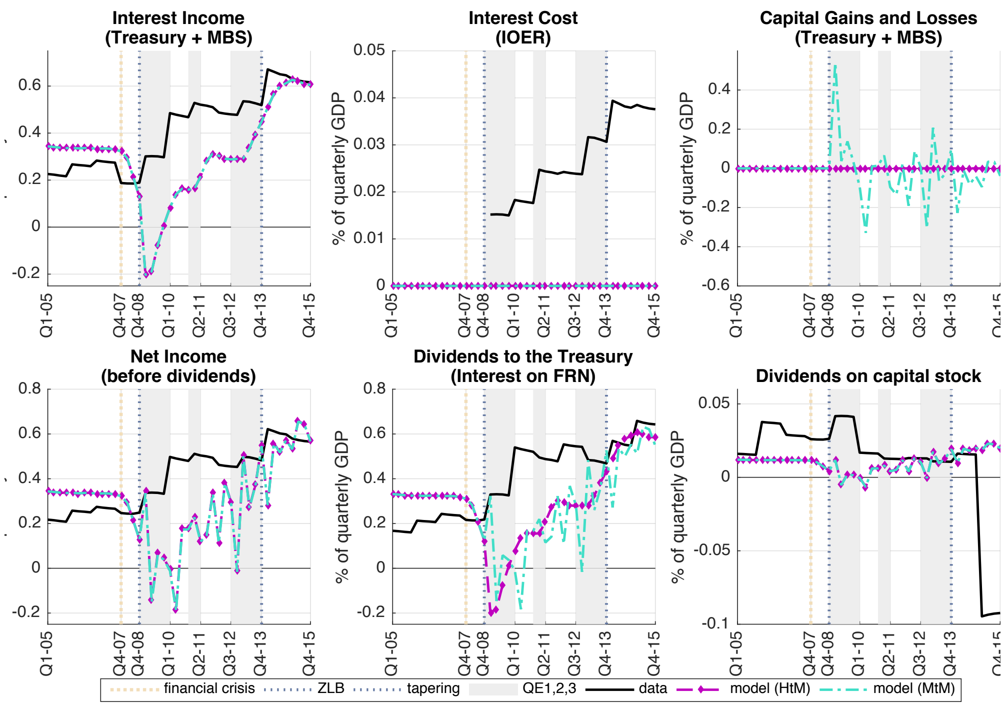


Figure 2.10: **Actual and Model-Based Fed's Income and Expenses: Smoothed Variables from the Commitment Model (benchmark calibration & full fiscal backing case).** *Notes:* estimation based on the adapted OLS filter for the pre-ZLB sample 1984:Q1 - 2008:Q1 and based on the filter developed by Guerrieri & Iacoviello (2014) for the post-ZLB sample 2008:Q2 - 2015:Q4. *Data source:* Board of Governors of the Federal Reserve System (US).

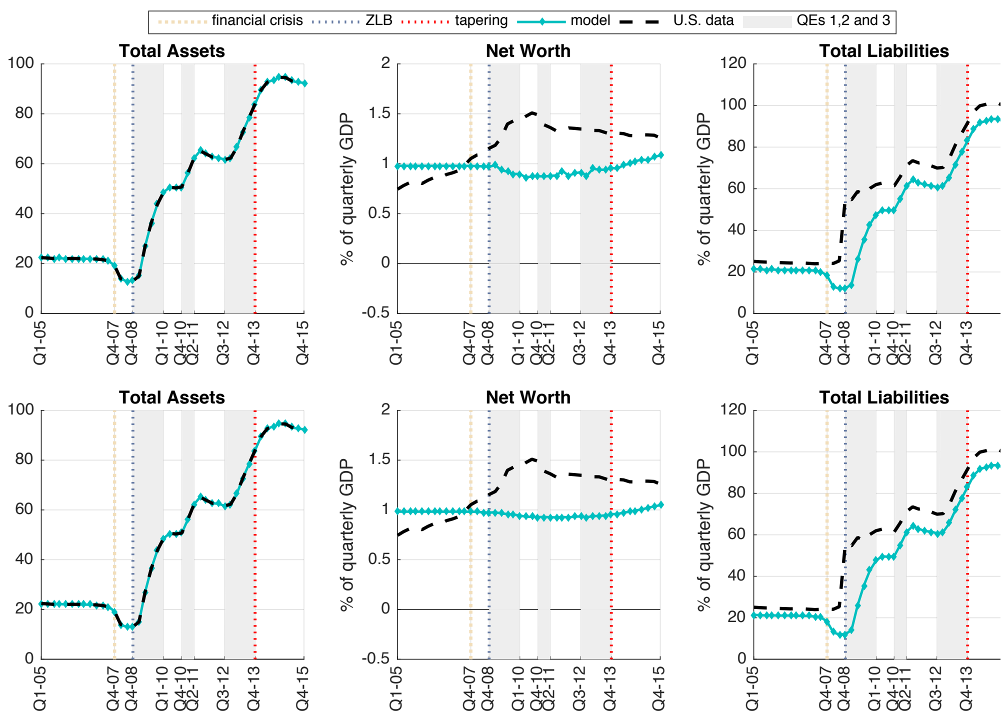
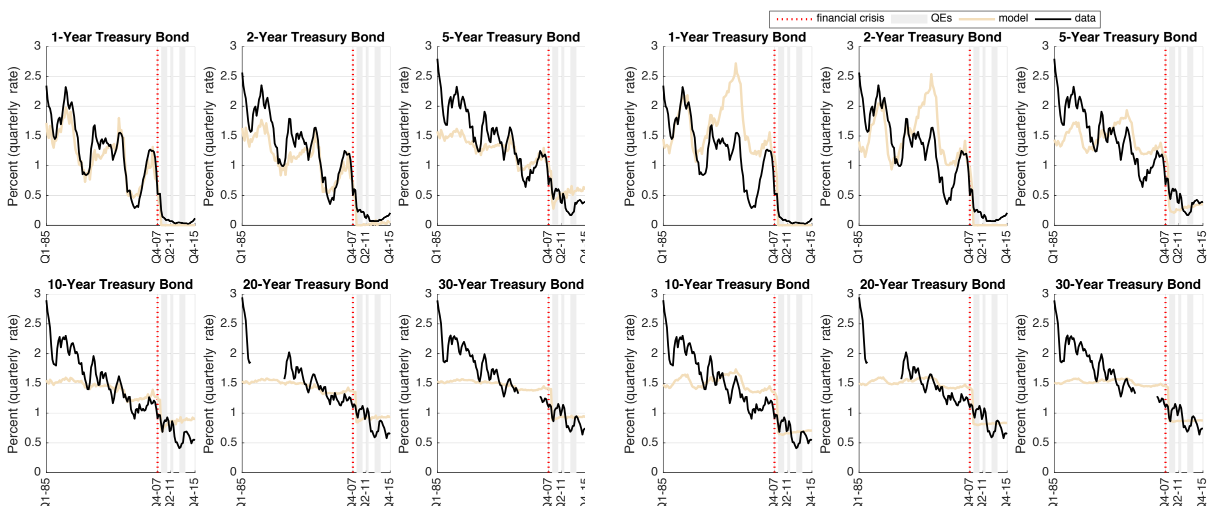
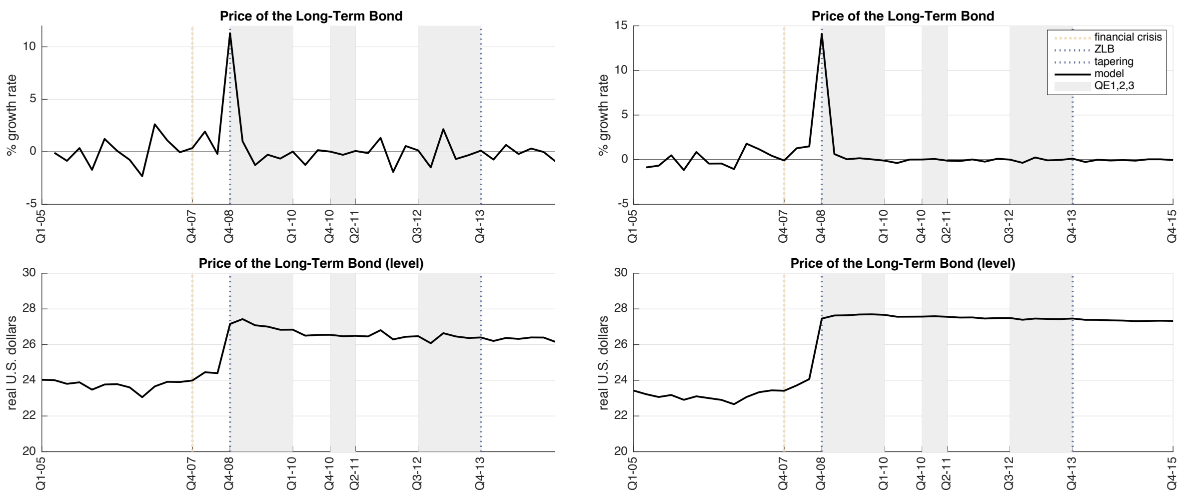


Figure 2.11: **Actual and Model-Based Fed's Assets and Liabilities: Smoothed Variables from the Discretion (Upper Panel) and Commitment (Lower Panel) Models (baseline calibration & full fiscal backing case).** *Notes:* estimation based on the adapted OLS filter for the pre-ZLB sample 1984:Q1 - 2008:Q1 and based on the filter developed by Guerrieri & Iacoviello (2014) for the post-ZLB sample 2008:Q2 - 2015:Q4. *Data source:* Board of Governors of the Federal Reserve System (US).



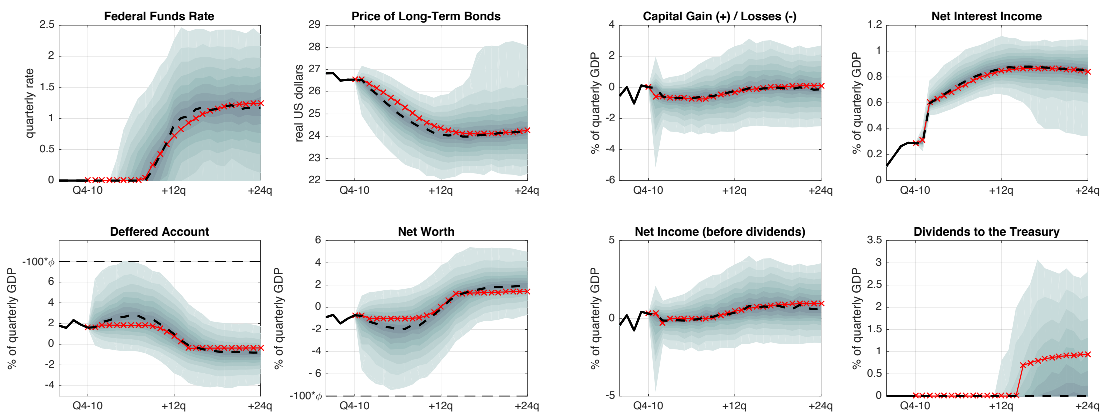
(a) Discretion **(b) Commitment**

Figure 2.12: **Actual and Model-Based Yields on U.S. Treasury Bonds by Maturities. Smoothed Variables from Discretion and Commitment Models.** *Notes:* estimation based on the adapted OLS filter for the pre-ZLB sample 1984:Q1 - 2008:Q1 and based on the filter developed by Guerrieri & Iacoviello (2014) for the post-ZLB sample 2008:Q2 - 2015:Q4. *Data source:* Board of Governors of the Federal Reserve System (US).

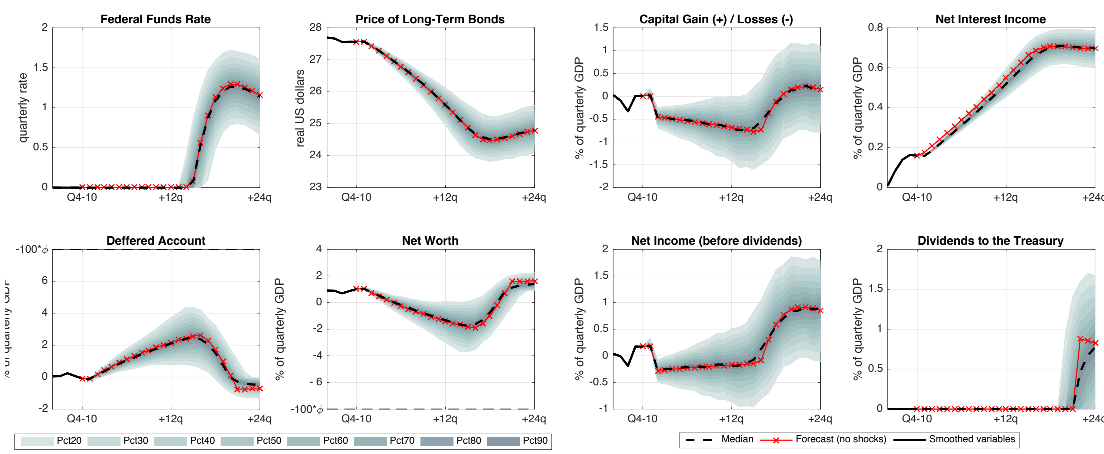


(2.13(a)) **Discretion** (2.13(b)) **Commitment**

Figure 2.13: **Smoothed Price of Long-term Government Bonds (7.8-years duration) from Discretion and Commitment Models.** *Notes:* estimation based on the adapted OLS filter for the pre-ZLB sample 1984:Q1 - 2008:Q1 and based on the filter developed by Guerrieri & Iacoviello (2014) for the post-ZLB sample 2008:Q2 - 2015:Q4.



(a) Discretion



(b) Commitment

Figure 2.14: **Monte Carlo-Based Forecasts of the Fed's Balance Sheet at the QE 2 Announcement Date:** Net Income-Based Dividend Rule & Baseline Calibration. *Notes:* smoothed variables until 2010:Q4, shaded gray areas and the dashed black line represent the percentiles and the median of the forecast distribution, respectively. The red line corresponds to forecasts assuming no further shocks hit the economy over the forecast period.

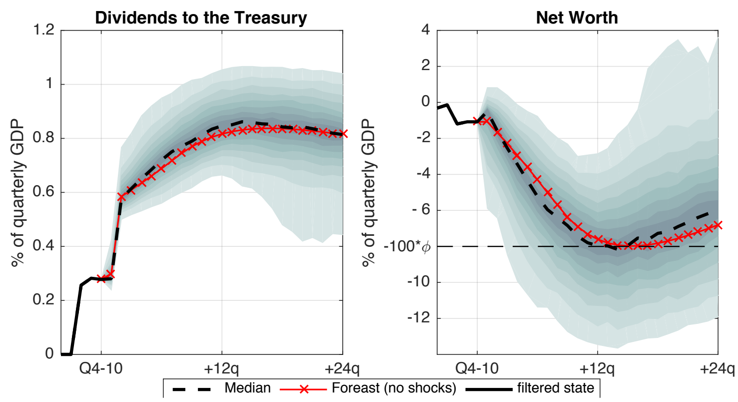


Figure 2.15: **Monte Carlo-Based Forecasts of the Fed's Net Worth and Dividends in the QE 2 Announcement Date:** Discretion & Net Interest Income-Based Dividend Rule (baseline calibration). *Notes:* smoothed variables until 2010:Q4, shaded gray areas and the dashed black line represent the percentiles and the median of the forecast distribution, respectively. The red line corresponds to forecasts assuming no further shocks hit the economy over the forecast period.

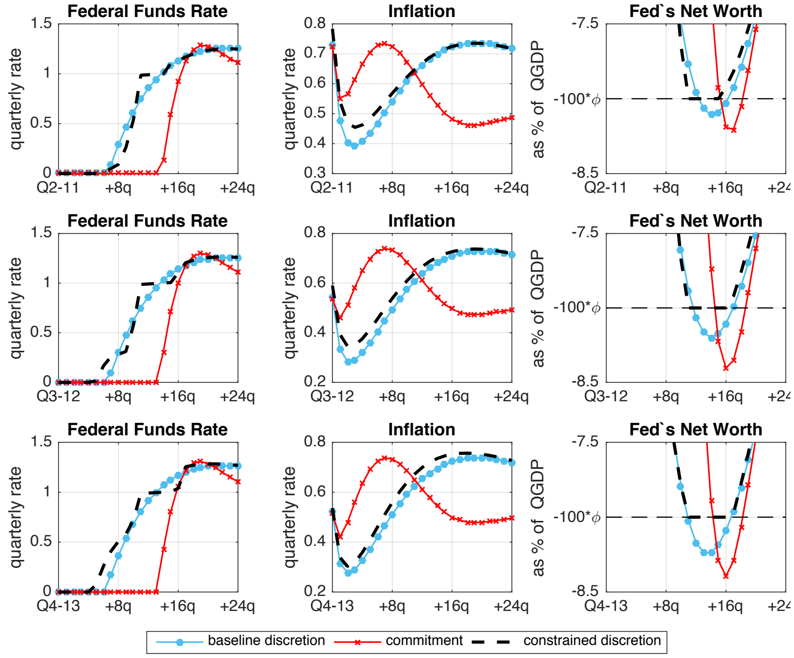


Figure 2.16: **The Effects of the Solvency Constraint on the Federal Funds Rate and Inflation Dynamics:** Baseline Discretion, Commitment and Constrained Discretion. *Notes:* Projections with information available at the end of QE 2 (Frist Row - 2011:Q2), announcement date of QE 3 (Second Row - 2012:Q3) and Ben Benanke's tapering announcement date (Third Row - 2013:Q4).

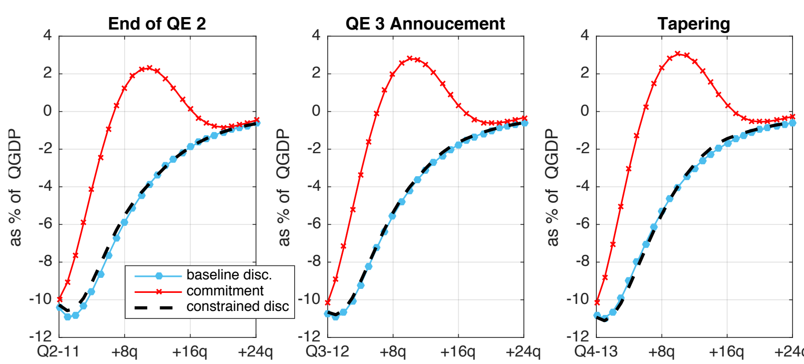


Figure 2.17: **The Effects of the Solvency Constraint on the Output Gap:** Baseline Discretion, Commitment and Constrained Discretion. *Notes:* Projections with information available at the end of QE 2 (Left panel - 2011:Q2), announcement date of QE 3 (Middle panel - 2012:Q3) and Ben Benanke's tapering announcement date (Right panel - 2013:Q4).

3
Consumption Smoothing and Shock Persistence: Optimal Simple Fiscal Rules for Commodity Exporters

3.1
Introduction

Commodity-exporting economies are often characterized as having needlessly pro-cyclical fiscal policy: spending when commodity prices are high, and then cutting back when commodity prices fall (i.e. a balanced budget rule, BBR). One of the purposes of Sovereign Wealth Funds (SWFs), combined with structural surplus fiscal rules (SSRs), is to smooth out government expenditure over time: to *save* when commodity prices are high and build up a buffer which can be drawn upon in times of lower prices.[1]

Two widely admired countries in this literature are Norway and Chile. Norway's fiscal rule involves storing its oil revenue in a SWF, and withdrawing around 4% (the long run rate of return) per year to fund public expenditure (Gonzalez et al. (2012)). Pieschacon (2012) finds that if Mexico had adopted Norway's fiscal rule, it would have been better off by around 7.5% of steady state consumption. Chile's celebrated structural surplus rule (SSR) involves saving copper revenues that are above their perceived long-run level and drawing upon these savings when copper prices are low.[2] Based on a small open economy New Keynesian model calibrated to Chile, Kumhof and Laxton (2013) find that the welfare gains from adopting a SSR, relative to a BBR, are around 5 times the gains from adopting optimal monetary policy.[3] The

[1]Of course SWFs also have other functions, such as intergenerational equity, but in this paper we focus on their role in smoothing macroeconomic shocks.

[2]The current formulation of Chile's fiscal rule is based on the deviation of the current copper price from a long-run "reference" price formed by a committee of experts, rather than on the deviation from the long-run average price Fornero and Kirchner (2014). An important difference is that the reference price of copper can (and does) change, as it did over 2005-13 when it tripled in USD terms (Fornero and Kirchner (2014) Figure 7). This means that *in practice* the Chilean fiscal rule is more pro-cyclical over the medium term than it is characterized in the literature, and our critique is more based on this characterization than the operation of the rule in practice.

[3]The gain is around 0.13% of steady state consumption, which is relatively large given then well-known low welfare costs of business cycles. For example, with log utility, consumers in the US are only willing to spend 0.05% of steady state consumption to avoid all business cycle fluctuations (according to Lucas's formula).

welfare gains are even larger when Chilean government expenditure responds counter-cyclically to non-resource tax revenues. The authors argue that the reason for these large gains is that the "key task of fiscal policy is stabilization of LIQ [liquidity constrained] household's income" — because those households cannot save/borrow to smooth commodity revenues for themselves.

In this paper, we show that the optimal fiscal rule for commodity export revenue is surprisingly pro-cyclical. Moreover, we find that simple balanced budget rules are often preferred to the structural surplus rules in reducing consumption volatility. To reach these conclusions we use several simple models, each with a share of hand-to-mouth (liquidity constrained) households and, like Kumhof and Laxton (2013), we assume that transfers are the key fiscal variable that adjusts to shocks. The fully optimal fiscal rule in our benchmark results involves spending around 70% of commodity revenues above their long-run level (with the remaining 30% saved in a SWF). In contrast, Kumhof and Laxton (2013) find that almost all of windfall commodity revenue should be saved.

Shock Persistence and Pro-cyclicality The most important factor driving our results is the persistence of commodity price shocks. When commodity price shocks are transitory, we get the same results as others in the literature that the optimal fiscal rule is closely approximated by a structural surplus rule (where all deviations from the long-run value of the commodity prices are saved). However, commodity price shocks are not transitory, they are highly persistent. Figure 3.1 shows the time path of real prices for selected commodities over 1960 - 2017: crude oil, natural gas, copper, gold, and iron. One can see that prices in each case show little tendency to revert to their mean in the short-to-medium run. The *annual* persistence coefficients are 0.95, 0.94, 0.9, 0.98, and 0.9, or half-lifes of 12, 12, 7, 32, and 7 *years*, respectively. Other papers in the literature usually assume a much less persistent process for commodity prices.[4] For example Kumhof and Laxton (2013) estimate copper prices to have a half-life of 2 years based on a short 8 year sample (1999-2007), and Garcia-Cicco and Kawamura (2015) estimate a half-life of copper prices to be one year after removing a structural break in 2005.[5]

[4]One reason for this is that as shocks become persistent, shock variances increase which creates computational problems.

[5]There is a vast literature testing whether commodity prices follow a random walk (which generally cannot be rejected), and trying to estimate more sophisticated models with temporary variations and structural breaks. In section (7.1) of the technical appendix, we compare our estimates of commodity price persistence for different commodities to those in Cashin et al. (2000) (who use a median-unbiased estimator) and find similar results. From a policy perspective, we argue the break down into permanent vs temporary components to be unhelpful for most countries. Identifying permanent vs temporary changes is difficult enough in hindsight, even harder in real time (when fiscal decisions must be made), and

When shocks are highly persistent, balanced budget rules perform well because *shocks don't need smoothing*. The permanent income hypothesis suggests that households should only consume out of their permanent income. Temporary shocks need to be smoothed by saving/borrowing because permanent income differs from current income. But for highly persistent shocks, current income is similar to permanent income, so simply spending current income is close to optimal.

While the optimal rule is pro-cyclical with respect to highly persistent commodity revenues, it is strongly countercyclical with respect to non-resource shocks, which are much less persistent (and affect households' other income). In our main results, we find that the optimal simple rule insures hand-to-mouth (HtM) households from the vast majority of variation in non-resource income by increasing (decreasing) transfers when non-resource income falls (rises).

Literature Overview. Standard economic theory prescribes that fiscal policy should pursue the stabilization of output by following a counter-cyclical fiscal policy. In the neoclassical model of Barro and Gordon (1983), a government should optimally run surpluses in good times and deficits in bad times to smooth the variance of the tax rate over the business cycle. A government should do the same according to the Keynesian tradition, although for different reasons. Price stickiness and other frictions prevent the economy from achieving efficient allocations in the short-medium term, and the government should use counter-cyclical fiscal (and monetary) policy to promote full employment, especially during downturns. Both views imply that the government should run surpluses in good times and deficit in bad times.

Nevertheless, governments tend to follow a procyclical fiscal stance, especially in emerging markets. Governments tend to increase expenditure in economic expansions and cut back in downturns, exacerbating the economic cycles. Talvi and Vegh (2005) documents that govern- ments do not save or even run deficits during economic expansions. Gavin and Perotti (1997) show that procyclicality is particularly pronounced in Latin America. Arreaza et al. (1998) and Talvi and Vegh (2005) provide evidence that procyclicality is also present in OECD countries. Ilzetzki and Vegh (2008) show that the co-movement between output and government expendi- ture is not entirely due to the fiscal multiplier and that procyclicality is stronger in developing countries.

often can be counterproductive. For example, M and Engel (1993) argue that in the 1970s and 1980s, Latin American countries generally considered positive shocks to be permanent, and negative shocks to be temporary, when in fact the opposite was true. Moreover, the welfare losses from over-reacting to a temporary shocks are *much* lower than under-reacting to a permanent one. See Fornero and Krichner (2014) for a model where agents learn about the true persistence of commodity prices.

Two main strands of the literature try to explain why countries, especially emerging and developing countries, follow a procyclical fiscal stance that exacerbates the volatility of the business cycle. Gavin and Perotti (1997), Riascos and Vegh (2003) and Caballero and Krishnamurthy (2004) argue that imperfect international credit markets prevent developing countries from borrowing during economic downturns. Although financial frictions can potentially explain procyclicality in bad times, it cannot account for procyclicality during economic expansions. In particular, it is not clear why governments would not build a buffer stock in good times that would prevent the borrowing constraint from binding in bad times. The second strand of the literature turns to political economy explanations, typically based on the idea that prosperity encourages fiscal indiscipline and rent-seeking activities. Talvi and Vegh (2005) argue that run- ning budget surpluses is costly because they create pressures to increase public spending. Given this distortion, a government that faces large fluctuations in the tax base will find it optimal to run procyclical fiscal policy.

Cespedes and Velasco (2014) find the problem of procyclicality is especially sensitive for com- modity exporting countries because (i) commodity revenues can represent a significant fraction of government revenues and (ii) commodity prices are highly volatile. When a commodity-rich country follows a procyclical fiscal policy, government expenditure is cut off drastically when commodity prices plunge, often triggering severe recessions.

This paper contributes to this branch of the literature by providing a new mechanism that can explain, at least partially, why the fiscal authority in commodity-exporting countries tend to pursue apparently subotimal procyclical fiscal policy.

Relation to Fiscal-Rules Literature. As discussed above, most recent quantitative models of fiscal rules for commodity exporters have argued that commodity revenues should be saved, and balanced budget rules are suboptimal. To our knowledge, there are no recent papers that challenge that view. While it has been known for some time that (i) commodity prices are close to a random walk, and (ii) permanent changes in income should be spent (for example, see M and Engle (1993)), researchers have generally avoided incorporating highly persistent shocks into quantitative models of optimal fiscal rules. This has led to the current consensus in favor of saving commodity revenues.[6]

[6]Other researchers have modeled government spending as not valued by households, such as Garcia-Cicco and Kawamura (2015). BJS2013 and Bems and de Carvalho Filho (2011) consider the welfare gains of hedging and the role of precautionary savings in models where commodity exporters face persistent commodity price shocks, but neither paper discusses fiscal rules.

Our results are also related to several papers which treat commodity price shocks as highly persistent, though none of these papers calculate optimal fiscal rules. First, our findings are consistent with the policy discussion in Cashin et al. (2000) who argue that highly persistent commodity price shocks are likely to undermine commodity price stabilization schemes. Cashin et al. (2000) focus on estimating the persistence of commodity prices and make their policy argument descriptively, rather than calculating the welfare consequences of different rules as we do here. Second, our results are related to JS2016. Although our results are consistent with theirs (suitably modified to a consistent framework as discussed above and in Section 3.5), JS2016 evaluate optimal reserve management rather than optimal fiscal rules and do not include Hand-to-Mouth (liquidity constrained) households. Finally, Fornero and Kirchner (2014) decompose copper prices into temporary and persistent components and find that the latter is highly persistent. They build a New Keynesian model where agents learn about the true persistence of commodity prices, and show impulse responses to persistent commodity price shocks under different fiscal rules — although they don't calculate the welfare consequences of those different rules.

Naturally, there are a number of important real world issues we have abstracted from — such as irreversible public investment and political constraints — which are discussed in the conclusion. Nonetheless, our paper does clarify that if a structural surplus rule is optimal, it should be justified along those lines, rather than in order to smooth consumption of constrained households.

Optimal rules by commodity. Although many commodity prices are quite persistent, there is substantial variation across individual commodities and the optimal fiscal rule is sensitive to this variation. For example, while around three-quarters of windfall oil revenues should be spent, only half of above-average gas revenues should be spent, and around a quarter of sugar revenues (see Table 7.1 in the section (7.1) of the technical appendix for a full list). As such, superficially similar commodity exporting countries can have very different optimal rules. The reason is that the optimal degree of pro-cyclicality increases non-linearly with the persistence of commodity price shocks.

Debt-elastic interest rates and precautionary savings. In our quantitative model we follow Schmitt-Grohe and Uribe (2003) and others in assuming that as a country's assets become larger (smaller) the country's interest rate premium declines (increases), reflecting a lack of investment opportunities on the upside, and greater financial risks on the downside. This variation in interest rates makes large variation in the size of the SWF very costly to households in terms of interest income. We find that there is a positive relation between the debt-elastic interest spread of the country and the optimal speed of convergence of the SWF towards its target size. In our baseline calibration, the optimal rule is for governments to spend around 10% of the deviation of the value of the SWF from its target size each year, which is well above the real rate of return of the SWF (around 4% per year, as with Norway's SWF).

A real world concern for optimal fiscal rules is that countries face a borrowing limit which inhibits the ability of the government to smooth spending after a long period of low commodity prices. Jeanne and Sandri (2016) (henceforth JS2016) model an economy with non-linear constraints like this, and derive optimal precautionary holdings of reserves (equivalent to an SWF). Since our main results rely on a linear model, it is natural to ask if we are overestimating the pro-cyclicality of optimal fiscal rules because we abstract from non-linear constraints. In fact, JS2016 find that an optimal simple linear rule similar to the one used in this paper is able to deliver the vast majority of the welfare gains from optimal non-linear reserve management. It turns out that our simple linear model is able to capture much of the dynamics of precautionary savings in JS2016 when we increase the debt-elastic interest spread. When we choose the debt-elastic interest spread to match the autocorrelation of net assets/reserves in JS2016 (with their calibration) we find almost exactly the same degree of pro-cyclicality of the fiscal rule — as well as the first-order autocorrelation of the trade balance-to-output ratio observed in the data — even though neither of these were calibration targets. When we use that debt-elasticity with our default calibration, we find that it actually *increases* the fraction of above-average commodity revenues that are spent. These results suggest that (i) our simple linear model with a reduced form financial friction is able to capture much of the dynamics of more complicated non-linear models, and (ii) if anything our simple linear model *understates* the pro-cyclicality of optimal fiscal rules.

Commodity price spillovers and endogenous GDP Another real-world concern is that in commodity intensive economies, shocks to commodity prices spill over into the non-resource economy, potentially complicating the optimal simple rule and motivating greater smoothing of commodity price shocks. In two extensions, first with exogenous spillovers and second in a real business cycle (RBC) model, we show that actually spillovers make the variation in the non-resource economy more persistent, which *increases* the pro-cyclicality of the optimal fiscal rule with respect to commodity revenues. In these economies, the optimal rule involves spending *all* of commodity revenues

— as these are the ultimate cause of the increase in persistence in non-resource GDP — but still responding counter-cyclically to temporary non-resource GDP shocks as before.

Government objectives, untargeted transfers and an irrelevance result As there are two types of households, the government might care more about some households than others. For example, the government might care more about the welfare of HtM HHs because they are poorer.[7] In the extreme case that the government *only* cares about HtM HHs, it possible to completely insure HtM HHs from all risk, though this results implies large welfare losses for the Ricardian (unconstrained) HH, and so some risk sharing across households is generally optimal if the government cares about the welfare of both households.

Although the government might be able to target transfers to particular groups, it is unlikely that they would be able to do so perfectly. Practically, this is not a huge problem, because Ricardian households are indifferent to changes in the timing of transfers, so long as their present value remains the same. In the paper, we show that under some conditions, the fiscal rule followed will be *irrelevant* for Ricardian HH consumption and welfare. Even if those strict conditions are not met, Ricardian HHs welfare is fairly insensitive to many changes in fiscal rules, which means that government can choose the *untargeted* fiscal rule to be fairly similar to the rule they would like to target at the HtM HHs. This rule is similar to the one that maximizes HtM HH welfare, *conditional* on transfers being untargeted.

Structure of the paper Our paper is organized as follows. In Section 3.2, we solve for the optimal simple rule analytically in a model with only hand-to-mouth (liquidity constrained) households. In Section 3.3 we present our main quantitative model, which includes two types of households but where output and commodity prices are exogenous. In Section 3.4 we present the main numerical results in terms of the welfare loss under different popular fiscal rules, and also the optimal fiscal rule. We then present three extensions to the baseline model. In Section 3.5 we show that the linear model of Section 3.4 comes close to replicating the optimal policy in the non-linear model of precautionary savings used by JS2016 with a higher debt-elastic interest spread. In Section 3.6, we generalize the results of the model by allowing commodity price shocks to spill over to non-resource GDP, which generally makes the response to commodity price shocks even more procyclical. In

[7]In our main results, we assume that all households have the same per capita income and their weights in the government's social welfare function are equal. Alternatively, if HtM households were poorer, then a utilitarian government would automatically put more weight on minimizing volatility in their consumption.

Section 3.7, we endogenize output in a Real Business Cycle (RBC) model which generally yields similar results (RBC model details in the Appendix). Section (3.8) analyzes the interaction between monetary and fiscal policy in a Small Open Economy New-Keynesian Model. Section 3.9 concludes.

3.2 Analytical Model

A common justification for saving commodity windfalls in sovereign wealth funds is a desire to *smooth consumption*. The idea is that households are risk averse, and so prefer a steady stream of consumption to a volatile one. If households are not able to borrow or lend for themselves — for example due to credit constraints, a lack of savings instruments or behavioral factors — then the government has a role to smooth commodity revenues on their behalf. In this section we focus on this mechanism in a model simple enough to solve *analytically*.

In order to do that, we assume that the *only* agent is a household who consumes his income hand-to-mouth each period, and that utility is *quadratic* (in the rest of the paper we assume more standard constant relative risk aversion (CRRA) utility).[8] The government taxes non-resource output $\tau_Y Y_t$, can save in or spend from a sovereign wealth fund A_t (if $A_t < 0$ then this is government debt) and receives a fraction τ_p of commodity revenues QP_t (for the rest of the paper we assume that $\tau_p = 1$ so the government receives all commodity revenues, as is standard in the literature). Commodity output is fixed at Q, but commodity prices P_t vary. The household's income each period consists of transfers, before tax non-resource GDP $(1-\tau_Y)Y_t$ and a fraction $(1-\tau_p)$ of commodity revenues ($1-\tau_p = 0$ in the rest of the paper). The government then chooses a transfer policy (equivalent to choosing consumption) to maximize the household's utility. More formally, the problem is:

$$max_{\{c''\}} E_0 \sum_{t=0}^{\infty} \beta^t U(c''_t) \tag{3-1}$$

such that:

[HH's budget constraint] $c''_t = (1-\tau_Y)Y_t + (1-\tau_p)QP_t + Tr''_t$

[Government's budget constraint] $A_t = (1+r)A_{t-1} + \tau_Y Y_t + \tau_p P_t Q - Tr''_t$

[Exogenous shocks] $P_t - \bar{P} = \rho_P(P_{t-1} - \bar{P}) + e_{Pt}$ *and* $Y_t - \bar{Y} = \rho_Y(Y_{t-1} - \bar{Y}) + e_{Yt}$

[8]The linear-quadratic approach is an extension of that in Basch and Engel (1993).

where $u(c_t) = -(c_t - \gamma)^2$ and $\beta = (1+r)^{-1}$. The Euler equation implies $c_t = E_t c_{t+1}$, and combined with the transversality condition, and some algebra yields the consumption function for households. This can be rearranged to give the government transfer rule where transfers respond (i) the deviation of the sovereign wealth fund from its long run level $(A_{t-1} - \overline{A})$, (ii) the deviation of commodity prices from their long run level $(P_t - \overline{P})$ and (iii) the deviation of non-resource output from potential $\left(Y_t - \overline{Y}\right)$.

In the analytical model (with only HtM HHs) the Optimal Simple Rule (OSR) is:

$$Tr''_t = \overline{Tr''} + \theta_A(A_{t-1} - \overline{A}) + \theta_P Q(P_t - \overline{P}) + \theta_Y \left(Y_t - \overline{Y}\right) \tag{3-2}$$

where $\overline{Tr''} = \tau_Y \overline{Y} + \tau_p Q\overline{P} + r\overline{A}$ and

$$\begin{aligned}\theta_P &= \tau_p \frac{r}{1+r-\rho_p} - (1-\tau_p)(1 - \frac{r}{1+r-\rho_p}) \\ \theta_Y &= \tau_Y \frac{r}{1+r-\rho_Y} - (1-\tau_Y)(1 - \frac{r}{1+r-\rho_Y}) \\ \theta_A &= r\end{aligned} \tag{3-3}$$

Each of the fiscal rule coefficients θ_P and θ_Y in Equation 3-3 has two components: (i) how the government spends above-average revenues ($\tau_p Q(P_t - \overline{P})$ for commodities or $\tau_y(Y_t - \overline{Y})$ for non-resource GDP), and (ii) the countercyclical transfers the government provides to smooth non-transfer income on behalf of the household ($(1-\tau_p)Q(P_t - \overline{P})$ or $(1-\tau_y)(Y_t - \overline{Y})$).

For commodity revenues: If oil shocks are transitory, $\rho_P = 0$, the optimal rule involves only spending $r/(1+r) \approx 4\%$ of any increase in oil revenues above trend. In contrast, as $\rho_P \to 1$, the government should transfer all of the above average oil revenues to households. With $\rho_P = 0.96$ and $r = 0.04$, as is close to the data (for oil), $r/(1+r-\rho_P) = 0.5$, so one should spend around half of excess oil revenues each period. This is (roughly) similar to the Optimal Simple Rule in Section 3.4, where around 70% of commodity revenues should be spent. With $\rho_P = 0.94$, around 40% of excess oil revenues should be spent. If $\tau_p < 1$, the government also wants to provide countercyclical transfers to help the HtM HH smooth their $1 - \tau_p$ share of non-commodity income. We remove that channel by assuming $\tau_p = 1$ (as is common in the literature), such that $\theta_p = r/(1+r-\rho_p)$.

For output, we calibrate $\rho_Y = 0$ and $\tau_y = 0.15$, which (with $r = 4\%$) imply $\theta_Y \approx -0.8$ (similar to numerical results in Section 3.4). This suggests that the government should increase transfers by 80% of any fall in GDP during a recession, a strongly countercyclical response. One can decompose this into the two components above. $r/(1+r-\rho_Y) \approx 0.04$ so the government should save almost all above-average non-commodity revenues. However $1-r/(1+r-\rho_Y) \approx 0.96$ which means that the government should respond counter-cyclically to non-resource GDP shocks to help the HtM HH smooth its own income, which is the main reason for the counter-cyclical fiscal response with respect to non-resource GDP.

Finally, the government should only spend the *interest* on any extra assets in the sovereign wealth fund (above the target level of the SWF $\bar{A}$). This is quite different from the rule in the quantitative model in the next section, where the optimal rule requires spending more than $r = 4\%$ of the SWF value for stability. With $\theta_A = r$, the value of the sovereign wealth fund exhibits almost a unit root. Not only does this mean that value of the SWF will eventually be exhausted (something we ignore here without government borrowing constraints), it also means consumption will exhibit a unit root and its variance will become very large. We revisit these issues in the next section.

3.3 Model (Description and Calibration)

In this section we build a simple exogenous-income model, which can be used to evaluate the quantitative welfare losses of alternative popular fiscal rules (e.g. balanced budget rule, structural surplus rules), and to calculate the optimal simple rule. Relative to the analytical model, we now include Ricardian households (who can borrow and save), change the utility function to the constant relative risk aversion (CRRA), and add a debt-elastic interest spread. The results are robust to endogenizing output in a Real Business Cycle model, discussed in Section 3.7, as well as other extensions.

3.3.1 Model Overview

Consider a small open economy disaggregated into resource and non-resource sectors. The sectoral decomposition is a key feature of the model because it allows us to account for the characteristics of the business cycle that are particular to each sector. For simplicity, we assume that production in each sector is exogenous. Each period the resource sector produces Q units of a commodity good that is not consumed domestically and only provides an additional source of income from export sales that accrues entirely to the government. The international price of the commodity follows an auto-regressive process in logs with persistence ρ_p and error standard deviation σ_p.[9] Production in the non-resource sector follows an auto-regressive process with

[9]The standard deviation of log commodity prices is $\sigma_P/\sqrt{1-\rho_P^2}$

persistence ρ_y and error standard deviation σ_y and can be either consumed domestically or traded internationally at constant price of one dollar. For now, commodity prices and output are independent, but we relax this assumption in Section 3.6. Time is measured in years.

Another key feature of this economy is that it is populated by two types of households. Ricardian households have full access to an international financial market, while liquidity-constrained (Hand-to-Mouth) households consume their after-tax income each period. The Ricardian/non-Ricardian framework generates a non-trivial role for fiscal policy and introduces household heterogeneity that will allow welfare evaluation from the perspective of two different households. In the calibration we assume that population, income and social welfare function weights of each household type ω are one-half.

3.3.1.1 Households

The fraction ω of Hand-to-Mouth households are denoted by the upper index ($''$) and the fraction $1-\omega$ of Ricardian households is denoted by ($'$). They value consumption paths according to Equation (3-4). There is no labor, leisure or public goods.

$$U_i = \sum_{t=0}^{\infty} \beta^t \frac{C_t^{i1-\sigma}}{1-\sigma} \qquad i \in \{', ''\} \tag{3-4}$$

where C_t^i is per household consumption in period t, β is the intertemporal discount factor and σ is the coefficient of risk aversion. Because the Ricardian household has access to an internationally traded riskless security, it chooses consumption and bond holdings to maximize Equation (3-4) subject to the budget constraint:

$$C_t' = R_{t-1}B_{t-1} + (1-\omega)^{-1}(1-\omega_y)(1-\tau)Y_t + Tr_t' - B_t \tag{3-5}$$

where B_t is the stock of international bonds held by the Ricardian household (private assets) at the end of period t, R_t is the domestic gross rate of return, Tr_t' are government transfers per Ricardian household, Y_t is the exogenous non-resource income and τ is the income tax rate. Utility maximization yields the following first-order condition for the Ricardian HH:

$$C_t'^{-\sigma} = \mathbb{E}_t R_t \beta C_{t+1}'^{-\sigma} \tag{3-6}$$

Since the HtM household does not participate in the international bond market, per HtM household consumption in period t is restricted to the share ω_y of the after-tax non-resource income, $(1-\tau)Y_t$, plus per HtM household transfers from the government, Tr_t'',

$$C_t'' = \omega^{-1}\omega_y(1-\tau)Y_t + Tr_t'' \tag{3-7}$$

3.3.1.2
The Government

The government receives exogenous resource income, $P_t Q$ (Q is the quantity of resource exports, which we assume to be constant), collect taxes, participates in the international bond market and makes transfers to both households. The government's budget constraint is:

$$A_t = R_{t-1}A_{t-1} + \tau Y_t + P_t Q - (1-\omega)Tr_t' - \omega Tr_t'' \tag{3-8}$$

where A_t is the stock of international bonds held by the government (public assets) in period $t+1$ and P_t is the exogenous commodity price.[10]

3.3.1.3
Debt-Elastic Interest Rate Spread

Following Schimitt-Groe and Uribe (2003), we induce stationarity in the model by assuming that the interest rate faced by domestic agents increases with the public + private level of debt in the economy

$$R_t = R^\star + \psi e^{-(A_t + (1-\omega)B_t - A_{ss} - (1-\omega)B_{ss})} - \psi \tag{3-9}$$

where B_{ss} and A_{ss} are steady state private and public assets, $R^\star$ is the constant world interest rate, and ψ is the debt-elasticity of the interest-rate spread.[11] Although we introduce this feature to the model for mostly technical

[10]Here the public sector budget surplus is $S_t = (R_{t-1}-1)A_{t-1} + \tau Y_t + P_t Q - (1-\omega)Tr_t' - \omega Tr_t''$

[11]B_t is measured in per-Ricardian HH terms, whereas other variables are in aggregate terms, so we need to multiply B_t by the share of Ricardian HHs $(1-\omega)$.

reasons, the debt elastic interest rate can be viewed as a reduced form way of introducing financial frictions in the model (see Section 3.5).

3.3.1.4
Exogenous Process

Commodity prices and non-resource endowments follow an autoregressive process of the form,

$$P_t = P_{t-1}^{\rho_p} exp(\epsilon_t^p) \tag{3-10}$$

$$Y_t = Y_{t-1}^{\rho_y} exp(\epsilon_t^y) \tag{3-11}$$

where $\begin{bmatrix} \epsilon_t^p \\ \epsilon_t^y \end{bmatrix} = N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_p^2 & \sigma_{py} \\ \sigma_{py} & \sigma_y^2 \end{bmatrix} \right)$

3.3.1.5
Welfare Approximation

Assume the government assigns weight ω_U to the HtM and $1 - \omega_U$ to the Ricardian HH so that different paths of consumption are ranked by the government according to the following social welfare function:

$$W = \mathbb{E} \sum_{t=0}^{\infty} \beta^t \left[(1 - \omega_U) \frac{C_t'^{1-\sigma}}{1 - \sigma} + \omega_U \frac{C_t''^{1-\sigma}}{1 - \sigma} \right] \tag{3-12}$$

Up to a second order, the problem of maximizing Equation (3-12) is equivalent to minimizing Equation (3-13), where $\hat{c}_t'$ and $\hat{c}_t''$ denote the percentual deviation of Ricardian and HtM consumption from their steady state values:

$$\mathbb{L} = \frac{\sigma}{2} \left\{ (1 - \Psi) \left[Var(\hat{c}_t') \right] + \Psi \left[Var(\hat{c}_t'') \right] \right\} \tag{3-13}$$

where

$$\Phi \equiv \left[\frac{C_{ss}''}{C_{ss}'} \right]^{1-\sigma} = \left[\frac{1 - \omega}{\omega} \frac{(1 - \tau)\omega_y + Tr_{ss}}{(1 - \tau)(1 - \omega_Y) + Tr_{ss}} \right]^{1-\sigma} \quad \text{and} \quad \Psi \equiv \frac{\Phi \omega_U}{(1 - \omega_U) + \Phi \omega_U}$$

One can interpret Equation 3-13 as the share of steady-state consumption that the household is willing to give up *each period* to eliminate the variance of consumption over the business cycle. In a world with complete markets, the household could sign a state-contingent contract with foreign investors so that equilibrium consumption is constant and welfare loss is zero. However, due to the less sophisticated financial structure assumed in this model, income shocks will lead to consumption volatility and welfare losses. The higher the variance of consumption, the greater the welfare loss.[12]

3.3.1.6 Fiscal Rules

The simple fiscal rule dictates how transfers to each type of household change in response to *observable* economic variables. In particular, we allow transfers to respond to deviations of public assets A_t, non-resource income Y_t and the international commodity price P_t from their respective long-term (steady state) levels. Note that transfers are written in per capita terms, so the total proportion of assets transferred (for example) is $(1-\omega)\theta'_a + \omega\theta''_a$

$$Tr'_t = Tr_{ss} + \theta'_a(A_{t-1} - A_{ss}) + \theta'_y(Y_t - Y_{ss}) + \theta'_p Q(P_t - P_{ss}) \tag{3-14}$$

$$Tr''_t = Tr_{ss} + \theta''_a(A_{t-1} - A_{ss}) + \theta''_y(Y_t - Y_{ss}) + \theta''_p Q(P_t - P_{ss}) \tag{3-15}$$

We consider seven types of fiscal rule, which are listed below. Countercyclical Rules (CCY) are where the government tries to smooth the business cycle by decreasing (increasing) transfers when output is above (below) potential. In rules (4) and (5) we combine countercyclical rules with Balanced Budget Rules (BBR) and Structural Surplus Rules (SSR), where the BBR/SSR refers to the treatment of commodity revenues, and CCY refers to the response to domestic non-resource income shocks.

1. **Full HtM Stabilization** is where the government completely smooths HtM's consumption by setting $\theta''_a = \theta''_p = 0$ and $\theta''_y = -(1-\tau)$. Following Lemma (7.2.2) (appendix section (7.2.2)), given $\theta''_a = 0$, the welfare of the Ricardian HH is independent of the coefficients $\{\theta'_a, \theta'_y, \theta'_p\}$.

2. The **Balanced Budget Rule (BBR)** suggests that the government should focus on minimizing the volatility of public assets around its long-term level. In this setup the government can perfectly stabilize public

[12] In our baseline calibration, we assume that $C''_{ss} = C'_{ss}$ (with $\omega = \omega_U = 0.5$), which results in equally-weighted variances of the two households. Alternatively, if HtM HHs had half the steady state income as Ricardian households (with $\sigma = 2$), then $\Phi = 2$ and $\Psi = 2/3$, which means the variance of HtM consumption would have twice the weight as variance of Ricardian consumption.

assets by pursuing the following fiscal rule by setting $\theta'_a = \theta''_a = \beta^{-1} - 1 + \epsilon$, $\theta'_y = \theta''_y = \tau$ (we sometimes assume $\theta'_y = \theta''_y = 0$) and $\theta'_p = \theta''_p = 1$.[13]

3. The **Structural Surplus Rule (SSR)** states that the role of government is to minimize the volatility of fiscal instruments. In this case, the government saves revenues in excess of its long-run level and draws down from the SWF when revenues fall below the long-run level. The value of the parameters that accomplish that are $\theta'_a = \theta''_a = \beta^{-1} - 1 + \epsilon$, $\theta'_y = \theta''_y = 0$ and $\theta'_p = \theta''_p = 0$.

4. The **Hybrid BBR-CCY** responds differently to commodity revenues and variations in non-resource GDP. Specifically the BBR-CCY *spends* all commodity revenues ($\theta'_p = \theta''_p = 1$), but *smooths* non-resource income ($\theta'_y = -(1-\omega)^{-1}(1-\tau)(1-\omega_y)$, $\theta''_y = -\omega^{-1}(1-\tau)\omega_y$). The response to government assets is unchanged $\theta'_a = \theta''_a = \beta^{-1} - 1 + \epsilon$

5. The **Hybrid SSR-CCY** responds differently to commodity revenues and variations in non-resource GDP. Specifically the SSR-CCY *saves* commodity revenues ($\theta'_p = \theta''_p = 0$), but *smooths* non-resource income ($\theta'_y = -(1-\omega)^{-1}(1-\tau)(1-\omega_y)$, $\theta''_y = -\omega^{-1}(1-\tau)\omega_y$). The response to government assets is unchanged $\theta'_a = \theta''_a = \beta^{-1} - 1 + \epsilon$

6. The **Optimal Simple Rule (OSR)** chooses *all* parameters $\{\theta'_a, \theta'_y, \theta'_p, \theta''_a, \theta''_y, \theta''_p\}$ optimally so that the loss function Equation 3-13 is minimized.

7. The **OSR-Equal** also chooses parameters to minimize the loss function Equation 3-13, with the restriction that the transfers are *untargeted.* This means that $\theta_a = \theta'_a = \theta''_a$, $\theta_y = \theta'_y = \theta''_y$,$\theta_p = \theta'_p = \theta''_p$.

3.3.2 Equilibrium, stability and an irrelevance result

We take a first-order Taylor expansion of the system of equations (3-5)-(3-15) around the steady state and consider an equilibrium driven by the two exogenous shocks: a commodity price shock (ϵ_t^p) and a non-resource GDP shock (ϵ_t^y). The equations of the linear system are presented in section 7.2.1 in the appendix.

[13] $\epsilon > 0$ is required for stability purposes – see Lemma (7.2.2) in the appendix section (7.2.2). In the table 3.2 we set $\theta'_a = \theta''_a = 0.1$ for HtMHH, BBR, SSR, BBR-CCY and SSR-CCY (i) to make sure we are well away from the unstable region and (ii) because that is close to the value in the optimal rules.

Lemma (7.2.2), presented in appendix 7.2, provides two conditions for the existence and uniqueness of an equilibrium in the exogenous-income model. To guarantee a stable path for the SWF, the first condition states that governments must transfer to households no less than 4% (the long-run real interest rate) of the deviations of the value of the SWF from its target size each year (but also no more than 104%). The second condition removes the unit root in the consumption of the Ricardian households by imposing a debt-elastic interest rate spread in the model ($\psi > 0$), as in Schimitt-Groe and Uribe (2003).

Lemma 2, also in Appendix 7.2, presents a version of the well-known Ricardian equivalence result in Barro(1974), adapted to this heterogeneous agent framework. It says that if the government commits to a transfers rule to the hand-to-mouth household that does not depend on the size of the SWF ($\theta''_a = 0$), then the equilibrium path of consumption of the Ricardian household is completely independent of the transfer rule coefficients for the Ricardian HH $\{\theta'_a, \theta'_y, \theta'_p\}$. The reason is that transfers to the Ricardian household do not alter the discounted flow of expected after-tax income of the Ricardian household when $\theta''_a = 0$.

3.3.3 Calibration

General Parameters. Many of the most important parameters are not country specific, and so we calibrate these to international data or take them from the literature. The most important are the persistence of commodity price shocks, which we calibrate to a weighted average of oil and gas prices taken from Borensztein at al. (2013). The overall persistence ($\rho = 0.93$) is a weighted average of oil prices ($\rho = 0.94$) and gas prices ($\rho = 0.89$) as many oil exporters also produce gas.[14] We set $\beta = 0.96$ so that long-run annual real rate of interest is 4%. The coefficient of risk aversion $\sigma = 2$, which is a standard parameter in the literature. We follow Gali et al. (2007) to set $\omega = \omega_y = \omega_U = 0.5$ (50% of the population is HtM). For simplicity we set $B_{ss} = 0$ so that steady-state consumption is equal across households.

We set the benchmark debt-elastic interest spread to be $\psi = 0.01$, which implies that a 100% of GDP increase in debt (or reduction in assets) increases interest rates by 1%. Schimitt-Groe and Uribe (2003) set $\psi = 0.001$ to match volatility of the observed current-account-to-GDP ratio for Canada. Schmitt-Grohe and Uribe (2016) argue that ψ should be set to match the

[14] The actual weights are a combination of country-specific persistence for Trinidad & Tobago (TTO) and Algeria, discussed below.

autocorrelation of the trade-balance-to-GDP ratio, and they estimate $\psi = 1$ for Argentina. We take $\psi = 0.01$ as compromise between these two approaches. In Section 3.5, we use an alternative calibration of the exogenous income model with $\psi = 0.45$ and find broadly similar results.

Country-specific calibration to Algeria and Trinidad & Tobago (TTO). For country-specific parameters, we chose to calibrate to Algeria and Trinidad & Tobago (TTO). This is mostly because Algeria is close to a "typical" oil producer, and TTO is close to a "typical" gas exporter — many countries export both — as measured by the size of resource exports relative to non-resource GDP. Specifically, BJS2013 (Table 1) lists the 2002-07 average export revenues/non-resource GDP for 21 petroleum exporting countries which have petroleum export revenues/non-resource GDP above 10% — running from Sudan (12%) to Saudi Arabia (82%). The cross-country average is 38%, which is fairly close to Algeria's 33% oil revenues and so it might be regarded as "typical". BJS2013 also list five countries with natural gas exports above 10% of non-resource GDP, and TTO's 20% is very close to the 21% average. For the calibration of the model, we also include TTO's oil exports (13% non-resource GDP) which bring total TTO resource exports to 33% non-resource GDP. As such we calibrate $Q = 1/3$ (and $Y_{ss} = 1$, $P_{ss} = 1$) so total natural resource exports are 1/3 of non-resource GDP ($QP/Y = 1/3$) to reflect the relative size of resource exports to non-resource GDP in Algeria and TTO.[15] However, we also chose these two countries based on a desire for geographic diversity (Middle East/North Africa for Algeria, Latin America/Caribbean for TTO), and diversity of country size (TTO has about 1.3m people with Algeria having 40 million). Finally, we also excluded a number of other countries with idiosyncratic features that would make optimal simple rules difficult to calculate, such as large numbers of migrant workers or political instability.[16]

Despite their other differences, in most cases Algeria and TTO have very similar characteristics relevant for the model (in the cases where parameters differ, we usually take the average). We calibrate the tax rate $\tau = 0.15$ as non-resource taxes are 15% of GDP in Trinidad and Tobago (IMF 2014 Article IV) and 16% of GDP for Algeria (IMF 2016 Article IV). Algeria and Trinidad & Tobago also have very similar sized SWFs. The Algerian SWF represents a share of 33% of the Algerian non-resource GDP and Trinidad & Tobago's SWF is about 28% non-resource GDP (data from the SWF Institute), so we

[15]Note that Algeria also exports natural gas, though these are not included in the calibration. In general, the results of the paper are fairly insensitive to the exact size of resource exports around a reasonable baseline.

[16]For example, Gulf Co-operation Countries (GCC) have a high share of migrant workers. This could mean that in times of low oil prices, governments reduce migration — a channel of adjustment not available to other countries — which might affect the optimal fiscal rule.

set $A_{ss} = 0.3$ as an intermediate value. We assume steady state transfers are set to pay out all steady state revenues. The persistence and volatility of non-resource GDP is taken from estimating an AR(1) process on HP-filtered log real per capita GDP (from the World Development Indicators). While it is true that HP filtering removes much of the persistence in log GDP by constructtion, this is necessary given log GDP per capita is close to a random walk, and this is a standard procedure in the literature.

3.4 Main Numerical Results

Table 3.2 summarizes the main results for optimal and classical rules.

HtM Stabilization: smoothing vs insurance In the first column of Table 3.2, the government follows a rule which provides full insurance for the HtM household against non-resource and commodity price shocks. As a result, HtM household consumption remains constant at its steady state level (a welfare loss of zero for that HH), which is the same result we would get if the household had access to state-contingent Arrow-Debreu securities. Given the HtM HHs lack access to financial markets, one might think this policy is similar to government borrowing/saving on behalf of the HtM household as it "fixes" market incompleteness.

However, from an aggregate perspective this rule is very inefficient because it concentrates risk with the Ricardian HH rather than sharing risk across households. In fact, the HtM household full stabilization rule has the worst aggregate welfare performance among the 7 rules considered in this section (a 4.9% of SS consumption welfare loss, *each period*). The full HtM stabilization rule leads to a very large consumption variance for the Ricardian HH (annual standard deviation of 31%). This is the difference between smoothing vs insurance — the Ricardian HH and government have the financial technology to *smooth* out anticipated changes in the time path of income, but persistent commodity price shocks also lead to large changes in the *present value* of future income, which is *uninsurable* for both Ricardian HHs and the government. Note that because $\theta''_a = 0$, the Ricardian Equivalence result (Lemma (7.2.2)) applies, and the government and the Ricardian HH act like one entity (we get the same allocation for *any* feasible combination of $\{\theta'_a, \theta'_y, \theta'_p\}$).

BBR outperforms SSR Columns 2 and 3 of Table 3.2 show an interesting and unexpected result: the BBR outperforms SSR. From the results of the literature summarized in the introduction, one would expect the opposite. Moreover, the difference in welfare is quite sizable: the welfare loss is

13% lower under BBR than under SSR. As argued in Section 3.2, when shocks are persistent, the SSR *over-saves* windfall revenues (and *overspends* when commodity prices are low), which means that in the short term, consumption of the HtM HH responds less than optimally to changes in commodity prices.

A related problem is that a SSR leads to a very large standard deviation of assets, both public and private. In our model, the main problem this creates is that the rate of return earned on the SWF assets will decrease as the SWF increases in size ("beating the market" is hard for large funds), or alternatively the interest rate increases when the SWF is small (fixed management costs become larger, there become worries about future solvency) — thus reducing the income available for consumption. In the real world, it would also mean the SWF would be exhausted, or the government/agents would eventually reach their debt limit, though with a linear model the occasionally binding constraints are excluded from our analysis.[17]

The importance of persistence The reason the BBR outperforms the SSR is that the oil price shocks are very persistent. This means that current income is very close to permanent income and so a BBR where households just consume current income is close to optimal. In Figure 3.2, we plot the welfare loss of the SSR and BBR against the persistence of the commodity shock. As the persistence of the shock increases, so does its standard deviation (which is equal to $\sigma_p/\sqrt{1-\rho_p^2}$). To isolate the effect of persistence on welfare, σ_p is adjusted as ρ_P increases so as to keep the total SD constant. (In Figure 7.1 in Appendix 7.2 we repeat this exercise with constant σ_p.) In the upper LHS of Figure 3.2 one can see that for $\rho_P < 0.90$ the SSR is preferred — which is benchmark result in the literature. However, for $\rho_P > 0.90$, which is the empirically relevant region for many commodity prices like oil, a BBR is preferred. The change in ranking of BBR and SSR is due entirely to the HtM HH (bottom LHS), who prefers a BBR for $\rho_P > 0.90$ (but SSR for $\rho_P < 0.90$) because, for persistent shocks, the current income is close to the permanent income. One can see that the HtM HH consumption SD is constant for the BBR (as variance of consumption equals that of income, which is constant by construction). Ricardian HHs are indifferent between the two rules because they can smooth income themselves, and so "undo" the effects of a sub-optimal fiscal rule. Note, however, that the Ricardian HH prefers less persistent shocks because they are easier to smooth.

[17]The standard deviation of a and b are relative to non-resource GDP, and are very large. However this large standard deviation comes from high persistence, rather than from large year-to-year variation. In fact, the "error" component of the standard deviation is only around 0.24, but because assets are very persistent, their variance is high. This suggests that it would probably take some years of persistently negative shocks for debt limits to be reached.

Countercyclical Transfers Kumhof and Laxton (2013) find that countercyclical fiscal policy — where a fall in output leads to an increase in transfers — leads to a substantial increase in welfare. Here we add counter-cyclical transfers to both BBR and SSR in Column 4 and 5 of Table 3.2, and find only a small improvement in welfare (by 0.03-0.05).[18] The reason is that shocks to non-resource GDP are relatively small and not very persistent, and so even with a sub-optimal policy they generate little welfare loss. Countercyclical transfers are optimal because the temporary nature of the non-resource shocks means that they should be smoothed by HtM HHs.[19]

Optimal Simple Rules (OSR) The optimal simple rule is shown in Column 6 of Table 3.2, which is the rule that chooses all six parameters $\{\theta'_a, \theta'_y, \theta'_p, \theta''_a, \theta''_y, \theta''_p\}$ to minimize the weighted average consumption variances in the loss function (Equation 3-13). One can see that the coefficients are *very* similar to those in the BBR-CCY rule — especially for HtM HHs where the details of the rule have the most effect. The optimal rule suggests that a 10% increase in oil revenues will lead to a 7% increase in transfers to HtM HHs, and so in a sense is a compromise between a BBR and a SSR. The optimal rule also means that temporary non-resource income shocks are almost completely insured for HtM HHs, and that an increase in SWF assets leads to an increase in transfers to HtM HHs by more than the interest earned on those extra assets. For Ricardian HHs, the coefficients are relatively similar to those of HtM HHs. Figure 7.2 in section 7.2.3 shows how the welfare loss changes as we change the fiscal rule coefficients one at a time around the OSR.

Equal Allocation OSR In Column 7 of Table 3.2, we also calculate the optimal fiscal coefficient assuming that the government cannot target transfers separately at HtM and Ricardian HHs (i.e. $\theta'_a = \theta''_a$, $\theta'_y = \theta''_y$, $\theta'_p = \theta''_p$), so the fiscal authority just has to choose a transfer-based rule for all households $\{\theta_a, \theta_y, \theta_p\}$(without primes). Despite this substantial restriction, the OSR-equal delivers *almost the exact same welfare as the fully OSR.* The reason is that the unrestricted OSR coefficients discussed above are quite similar for HtM and Ricardian HHs. Although Lemma 2 doesn't hold exactly (as $\theta''_a \neq 0$), welfare is generally less sensitive to variation in transfers to Ricardian HHs than to the HtM HHs. The policy implication is that for stabilization purposes, it doesn't matter if the government can target transfers at the HtM HH — they should just set the fiscal rule that is relatively optimal for HtM HHs, and this

[18] That is, commodity revenues are spent/saved according to the BBR/SSR, but shocks to non-resource GDP are smoothed with counter-cyclical transfers.

[19] The relationship between the persistence of the commodity shock and the ranking SSR-CCY vs BBR-CCY are almost identical to that of SSR vs BBR (not reported).

will be close to optimal for the Ricardian HHs as well.[20]

In the top row of Figure 3.3 one can see how the welfare loss changes in the neighborhood of the OSR-Equal rule. Most important, we find that the welfare loss increases substantially as the government saves more commodity revenue (i.e. $\theta_p < 0.68$). For θ_a, the welfare loss increases sharply below the optimal value of $\theta_a = 0.09$, because this causes SWF assets to become highly volatile. As above, the welfare loss increases slowly as the response to non-resource income shocks become more pro-cyclical above the optimum of around $\theta_y = -0.8$.

In the bottom row of Figure 3.3, we show that if $\rho_p = 0$ (commodity revenues are not persistent) general welfare losses are much lower, and even very suboptimal procyclical policies (spending all of resource revenues) generate relatively minor welfare loses of around 0.2% of steady state consumption. This leads us to conclude that the payoffs for making policy mistakes are *asymmetric*: if commodity price shocks are transitory, setting the fiscal rule as if they are permanent only leads to an additional welfare loss of 0.17% of steady state consumption. In contrast, if one sets optimal policy for a transitory shock when shocks are actually permanent, welfare losses are over 1% of steady state consumption (in the baseline calibration).

The optimal rule for shocks of different persistence Figure 3.4 shows how the untargeted optimal rule (OSR-equal) changes as the persistence of the commodity price shock increases. θ_p increases non-linearly with the persistence of the commodity price shocks. When the shock is very persistent, $\rho_p = 0.95$, the optimal rule prescribes that governments should spent 80 cents in the dollar of windfall commodity revenues in times of high prices, but only 8 cents when the shock is purely transitory, $\rho_p = 0$. An implication is that seemingly similar commodities can have quite different optimal fiscal rules. For example, oil is one of the most commodities with the most persistent price shocks, and so the optimal rule involves spending around three quarters of excess oil revenues. However, for gas and copper the slightly less persistent price process involves the government should only spend half of above average commodity revenues. For sugar, Arabica coffee and bananas, the government should spend around a a quarter to a third of average average commodity revenues. See Appendix Table 7.1 for a list of commodities, the persistence of their prices and the implied value of fiscal rule coefficient θ_p.

As in the analytical model of section 3.2, the persistence of the commodity price shocks does not affect θ_a and θ_y. The government should spend

[20]In additional results (not reported) we show that if the government is restricted to equal transfers, the fiscal rule that maximizes the welfare of the HtM HH is almost identical to the fiscal rule that maximizes total welfare.

annually 9% of the assets above the long-run target for the SWF and insure households of most ($\theta_y = -0.79$) of the variation in non-resource income.

3.5 Extension 1: Debt Elastic Interest Spread and Precautionary Savings

The debt-elastic interest spread (ψ) is needed to make the model stationary, but also can provide a reduced-form way for our simple linear model to capture non-linear precautionary savings (as in Jeanne and Sandri (2016)) and/or financial frictions which generate a realistic autocorrelation of the trade balance (Schmitt-Grohe and Uribe 2016). In our baseline simulations above we calibrate $\psi = 0.01$, which suggests that a 100% of GDP increase in SWF assets (debts) leads to a 1% decrease (increase) in interest rates. In general we need a *higher* debt elasticity spread to capture financial frictions or precautionary savings, but let's first consider the effect of a lower value of the debt-elastic interest spread.

A lower debt-elastic interest spread reduces the penalty of deviating from target assets as a share of GDP. As SSRs require large building up and drawing down of assets it also makes SSRs slightly more attractive. We can reduce the debt-elastic interest spread to $\psi = 0.001$ (the value used in Schimitt-Groe and Uribe (2003)), which means that a 100% of non-resource GDP increase in SWF assets (debts) leads to a 0.1% decrease (increase) in interest rates. With $\psi = 0.001$, the optimal policy involves drawing down (or building up) the SWF assets *at half the rate* of the baseline calibration, i.e. the optimal $\theta_a = 0.05$ (vs 0.09 the baseline OSR equal), though optimal θ_p and θ_y are mostly unchanged. BBR is still preferred to SSR, but the welfare loss difference is much smaller (around 1%).

Precautionary Savings. One caveat to the class of linear models considered in the previous sections is that they abstract from precautionary motives to save. In a model with precautionary savings and borrowing constraints, the government has an additional incentive to save in order to stay away from their borrowing limit and avoid drastic cuts in spending when a large shock causes the borrowing constraint to bind. As a result, there is a worry that linear models without precautionary savings might *overestimate* the optimal pro-cyclicality of fiscal policy. However, we show that the linear exogenous-income model of section 3.4 actually does a good job in capturing the main features of JS2016 non-linear model of precautionary savings and, if anything, *underestimates* (rather than overestimates) the optimal pro-cyclicality of fiscal policy.

JS2016 analyze the optimal management of reserves using an intertemporal fully optimal (non-linear) model of an open economy where a representative household consumes non-tradable goods and imported goods. The household can borrow from and lend to the government but does not have access to international financial markets. The government holds reserves (foreign assets equivalent to SWFs) to smooth the household's consumption path of imported goods. The government's problem involves trading off the opportunity cost of holding reserves (or carry cost) vs the risk of costly contractions in imports when negative external shocks cause debt constraints to bind. In the benchmark calibration they find that an optimal simple linear rule (similar to the one in this paper) can deliver more than 90% of the welfare gains from optimal non-linear reserve management. With their calibration to a lower level of shock persistence, the linear rule prescribes that the government should spend 24% of the reserves above the optimal level and 65% of export revenues above the estimated long-run level.

Comparing the exogenous income model with JS2016. In this section, we increase the debt-elastic spread ψ until our model is able to replicate the first-order autocorrelation of net assets/reserves (public+private) in JS2016 with the OSR-Equal fiscal rule with the same calibration. We then check if the linear model of section 3.4 is biased towards more procyclical rules by comparing our rule with that in JS2016, and re-generating Table 3.2 with the higher value of ψ. But before we do this we need to recalibrate the predetermined parameters of our model so that they are in line with JS2016.

Panel A of Table 3.3 shows the calibration of key parameters in JS2016 and the new calibration (aimed to "mimic" JS2016) of the linear exogenous income model of section 3.4.[21] Note that the estimated persistence of the value of exports ($\rho_p = 0.78$) is significantly lower than our baseline estimate for Algeria and Trinidad & Tobago ($\rho_p = 0.93$). We set $\tau = 0$ since there are no taxes in JS2016. We lowered the intertemporal discount factor ($\beta = 0.95$) and steady-state transfers ($Tr_{ss} = 0.33$) to match the long-run interest rate ($r_{ss} = 5.1\%$) and the optimal level of reserves ($A_{ss}/M_{ss} = .22 \sim 2.2$ months of imports) in JS2016, respectively (shown in Table 3.3 Panel B).[22]

[21] JS2016 use annual data from a group of 24 developing countries (sample 1960 to 2014) provided by the World Bank's World Development Indicators to calibrate the path of detrended non-tradable output, value of exports and the country's interest rate.

[22] Under the new calibration (aside from discrepancies generated by the linearity assumption) there are two main differences between the models: the elasticity of substitution between non-tradable and imported goods and the structure of the country's interest rates. In the exogenous income model, we implicitly assume that imported goods and non-tradable goods are perfect substitutes $\eta = \infty$. Second, the country's interest rate path is assumed to follow an AR(1) process in JS2016. In our model, the interest rate depends on the country's net debt, and its persistence and variance are endogenously determined. While relevant variables in JS2016 are expressed in terms of imported goods, this is comparable with our

Table 3.1: Exogenous-Income Model Calibration to Algeria and Trinidad & Tobago

Param.	Value	Description	Algeria	TTO	Target/Source
β	0.96	Discount Factor	-	-	s.s. 4% annual real interest rate
σ	2	Coefficient of risk aversion	-	-	Common value in literature
ω	0.5	HtM HH share	-	-	Galí *et al* (2007)
Y_{ss}	1	SS non-resource GDP.	-	-	Normalization
$P_{ss}Q$	0.33	SS resource GDP	0.33	0.33	resource GDP / non-resource GDP (BJS2013)
A_{ss}	0.3	S.s. SWF	0.33	0.28	SWF / non-resource GDP (SWF institute)
B_{ss}	0	S.s. private assets	-	-	Symmetric s.s. consumption
ψ	0.01	Debt-elasticity of interest spread	-	-	Schmitt-Grohe & Uribe (2003,2016) (see text)
τ	0.15	Income tax rate	0.16	0.15	Tax revenue / non-resource GDP (IMF)
ρ_y	0	Persistence of non-resource income shocks	Insig	0.3*	Estimate of ρ based on HP Filtered data.**
ρ_p	0.93	Persistence of commodity export prices	0.94	0.91	TTO: weighted ave of $\rho_{oil} = 0.94$ and $\rho_{gas} = 0.89$ Algeria: $\rho_{oil} = 0.94$. (from BJS2013)
σ_y	4%	Std. deviation of non-resource income	4%	2%	SD of error from AR(1) reg on filtered data**
σ_p	24%	Std. deviation of resource prices	23%	26%	Average of SD export prices (BJS2013)

Notes: * In part due to oil prices, see Section 3.6 on "correlated shocks" **Regression of $lnX_t = \alpha + \rho lnX_{t-1} + e_t$

Table 3.2: Welfare Performance of Optimal and Classical Rules.

$\psi = 0.01$	(1)	(2)	(3)	(4)	(5)	(6)	(7)
$\rho_P = 0.93$	**HtMHH**	**BBR**	**SSR**	**BBR CCY**	**SSR CCY**	**OSR**	**OSR Equal**
θ'_a	0.10	0.10	0.10	0.10	0.10	0.08	0.09
θ'_y	-0.53	0.15	0.00	-0.85	-0.85	-0.80	-0.77
θ'_p	1.72	1.00	0.00	1.00	0.00	0.70	0.68
θ''_a	0.00	0.10	0.10	0.10	0.10	0.09	0.09
θ''_y	-0.85	0.15	0.00	-0.85	-0.85	-0.80	-0.77
θ''_p	0.00	1.00	0.00	1.00	0.00	0.70	0.68
$sd(\hat{c}')$	0.31	0.16	0.16	0.16	0.16	0.15	0.15
$sd(\hat{c}'')$	0.00	0.16	0.18	0.16	0.18	0.16	0.15
$sd(\tilde{a})$	0.93	0.03	2.46	0.12	2.47	0.87	0.91
$sd(\tilde{b})$	1.85	1.57	2.39	1.57	2.39	0.22	0.17
Loss (% of C_{ss})	**4.87**	**2.58**	**2.96**	**2.54**	**2.93**	**2.38**	**2.38**

Table 3.3: The Role of Precautionary Savings: Calibration and OSR in Linear and Non-Linear Models.

	symbol	Exogenous Income Model (Linear Model of Section 4)	Jeanne and Sandri (2016) (Global Fully Optimal)
Panel A: Calibration			
Coefficient of risk aversion	σ	2	2
Elasticity of substitution (imports/nontradables)	η	∞	1
Annual persistence of real interest rates	ρ_r	0	0.19
Annual std. dev of real interest shocks	σ_r	0	0.13
Annual persistence of commodity revenues	ρ_p	0.78	0.78
Annual std. dev of commodity revenues shocks	σ_p	16%	16%
Tax rate	τ	0%	0%
Intertemporal discount factor	β	0.951	0.99
Long-run growth rate	G	0%	4.6%
Long-run transfers (share of annual non-res GDP)*	Tr_{ss}	33%	–
Debt-elasticity of country premia**	ψ	0.45	–
Panel B: Calibration Targets			
Long-run real interest rate	r_{ss}	5.1%	5.1%
1st order autocorrelation of assets (public + private)	$\rho_{ab_t,ab_{t-1}}$	0.81	0.8
Long-run SWF/Reserves (share of annual imports)	A_{ss}/M_{ss}	22%	22%
Panel C: Untargeted Moments and OSR			
1st order autocorrelation of trade balance-to-output ratio	ρ_{tb}	0.55***	–
Optimal change in transfers given a \$1 increase in SWF	θ_a	**0.33**	**0.24**
Optimal change in transfers given a \$1 increase in commodity revenues	θ_p	**0.67**	**0.65**

*Calibrated to match the estimated optimal long-run target of reserves in JS2016.

**Calibrated to match the 1st order autocorrelation of foreign reserves in JS2016.

*** Our estimates of the 1st autocorrelation of the trade balance-to-output ratio for Algeria and T&T are 0.52 and 0.56 respectively.

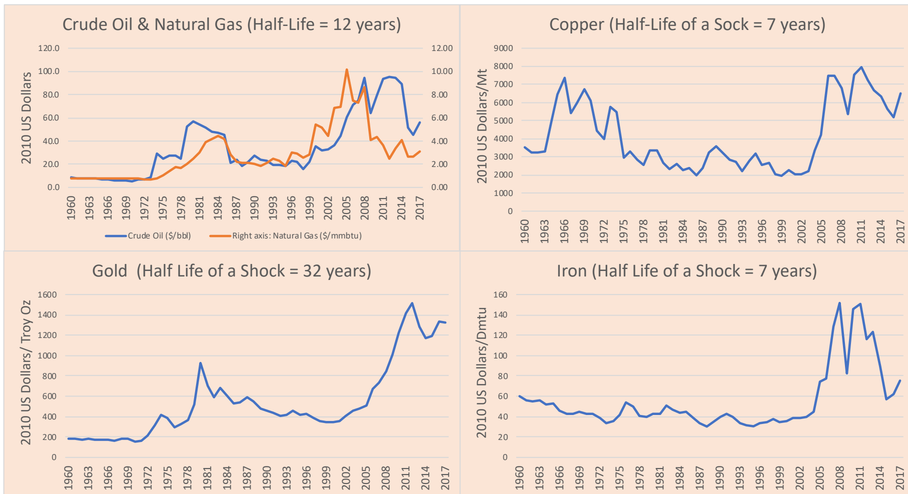


Figure 3.1: **The Persistence of Selected Commodity Prices**: Price Trajectory of Crude Oil, Natural Gas, Copper, Gold and Iron. *Source*: The World Bank's Commodity Price Dataset (The Pink Sheet). *Notes:* The half life of shocks is calculated as $ln(1/2)/ln(\rho)$, where ρ is autoregressive coefficient from an AR(1) regression on each commodity price time series over 1960 to 2017.

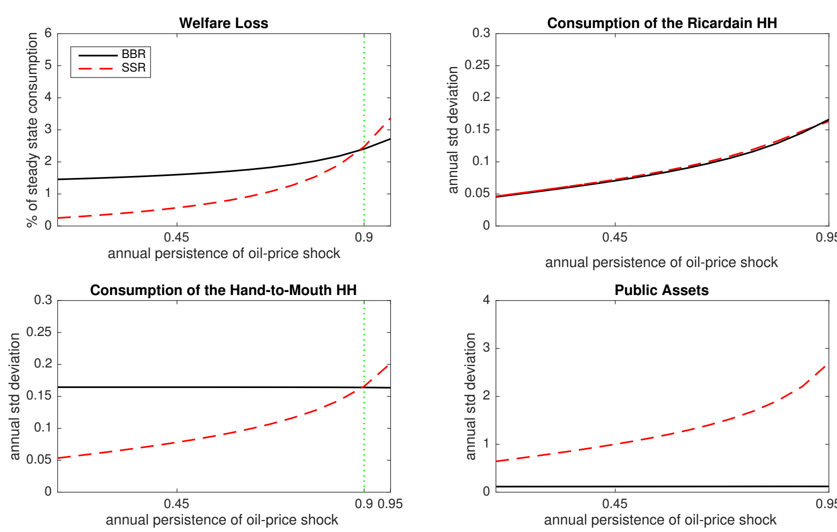


Figure 3.2: **Welfare loss and shock persistence with BBR and SSR:** baseline calibration and constant variance of commodity price shocks

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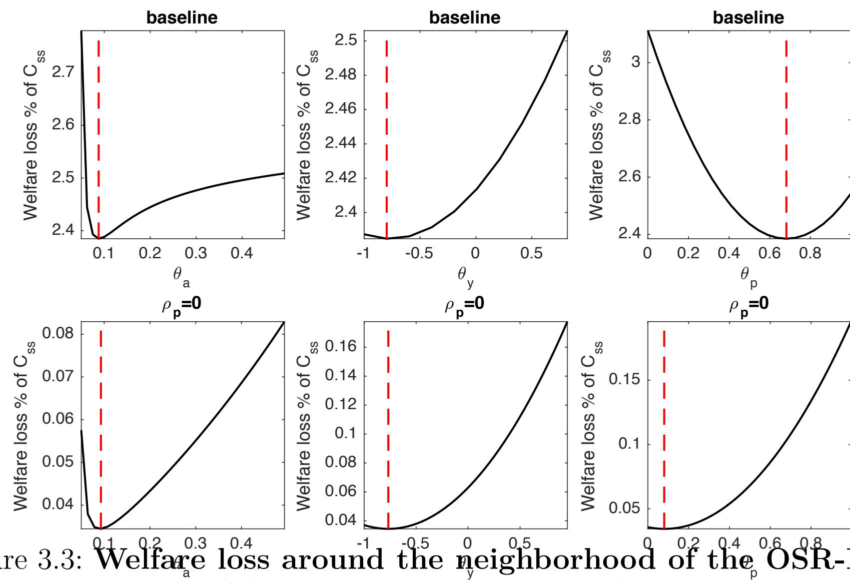

Figure 3.3: **Welfare loss around the neighborhood of the OSR-Equal:** vertical line indicates OSR Equal. Top row: Baseline Calibration. Bottom Row: Transitory commodity price shocks ($\rho_p = 0$).

With this new calibration, we have to set $\psi = 0.45$ to match the first-order autocorrelation of total assets of $\rho_A = 0.8$ in JS2016. As the debt-elastic interest spread increases, assets become less persistent (Figure 3.5 Panel A) because there is a strong incentive to avoid moving assets away from their steady state value. This is reflected in an increasing θ_A (Figure 3.5 Panel B), which means the government increases (decreases) transfers more quickly as assets are above (below) their steady state level. This value of ψ is very large — 45 times larger than the baseline value of 0.01 and implies that a 10% of GDP increase in debt (or reduction in assets) increases interest rates by 4.5ppts.

With $\psi = 0.45$ and OSR-Equal, our model matches very closely the optimal degree of pro-cyclical spending of commodity revenues in JS2016 and the first-order autocorrelation of the trade balance-to-output ratio in the data (see Panel C of Table 3.3). This is remarkable given neither of these were calibration targets. Specifically, our model predicts that the government

model because we assume all goods are perfectly substitutable.

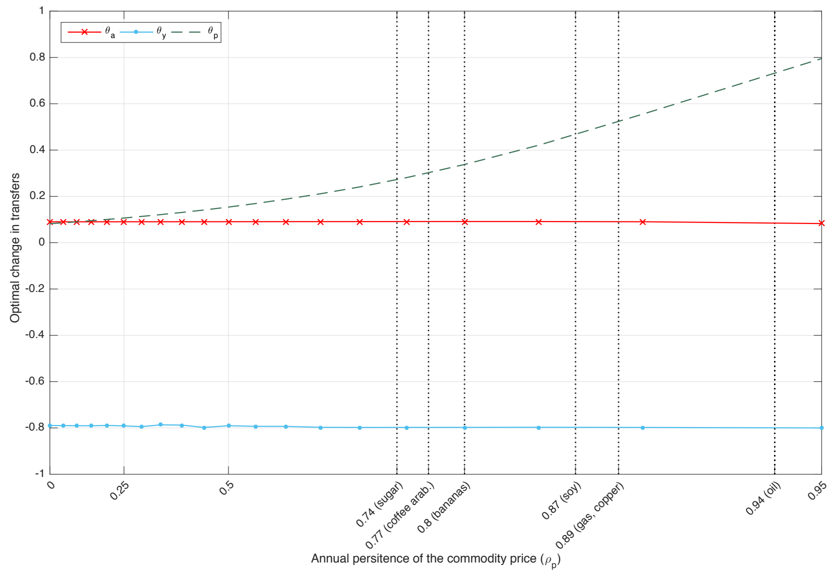

Figure 3.4: **Untargeted Optimal Simple Rule in the Baseline Calibration** *Notes:* change in untargeted government transfers to HHs given an one-dollar increase in (i) commodity revenues (green line, θ_p), (ii) non-resource GDP (blue line θ_y), and (iii) government assets (red line, θ_A) as a function of the persistence of commodity prices. See Appendix Table 7.1 for estimates of persistence of commodities and data sources. To isolate the effect of the shock persistence, we adjust σ_p as ρ_p increases so that the total variance of the commodity price, $\sigma_p/(1-\rho_p)$, is kept constant.

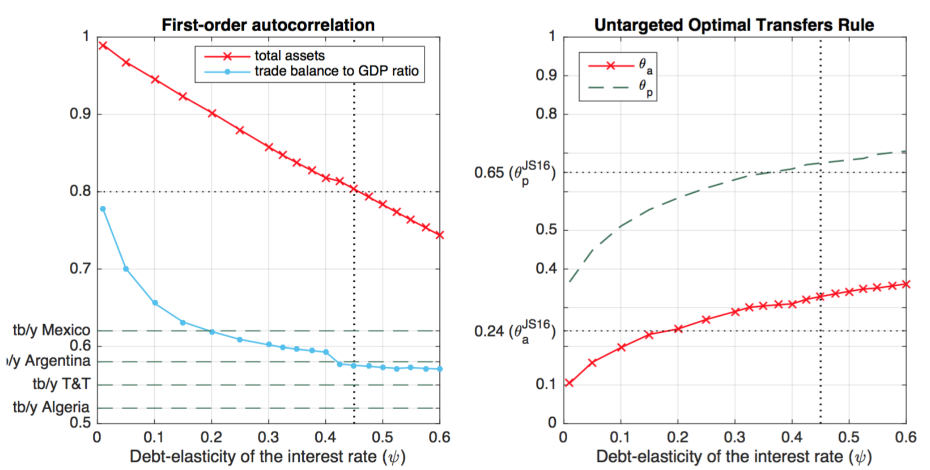

Figure 3.5: **Panel A** (LHS): Autocorrelation of the trade balance-to-output ratio and total assets vs debt elastic interest rate (ψ). **Panel B** (RHS): OSR-Equal coefficients on commodity revenues θ_P and government asset θ_A vs debt-elastic interest rate (ψ) (and comparison with JS2016)

should spend $\theta_p = 0.67$ of export revenues above the long-run level — very close to the value in JS2016 — and 33% of accumulated assets, slightly *more procyclical* than JS2016's estimate of 24%. A higher debt-elastic interest spread increases the degree of pro-cyclicality of θ_P because saving commodity revenues now leads to more unfavorable movements in interest rates. Moreover, the model predicts a first-order autocorrelation of trade balance-to-output ratio of $\rho_{TB} = 0.55$ that is in line with the empirical evidence for many countries — we estimate 0.52 for Algeria, 0.55 for Trinidad and Tobago and Garc ıa-Cicco and Kawamura (2015) estimate 0.58 for Argentina and 0.62 for Mexico (table 3.3 Panel A).

Table 3.4 shows the welfare performance of optimal and classical rules when we use the higher debt-elastic interest spread consistent with JS2016 ($\psi = 0.45$), but applied to the baseline calibration which has higher commodity price persistence ($\rho_p = 0.93$). Column 6 shows that adding a debt-elastic interest rate spread makes the OSR-Equal significantly *more procyclical* than the baseline case with low ψ. Most important, the government spends 77% of above-average commodity revenues here against 68% in the baseline in Table 3.2. Because more higher debt-elastic of interest rates make deviations of public assets from their long-run level more costly, the speed of convergence of public assets towards the target is much faster relative to the baseline case ($\theta_a = 0.31$ vs $\theta_a = 0.09$ in the baseline). Column 5 also shows that the targeted optimal simple rule is more procyclical relative to the baseline calibration and columns 1-4 show that the welfare gains of adopting a BBR (relative to SSR) increase with ψ.

In sum, we find that (i) the linear model of Section 3.4 with a reduced-form financial friction is able to capture much of the dynamics of more complicated non-linear models, and (ii) if anything our baseline simple linear model *slightly understates* the optimal pro-cyclicality of fiscal rules.

3.6 Extension 2: Spillovers from Commodity Prices to non-resource GDP

In commodity exporting countries, changes in commodity prices often have a large impact on GDP. In both Trinidad & Tobago and Algeria, real GDP per capita rose as oil prices increased in the 1970s and early 1980s, fell during the period of low oil prices from the mid-1980s, and then growth returned as oil prices increased from around 2000 (Figure 3.6).[23] After detrending both series,

[23]Ideally one would like to use non-resource GDP per capita, rather than GDP per capita, which of course includes oil and gas production. As GDP is in real terms, there should not be any mechanical effect of commodity prices on output. Moreover, oil and gas production are known to be fairly inelastic to oil price movements in the short term, given the large

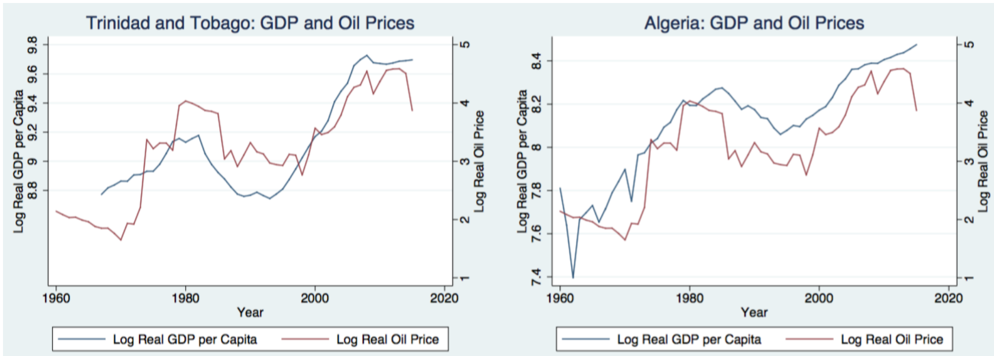

Figure 3.6: **Spillovers from Oil Prices to GDP**

the correlation between log real GDP and log real oil prices is around 0.5 for Trinidad & Tobago and 0.7 for Algeria. A simple regression of log real GDP per capita on log real oil prices and a time trend yields a coefficient of around $\beta_{YP} = 0.2$ on oil prices, and is highly statistically significant. This suggests that in each of Trinidad & Tobago and Algeria, a 1% increase in real oil prices increases the level of GDP per capita by about 0.2%. As both countries are a fairly small share of global oil production, it is unlikely that causality runs from GDP shocks to oil prices.

We implement this spillover from oil prices to GDP in the exogenous income model in the simplest way: by assuming by that non-resource GDP increases by 0.2% when oil prices increase by 1%. We keep pure non-resource shocks as iid ($\rho_Y = 0$), though non-resource GDP inherits much of the persistence of oil prices.[24] Note, however, that because the oil price shocks are much larger, they completely swamp variation in non-resource GDP with our default calibration of σ_y, leading to a 95% correlation between output and oil prices (which is much higher than in the data). To reduce this correlation to around 50% (the value in the data), we also consider an alternative calibration with $\sigma_y = 0.22$.

From the analytical model, we know that as shock persistence increases ($\rho_Y \to 1$) the fiscal response to non-resource GDP shocks should become more pro-cyclical (less countercyclical) (Equation 3-3, $\theta_Y \to \tau$). Indeed, keeping other fiscal rule components fixed at their values in the baseline model from the previous section, an increase in θ_Y tends to improve welfare in the model with spillovers. However, as the increased persistence of non-resource GDP can be traced to commodity price shocks, the optimal policy is able to address

fixed costs of oil and gas production.

[24]We calibrate non-resource GDP shocks to be transitory in Section 3.1 based on HP-filtered data (which removes much of the persistence mechanically). Without HP filtering, GDP per capita is highly persist in both Trinidad & Tobago and Algeria.

its welfare effects through an *increase* in θ_p, without changing θ_Y. That is, when non-resource GDP responds to commodity shocks, it makes the optimal response of fiscal policy to commodity shocks *even more pro-cyclical*. This leaves θ_Y to respond counter-cyclically to the temporary non-resource GDP shocks as before.

To this, Panels B and C of Table 3.5 reports untargeted OSRs and constrained-optimal simple rules — where $\theta_y = -0.5$ is fixed but the other coefficients are chosen optimally— for two alternative calibrations. Columns 6 and 8 show that the fully optimal response to more persistent non-resource GDP shocks is to increase the pro-cyclicality of the response to commodity shocks ($\uparrow \theta_P$), and keep other aspects of the fiscal rule unchanged ($\theta_a = 0.09$ and $\theta_y = -0.8$) relative to the uncorrelated baseline in Table 3.2. Specifically, the optimal response to commodity price shock increases from $\theta_p = 0.68$ in the baseline above to around $\theta_p = 1.05$ when commodity shocks spill over to the non-resource economy.[25]

A caveat is that an optimal counter-cyclical response to non-resource GDP is only important if σ_y is not too small. Otherwise, the household is close to indifferent along a locus of points with a higher θ_Y and lower θ_P which generate the same fiscal response to commodity price shocks. For our default $\beta_{YP} = 0.2$, Figure 7.3 (in Appendix 7.3), shows that the locus of points forms the line $\theta_P = intercept - 0.6 \times \theta_Y$, where an increase in commodity price persistence increases the intercept. In Column 7 of Table 3.5, we show this numerically, by fixing the coefficient $\theta_Y = -0.5$ and choosing the other fiscal rule coefficients optimally (with a low σ_Y). This results in a fall in θ_p from 1.05 to 0.88, but there is no change in the welfare loss (6.09% of steady state consumption). In Column 9 of the same table we repeat the exercise (fixed $\theta_Y = -0.5$ with optimal θ_P and θ_A) with a higher standard deviation of non-resource GDP shocks ($\sigma_y = 0.22$) and find the same fall in θ_p from 1.05 to 0.88, but a higher welfare loss of 0.11% (from 6.31% to 6.42%), as responding counter-cyclically to non-resource GDP shocks is now quantitatively important.

Adding spillovers from commodity prices to non-resource GDP only has a small effect on the performance of classical rules (Panel A, of Table 3.5 with $\sigma_y = 0.04$). Specifically, we still find that balanced budget rules are at least as good as structural surplus rules — in contrast to the literature where the SSRs is strictly preferred — and that BBR-CCY is close to optimal. Here the BBR

[25] Comparing Table 3.5 Columns 6 and 8, note that the OSR-Equal does not depend on σ_y. This is because θ_y responds only to non-resource GDP shocks (with θ_P targeting commodity price shocks and their spillovers), and hence the optimal coefficient only depends on persistence, not volatility.

and SSR generate a very similar welfare loss, whereas in Section 3.4 BBRs are strictly preferred.[26] One can see that the BBR is too pro-cyclical (when $\theta_y = 0$, θ_p should be around 0.6), and as such introducing a countercyclical response to non-resource shocks substantially reduces welfare losses (close to the loss achieved by OSR-Equal). In contrast, SSRs are not pro-cyclical enough and so reducing θ_Y increases welfare losses substantially, leading to a very large welfare gap between between BBR-CCY and SSR-CCY.[27]

3.7 Extension 3: Real Business Cycle model

In the models of Sections 3.4-3.6 we assumed that non-resource GDP is exogenous. In this section, we set up a small open economy (SOE) real business cycle (RBC) model and show that our main results are robust in a model with *endogenous* output in the non-resource sector.

Model Overview. The SOE RBC model maintains the basic structure of the exogenous-income model of the Section 3.4 — two types of households, exogenous commodity income, a debt-elastic interest rate spread and fiscal rules based on transfers to HHs — but introduces endogenous capital accumulation, labor supply and production in the non-resource sector of the economy. Output in this sector is produced by competitive firms by combining labor hired from both types of households and capital rented from the Ricardian HH (HtM HHs don't own capital) using a Cobb-Douglas production technology. Volatility in non-resource GDP is driven by temporary TFP shocks ($\rho_z = 0$). As is standard in the SOE literature, we assume GHH preferences which imply that labor supply is unaffected by variations in household wealth.[28] The capital share of income is $\alpha = 1/3$, and the Frisch elasticity of labor supply is $(\eta - 1)^{-1} = 2.2$ (see Appendix 7.4.1 for further details on the calibration and a description of the RBC model).

There are two main changes in the RBC model relative to the exogenous income model: the endogeneity of labor and the endogeneity of capital. To isolate each of these effects we first add endogenous labor in Panel A of Table 3.6 keeping the capital stock fixed at its steady state level (by assuming very

[26] Specifically, BBRs are marginally preferred to SSRs excluding tax revenues (Column 2, a difference of 0.08%), BBRs generate slightly higher welfare loss than SSRs including tax revenues (Column 1, a gap of 0.05%). We view these differences as small enough that the household is effectively indifferent across the rules, especially given that the welfare loss calculation is a second order approximation (rather than being exact).

[27] The gap is much larger than before because non-resource GDP is much more volatile.

[28] This is an important assumption. With standard (separable) preferences, an increase in commodity prices would cause households to want to consume more leisure, reducing labor supply and GDP. This would be contrary to the evidence presented above that GDP and commodity prices are generally positively correlated for commodity exporters.

high capital adjustment costs). Then in Panel B, we allow both labor and capital to vary. For all the results in Table 3.6 we also fix $\theta_A = 0.1$, which simplifies the exposition but usually has little effect on welfare (see the end of this section for a further discussion).

Endogenous labor supply (and fixed capital) The RBC model with fixed capital has very similar optimal rules as the exogenous income model without spillovers, allowing for some scaling of the coefficient on non-resource GDP. In Table 3.6, Panel A Column 1 we present OSR-Equal, when instead of responding to non-resource GDP, the fiscal rule responds to the fundamental TFP shock Z. One can see that these results are *almost identical* to the baseline exogenous income results in Table 3.2 without spillovers. The optimal rule involves spending 68% of non-resource GDP shocks (the same as in Table 3.2), and transfers are strongly countercyclical with respect to TFP, where the coefficient on TFP is $\theta_Z = -0.9$. With fixed capital, non-resource GDP is perfectly correlated with TFP, and uncorrelated with commodity price shocks. When the countercyclical term is expressed in terms of non-resource GDP, we get an *identical* allocation and value for θ_P, but with $\theta_Y = -0.5$ (relative to -0.8 in Table 3.2).

The difference in the value of θ_Z and θ_Y from the baseline model is due to (i) the distribution of income across households; and, (ii) the fact that endogenous labor supply amplifies TFP shocks. First, note that even though the capital stock is fixed, a fraction α of any change in GDP accrues to capital owners (the Ricardian HHs). Hence a 1% decrease in non-resource GDP reduces the after-tax incomes of the HtM HHs by $(1-\alpha)(1-\tau) = (1-\frac{1}{3})(1-0.15) = 0.57$, which is $\frac{2}{3}$ of the fall in income of 0.85 in the exogenous income model in Table 3.2. This explains why instead of getting a coefficient on $\theta_Y = -0.8$ in Table 3.2, we get a coefficient of $\theta_Y = -0.5$ here (which is $\frac{2}{3}$ as large). Second, a 1% TFP shock will increase labor supply by $(\eta - 1 + \alpha)^{-1}$ (which is equal to 1.26% with our calibration) leading to an increase in GDP of $\eta/(\eta - 1 + \alpha)$ which is 1.85% with our calibration. Hence, the fiscal response to a TFP shock (θ_Z) needs to be 1.85 times as large as the fiscal response to deviations in non-resource GDP (θ_y) to generate the same sized transfer.[29] One can see that as in the main exogenous income model, the BBR is preferred to the SSR, and both rules improve marginally when they respond counter-cyclically to non-resource GDP/TFP shocks (Table 3.6 Panel A, Columns 3-6).

Variable Capital and Endogenous Correlation Between Non-Resource GDP and Commodity Prices. When capital is allowed to

[29] As such, the coefficients on the HtM HH rule (which perfectly smooths HtM consumption) are $\theta_y = -0.57$ and $\theta_Z = -1.05$ respectively (1.85 times as large).

Table 3.4: Fiscal Rule with ψ to capture precautionary savings.

$\psi = 0.45$ $\rho_P = 0.93$	(1) **BBR**	(2) **SSR**	(3) **BBR CCY**	(4) **SSR CCY**	(5) **OSR**	(6) **OSR Equal**
θ'_a	0.10	0.10	0.10	0.10	0.46	0.31
θ'_y	0.15	0.00	-0.85	-0.85	-0.60	-0.60
θ'_p	1.00	0.00	1.00	0.00	0.61	0.77
θ''_a	0.10	0.10	0.10	0.10	0.26	0.31
θ''_y	0.15	0.00	-0.85	-0.85	-0.60	-0.60
θ''_p	1.00	0.00	1.00	0.00	0.74	0.77
$sd(\hat{c}')$	0.16	0.22	0.16	0.22	0.16	0.16
$sd(\hat{c}'')$	0.16	0.18	0.16	0.18	0.15	0.16
$sd(\tilde{a})$	0.07	2.37	0.12	2.37	0.19	0.15
$sd(\tilde{b})$	0.20	4.61	0.25	4.61	0.27	0.20
Loss (% of C_{ss})	2.54	4.02	2.51	4.01	2.48	2.48

Table 3.5: Welfare Properties of Optimal Rules - Spillovers from Commodity Prices to Non-Resource GDP.

$\beta_{YP} = 0.2$	**(A) Classical Rules** ($\sigma_y = 0.04$)					**(B) OSR-Equal** ($\sigma_y = 0.04$)		**(C) OSR-Equal** ($\sigma_y = 0.22$)	
$\psi = 0.01$	(1) BBR	(2) BBR#	(3) SSR	(4) BBR-CCY	(5) SSR-CCY	(6) Optimal	(7) θ_y =-0.5 fixed	(8) Optimal	(9) θ_y =-0.5 fixed
θ_a	0.10	0.10	0.10	0.10	0.10	0.09	0.09	0.09	0.09
θ_y	0.15	0	0.00	-0.85	-0.85	-0.80	-0.50	-0.77	-0.50
θ_p	1.00	1.00	0.00	1.00	0.00	1.06	0.88	1.04	0.88
$sd(\hat{c}')$	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25
$sd(\hat{c}'')$	0.26	0.26	0.26	0.25	0.29	0.25	0.25	0.25	0.25
$sd(\tilde{a})$	0.05	0.19	2.67	1.46	3.94	1.45	1.44	1.59	1.52
$sd(\tilde{b})$	2.51	2.15	1.81	0.14	3.82	0.28	0.28	0.33	0.31
Loss	6.54	6.41	6.49	6.10	7.49	6.09	6.09	6.31	6.42
ρ_{yp}			0.96			0.96		0.50	
$\rho_{y_t,y_{t-1}}$			0.85			0.85		0.23	

Balance Budget Rule which doesn't respond to tax revenues. Fiscal rule coefficients are the same for both households.

Table 3.6: Welfare Properties of Optimal and Simple Rules - RBC Model

$\psi = 0.01$, $\rho_P = 0.93$	**Panel A. Fixed Capital (variable labor supply)** ($\theta_a = 0.1$ fixed) (1) OSR EQ	(2) OSR EQ	(3) BBR	(4) SSR	(5) BBR-CCY	(6) SSR-CCY
θ_Z	-0.9	-	0		$\theta_Z = -1.05$ or $\theta_y = -0.57$	
θ_y	-	-0.5	0		$\theta_Z = -1.05$ or $\theta_y = -0.57$	
θ_p	0.68	0.68	1	0	1	0
$sd(\hat{c}')$	0.170	0.170	0.172	0.177	0.172	0.177
$sd(\hat{c}'')$	0.195	0.195	0.205	0.233	0.204	0.232
$sd(\hat{l})$	0.016	0.016	0.016	0.016	0.016	0.016
$sd(\tilde{a})$	0.84	0.84	0.032	2.468	0.058	2.469
$sd(\tilde{b})$	0.20	0.20	1.395	2.657	1.393	2.657
$sd(k)$	0	0	0	0	0	0
Loss (% of C_{ss})	5.72	5.72	6.105	7.342	6.093	7.330
$\rho_{y_t,y_{t-1}}$	0	0	0	0	0	0
$\rho_{y,p}$	0	0	0	0	0	0

$\psi = 0.01$, $\rho_P = 0.93$	**Panel B. Variable Capital & Labor Supply** ($\theta_a = 0.1$ fixed) (1) OSR EQ	(2) OSR EQ	(3) BBR	(4) SSR	(5) BBR-CCZ	(6) SSR-CCZ
θ_Z	-0.83	-	0		-1.03	-1.03
θ_y	-	-0.05	0		-	-
θ_p	1.01	1.01	1	0	1	0
$sd(\hat{c}')$	0.19	0.191	0.19	0.20	0.19	0.20
$sd(\hat{c}'')$	0.23	0.23	0.22	0.31	0.22	0.31
$sd(\hat{l})$	0.05	0.05	0.05	0.08	0.05	0.078
$sd(\tilde{a})$	0.06	0.10	0.11	2.66	0.12	2.66
$sd(\tilde{b})$	1.15	1.07	1.07	3.24	1.07	3.24
$sd(k)$	0.20	0.20	0.20	0.34	0.20	0.34
Loss (% of C_{ss})	7.60	7.61	7.61	11.98	7.60	11.97
$\rho_{y_t,y_{t-1}}$	0.99	0.99	0.99	0.995	0.990	0.995
$\rho_{y,p}$	0.44	0.44	0.45	0.58	0.44	0.58

vary, the RBC model performs similarly to the exogenous income model with spillovers (in Section 3.6), though with a more subtle form of counter-cyclical response to transitory non-resource GDP/TFP shocks. Capital accumulation also amplifies the persistence of the commodity price shock, which makes variation in non-resource GDP highly persistent and means it is optimal to spend *all* of the commodity windfall ($\theta_p = 1$).

As shown in Figure 3.7, the endogenous spillover from commodity price shocks to non-resource GDP is driven by the debt-elasticity of interest rates. A positive shock to commodity prices leads to an increase of aggregate (public+private) assets in general and consequently to a fall in interest rates in the home country (due to debt-elastic interest rates). Lower returns on international bonds provide the Ricardian household with an incentive to invest in physical capital which generates an output boom in the non-resource sector. The correlation is generally higher for SSR than BBR, as there is a greater accumulation of assets, and increases with ψ. In the baseline calibration ($\psi = 0.01$) the correlation between non-resource GDP and commodity prices is $\rho_{y,p} = 0.45$ under BBR and $\rho_{y,p} = 0.58$ under SSR, which is similar to a correlation of 0.5 for Trinidad & Tobago and 0.7 for Algeria in the data (Section 3.6). Figure 7.4 in the Appendix highlights this mechanism by showing the impulse response of key variables to a commodity price shock in the RBC model.

Panel B of Table 3.6 shows that the results of the exogenous income model with spillovers are quantitatively robust in full RBC model with variable capital, conditional on the fiscal rule responding to TFP (rather than non-resource GDP). The first column of Panel B shows that the optimal rule in the RBC model is extremely close to the optimal rule in columns 6 and 8 of Table 3.5: the government should spend all of their commodity revenue ($\theta_p \approx 1$) and respond counter-cyclically to temporary TFP shocks with a coefficient of $\theta_Z \approx -0.8$. This is almost identical to the BBR-CCZ rule, which is preferred to SSR-CCZ (BBR-CCY is also close to optimal in Table 3.5 and is preferred to SSR-CCY). In the full RBC model, the household strongly prefers the BBR to SSR with respect to commodity revenues (the welfare loss is around more than 50% larger with SSR than BBR). This is what we would expect given the high persistence of commodity prices, but Figure 7.5 in the Appendix shows that the cut-off above which the BBR is preferred is much lower in the RBC model ($\rho_p > 0.74$ in the RBC model, relative to $\rho_p > 0.9$ in the exogenous-income model), perhaps due to extra persistence of non-resource GDP through capital accumulation.[30]

[30]One difference between models is that in the exogenous spillovers model, HHs are close

When the fiscal rule is expressed in terms of endogenous non-resource GDP rather than TFP (Table 3.6, Panel B, Column 2), the optimal $\theta_P \approx 1$ is unchanged, but the response to non-resource GDP is almost *acyclical* ($\theta_y = -0.05$), and the welfare loss is slightly larger than for the optimal rule in terms of TFP. Capital accumulation — due to changes in interest rates, and assets and ultimately commodity prices — means that non-resource GDP is no longer as highly correlated with temporary non-resource TFP shocks (the correlation is now around 0.25). In addition, because endogenous capital accumulation makes non-resource GDP is *even more* persistent than commodity prices, the government cannot just increase pro-cyclicality of the response to commodity shocks and then respond counter-cyclically to non-resource GDP (as in the exogenous income with spillovers).[31] This suggests that policymakers need to be careful to respond to the fundamental shocks affecting the economy, rather than noisy proxies like non-resource GDP.

Optimal spending of SWF assets (θ_A) For most of this section, we fixed $\theta_A = 0.1$, which has almost no effect on the results with fixed capital (Panel A of Table 3.6) or when the fiscal rule responds to TFP shocks.[32] However, in the full RBC model (flexible capital) and when the fiscal rule is in terms of deviations in non-resource GDP, the optimal rule suggests the government should spend more than a third of SWF assets each year ($\theta_A = 0.36$), mainly to keep public assets and interest rates near their steady state level (not reported). With less variation in public assets and interest rates, non-resource GDP becomes is a better proxy for transitory TFP shocks, and hence the coefficient on θ_y becomes more countercyclical and similar to the value in the exogenous income model.[33] The optimal response to commodity price shocks remains strongly pro-cyclical $\theta_P \approx 1$.

3.8 Extension 4: Fixed and Flexible Exchange Rate Regimes.

This extension analyzes the interaction between monetary and fiscal policy. To do so we build a Small Open Economy New Keynesian model with price

to indifferent between BBR and SSR (rather than strongly preferring BBR as they do in the RBC model). This could be because in the RBC model, the SSR increases spillovers from commodity prices to non-resource GDP — which then makes the economy more persistent and hence the SSR worse.

[31] This is, even after removing the effect of commodity prices, non-resource GDP is not that highly correlated with TFP shocks in the RBC model with variable capital.

[32] Specifically the optimal θ_A in these cases is $0.09 - 0.13$ and the welfare loss is the same as optimal θ_A to three significant figures.

[33] Although $\theta_y = -0.36$ is still much lower than $\theta_y = -0.8$ in the exogenous income models, recall that much of this is because the HtM HH is shielded from over 40% of variations in non-resource GDP due to capital income (which accrues to Ricardian HHs) as well as the tax system.

stickiness and non-neutral monetary policy, and evaluate the welfare consequences of simple and optimal fiscal rules in economies with fixed and flexible exchange-rate regimes. The New-Keynesian model is similar to the RBC model from (3.7) (without physical capital), augmented to incorporate heterogenous domestic and foreign goods, sticky prices, and non-trivial monetary policy and exchange rate regimes. A detailed description of the model is provided in the appendix section (7.5).

The main insight from the exogenous-income and RBC models goes through in the New Keynesian model with flexible exchange rates. Specifically, in the model calibrated to Algeria, the optimal rule prescribes that transfers should increase by 6.6% when oil revenues increase by 10% and decrease by 2.5% when a transitory 10% productivity shock hits the economy. This is not surprising since flexible exchange rates can replicate very well the main features of RBC models (see Gali and Monacelli (2002)).

However, the OSR depends crucially on the type of exchange-rate regime in place. The optimal rule is countercyclical with respect to oil-price shocks and procyclical with respect to productivity shocks in the model with fixed exchange rates. It suggests that a 10% increase in productivity will lead to a 17% increase in transfers to HtM HHs while a 10% increase in oil revenues will lead to an exact 10% decrease in transfers.

As emphasized above, because some households are not able to borrow or lend for themselves the government has a role to smooth commodity revenues on their behalf. Under flexible exchange rate the fiscal authority can focus on its goals of intertemporal consumption smoothing because the central bank can use monetary policy to stabilize inflation and the output gap. Under fixed exchange rates, the central bank is no longer able to provide price stability. Welfare losses associated with inflation and output gap are larger than the benefits of optimal consumption smoothing over time and the fiscal authority acts to reduce inflation and output gap volatility. This explains the reversal of results in the fixed exchange rate regime. OSR is countercyclical with respect to commodity price shocks because they are inflationary and procyclical with respect to productivity shocks because they are deflationary.

Figure (3.8) shows how the optimal response to commodity-price shocks changes as the persistence of the shocks increase in the New Keynesian model with fixed and flexible exchange rates. The RHS of the figure shows that when the monetary authority pegs the nominal exchange rate, the OSR become more countercyclical as the persistence of the commodity price shock increases. Even though intertemporal consumption smoothing calls for procyclical transfers, price dispersion and labor allocation costs dominate and drive the OSR countercyclical.

The key difference between fixed and flexible exchange rate regimes that generates this reversal in the shape of the OSR is that commodity-price shocks cause inflation instead of deflation. This happens because the terms of trade move too slowly under the fixed exchange rate regime and fails to completely offset the extra demand for domestic goods. The welfare costs of deviating employment and production from their efficient values overcomes the gains of intertemporal consumption smoothing inducing the fiscal authority to cut transfers to hand-to-mouth households in order to reduce demand and inflationary pressures. This result is in line with Gali and Monacelli (2002).

3.9 Conclusions

In this paper, we re-evaluate the result that commodity exporters should save most commodity price windfalls using several simple models where a share of the population is unable to borrow or save, and fiscal policy takes the form of transfers directly to households. Unlike much of the literature, we find that the optimal fiscal rule is surprisingly procyclical, at least with respect to commodity revenues. Specifically we find that the optimal rule involves spending around two-thirds (or even up to 100%) of above-average oil revenues, though only around half of above-average gas or copper revenues (as those commodities have less persistent price shocks). As the rule is symmetric, this also means cuts in transfers of the same size when commodity prices fall. The reason for the relatively pro-cyclical rule is that many commodity price shocks are highly persistent, and this means permanent income (which should drive consumption) is quite similar to current income. In contrast, we find that non-resource shocks (which are less persistent) should generally be smoothed with counter-cyclical transfers, though transitory non-resource income shocks only drive a small proportion of consumption volatility in the model.

Nonetheless, our results are subject to several caveats, with the most important being that we only look at the effect of fiscal rules on consumption volatility.[34] While consumption volatility is an important determinant of the optimal rule in reality, it is not the only one. Chile's original fiscal rule, for example, was largely designed to smooth the volatility of fiscal instruments

[34]Other caveats are on fiscal multipliers/spillovers and non-linearities, though we partly address these concerns in model extensions. Specifically, while our RBC model has endogenous spillovers from commodity price shocks to output, it also has small fiscal multipliers due to the lack of Keynesian channels, which might affect the optimal rule. While our model does not include non-linearities (like debt limits), we show that we can capture much of their impact on optimal rules through a higher debt-elastic interest spread which, if anything, increases the pro-cyclicality of the optimal rule.

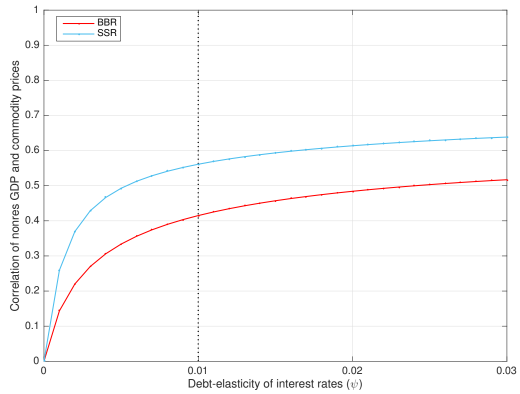

Figure 3.7: **Endogenous Correlation of Non-Resource GDP with Commodity Prices Vs Debt-Elasticity of the Interest-Rates Spread.**

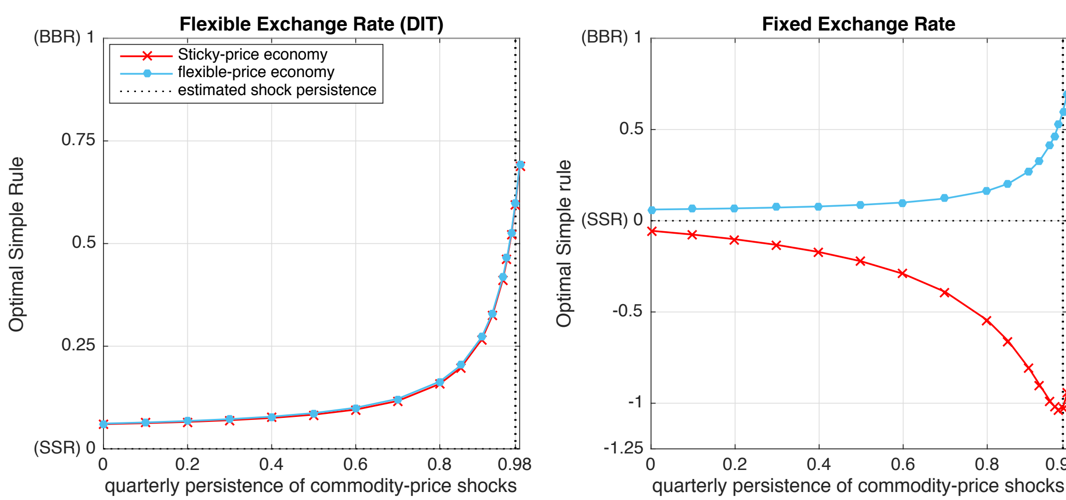

Figure 3.8: **The Optimal Simple Rule in the New-Keynesian Model:** Fexible and Fixed Exchange Rate Regime.

rather than consumption, presumably because fiscal volatility interrupted the efficient operation of government. If public investment is funded from commodity revenues, then balanced-budget-type rules will lead to inefficient volatility in public investment: a fall in commodity revenues will leave half-finished dams, roads and bridges. There might also be important political economy consequences of fiscal volatility — for example it is generally more politically chalenging to cut spending than to increase it, meaning that variable commodity revenues can lead to a "ratcheting" up of spending that quickly becomes unsustainable. While our model (and results) include none of these forces, they do clarify that if structural surplus fiscal rules are optimal, they should be justified along those lines rather than in order to smooth consumption of liquidity constrained (HtM) HHs.

4
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5
Appendix for Chapter 1

5.1
Endowment Economy: Linearized Equations

The endowment-economy model contains 9 endogenous variables, $\{\hat{c}_t, \hat{q}_t, \tilde{m}_t, \tilde{d}_t, \tilde{b}_t^{cb}, \tilde{nw}_t, \tilde{ni}_t, \tilde{b}_t^{l,cb}, \hat{i}_t^m\}$, one exognoeus variable $\hat{y}_t$, and one control variable, i_t. Consider the set of equations,[1]

$$\hat{c}_t = \hat{y}_t \tag{5-1}$$

$$\hat{c}_t = \hat{c}_{t+1|t} - \sigma^{-1}[i_t - (\hat{p}_{t+1|t} - \hat{p}_t) - \rho] \tag{5-2}$$

$$\hat{q}_t = \beta(1-\delta_b)\hat{q}_{t+1|t} - (i_t - \rho) \tag{5-3}$$

$$m_*\tilde{m}_t \begin{cases} = \hat{p}_t + \hat{y}_t & \text{if } i_t - i_t^m > 0 \\ \geq m_*\delta_m & \text{if } i_t - i_t^m = 0 \end{cases} \tag{5-4}$$

$$\tilde{d}_t = \begin{cases} \tilde{ni}_t & \text{if } \tilde{ni}_t \geq -(\xi + d_*) \\ -d_* & \text{otherwise} \end{cases} \tag{5-5}$$

$$\tilde{d}_t = max(\tilde{ni}_t, -(\phi + d_*)) \tag{5-6}$$

$$\tilde{m}_t = \tilde{b}_t^{cb} + \tilde{b}_t^{cb,l} - \tilde{nw}_t \tag{5-7}$$

$$\tilde{nw}_t = \tilde{nw}_{t-1} + \tilde{ni}_t - \tilde{d}_t \tag{5-8}$$

$$\tilde{ni}_t = $$
$$\rho\left(\tilde{b}_{t-1}^{cb} + \tilde{b}_{t-1}^{l,cb}\right) + \rho(b_*^{cb,l} + b_*^{cb})\hat{i}_{t-1} - \rho\tilde{m}_{t-1} - \rho m_*\hat{i}_{t-1}^m + b_*^l\left(\hat{q}_t - (1+\rho-\delta_b)\hat{q}_{t-1}\right) \tag{5-9}$$

$$\hat{i}_t^m = \hat{i}_t \tag{5-10}$$

$$\tilde{b}_{t-1}^{l,cb} = 0 \tag{5-11}$$

Together with the non-linear constraints,

$$\tilde{nw}_t \geq -(\phi + nw_*) \tag{5-12}$$

$$i_t \geq 0 \tag{5-13}$$

[1]Where $\hat{x}_t$ is the log-deviation of variable X around its zero-inflation steady state, i_t is the nominal interest rate ($log(1+i_t)$) and $\rho \equiv log(\beta^{-1})$. The Appendix provides a detailed derivation of the zero-inflation steady and log-linearized equations

Note that assuming $\xi = \phi$ implies that $\tilde{d}_t = \tilde{ni}_t$, $\tilde{nw}_t = 0$, and $\tilde{m}_t = \tilde{b}_t^{cb} \geq m_*$. Moreover, since $m_* = b_*^{cb,l} + b_*^{cb} \Rightarrow \tilde{ni}_t = b_*^l\left(\hat{q}_t - (1+\rho-\delta_b)\hat{q}_{t-1}\right)$ and hence the system reduces to three equations,

$$\begin{aligned}
\tilde{ni}_t &= b_*^l\left(\hat{q}_t - (1+\rho-\delta_b)\hat{q}_{t-1}\right)\\
\hat{y}_t &= \hat{y}_{t+1|t} - \sigma^{-1}[i_t - (\hat{p}_{t+1|t} - \hat{p}_t) - \rho]\\
\hat{q}_t &= \beta(1-\delta_b)\hat{q}_{t+1|t} - (i_t - \rho)
\end{aligned}$$

5.2
Proof of Lemmas, Propositions, and Theorems.

Lemma 2. *Assume A1, $\phi = \xi$ and $N \to \infty$. The equilibrium under discretionary monetary policy is characterized by $\{\hat{p}_t(s_i), i_t(s_i), \hat{q}_t(s_i)\} = \{0, \rho, 0\}$ for $t \geq 3$ and $s^i = \{s^l, s^h\}$.*

Prova. Since the policy plan $i_t(si) = \rho$ for all $t \geq 3$ and $i \in \{h, l\}$ yields the lowest loss possible, $L_3 = 0$, it will be the equilibirum under discretion unless the solvency constraint prevents that allocation. We show that this is not the case if $\frac{\phi}{b_*^l(1+\rho-\delta_b)} \geq \rho$,

$$\begin{aligned}
\tilde{nw}_t &= b_*^l(\hat{q}_3 - (1+\rho-\delta_b)\hat{q}_2)\\
&= -b_*^l(1+\rho-\delta_b)\hat{q}_2\\
&= -b_*^l(1+\rho-\delta_b)\left(\left(\frac{\beta}{1-\beta(1-\delta_b)}\right)\hat{q}_{3|2} - (i_2 - \rho)\right)\\
&= b_*^l(1+\rho-\delta_b)\left(i_2 - \rho\right)\\
&\geq= -b_*^l(1+\rho-\delta_b)\rho\\
&\geq= -\phi \qquad \text{if } \frac{\phi}{b_*^l(1+\rho-\delta_b)} \geq \rho
\end{aligned}$$

■

Lemma 3. *Assume A1, $\phi = \xi$ and $N \to \infty$. The equilibrium under discretion in the high-income state of period 2 is characterized by*

$$\{\hat{p}_2(\hat{q}_1, s_l), i_2(\hat{q}_1, s_l), \hat{q}_2(\hat{q}_1, s_l)\} = \{\rho + \sigma\underline{y}, 0, \rho\}$$

Prova. It is easy to see that $\partial_i L_2(\hat{q}_1, s_l) > 0$ when $i_2(\hat{q}_1, s_l) = 0$. It follows that

$i_2(\hat{q}_1, s_l) = 0$ is in fact optimal unless the solvency constraint prevents that policy choice. However, note that if $i_2(\hat{q}_1, s_l) = 0$,

$$\tilde{nw}_2 = b^l_* \left(\hat{q}^l_2 - (1+\rho-\delta_b)\hat{q}_1\right) = b^l_* \left(\rho - (1+\rho-\delta_b)\hat{q}_1\right) \geq -\phi$$
$$\text{if } \hat{q}_1 \leq \tilde{\phi}_b + \frac{\rho}{(1+\rho-\delta_b)}$$

Proposition 1 shows that $\hat{q}_1 \leq \tilde{\phi}_b + \frac{\rho}{(1+\rho-\delta_b)}$.

■

Lemma 3 *Assume A1, $\phi = \xi$ and $N \to \infty$. The equilibrium under discretionary monetary policy in the high-income state of the second period is characterized by*

$$i_2(\hat{q}_1, s_h) = \begin{cases} \rho & \text{if} \quad \hat{q}_1 \leq \tilde{\phi}_b \\ \rho - (1+\rho-\delta_b)(\hat{q}_1 - \tilde{\phi}_b) & \text{if} \quad \tilde{\phi}_b < \hat{q}_1 \leq \tilde{\phi}_b + \frac{\rho}{1+\rho-\delta_b} \\ 0 & \text{if} \quad \tilde{\phi}_b + \frac{\rho}{1+\rho-\delta_b} < \hat{q}_1 \end{cases}$$

$$\hat{p}_2(q_1, s_h) = \hat{q}_2(q_1, s_h) = \rho - i_2(q_1, s_h)$$

$$\tilde{nw}_2(q_1, s_h) = \begin{cases} -\left(\frac{\phi}{\tilde{\phi}_b}\right)\hat{q}_1 & \text{if} \quad \hat{q}_1 \leq \tilde{\phi}_b \\ -\phi & \text{if} \quad \tilde{\phi}_b < \hat{q}_1 \leq \tilde{\phi}_b + \frac{\rho}{1+\rho-\delta_b} \end{cases}$$

Prova. Note that $\hat{y}_{3|2} = \hat{y}_2(\hat{q}_1, s^h) = y_*$ and $\hat{p}_{3|2} = 0$ imply that $\hat{p}_2(\hat{q}_1, s^h) = \rho - i_2(\hat{q}_1, s^h)$. Hence, $i_2(\hat{q}_1, s^h) = \rho$ unless the solvency constraint prevents the central bank from implementing that policy. Since $\hat{q}_{3|2} = 0 \Rightarrow \hat{q}_2(\hat{q}_1, s^h) = \rho - i_2(\hat{q}_1, s^h)$. Plugging that last relation in the central bank's net income yields $\tilde{nw}_2(\hat{q}_1, s^h) = b^l_* \left(\rho - i_2(\hat{q}_1, s^h) - (1+\rho-\delta_b)\hat{q}_1\right)$ and hence $\tilde{nw}_2(\hat{q}_1, s^h) \geq -\phi$ if and only if,

$$\rho - i_2(\hat{q}_1, s^h) - (1+\rho-\delta_b)\hat{q}_1 \geq -\frac{\phi}{b^l_*}$$

and hence,

$$i_2(\hat{q}_1, s_h) = \begin{cases} \rho & \text{if} \quad \hat{q}_1 \leq \tilde{\phi}_b \\ \rho - (1+\rho-\delta_b)(\hat{q}_1 - \tilde{\phi}_b) & \text{if} \quad \tilde{\phi}_b < \hat{q}_1 \leq \tilde{\phi}_b + \frac{\rho}{1+\rho-\delta_b} \\ 0 & \text{if} \quad \tilde{\phi}_b + \frac{\rho}{1+\rho-\delta_b} < \hat{q}_1 \end{cases}$$

■

Proposition 1 *Assume A1, $\phi = \xi$ and $N \to \infty$. The equilibrium under discretionary monetary policy in the first period is characterized by*

$$
\begin{aligned}
&i_1 = 0\\
&\hat{q}_1 = \rho\left(\frac{(1+\mu\theta_q)}{1-(1-\mu)(1+\rho-\delta_b)\theta_q}\right) - \left(\frac{(1-\mu)(1+\rho-\delta_b)\theta_q}{1-(1-\mu)(1+\rho-\delta_b)\theta_q}\right)\tilde{\phi}_b\\
&\hat{p}_1 = \underbrace{\rho-\sigma\bar{y}+\mu(\rho+\sigma\underline{y})}_{\text{baseline discretion}} + \underbrace{(1-\mu)\left(\frac{(1+\rho-\delta_b)}{1-(1-\mu)(1+\rho-\delta_b)\theta_q}\right)\left(\rho(1+\mu\theta_q)-\tilde{\phi}_b\right)}_{\text{QE effect}}
\end{aligned}
$$

Prova. Begin assuming $i_1 = 0$ and $\tilde{\phi}_b < \hat{q}_1 \leq \tilde{\phi}_b + \frac{\rho}{1+\rho-\delta_b}$. We will solve the problem relying on these 2 assumptions and then we will check under which conditions they hold in equilibrium. Using (1-18) and (1-19) to calculate expectations given $\hat{q}_1$ yields

$$
\begin{aligned}
\hat{p}_{2|1}(q_1) &= \mu(\rho+\underline{y}) + (1-\mu)(1+\rho-\delta_b)(\hat{q}_1-\tilde{\phi}_b)\\
\hat{q}_{2|1}(q_1) &= \mu\rho + (1-\mu)(1+\rho-\delta_b)(\hat{q}_1-\tilde{\phi}_b)
\end{aligned}
$$

We can now sove the fixed point, $\hat{q}_1 = \theta_q\hat{q}_{2|1}(q_1)+\rho$, and find $\hat{q}_1$,

$$
\begin{aligned}
\hat{q}_1 &= \theta_q\left(\mu\rho + (1-\mu)(1+\rho-\delta_b)(\hat{q}_1-\tilde{\phi}_b)\right) + \rho\\
&= \rho\left(\frac{(1+\mu\theta_q)}{1-(1-\mu)(1+\rho-\delta_b)\theta_q}\right) - \left(\frac{(1-\mu)(1+\rho-\delta_b)\theta_q}{1-(1-\mu)(1+\rho-\delta_b)\theta_q}\right)\tilde{\phi}_b
\end{aligned}
$$

and hence,

$$
\hat{p}_1 = \rho-\sigma\bar{y}+\mu(\rho+\sigma\underline{y}) + (1-\mu)\left(\frac{(1+\rho-\delta_b)}{1-(1-\mu)(1+\rho-\delta_b)\theta_q}\right)\left(\rho(1+\mu\theta_q)-\tilde{\phi}_b\right)
$$

and

$$
n\tilde{w}_1 = \left[\rho\left(\frac{(1+\mu\theta_q)}{1-(1-\mu)(1+\rho-\delta_b)\theta_q}\right) - \left(\frac{(1-\mu)(1+\rho-\delta_b)\theta_q}{1-(1-\mu)(1+\rho-\delta_b)\theta_q}\right)\tilde{\phi}_b\right] b^l_*
$$

For the trio $\{\hat{p}_1, \hat{q}_1, n\tilde{w}_1\}$ be part of an equilibrium they must obey 4 conditions: (i) $\hat{q}_1 > \tilde{\phi}_b$ (ii) $\hat{q}_1 \leq \tilde{\phi}_b + \frac{\rho}{1+\rho-\delta_b}$, (iii) $n\tilde{w}_1 \geq -\phi$ and (iv) $\partial_{i_1} L^*_1 \leq 0$

1. Condition (i) holds if $\tilde{\phi}_b \leq \rho(1+\mu\theta_q)$.
2. Condition (ii) holds if $\tilde{\phi}_b \geq \left(\frac{\rho}{1+\rho-\delta_b}\right)$.

3. Note that condition (iii) holds if condition (i) holds

$$
\begin{aligned}
n\tilde{w}_1 &= \left[\rho\left(\frac{(1+\mu\theta_q)}{1-(1-\mu)(1+\rho-\delta_b)\theta_q}\right) - \left(\frac{(1-\mu)(1+\rho-\delta_b)\theta_q}{1-(1-\mu)(1+\rho-\delta_b)\theta_q}\right)\tilde{\phi}_b\right] b_*^l \\
&= b_*^l \left(\frac{1}{1-(1-\mu)(1+\rho-\delta_b)\theta_q}\right)\left(\rho(1+\mu\theta_q) - (1-\mu)(1+\rho-\delta_b)\theta_q\tilde{\phi}_b\right) \\
&\geq b_*^l \left(\frac{1}{1-(1-\mu)(1+\rho-\delta_b)\theta_q}\right) \\
&(\rho(1+\mu\theta_q) - (1-\mu)(1+\rho-\delta_b)\theta_q\rho(1+\mu\theta_p)), \quad \text{due to } (i) \\
&= b_*^l \left(\frac{1}{1-(1-\mu)\theta_q}\right)\rho(1+\mu\theta_p)\left(1-(1-\mu)(1+\rho-\delta_b)\theta_p\right) \\
&\geq 0 > -\phi
\end{aligned}
$$

4. Condition (iv) holds if $(1+\beta)^{-1}\bar{y} - \mu\underline{y} \geq \rho[\mu + (1+\beta)^{-1} + \theta_q]$.

$$
\begin{aligned}
L_1 &= \frac{1}{2}\left[\hat{p}_1^2 + \beta\hat{p}_{2|1}^2(\hat{q}_1)\right] \\
&= \frac{1}{2}\left[\left(\rho - \bar{y} + \hat{p}_{2|1}(\hat{q}_1)\right)^2 + \beta\hat{p}_{2|1}^2(\hat{q}_1)\right]
\end{aligned}
$$

taking the first derivate with respect to $\hat{i}_1$ yields

$$
\left[\left(\rho - \sigma\bar{y} + \hat{p}_{2|1}(\hat{q}_1)\right) + \beta\hat{p}_{2|1}(\hat{q}_1)\right]\underbrace{\partial_{i_1}\hat{q}_1\partial_q\hat{p}_{2|1}}_{\leq 0} \geq 0 \Rightarrow \hat{p}_{2|1}(\hat{q}_1) \leq \frac{\sigma\bar{y} - \rho}{1+\beta}
$$

and hence

$$
\begin{aligned}
&\hat{p}_{2|1}(\hat{q}_1) = \mu(\rho + \sigma\underline{y}) + (1-\mu)(1+\rho-\delta_b)(\hat{q}_1^* - \tilde{\phi}_b) \leq \frac{\sigma\bar{y} - \rho}{1+\beta} \Rightarrow \\
&\Rightarrow \hat{q}_1^* \leq \tilde{\phi}_b + \frac{1}{(1-\mu)(1+\rho-\delta_b)}\left[\frac{\sigma\bar{y} - \rho}{1+\beta} - \mu(\rho + \sigma\underline{y})\right]
\end{aligned}
$$

Note, however, that the last expression will hold if condition (ii) holds and the expected fall in income is large enough: $\bar{y} - (1+\beta)\mu\underline{y} \geq \rho\sigma^{-1}(2+\beta)$

■

6
Appendix for Chapter 2

6.1
Data Appendix

6.1.1
Income and Expenses of the Federal Reserve Bank.

- **Total Income**. Board of Governors of the Federal Reserve System. Annual Report, Statistical Tables: Income and Expenses of the Federal Reserve Banks, by Bank. Interest Income/Total current income, Annual data collected year-by-year 1970-2015.
  https://www.federalreserve.gov/publications/annual-report.htm

  - **Interest Income from Treasury securities**. Board of Governors of the Federal Reserve System. Annual Report, Statistical Tables: Income and Expenses of the Federal Reserve Banks, by Bank. Interest Income/Treasury securities, Annual data collected year-by-year 2005-2015. https://www.federalreserve.gov/publications/annual-report.htm
  - **Interest Income from Federal agency and government-sponsored enterprise mortgage-backed securities**. Board of Governors of the Federal Reserve System. Annual Report, Statistical Tables: Income and Expenses of the Federal Reserve Banks, by Bank. Interest Income/Federal agency and government-sponsored enterprise mortgage-backed securities, Annual data collected year-by-year 2005-2015.
    https://www.federalreserve.gov/publications/annual-report.htm
  - **Other Income.**. Board of Governors of the Federal Reserve System. Annual Report, Statistical Tables: Income and Expenses of the Federal Reserve Banks, by Bank. Interest Income/Other income, Annual data collected year-by-year 2005-2015.
    https://www.federalreserve.gov/publications/annual-report.htm

- **Net Expenses**. Board of Governors of the Federal Reserve System. Annual Report, Statistical Tables: Income and Expenses of the Federal

Reserve Banks, by Bank. Total current expenses/Net expenses, Annual data collected year-by-year 1970-2015.

- **Interest on Reserves**. Board of Governors of the Federal Reserve System. Annual Report, Statistical Tables: Income and Expenses of the Federal Reserve Banks, by Bank. Current Expenses/Interest on Reserves, Annual data collected year-by-year 2009-2015. https://www.federalreserve.gov/publications/annual-report.htm
- **Other Expenses**. Board of Governors of the Federal Reserve System. Annual Report, Statistical Tables: Income and Expenses of the Federal Reserve Banks, by Bank. All other expenses, Annual data collected year-by-year.
  https://www.federalreserve.gov/publications/annual-report.htm

- **Net Income**. Total Income - Net Expenses (1970-2015).
- **Capital retained at the Fed** = Surplus + Dividends on Capital Stock (1970-2015)

  - **Surplus retained at the Fed**. Board of Governors of the Federal Reserve System. Annual Report, Statistical Tables: Income and Expenses of the Federal Reserve Banks, by Bank. Distribution of comprehensive income/Transferred to/from surplus and change in accumulated other comprehensive income, Annual data collected year-by-year 1970-2015.
    https://www.federalreserve.gov/publications/annual-report.htm
  - **Dividends on Capital Stock**. Board of Governors of the Federal Reserve System. Annual Report, Statistical Tables: Income and Expenses of the Federal Reserve Banks, by Bank. Distribution of comprehensive income/Dividends on capital stock, Annual data collected year-by-year 1970-2015.
    https://www.federalreserve.gov/publications/annual-report.htm

- **Remittances to the Treasury**. Board of Governors of the Federal Reserve System. Annual Report, Statistical Tables: Income and Expenses of the Federal Reserve Banks, by Bank. Distribution of comprehensive income/Interest on Federal Reserve notes expense remitted to Treasury, Annual data collected year-by-year 1970-2015.
  https://www.federalreserve.gov/publications/annual-report.htm

6.1.2 Assets, Liabilities and Net Worth of the Federal Reserve Bank.

- **Total Assets.** U.S. Treasury securities held by the Federal Reserve: All Maturities + Mortgage-backed securities held by the Federal Reserve: All Maturities.
  - Board of Governors of the Federal Reserve System (US), U.S. Treasury securities held by the Federal Reserve: All Maturities [TREAST], retrieved from FRED, Federal Reserve Bank of St. Louis; https://fred.stlouisfed.org/series/TREAST.
  - Board of Governors of the Federal Reserve System (US), Mortgage-backed securities held by the Federal Reserve: All Maturities [MBST], retrieved from FRED, Federal Reserve Bank of St. Louis; https://fred.stlouisfed.org/series/MBST.

- **Total Liabilities.** Board of Governors of the Federal Reserve System (US), Monetary Base; Total [BOGMBASEW], retrieved from FRED, Federal Reserve Bank of St. Louis; https://fred.stlouisfed.org/series/BOGMBASEW.

- **Net Worth.** Board of Governors of the Federal Reserve System (US), Capital: Total Capital [WCTCL], retrieved from FRED, Federal Reserve Bank of St. Louis;
  https://fred.stlouisfed.org/series/WCTCL.

6.1.3 U.S. Treasury Yields and IOER

- **Yields** Board of Governors of the Federal Reserve System (US), 10-Year Treasury Constant Maturity Rate [GS10], retrieved from FRED, Federal Reserve Bank of St. Louis; https://fred.stlouisfed.org/series/GS10. *Note: same source for yields on 1, 2, 5, 10, 20 and 30 year Treasury securities.*
- **IOER** Board of Governors of the Federal Reserve System (US), Interest Rate on Excess Reserves [IOER], retrieved from FRED, Federal Reserve Bank of St. Louis;
  https://fred.stlouisfed.org/series/IOER, March 29, 2018.

6.2
The Quantitative Model Equations

The quantitative model can be divided into two blocks: the standard SW07 DSGE block, and the public sector block. The public sector block contains the central bank and the treasury.

6.2.1
The Standard Smets and Wouters 2007 Block

The SW07 lock of the model is a system of 15 equations, 15 endogenous variables $\{y_t, c_t, inv_t, Q_t^k, k_t^s\}$, $\{k_t, L_t, \xi_t, R_t, \pi_t, inv_t, Z_t, k_t, r_t^k, L_t, \mu_t^p, \pi_t, \mu_t^w, w_t, Z_t, R_t, \xi_t\}$ and seven exogenous shocks $\{\epsilon_t^a, \epsilon_t^b, \epsilon_t^g\}$, $\{\epsilon_t^i, \epsilon_t^p, \epsilon_t^w, \epsilon_t^r\}$, where each shock is assumed to follow an AR(1) process with IID-Normal error term with zero mean, estimated persistence and standard deviation.

List of variables: output (y_t), consumption (c_t), investment (inv_t), value of capital (Q_t^k), capital services (k_t^s), installed capital (k_t), capital utilization rate (U_t), rental rate of capital (r_t^k), hours worked (L_t), price mark-up (μ_t^p), inflation rate (π_t), wage mark-up (μ_t^w), wages (w_t), gross interest rate (R_t), and marginal utility of consumption (ξ_t). *List of shocks:* technology shock (ϵ_t^a), premium-risk shock (ϵ_t^b), autonomous expenditure shock (ϵ_t^g), investment-specific technology shock (ϵ_t^i), price mark-up (ϵ_t^p), wage mark-up shock (ϵ_t^w), and monetary policy shock (ϵ_t^r).

Lower case variables denote detrended real variables $x_t \equiv X_t/\gamma^t P_t$.

Eq1 (Production Functions): $y_t = \epsilon_t^a (k_t^s)^\alpha (L_t)^{1-\alpha} - \Phi$

Eq2 (Euler Equation 1): $1 = \bar{\beta}\epsilon_t^b \mathbb{E}_t \left[\frac{\xi_{t+1}}{\xi_t} \frac{R_t}{\pi_{t+1}} \right]$

Eq3 (Investment Dynamics): $1 = Q_t^k \epsilon_t^i \left(1 - S(\frac{\gamma inv_t}{inv_{t-1}}) - S'(\frac{\gamma inv_t}{inv_{t-1}})(\frac{\gamma inv_t}{inv_{t-1}}) \right) +$

$+ \bar{\beta}\mathbb{E}_t \left[\frac{\xi_{t+1}}{\xi_t} Q_{t+1}^k \epsilon_{t+1}^i S'(\frac{\gamma inv_{t+1}}{inv_t})(\frac{inv_{t+1}}{inv_t})^2 \right]$

Eq4 (Euler Equation 2): $1 = \bar{\beta}\mathbb{E}_t \left[\frac{\xi_{t+1}}{\xi_t} \frac{r_{t+1}^k Z_{t+1} + (1-\delta) Q_{t+1}^k}{Q_t^k} \right]$

Eq5 (Capital Services): $k_t^s = U_t k_{t-1} \gamma^{-1}$

Eq6 (Capital Dynamics): $k_t = (1-\delta)\gamma^{-1} k_{t-1} + \epsilon_t^i \left[1 - S\left(\frac{\gamma inv_t}{inv_{t-1}} \right) \right] inv_t$

Eq7 (Capital Rental Rate): $r_t^k = a'(U_t)$

Eq8 (Capital-Labor FOC): $k_t^s = \frac{\alpha}{1-\alpha} \frac{w_t}{r_t^\alpha} L_t$

Eq9 (Price Markup): $\mu_t^p = \frac{w_t}{(1-\alpha)\epsilon_t^a (k_t^s)^\alpha (L_t)^{-\alpha}}$

Eq10 (Phillips Curve - Linear): $\pi_t = \pi_1 \pi_{t-1} + \pi_2 \mathbb{E}_t \pi_{t+1} - \pi_3 \mu_t^p + \epsilon_t^p$

Eq11 (Wage Markup): $\mu_t^w = \frac{w_t}{(c_t - \lambda\gamma^{-1} c_{t-1}) L_t^{\sigma_l}}$

Eq12 (Wage Dynamics - Linear):

$\hat{w}_t = w_1 \hat{w}_{t-1} + (1-w_1)(\mathbb{E}_t \hat{w}_{t+1} + \mathbb{E}\hat{\pi}_{t+1}) - w_2 \hat{\pi}_t - w_3 \hat{\pi}_{t-1} - w_4 \hat{\mu}_t^w + \epsilon_t^w$

Eq13 (Agg. Resource Constraint): $c_t + inv_t + y^* \epsilon_t^g + a(U_t) k_{t-1} \gamma^{-1} = y_t$

Eq14 (Taylor Rule):

$$\frac{R_t}{R^*} = \left(\frac{R_{t-1}}{R^*} \right)^{\rho_R} \left[\left(\frac{\pi_t}{\pi^*} \right)^{\phi_1} \left(\frac{y_t}{y_t^n} \right)^{\phi_2} \right]^{1-\rho_R} \left(\frac{y_t / y_{t-1}}{y_t^n / y_{t-1}^n} \right)^{\phi_3} \epsilon_t^r, \quad R_t = 1 + i_t$$

Eq15 (Marginal Utility): $\xi_t = exp\left(\frac{\sigma_c - 1}{1+\sigma_l} L_t(j)^{1+\sigma_l} \right) \left(c_t - \frac{\lambda}{\gamma} c_{t-1} \right)^{-\sigma_c}$

Where y_t^n denotes potential output, defined as the level of output that would prevail under flexible prices and wages in the absence of the two “mark-up” shocks. For more details of the model refer to SW07.

Table 6.1: Summary of Structural Parameters

σ_c	parameter of risk aversion	λ	habit formation
ξ_w	wage stickness	σ_l	inverse of the Frisch elasticity
ξ_p	price stickness	ι_w	wage indexation
ι_p	price indexation	a	function of the elasticity of the capital utilization
ρ_R	perisistence of taylor rule	Φ	fixed cost
ϕ_1	response to inflation (taylor rule)	β	intertemporal discount
ϕ_2	response to output gap (taylor rule)	$\bar{\beta}$	$\equiv \beta\gamma^{-\sigma_c}$
ϕ_3	response to change in output gap (taylor rule)	y_*	steady state detrended output
π_*	inflation s.s.	l_*	steady state hours worked
γ	trend growth	α	capital share of GDP

6.2.2
The Public Sector Block: The Central Bank and the Treasury.

The public sector of the model is a system of 13 equations, 13 endogenous variables $\{nw_t, nii_t, \varsigma_t, ni_t\}$, $\{z_t, d_t, i^m_t, b^{l,cb}_t, b^{cb}_t, m_t, Q^b_t, b^{hh}_t, \tau_t\}$, two exogenous shocks $\{\epsilon^{l,cb}_t, \epsilon^{cb}_t\}$ and three variables that are determined in the main block of the model (assuming that the solvency constraint is not binding), $\{i_t, \pi_t, \xi_t\}$.

Eq1 (Fed's Net Worth): $nw_t = \dfrac{nw_{t-1}}{\gamma\pi_t} + ni_t - d_t$

Eq2 (Fed's Net Interest Income): $nii_t = i_{t-1}\dfrac{b^{cb}_{t-1}}{\gamma\pi_t} + \left(\dfrac{1-\delta_b Q_t}{Q_{t-1}}\right)\dfrac{b^{l,cb}_{t-1}}{\gamma\pi_t} - i^m_{t-1}\dfrac{m_{t-1}}{\gamma\pi_t}$

Eq3 (Fed's Capital Gain): $\varsigma_t = \left(\dfrac{Q^b_t}{Q^b_{t-1}} - 1\right)\dfrac{b^{l,cb}_{t-1}}{\gamma\pi_t}$

Eq4 (Fed's Net Income): $ni_t = niit_t + \varsigma_t$

Eq5 (Deferred Assets Account): $z_t = \dfrac{z_{t-1}}{\gamma\pi_t} + (d_t - (1-\zeta)\Theta_t)$

Eq6 (Dividends): $d_t = \begin{cases} 0 & \text{if } \Theta_t < 0 \text{ or } z_t > 0 \\ (1-\zeta)\Theta_t & \text{otherwise} \end{cases}$

Eq7 (IOER): $i^m_t = i_t$

Eq8 (QE 1): $b^{l,cb}_t = (\gamma\pi_*)^{\rho_{cb}}\,(b^{l,cb}_*)^{1-\rho_{cb}}\left(\dfrac{b^{l,cb}_{t-1}}{\gamma\pi_*}\right)^{\rho_{cb}} exp(\epsilon^{l,cb}_t)$

Eq9 (QE 2): $b^{cb}_t = (\gamma\pi_*)^{\rho_{cb}}\,(b^{cb}_*)^{1-\rho_{cb}}\left(\dfrac{b^{cb}_{t-1}}{\gamma\pi_*}\right)^{\rho_{cb}} exp(\epsilon^{cb}_t)$

Eq10 (Fed's Liabilities): $m_t = b^{l,cb}_t + b^{cb}_t - nw_t$

Eq11 (Pricing of Long-term Bonds): $1 = \bar{\beta}\mathbb{E}_t\left[\dfrac{\xi_{t+1}}{\xi_t}\dfrac{1+(1-\delta_b)Q^b_{t+1}}{Q^b_t}\dfrac{1}{\pi_{t+1}}\right]$

Eq12 (Treasury's Budget): $d_t + \tau_t + b^{cb}_t + b^{l,cb}_t + b^{hh}_t =$

$$= (1+i_{t-1})\left(\frac{b^{cb}_{t-1} + b^{hh}_{t-1}}{\gamma\pi_t}\right) + \left(\frac{1+(1-\delta_b)Q^b_t}{Q^b_{t-1}}\right)\left(\frac{b^{l,cb}_{t-1} + b^{l,hh}_{t-1}}{\gamma\pi_t}\right)$$

Eq13 (Primary Fiscal Surplus): $d_t + \tau_t = exp\left\{\phi_z\left(\dfrac{b^{hh}_{t-1} + b_{t-1} + b^{hh,l}_{t-1} + b^l_{t-1}}{\gamma\pi_t}\right)\right\}$

6.2.3
List of Linearized Equations

We present here the list of linearized equations for the public sector block of the model. The linearized equations of the SW07 block can be found in SW07. Let hatted variables denote percentage deviation from steady state $\hat{x}_t \equiv \frac{x_t - x_*}{x_*}$ and tilded variables denote deviation from steady state as a share of steady state (detrended) GDP $\tilde{x}_t \equiv \frac{x_t - x_*}{y_*}$.

L.Eq1 (Fed's Net Worth): $\tilde{nw}_t = \left(\frac{1}{\gamma\pi_*}\right)\left[\tilde{nw}_{t-1} - \left(\frac{nw_*}{y_*}\right)\hat{\pi}_t\right] + \tilde{ni}_t - \tilde{d}_t$

L.Eq2 (Fed's Net Interest Income): $\tilde{nii}_t = \left(\frac{i_*}{\gamma\pi^*}\right)\left(\tilde{b}^{cb}_{t-1} + b^{cb}_*\left(\hat{i}_{t-1} - \hat{\pi}_t\right)\right)$

$-\left(\frac{1-\delta_b Q^b_*}{Q^b_*}\right)\left(\frac{b^l_*}{\gamma\pi_*}\right)\hat{q}^b_{t-1} - \delta_b\left(\frac{b^l_*}{\gamma\pi_*}\right)\hat{q}_t - \left(\frac{i_*}{\gamma\pi_*}\right)\left(\tilde{m}_t + m_*\left(\hat{i}^m_t - \hat{\pi}_t\right)\right)$

L.Eq3 (Fed's Capital Gain): $\tilde{\varsigma}_t = \left(\frac{b^l_*}{\gamma\pi_*}\right)(\hat{q}_t - \hat{q}_{t-1})$

L.Eq4 (Fed's Net Income): $\tilde{ni}_t = \tilde{nii}_t + \tilde{\varsigma}_t$

L.Eq5 (Deferred Assets Account): $\tilde{z}_t = \left(\frac{1}{\gamma\pi_*}\right)(\tilde{z}_{t-1} - z_*\hat{\pi}_t) + \left(\tilde{d}_t - (1-\zeta)\tilde{\Theta}_t\right)$

Eq6 (Dividends): $\tilde{d}_t = \begin{cases} 0 & \text{if } \tilde{\Theta}_t < \Theta_* \text{ or } \tilde{z}_t > 0 \\ (1-\zeta)\tilde{\Theta}_t & \text{otherwise} \end{cases}$

L.Eq7 (IOER): $\hat{i}^m_t = \left(\frac{i_*}{i^m_*}\right)\hat{i}_t$

L.Eq8 (QE 1): $\tilde{b}^{l,cb}_t = \rho_b\tilde{b}^{l,cb}_{t-1} - \rho_{cb}b_*\hat{\pi}_t + \epsilon^{bl}_t$

L.Eq9 (QE 2): $\tilde{b}^{cb}_t = \rho_b\tilde{b}^{cb}_{t-1} - \rho_{cb}b_*\hat{\pi}_t + \epsilon^{bs}_t$

L.Eq10 (Fed's Liabilities): $\tilde{m}_t = \tilde{b}^{l,cb}_{t-1} + \tilde{b}^{cb}_t - \tilde{nw}_t$

L.Eq11 (Pricing of Long-term Bonds): $\hat{q}^b_t = \left(\frac{(1-\delta_b)Q^b_*}{1+(1-\delta_b)Q^b_*}\right)\mathbb{E}_t\hat{q}^b_{t+1} - \hat{R}_t$

L.Eq12 (Treasury's Budget): $\tilde{d}_t + \tilde{\tau}_t + \tilde{b}^{cb}_t + \tilde{b}^{l,cb}_t + \tilde{b}^{hh}_t =$

$\left(\frac{1+i_*}{\gamma\pi_*}\right)\left(\tilde{b}^{cb}_{t-1} + \tilde{b}^{hh}_{t-1} - (b^{cb}_* + b^{l,cb}_*)\left(\hat{\pi}_t + \left(\frac{i_*}{1+i_*}\right)\hat{i}_t\right)\right) +$

$+\left(\frac{1+(1-\delta_b)Q^b_*}{Q^b_*}\right)\left(\frac{1}{\gamma\pi_*}\right)\left(\tilde{b}^{l,cb}_{t-1} + \tilde{b}^{l,hh}_{t-1} - (b^{l,cb}_* + b^{l,hh}_*)\hat{\pi}_t\right) +$

$+\left(\frac{b^{l,cb}_* + b^{l,hh}_*}{\gamma\pi_*}\right)\left((1-\delta)(\hat{q}_t - \hat{q}_{t-1}) - \left(\frac{1}{Q^b_*}\right)\hat{q}_{t-1}\right)$

Eq13 (Primary Fiscal Surplus): $\left(\frac{1}{d_* + \tau_*}\right)\left(\tilde{d}_t + \tilde{\tau}_t\right) =$

$= \left(\frac{\phi_z}{\gamma\pi_*}\right)\left(\tilde{b}^{hh}_{t-1} + \tilde{b}_{t-1} + \tilde{b}^{hh,l}_{t-1} + \tilde{b}^l_{t-1} - \left(\tilde{b}^{hh}_* + \tilde{b}_* + \tilde{b}^{hh,l}_* + \tilde{b}^l_*\right)\hat{\pi}_t\right)$

6.3 Measurement Equations

Quarterly percentage ouput growth $= 100(\gamma - 1) + 100(\hat{y}_t - \hat{y}_{t-1})$

Quarterly percentage consumption growth $= 100(\gamma - 1) + 100(\hat{c}_t - \hat{c}_{t-1})$

Quarterly percentage investment growth $= 100(\gamma - 1) + 100(\hat{i}_t - \hat{i}_{t-1})$

Quarterly percentage wage growth $= 100(\gamma - 1) + 100(\hat{w}_t - \hat{w}_{t-1})$

Quarterly inflation rate $= 100(\pi_* - 1) + 100\hat{\pi}_t$

Quarterly Federal Funds Rate $= 100(R_* - 1) + 100\hat{R}_t$

Hours worked index $= l_* + 100\hat{l}_t$

Fed's Holdings of MBS + Treasury securities (as % of QGDP) $= 100\tilde{b}_*^{l,cb} + 100\tilde{b}_t^{l,cb}$

6.4 Optimal Policy with Non-Linear Constraints - Solution Method

In this section we describe the method developed the compute the impulse response functions in the model with monetary policy under commitment and discretion. The trick part of the solution is the presence of the occassionally binding zero lower bound and solvency constraints. We adapt the solution method used in Eggertsson and Woodford (2003), which was further generalized in Guerrieri and Iacoviello (2015), that adjusts a first-order perturbation approach and applies it to handle occasionally binding constraints in dynamic models. The strategy to solve this problem is to consider it as a model with 4 regimes, depending if each non-linear constraints is binding or not. Let's define each regime as follows,

$R1$. ZLB is binding & Solvency Constraint is slack

$R2$. ZLB is slack & Solvency Constraint is slack (not absorving regime).

$R3$. ZLB is slack & Solvency Constraint is biding

$R4$. ZLB is slack & Solvency Constraint is slack

Note that the model was linearized around the stationary regime 4 in which Blanchard and Kahn (1980) conditions apply. The advantage of this approach is that in each regime the system of necessary conditions for equilibrium is linear so we can use standard methods to characterize the solution. The tricky part is to deal with expectations when transitioning from one regime to another. To deal with that, we employ a guess-and-verify approach. First, we guess the period in which each regime applies. Second, we proceed and verify, and if necessary update, the initial guess. Sections (6.5.1) to (6.5.4) describe the solution method in each regime for a given guess of the regime structure of the IRF. Section (6.5.5) describes the algorithm used to update the guess and find the equilibrium.

Before proceeding to the solution method, it will be useful to recall some definitions. First, remember that we can cast the quantitative model into the following matrix from,

$$\begin{bmatrix} H_{XX} & 0_{nX,nx} \\ H_{xX} & H_{xx} \end{bmatrix} \begin{bmatrix} X_{t+1} \\ x_{t+1|t} \end{bmatrix} = \begin{bmatrix} A_{XX} & A_{Xx} \\ A_{xX} & A_{xx} \end{bmatrix} \begin{bmatrix} X_t \\ x_t \end{bmatrix} + \begin{bmatrix} B_X \\ B_x \end{bmatrix} i_t + \begin{bmatrix} C_X \\ C_x \end{bmatrix} \epsilon_t \tag{6-1}$$

where the vector X_t is an $(nX, 1)$ vector of predetermined variables, x_t an $(nx, 1)$ vector of non-predetermined variables, i_t is the nominal interest rate. The intertemporal loss function in period 0 is given by,

$$\mathbb{E}_t \sum_{\tau=0}^{\infty} \beta^\tau L_{t+\tau} \tag{6-2}$$

where L_τ is the period loss function,

$$L_\tau = \frac{1}{2} x_\tau' W x_\tau \tag{6-3}$$

6.5 Monetary Policy under Discretion

In the stationary regime (R4), the central bank's problem is to choose a sequence $\{i_t\}_{t\geq 0}$ as function of the exogenous process $\{\epsilon_t\}_{t\geq 0}$ so as to minimize period-by-period the intertemporal loss function (6-2), subject to (6-1), the ZLB, the solvency constraint, and initial conditions X_0. The solution to this problem satisfies the Bellman Equation,

$$v_t(X_t, \epsilon_t) = \min \left\{ \frac{1}{2} x_t' W x_t + \beta \mathbb{E}_t v_{t+1}(X_{t+1}, \epsilon_{t+1}) \right\} \tag{6-4}$$
$$\text{s.t. (6-1)}, \ i_t \geq 0, \ nw_t \geq -\phi \text{ and } X_0$$

The central banker's decision problem in period t is to choose i_t to minimize the period loss function plus the discounted expected continuation value, taking into account that its current choice of policy will change the endogenous state of the economy in the next period, expectations about the future and therefore the curret outcome. Assuming that there is a solution in period $t+1$, we can write i_{t+1} and x_{t+1} as a linear function of the endogenous state, X_{t+1} and the comtemporaneous shocks, ϵ_{t+1},

$$i_{t+1} = F_{t+1}X_{t+1} + F^s_{t+1}\epsilon_{t+1} \tag{6-5}$$
$$x_{t+1} = G_{t+1}X_{t+1} + G^s_{t+1}\epsilon_{t+1} \tag{6-6}$$

Where F_{t+1}, F^s_{t+1}, G_{t+1} and G^s_{t+1} are determined by the decision problem in period $t+1$. Both F_{t+1} and G_{t+1} are assumed to be known in period t; only G_{t+1} will matter for the decision problem in period t. Take the expectation operator conditional to available information in period t on both sides of (6-6) and on the upper block of (6-1) to yield the following expression,

$$\begin{aligned} x_{t+1|t} &= G_{t+1}X_{t+1|t} \\ &= G_{t+1}H^{-1}_{XX}(A_{XX}X_t + A_{Xx}x_t + C_X\epsilon_t) \end{aligned}$$

Substituting into the lower block of (6-1) yields,

$$x_t = \bar{A}X_t + \bar{B}_t i_t + \bar{C}\epsilon_t \tag{6-7}$$
$$X_{t+1} = \tilde{A}_t X_t + \tilde{B}_t i_t + \tilde{C}\epsilon_t \tag{6-8}$$

where

$$\bar{A}_t = ((H_{xX} + H_{xx}G_{t+1})H^{-1}_{XX}A_{Xx} - A_{xx})^{-1}(A_{xX} - (H_{xX} + H_{xx}G_{t+1})H^{-1}_{XX}A_{XX}) \tag{6-9}$$
$$\bar{B}_t = ((H_{xX} + H_{xx}G_{t+1})H^{-1}_{XX}A_{Xx} - A_{xx})^{-1}(B_x - (H_{xX} + H_{xx}G_{t+1})H^{-1}_{XX}B_X) \tag{6-10}$$
$$\bar{C}_t = ((H_{xX} + H_{xx}G_{t+1})H^{-1}_{XX}A_{Xx} - A_{xx})^{-1}(C_x - (H_{xX} + H_{xx}G_{t+1})H^{-1}_{XX}C_X) \tag{6-11}$$
$$\tilde{A}_t = H^{-1}_{XX}(A_{XX} + A_{Xx}\bar{A}_t) \tag{6-12}$$
$$\tilde{B}_t = H^{-1}_{XX}(A_{Xx}\bar{B}_t + B_X) \tag{6-13}$$
$$\tilde{C}_t = H^{-1}_{XX}(A_{Xx}\bar{C}_t + C_X) \tag{6-14}$$

Rewrite the period loss function as a function of the predetermined X_t

and the interest rate i_t,

$$L_t = \frac{1}{2}\begin{bmatrix} X_t \\ i_t \\ \epsilon_t \end{bmatrix}' \begin{bmatrix} W_{XX} & W_{iX} & W_{eX} \\ W'_{iX} & W_{ii} & W_{ei} \\ W'_{eX} & W'_{ei} & W_{ee} \end{bmatrix} \begin{bmatrix} X_t \\ i_t \\ \epsilon_t \end{bmatrix} \tag{6-15}$$

where

$$W_t^{XX} = \bar{A}'_t W \bar{A}_t \tag{6-16}$$
$$W_t^{Xi} = \bar{A}'_t W \bar{B}_t \tag{6-17}$$
$$W_t^{Xe} = \bar{A}'_t W \bar{C}_t \tag{6-18}$$
$$W_t^{ii} = \bar{B}'_t W \bar{B}_t \tag{6-19}$$
$$W_t^{ie} = \bar{B}'_t W \bar{C}_t \tag{6-20}$$
$$W_t^{ee} = \bar{C}'_t W \bar{C}_t \tag{6-21}$$

6.5.1
R4: The Stationary Regime

Assume that from T onwards the model returns, and remains indeterminately, to the stationary regime (in which BK applies). In this case, since the loss function is quadratic and the all constraints are linear (ZLB and SC are dropped in this regime), it follows that the optimal value of the problem will be quadratic. In period $t+1$, the optimal values will depend on X_{t+1} and can hence be written $\frac{1}{2}[\beta X'_{t+1} V_{t+1} X_{t+1} + (1-\beta) w_{t+1}]$, where V_{t+1} is a positive semi-definite matrix and w_{t+1} is a scalar independent of X_{t+1}. Both V_{t+1} and w_{t+1} are assumed known in period t. Them the optimal value of the problem in period t is associated with the positive semidefinite matrix V_t and the scalar w_t, and satisfies the Bellman equation,[1]

$$\begin{aligned} \frac{1}{2}\left[\begin{bmatrix} X_t \\ \epsilon_t \end{bmatrix}' \begin{bmatrix} V_t^{XX} & V_t^{\epsilon X} \\ V_t^{X\epsilon} & V_t^{\epsilon\epsilon} \end{bmatrix} \begin{bmatrix} X_t \\ \epsilon_t \end{bmatrix} + \beta w_t \right] \equiv \\ \equiv \min_{i_t} \left\{ L_t + \beta \mathbb{E}_t \frac{1}{2}\left[\begin{bmatrix} X_{t+1} \\ \epsilon_{t+1} \end{bmatrix}' \begin{bmatrix} V_{t+1}^{XX} & V_{t+1}^{\epsilon X} \\ V_{t+1}^{X\epsilon} & V_{t+1}^{\epsilon\epsilon} \end{bmatrix} \begin{bmatrix} X_{t+1} \\ \epsilon_{t+1} \end{bmatrix} + \beta w_{t+1} + \frac{1-\beta}{\beta} w_{t+1} \right] \right\} \\ \text{s.t. to (6-15) and (6-8), given } X_0 \end{aligned} \tag{6-22}$$

The first-order condition of (6-22),

[1]For a detailed derivation of this result see Ljungqvist and Sargent (2004):Chapter 18, Dynamic Stackelberg Problems.

$$0 = i_t'(W_t^{ii} + \beta \tilde{B}_t' V_{t+1}^{XX} \tilde{B}_t) + X_t'(W_{Xi,t}' + \beta \tilde{B}_t' V_{XX,t+1}' \tilde{A}_t) + \epsilon_t'(W_{ie,t}' + \beta \tilde{B}_t' V_{XX,t+1}' \tilde{C}_t)$$

can be solved to yield the central bank's reaction function,

$$i_t = F_t X_t + F_t^s \epsilon_t \tag{6-23}$$

where

$$F_t \equiv -(W_t^{ii} + \beta \tilde{B}_t' V_{XX,t+1}' \tilde{B}_t)^{-1}(W_{Xi,t}' + \beta \tilde{B}_t' V_{XX,t+1}' \tilde{A}_t) \tag{6-24}$$

$$F_t^s \equiv -(W_t^{ii} + \beta \tilde{B}_t' V_{XX,t+1}' \tilde{B}_t)^{-1}(W_t^{ie} + \beta \tilde{B}_t' V_{XX,t+1}' \tilde{C}_t) \tag{6-25}$$

Pluging (6-23) back in (6-7) and (6-8) yields the following expressions,

$$x_t = G_t X_t + G_t^s \epsilon_t$$
$$X_{t+1} = M_t X_t + M_t^s \epsilon_t$$

where

$$G_t \equiv \bar{A}_t + \bar{B}_t F_t \tag{6-26}$$

$$G_t^s \equiv \bar{C}_t + \bar{B}_t F_t^s \tag{6-27}$$

$$M_t \equiv \tilde{A}_t + \tilde{B}_t F_t \tag{6-28}$$

$$M_t^s \equiv \tilde{C}_t + \tilde{B}_t F_t^s \tag{6-29}$$

Furthermore, using (6-23) in (6-22) and identifying terms results in

$$V_{XX} = M' V_{XX} M + W_{XX} + F' W_{Xi}' + W_{Xi} F + F' W_{ii} F \tag{6-30}$$

$$V_{Xe} = W_{Xi} F^s + F' W_{ii} F^s + W_{Xe} + F' W_{ie} + M' V_{XX} M^s \tag{6-31}$$

$$V_{ee} = V_{ee} + (M^s)' V_{XX} M^s + (F^s)' W_{ii} F^s + W_{ie}' F^s + (F^s)' W_{ie} + W_{ee} \tag{6-32}$$

Finally, the above set of equations (6-9)-(6-14), (6-16)-(6-21) and (6-24)-(6-32) define a mapping from (G_{t+1}, V_{t+1}) to (G_t, V_t) which also determines F_t. The solution to the problem is a fixed point (G, V) of the mapping and a corresponding F. It can be obtained as the limit of (G_t, V_t) when $t \to \infty$. The solution thus satisfies the corresponding steady state matrix equations,

$$X_{t+1} = M_4 X_t + M_4^s \epsilon_t \tag{6-33}$$

$$x_t = G_4 X_t + G_4^s \epsilon_t \tag{6-34}$$

$$i_t = F_4 X_t + F_4^s \epsilon_t \tag{6-35}$$

Note that the stationarity of this regime implies that the transition matrices M_4, G_4, F_4, M_4^s, G_4^s, and F_4^s are time-independent.

6.5.2
R3: The Balance Sheet Crisis Regime.

In this regime, the solvency constraint binds and restrict the ability of the central bank to choose the interest rate optimally. Instead, the interest rate becomes an forward looking variable that is pinned down by the solvency constraint: $nw_t = -\phi$. The augmented system,

$$\begin{bmatrix} H_{XX} & 0_{nX,nx+1} \\ \tilde{H}_{xX} & \tilde{H}_{xx} \end{bmatrix} \begin{bmatrix} X_{t+1} \\ \tilde{x}_{t+1|t} \end{bmatrix} = \begin{bmatrix} A_{XX} & A_{Xx} \\ \tilde{A}_{xX} & \tilde{A}_{xx} \end{bmatrix} \begin{bmatrix} X_t \\ \tilde{x}_t \end{bmatrix} + \begin{bmatrix} C_X \\ \tilde{C}_x \end{bmatrix} \epsilon_t \tag{6-36}$$

where

$$\tilde{H}_{xX} \equiv \begin{bmatrix} H_{xX} \\ H_{iX} \end{bmatrix}, \quad \tilde{H}_{xx} \equiv \begin{bmatrix} H_{xx} & H_{xi} \\ H_{ix} & H_{ii} \end{bmatrix}, \quad \tilde{A}_{xX} \equiv \begin{bmatrix} A_{xX} \\ A_{iX} \end{bmatrix}$$
$$\tilde{A}_{xx} \equiv \begin{bmatrix} A_{xx} & A_{xi} \\ A_{ix} & A_{ii} \end{bmatrix}, \quad \tilde{C}_x \equiv \begin{bmatrix} C_x \\ C_i \end{bmatrix}, \quad \tilde{x}_{t+1|t} \equiv \begin{bmatrix} x_{t+1|t} \\ i_{t+1|t} \end{bmatrix}, \quad \tilde{x}_t \equiv \begin{bmatrix} x_t \\ i_{|t} \end{bmatrix}$$

For the sake of exposition, assume that under the current guess of regimes, regime 3 applies to period $T-1$. In this case, since the private sector correctly expects the the economy will enter the stationary regime in period T, it follows that,

$$\tilde{x}_{T|T-1} = \tilde{G}_4 X_{T|T-1} + \tilde{G}_4^s \epsilon_{T|T-1} = \tilde{G}_4 X_T$$

where $\tilde{G}_4 \equiv [G_4; F_4]$ and $\tilde{G}_4^s \equiv [G_4^s; F_4^s]$. Hereafter, we drop all the tildes for convenience. By pluging the last expression into (6-36), we transform a rational-expectations model into a simple system of $nX + nx + 1$ variables and $nX + nx + 1$ linear equations that we can easily solve for x_{T-1} and X_T as functions of X_{T-1} and ϵ_T,

$$
\begin{aligned}
X_T &= M_{T-1}X_{T-1} + M^s_{T-1}\epsilon_{T-1}\\
x_{T-1} &= G_{T-1}X_{T-1} + G^s_{T-1}\epsilon_{T-1}
\end{aligned}
$$

where,

$$
\begin{aligned}
G_{T-1} &= ((H_{xX} + H_{xx}G_4)H_{XX}^{-1}A_{Xx} - A_{xx})^{-1}(A_{xX} - (H_{xX} + H_{xx}G_4)H_{XX}^{-1}A_{XX})\\
G^s_{T-1} &= ((H_{xX} + H_{xx}G_4)H_{XX}^{-1}A_{Xx} - A_{xx})^{-1}(C_x - (H_{xX} + H_{xx}G_4)H_{XX}^{-1}C_X)\\
M_{T-1} &= H_{XX}^{-1}(A_{XX} + A_{Xx}\bar{A}_t)\\
M^s_{T-1} &= H_{XX}^{-1}(A_{Xx}\bar{C}_t + C_X)
\end{aligned}
$$

Furthermore, we can retrieve V_{T-1} from,

$$
\begin{aligned}
&\frac{1}{2}\left[\begin{bmatrix} X'_{T-1} & \epsilon'_{T-1}\end{bmatrix}\begin{bmatrix} V^{XX}_{T-1} & V^{eX}_{T-1}\\ V^{Xe}_{T-1} & V^{ee}_{T-1}\end{bmatrix}\begin{bmatrix} X_{T-1}\\ \epsilon_{T-1}\end{bmatrix} + \beta w_{T-1}\right] =\\
&= \min_{i_{T-1}}\left\{x'_{T-1}Wx_{T-1} + \beta\frac{1}{2}\mathbb{E}_{T-1}\left[\begin{bmatrix} X'_T & \epsilon'_T\end{bmatrix}\begin{bmatrix} V^{XX}_4 & V^{eX}_4\\ V^{Xe}_4 & V^{XX}_4\end{bmatrix}\begin{bmatrix} X_T\\ \epsilon_T\end{bmatrix} + \beta w_{4,T}\right]\right\}\\
&= \min_{i_{T-1}}\left\{x'_{T-1}Wx_{T-1} + \beta\frac{1}{2}\mathbb{E}_{T-1}\left[X'_TV^{XX}_4X_T + \beta\frac{1}{2}\epsilon'_{T|T-1}V^{eX}_4X_T + X'_TV^{Xe}_4\epsilon_T + \epsilon'_TV^{ee}_4\epsilon_T + \beta w_{4,T}\right]\right\}\\
&= (G_{T-1}X_t + G^s_{T-1}\epsilon_t)'W(G_{T-1}X_{T-1} + G^s_{T-1}\epsilon_t) + \beta\frac{1}{2}\Big[(M_{T-1}X_{T-1} + M^s_{T-1}\epsilon_{T-1})'V^{XX}_4(M_{T-1}X_{T-1} + M^s_{T-1}\epsilon_{T-1}) +\\
&+ \underbrace{\beta\frac{1}{2}\epsilon'_{T|T-1}V^{eX}_4X_T}_{=0} + \underbrace{X'_TV^{Xe}_4\epsilon_{T|T-1}}_{=0} + \underbrace{\epsilon'_{T|T-1}V^{ee}_4\epsilon_{T|T-1}}_{=\tilde{\sigma_\epsilon}} + \beta w_4\Big]\Big\}\\
&= X'_{T-1}(G'_{T-1}WG_{T-1} + \beta\frac{1}{2}M'_{T-1}V^{XX}_4M_{T-1})X_{T-1} + \epsilon'_{T-1}(G^{s'}_{T-1}WG^s_{T-1} + \beta\frac{1}{2}M^{s'}_{T-1}V^{XX}_4M^s_{T-1})\epsilon_{T-1} + \beta\frac{1}{2}[\tilde{\sigma_\epsilon} + \beta w_4]
\end{aligned}
$$

Hence,

$$
\begin{aligned}
V^{XX}_{T-1} &= G'_{T-1}WG_{T-1} + \beta\frac{1}{2}M'_{T-1}V^{XX}_RM_{T-1}\\
V^{ee}_{T-1} &= G^{s'}_{T-1}WG^s_{T-1} + \beta\frac{1}{2}M^{s'}_{T-1}V^{XX}_RM^s_{T-1}\\
V^{Xe}_{T-1} &= 0\\
V^{eX}_{T-1} &= 0\\
w_t &= \tilde{\sigma}_\epsilon + \beta w_4
\end{aligned}
$$

6.5.3
R2: Policy Normalization Regime (Lift-off)

In the regime R2, neither the ZLB or the solvency constraint is biding so that the central bank is able to choose the policy rate optimally. The difference from R2 to R4 is that it is not absorbing, i.e., it is a transitory state and the model will eventually shift to another regime. The solution method in this regime is analogous to R4 and must also satisfy the set of equations (6-9)-(6-14), (6-16)-(6-21) and (6-24)-(6-32). The difference from R4 is that, under the current guess of regimes, we know beforehand matrices G_{t+1} and V_{t+1} and therefore it is not necesssary to find a fixed point from the maping (G_{t+1}, V_{t+1}) to (G_t, V_t).

6.5.4
R1: The Zero Lower Bound Regime.

In the regime 1 the zero lower bound binds and restricts the ability of the central bank to set the policy rate. The solution in this regime is similar to R3 but in this case the interest rate becomes a predetermined variable: $i_t = 0$. The upper block of the system (6-1) is augmented,

$$\begin{bmatrix} \tilde{H}_{XX} & 0_{nX+1,nx+1} \\ H_{xX} & H_{xx} \end{bmatrix} \begin{bmatrix} \tilde{X}_{t+1} \\ x_{t+1|t} \end{bmatrix} = \begin{bmatrix} \tilde{A}_{XX} & \tilde{A}_{Xx} \\ A_{xX} & A_{xx} \end{bmatrix} \begin{bmatrix} \tilde{X}_t \\ x_t \end{bmatrix} + \begin{bmatrix} \tilde{C}_X \\ C_x \end{bmatrix} \epsilon_t$$

where

$$\tilde{H}_{XX} \equiv \begin{bmatrix} H_{XX} \\ H_{iX} \end{bmatrix}, \quad \tilde{A}_{Xx} \equiv \begin{bmatrix} A_{Xx} \\ A_{ix} \end{bmatrix}$$
$$\tilde{A}_{XX} \equiv \begin{bmatrix} A_{XX} & A_{Xi} \\ A_{iX} & A_{ii} \end{bmatrix}, \quad \tilde{C}_X \equiv \begin{bmatrix} C_X \\ C_i \end{bmatrix}, \quad \tilde{X}_{t+1} \equiv \begin{bmatrix} X_{t+1} \\ i_{t+1} \end{bmatrix}, \quad \tilde{X}_t \equiv \begin{bmatrix} X_t \\ i_t \end{bmatrix}$$

This system is analogous to R3 and we can apply the same solution method described in the section (6.5.2) to find the trasitioning matrices.

6.5.5
The Shooting Algorithm

Given the current guess of regimes, let T denote the period in which the model switches to R4. Iterate back from T until X_0 is reached, applying regime 1 to 4 at each iteration, as implied by the current guess of regimes. This process will provide us with all transition matrices and value functions for $t \geq 0$. Given an initial state X_0 and a sequence of shocks $\{\epsilon_t\}_{t\geq 0}$, we can compute IRFs and check if the ZLB and the solvency constraint are satisfied under the current guess of regimes. If it is correct we stop and check (6-4), otherwise we update the guess and repeat the process.

1. Dealing with the ZLB. There is straighforward way to update the guess when dealing only with the ZLB. In this case, it is first assumed that the model is in the R4 at all periods. We then generate IRFs and check the validity of the guess. If the ZLB is satisfied at all periods we stop. Otherwise, we change the first period in which the ZLB is violated to R1. We repeat this processe until the ZLB is satisfied at all periods. This method is fast and works very well.

2. Dealing with the ZLB and the Solvency Constraint. Introducing the solvency constraint adds a lot of complexness to the problem due to the iteraction with the ZLB. We assume the follwing:

1. $0 \geq t \leq \tau_1$ the IRF is in R1.
2. $\tau_1 + 1 \geq t \leq \tau_1 + \tau_2$ the IRF is in R2.
3. $\tau_1 + \tau_2 + 1 \geq t \leq \tau_1 + \tau_2 + \tau_3$ the IRF is in R3.
4. $t \geq \tau_1 + \tau_2 + \tau_3 + 1$ the IRF is in R4.

We allow $\tau_1 = 0, ..., 15$ and $\tau_2 = 10, ..., 15$. For each pair (τ_1, τ_2) we compute the minimum $\tau_3(\tau_1, \tau_2)$ such that the solvency constraint is satisfied for all $t \geq \tau_1 + \tau_2 + 1$. We select the set of trios $(\tau_1, \tau_2, \tau_3(\tau_1, \tau_2))$ in which both ZLB and the solvency constraints are satisfied. From this set we, choose the trio that minimizes the intertemporal loss function subject to $max\{|\Delta i_t|, t \geq 0\} < 0.5$ (the FFR cannot vary more than 2 percentage points (in annualized term) within a single quarter).

3. Dealing with the ZLB and the Assimetric Dividend Rule. Dealing with the dividend rule and the ZLB simultaneosly is simple because the policy rate is determined independently of the variables related to the central bank's balance sheet. We employ method 1 to solve first for the ZLB, and then for the dividends. In this case we must assume that the model has the following constraints,

$R1$. ZLB is binding & Divideds are positive

$R2$. ZLB is binding & Divideds are zero

$R3$. ZLB is slack & Divideds are positive

$R4$. ZLB is slack & Divideds are zero

Note that in section 7.2 of the main text we don't need to deal with the interaction fo the (i) ZLB, (ii) solvency constraint and (iii) the assimetric dividend rule because dividends are always positive when the dividend rule is based on the central bank's interest income.

6.6 Monetary Policy under Commitment - Solution Method

Consider minimizing the intertemporal loss function, (6-2), under commitment once-and-for-all in period $t = 0$, subject to (6-1) for $t \geq 0$ and X_0 given. The method described in Svensson (2010) consists in setting-up the Lagrangian, deriving the first-order conditions, combine these with dynamic equations, and solve the resulting difference equation system.

Set up the Lagrangian,

$$
\begin{aligned}
\mathbb{L} &= \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left\{ x_t' W x_t + \begin{bmatrix} \phi_{t+1}^{X\prime} & \phi_t^{x\prime} \end{bmatrix} \left\{ H \begin{bmatrix} X_{t+1} \\ x_{t+1|t} \end{bmatrix} - A \begin{bmatrix} X_t \\ x_t \end{bmatrix} - B i_t - C\epsilon_t \right\} - \phi_t^i i_t \right\} \\
&= \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left\{ x_t' W x_t + \begin{bmatrix} \phi_{t+1}^{X\prime} & \phi_t^{x\prime} \end{bmatrix} \left\{ H \begin{bmatrix} X_{t+1} \\ x_{t+1} \end{bmatrix} - A \begin{bmatrix} X_t \\ x_t \end{bmatrix} - B i_t - C\epsilon_t \right\} - \phi_t^i i_t \right\}
\end{aligned}
$$

where ϕ_t^X, ϕ_t^x and ϕ_t^i are vectors of n_X, n_x and 1 Lagrange multipliers of the upper, lower block and the policy rate, respectively, of the model equations. The second equality uses the law of iterate expectations. Take the first order conditions with respect to X_t, x_t and i_t,

$$
\begin{bmatrix} 0_{XX} & 0_{Xx} \\ 0'_{xX} & W \end{bmatrix} \begin{bmatrix} X_t \\ x_t \end{bmatrix} + \beta^{-1} \begin{bmatrix} H'_{XX} & H'_{xX} \\ H'_{Xx} & H'_{xx} \end{bmatrix} \begin{bmatrix} \phi_t^X \\ \phi_{t-1}^x \end{bmatrix} - \begin{bmatrix} A'_{XX} & A'_{xX} \\ A'_{Xx} & A'_{xx} \end{bmatrix} \begin{bmatrix} \phi_{t+1|t}^X \\ \phi_t^x \end{bmatrix} = 0 \tag{6-37}
$$

$$
- B'_X \phi_{t+1|t}^X - B'_x \phi_t^x - \phi_t^i = 0 \tag{6-38}
$$

System (6-1):

$$
\begin{bmatrix} H_{XX} & 0_{nX,nx} \\ H_{xX} & H_{xx} \end{bmatrix} \begin{bmatrix} X_{t+1} \\ x_{t+1|t} \end{bmatrix} = \begin{bmatrix} A_{XX} & A_{Xx} \\ A_{xX} & A_{xx} \end{bmatrix} \begin{bmatrix} X_t \\ x_t \end{bmatrix} + \begin{bmatrix} B_X \\ B_x \end{bmatrix} i_t + \begin{bmatrix} C_X \\ C_x \end{bmatrix} \epsilon_t
$$

Slackness Conditions:

$$\phi_t^i i_t = 0$$

$$\phi_t^X, \phi_t^x, \phi_i \geq 0$$

Initial Condition:

$$\phi_{-1}^X = \phi_{-1}^x = \phi_{-1}^i = X_{-1} = 0$$

The Stationary Regime. The first n_X equations of system (6-38), associated with the Lagrange multipliers of the predetermined variables X_t, are forward-looking variables; while the next n_x equations, associated with the Lagrange multipliers of the forward looking variables x_t, are predetermined. In the stationary regime $\phi_i = 0$ is a state variable and hence I can rewrite system (6-38) as,

$$
\begin{bmatrix}
H_{XX} & 0_{Xx} & 0_{X1} & H_{Xx} & 0_{XX} & 0_{X1} \\
0_{xX} & A'_{xx} & 0_{x1} & 0_{xx} & A'_{Xx} & 0_{x1} \\
0_{1X} & 0_{1x} & 1 & 0_{1x} & 0_{1X} & 0_{11} \\
H_{xX} & 0_{xx} & 0_{x1} & H_{xx} & 0_{xX} & 0_{x1} \\
0_{XX} & A'_{xX} & 0_{X1} & 0_{Xx} & A'_{XX} & 0_{X1} \\
0_{1X} & B'_x & 1 & 0_{1x} & B'_X & 0_{11}
\end{bmatrix}
\begin{bmatrix} X_{t+1} \\ \phi_t^x \\ \phi_t^i \\ x_{t+1|t} \\ \phi_{t+1|t}^X \\ i_{t+1|t} \end{bmatrix} =
$$

$$
= \begin{bmatrix}
A_{XX} & 0_{Xx} & 0_{X1} & A_{Xx} & 0_{XX} & B_X \\
0_{xX} & \beta^{-1} H'_{xx} & 0_{x1} & W_{xx} & \beta^{-1} H'_{Xx} & 0_{x1} \\
0_{1X} & 0_{1x} & 0 & 0_{1x} & 0_{1X} & 0_{11} \\
A_{xX} & 0_{xx} & 0_{x1} & A_{xx} & 0_{xX} & B_x \\
0_{XX} & \beta^{-1} H'_{xX} & 0_{X1} & 0_{Xx} & \beta^{-1} H'_{XX} & 0_{X1} \\
0_{1X} & 0_{1x} & 0 & 0_{1x} & 0_{1X} & 0_{11}
\end{bmatrix}
\begin{bmatrix} X_t \\ \phi_{t-1}^x \\ \phi_{t-1}^i \\ x_t \\ \phi_t^X \\ i_t \end{bmatrix} +
\begin{bmatrix} C_X \\ 0_x \\ 0 \\ C_x \\ 0_X \\ 0 \end{bmatrix} \epsilon_t
$$

which is a linear system of rational expectations that can be solved with standard methods.

The ZLB Regime. When the ZLB binds, i_t is a state variable and ϕ_t^i is a forward looking variable,

$$\begin{bmatrix} H_{XX} & 0_{Xx} & -B_X & H_{Xx} & 0_{XX} & 0_{X1} \\ 0_{xX} & A'_{xx} & 0_{x1} & 0_{xx} & A'_{Xx} & 0_{x1} \\ 0_{1X} & 0_{1x} & 1 & 0_{1x} & 0_{1X} & 0_{11} \\ H_{xX} & 0_{xx} & -B_x & H_{xx} & 0_{xX} & 0_{x1} \\ 0_{XX} & A'_{xX} & 0_{X1} & 0_{Xx} & A'_{XX} & 0_{X1} \\ 0_{1X} & B'_x & 0 & 0_{1x} & B'_X & 0_{11} \end{bmatrix} \begin{bmatrix} X_{t+1} \\ \phi_t^x \\ i_t \\ x_{t+1|t} \\ \phi_{t+1|t}^X \\ \phi_{t+1|t}^i \end{bmatrix} =$$

$$= \begin{bmatrix} A_{XX} & 0_{Xx} & 0_{X1} & A_{Xx} & 0_{XX} & 0_{X1} \\ 0_{xX} & \beta^{-1}H'_{xx} & 0_{x1} & W_{xx} & \beta^{-1}H'_{Xx} & 0_{x1} \\ 0_{1X} & 0_{1x} & 0 & 0_{1x} & 0_{1X} & 0_{11} \\ A_{xX} & 0_{xx} & 0_{x1} & A_{xx} & 0_{xX} & 0_{x1} \\ 0_{XX} & \beta^{-1}H'_{xX} & 0_{X1} & 0_{Xx} & \beta^{-1}H'_{XX} & 0_{X1} \\ 0_{1X} & 0_{1x} & 0 & 0_{1x} & 0_{1X} & -1 \end{bmatrix} \begin{bmatrix} X_t \\ \phi_{t-1}^x \\ i_{t-1} \\ x_t \\ \phi_t^X \\ \phi_t^i \end{bmatrix} + \begin{bmatrix} C_X \\ 0_x \\ 0 \\ C_x \\ 0_X \\ 0 \end{bmatrix} \epsilon_t$$

More compactly,

$$\begin{bmatrix} \bar{H}_{XX} & \bar{H}_{Xx} \\ \bar{H}_{xX} & \bar{H}_{xx} \end{bmatrix} \begin{bmatrix} \bar{X}_{t+1} \\ \bar{x}_{t+1|t} \end{bmatrix} = \begin{bmatrix} \bar{A}_{XX} & \bar{A}_{Xx} \\ \bar{A}_{xX} & \bar{A}_{xx} \end{bmatrix} \begin{bmatrix} \bar{X}_t \\ \bar{x}_t \end{bmatrix} + \begin{bmatrix} \bar{C}_X \\ \bar{C}_x \end{bmatrix} \epsilon_t \tag{6-39}$$

Using the fact that $\bar{x}_{t+1|t} = G_{t+1}\bar{X}_{t+1}$, where G_{t+1} is assumed to be known in period t, we can solve (6-39),

$$\begin{aligned} \bar{x}_t &= G_t\bar{X}_t + G_t^s\epsilon_t \\ \bar{X}_{t+1} &= M_t\bar{X}_t + M^s t\epsilon_t \end{aligned}$$

where,

$$\begin{aligned} G_t &= (\bar{A}_{xx} - D\bar{A}_{Xx})^{-1}(D\bar{A}_{XX} - \bar{A}_{xx}) \\ G_t^s &= (\bar{A}_{xx} - D\bar{A}_{Xx})^{-1}(D\bar{C}_X - \bar{C}_x) \\ M_t &= (\bar{H}_{XX} + \bar{H}_{Xx}G_{t+1})^{-1}(\bar{A}_{XX} + \bar{A}_{Xx}G_t) \\ M_t^s &= (\bar{H}_{XX} + \bar{H}_{Xx}G_{t+1})^{-1}(C_X + \bar{A}_{Xx}G_t^s) \\ D_t &= (\bar{H}_{xX} + \bar{H}_{xx}G_{t+1})(\bar{H}_{XX} + \bar{H}_{Xx} * G_{t+1})^{-1} \end{aligned}$$

note the model's solution is no longer linear since each transition matrix depends on period subscript t.

The Shooting Algorithm. Analogous to (6.5.5) (1. Dealing with the ZLB.)

6.7
Extra: Calculating the Yield Curve

We compute the yield to maturity of the delta bonds as the constat rate that yields the same expected present value of the pertetuities. Denote y_t^m the yield of a delta bond that pays a perpetual coupon with decay rate δ_b^m. The yield must satisfy the following expression,

$$\left(\frac{1}{1+y_t^m}\right)\mathbb{E}_t\sum_{s=0}^{\infty}\left(\frac{1-\delta_b^m}{1+y_t^m}\right)^s = \sum_{s=0}^{\infty}\frac{(1-\delta_b^m)^s}{\Pi_{j=0}^{s}R_{t+j}}$$

or

$$\frac{1}{y_t^m+\delta_b^m} = \mathbb{E}_t\sum_{s=0}^{\infty}\frac{(1-\delta_b^m)^s}{\Pi_{j=0}^{s}R_{t+j}}$$

Taking a first order Taylor expansion,

$$-\left(\frac{y_*^m}{(y_*^m+\delta_b^m)^2}\right)\hat{y}_t^m = -\mathbb{E}_t\sum_{k=0}^{\infty}\sum_{s=k}^{\infty}\frac{(1-\delta_b^m)^s}{R_*^{s+1}}\hat{R}_{t+k}$$

another way to do this is

$$(1+y_t^m)^n = \Pi_{j=1}^{n}\left(\frac{(1-\delta_b^m)^{j-1}+(1-\delta_b^m)^j Q_{t+j}^b}{Q_{t+j-1}^j}\right) \tag{6-40}$$

taking logs

$$nlog(y_t^m) = \mathbb{E}_t\sum_{j=1}^{n}log\left(\frac{(1-\delta_b^m)^{j-1}+(1-\delta_b^m)^j Q_{t+j}^b}{Q_{t+j-1}^b}\right) \tag{6-41}$$

first order taylor expansion yields

$$\hat{y}_t^m = \left(\frac{1}{ny_*^m}\right)\mathbb{E}_t\sum_{j=1}^{n}\left[\left(\frac{(1-\delta_b^m)^j Q_*^b}{(1-\delta_b^m)^{j-1}+(1-\delta_b^m)^j Q_*^b}\right)\hat{q}_{t+j}^b - \hat{q}_{t+j-1}^b\right] \tag{6-42}$$

$$= \left(\frac{1}{ny_*^m}\right)\mathbb{E}_t\sum_{j=1}^{n}\left[\left(\frac{(1-\delta_b^m)Q_*^m}{1+(1-\delta_b^m)Q_*^m}\right)\hat{q}_{t+j}^b - \hat{q}_{t+j-1}^b\right] \tag{6-43}$$

assuming that no further shocks hit the economy after period t,

$$\hat{y}_t^m = \left(\frac{1}{n y_*^m}\right) \mathbb{E}_t \sum_{j=0}^{n-1} \hat{R}_{t+j}$$

or

$$y_t^m \equiv 100(R_* - 1) + y_*^m \hat{y}_t^m = \left(\frac{1}{n}\right) \mathbb{E}_t \sum_{j=0}^{n-1} \left(100(R_* - 1) + 100\hat{R}_{t+j}\right)$$

taking the limit as $n \to \infty$

$$\hat{y}_t^m \to 0 \quad \Rightarrow \quad y_t^m = \frac{\beta^{-1} - 1}{\gamma \pi_*}$$

which is the yield in case the security is held until maturity.

7
Appendix for Chapter 3

7.1
The Persistence of Commodity Price Shocks

The persistence and volatility of the price shocks are calibrated to different commodity prices, reported in Table 7.1. We estimate an AR(1) process for the natural log of real prices of beef, coffee (Robusta), soy beans, bananas and coffee (Arabica) using annual data from World Bank's "Pink Sheet" on commodity prices.[1] We complement our estimates with those of BJS2013 who use commodity price data from the International Finance Statistics (IFS) to estimate the persistence of petroleum, copper, gold and sugar. For the sake of comparison, we report the estimates in Cashin *et al* (2000) (their Table 3) in the last column. They estimate the persistence of the commodity price shocks using monthly IMF data between 1957-1998 and a median-unbiased estimator, including a time trend. One can see that for most commodities — and especially the most persistent commodities — our estimates of persistence are generally lower than those in Cashin *et al* (2000).[2] This means that, if anything, our estimates of the optimal fiscal rule are not pro-cyclical enough.

7.2
Exogenous Income Model

7.2.1
Equilibrium

Let $\hat{x}$ denote the percentage deviation of variable X from its steady state value and $\tilde{x}$ denote deviation of X from its steady state as a share of non-resource GDP.[3] We take a first-order Taylor expansion of the system of equations (3-5)-(3-15) around the benchmark calibration steady state.

[1]The estimates in Table 7.1 do not include a time trend, because (if taken literally) this would imply that *real* log prices would go to ∞ or zero in the (very) long run, neither of which are feasible.

[2]This is unsurprising as Cashin *et al* (2000) argue that least squares estimates of persistence are downward biased.

[3]Steady state non resource GDP is unity, so $\tilde{x}_t = X_t - X_{SS}$

$\hat{c}'_t, \hat{c}''_t, \tilde{b}_t, \tilde{tr}'_t, \tilde{tr}''_t$ are measured in per capita terms, whereas other variables are aggregate. The resulting system of linear equations is:

$$\hat{c}''_t = \left[\frac{\omega^{-1}\omega_y(1-\tau)}{\omega^{-1}\omega_y(1-\tau)+Tr_{ss}}\right]\hat{y}_t + \left[\frac{1}{\omega^{-1}\omega_y(1-\tau)+Tr_{ss}}\right]\tilde{tr}''_t \tag{7-1}$$

$$\hat{c}'_t = \mathbb{E}_t\hat{c}'_{t+1} - \sigma^{-1}\hat{R}_t \tag{7-2}$$

$$\tilde{b}_t = \beta^{-1}\tilde{b}_{t-1} + (1-\omega)^{-1}(1-\tau)(1-\omega_y)\hat{y}_t + \tilde{tr}'_t -$$

$$- [(1-\omega)^{-1}(1-\tau)(1-\omega_y) + Tr_{ss}]\hat{c}_t \tag{7-3}$$

$$\tilde{a}_t = \beta^{-1}\tilde{a}_{t-1} + \beta^{-1}A_{ss}\hat{R}_{t-1} + \tau\hat{y}_t + QP_{ss}\hat{p}_t - (1-\omega)\tilde{tr}'_t - \omega\tilde{tr}''_t \tag{7-4}$$

$$\hat{R}_t = -\beta\psi(\tilde{a}_{t-1} + (1-\omega)\tilde{b}_{t-1}) \tag{7-5}$$

Transfers,

$$\tilde{tr}'_t = \theta'_a\tilde{a}_{t-1} + \theta'_y\hat{y}_t + \theta'_pQP_{ss}\hat{p}_t \tag{7-6}$$

$$\tilde{tr}''_t = \theta''_a\tilde{a}_{t-1} + \theta''_y\hat{y}_t + \theta_{1p}QP_{ss}\hat{p}_t \tag{7-7}$$

AR(1) shock processes for commodity prices and

$$\hat{p}_t = \rho_p\hat{p}_{t-1} + \epsilon^p_t \tag{7-8}$$

$$\hat{y}_t = \rho_y\hat{y}_{t-1} + \epsilon^y_t \tag{7-9}$$

7.2.2
Condition for Stability and an Irrelevance Result

In this section we establish two results that help to get some intuition of how the choice of fiscal policy affects determinacy and uniqueness of equilibrium in the exogenous income model. Consider the model (7-1)-(7-9) and assume $\theta''_a = 0$. Conditions (i) and (ii) are necessary and sufficient for the existence of an unique and bounded equilibrium.

(i) $\beta^{-1} - 1 < \theta'_a < \beta^{-1} + 1$

(ii) $0 < \psi < \frac{\sigma(2+\beta^{-1})}{\beta((1-\omega)^{-1}(1-\omega_y)(1-\tau_1)+Tr_{ss})}$ Lemma 1 provides two conditions for the existence and uniqueness of a stable equilibrium in the exogenous income model. The first condition restricts the set of feasible transfers rules available to the fiscal authority. While size of the transfer to the HtM household does not depend on the size of the SWF ($\theta_a = 0$), it is required that accumulated public assets are transferred to the Ricardian household at a rate at least as large as the steady-state net interest rate, $\beta^{-1} - 1$, and no higher than $1 + \beta^{-1}$. With this condition in place, explosive paths of public assets are prevented. Note that any set of values for θ_y and θ_p can be consistent with stable paths for the public debt and highly counter-cyclical or pro-cyclical transfers rule are feasible policy choices for the government. The second condition adds a debt-elastic interest rate in the model to remove the random-walk behavior of the consumption of Ricardian household, as in Schmitt-Grohe & Uribe (2003). Consider the model (7-1)-(7-9), assume $\theta''_a = 0$ and $\psi \rightarrow 0$. In a unique and stable equilibrium, the consumption path and welfare of the Ricardian household is independent of the transfers it receives from the government $\{\theta'_a, \theta'_p, \theta'_y\}$. Lemma (7.2.2) is an adaptation of the classic Ricardian equivalence result in Barro (1974) for our heterogeneous-agent setup. Ricardian agents consume out of their permanent income, and to the extent that lump sum transfers financed by public debt issuance do not affect the agent's intertemporal budget constraint, fiscal policy cannot influence equilibrium allocations in a non-trivial fashion. The introduction of HtM households breaks the classical Ricardian equivalence (e.g. Galí *et al* (2007)). In our model, transfers to one type of agent affect the other through the effects on public assets (if $\theta_a > 0$). However, if the government commits to a particular transfer rule to the HtM HH which does not depend on the level of public assets ($\theta''_a = 0$), then transfers between the government and the Ricardian HH do not affect the intratemporal budget of the Ricardian HH and hence the Ricardian equivalence applies again.

7.2.3 Optimal Simple Rule

In Figure 7.2, we show how the welfare loss changes as we change the fiscal rule coefficients one at a time around the Optimal Simple Rule (which is indicated by a vertical line). First, one can see that optimal simple rule is optimal, in that it leads to the lowest welfare loss (at least locally). Most interesting is the bottom RHS plot which shows that as the government saves more commodity revenues on behalf of the HtM HH ($\theta''_P \rightarrow 0$), the welfare loss increases from about 2.4% of steady state consumption to almost 3%, reinforcing the point that it is optimal for HtM HHs to spend rather than save commodity revenues when they are highly persistent.

7.3 Spillovers from Commodity Prices to non-resource GDP

Figure 7.3 shows combinations of θ_Y and θ_P that approximately maximize welfare (minimize the welfare loss) when ρ_p is calibrated to match the persistence of the price shock of selected commodities (here $\beta_{YP} = 0.2$ and $\sigma_Y = 0.04$). We find that fixing $\theta_A = 0.09$ the welfare loss is minimal along *all points* of the line $\theta_P = intercept - 0.6 \times \theta_Y$ for all ρ_p. The intercept of each indifference curve is increasing in the persistence of the commodity price shock, because more persistent shocks requires more pro-cyclical fiscal policy (either in terms of θ_P or θ_Y). Also, the red dot ($\theta_Y = -0.77$ and $\theta_P = 0.68$) and the black dot ($\theta_Y = -0.77$ and $\theta_P = 1.03$) display the OSR equal in the baseline calibration and in the calibration with correlated shocks, respectively. One can see that the optimal response to commodity price shocks becomes much more pro-cyclical when shocks are correlated.

7.4 Real Business Cycle Model

7.4.1 RBC Model Description

The RBC model is based on the exogenous income model of Section 3.3 augmented to introduce endogenous production in the non-resource sector of the economy. To be brief, we will only discuss the main differences between the RBC model and the exogenous-income model.

First, preferences are different: here we use Greenwood–Hercowitz–Huffman (1988) (GHH hereafter) preferences to remove wealth effects on labor supply (common in small open economy RBC models). Each type of household chooses labor supply and consumption to maximize utility subject to a budget constraint,

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \frac{1}{1-\gamma} \left([C_t^i - \eta^{-1} (L_t^i)^\eta]^{1-\gamma} - 1 \right) \qquad \text{for } i \in \{\prime, \prime\prime\} \tag{7-10}$$

where $(\eta - 1)^{-1}$ is the Frisch elasticity of labor supply and γ the coefficient of risk aversion (equivalent to σ in the exogenous income model in the main text).

The Ricardian household can smooth consumption by accumulating two types of assets: physical capital and one-period bonds traded internationally at interest rate R_t. Physical capital accumulation is subject to a depreciation rate and adjustment costs according to equation (7-11),

$$K_t = (1-\delta) K_{t-1} + I_t - \frac{\phi}{2} (K_t - K_{t-1})^2 \tag{7-11}$$

the Ricardian household's budget constraint is given by,

$$C'_t = R_{t-1}B'_{t-1} - B'_t + (1-\omega)^{-1}(R^k_t - 1)K_{t-1} - (1-\omega)^{-1}I_t + W_tL'_t + Tr'_t \tag{7-12}$$

where C'_t is per Ricardian household consumption, B'_t is per Ricardian household stock of bonds, W_t is the wage rate and Tr'_t is per Ricardian government transfers and R^k is the rate of return on capital. The Ricardian household maximizes Equation (7-10) subject to (7-12) and (7-11). The first-order conditions of this problem are,

$$L_t^{\eta-1} = W_t \tag{7-13}$$

$$\left(C'_t - \eta^{-1}L_t'^{\eta}\right)^{-\gamma} = \beta R_t \mathbb{E}_t \left(C'_{t+1} - \eta^{-1}L_{t+1}'^{\eta}\right)^{-\gamma} \tag{7-14}$$

$$\begin{aligned}&[1+\phi(K_t - K_{t-1})]\left(C'_t - \eta^{-1}L_t'^{\eta}\right)^{-\gamma} = \\ &= \beta \mathbb{E}_t \left(C'_{t+1} - \eta^{-1}L_{t+1}'^{\eta}\right)^{-\gamma}\left[R^k_{t+1} - \delta - \phi(K_{t+1} - K_t)\right]\end{aligned} \tag{7-15}$$

where we dropped the upper index ($\prime$) from (7-13) because labor supply is the same across households in equilibrium. The Hand-to-Mouth household has no access to any sort of financial instrument and hence is subject to the period-by-period budget constraint,

$$C''_t = W_tL_t + Tr''_t \tag{7-16}$$

where Tr''_t is per HtM household government transfers.

Non-resource goods are produced competitively using labor and capital. Firms maximize profits choosing labor and capital inputs subject to a Cobb-Douglas production function,

$$Y_t = Z_t K^{\alpha}_{t-1} {L_t}^{1-\alpha} \tag{7-17}$$

where Total Factor Productivity, Z_t, follows an AR(1) process with persistence $\rho_z$and standard deviation σ_z. Note that K_{t-1} is a predetermined variable at time period t. Profit maximization yields the first-order conditions,

$$W_t = (1-\tau)(1-\alpha)\frac{Y_t}{L_t} \tag{7-18}$$

$$R_t^k - 1 = (1-\tau)\alpha\frac{Y_t}{K_{t-1}} \tag{7-19}$$

where τ is a sales tax rate levied on firms. As in exogenous income model, all income from commodity exports accrues to the government. The government's budget constraint is:

$$A_t = R_{t-1}A_{t-1} + QP_t + \tau Y_t - \omega Tr_t'' - (1-\omega)Tr_t' \tag{7-20}$$

and the debt-elastic interest-rate spread is

$$R_t = R_t^w + \psi(e^{-(1-\omega)B_t + A_t - (1-\omega)B_{ss} - A_{ss})} - 1)$$

It is useful to define GDP, trade balance and the current account,

$$GDP_t = QP_t + Y_t \tag{7-21}$$

$$TB_t = Y_t + QP_t - (1-\omega)C_t' - \omega C_t'' - I_t \tag{7-22}$$

$$CA_t = TB_t + (R_{t-1} - 1)((1-\omega)B_{t-1} + A_{t-1}) \tag{7-23}$$

Up to a second order, welfare losses using the GHH utility function can be approximated by ζ.

$$\begin{aligned}\zeta = &-\frac{\gamma}{2}(C_{ss} - \eta^{-1}l_{ss})^{-1}C_{ss}\left\{(1-\omega)\mathbb{V}(\hat{c}_t') + \omega\mathbb{V}(\hat{c}_t'')\right\} \\ &+\frac{1}{2}(\gamma l_{ss}^{\eta} + (\eta - 1))\frac{l_{ss}^{\eta}U_l}{C_{ss}}\left\{(1-\omega)\mathbb{V}(l_t') + \omega\mathbb{V}(l_t'')\right\}\end{aligned}$$

7.4.2 RBC Calibration

We follow Schmitt-Grohe & Uribe (2003) in calibrating η, γ, δ, ϕ, ρ_z and σ_z . We choose $\psi = 0.01$ and other parameters from the exogenous income model in the body of the text. Table 7.2 summarizes the calibration. Note that $C_{1,ss}^{PC} = 1.2292$, $C_{2,ss}^{PC} = 1.0625$, $Y_{ss} = 1$, $Y_R = 1/3$, $K_{SS} = 2$.

Table 7.1: Calibration of Commodity Price Persistence (in years)

	Persistence	**Half Life**	OSR θ_p	**Source**	**Sample**	**Half Life from Cashin et al** <1	1-4	5-8	9-18	∞
Petroleum	0.94	11.2	0.73	IFS/BJS(2013)	1970–2008					X
Beef	0.90	6.6	0.56	WB PinkSheet	1960-2016			X		
Natural Gas	0.89	6	0.53	IFS/BJS(2013)	1985–2008					X
Copper	0.89	6	0.53	IFS/BJS(2013)	1957–2008			X		
Gold	0.89	6	0.53	IFS/BJS(2013)	1970–2008					X
Coffee (Robusta)	0.89	6	0.53	WB PinkSheet	1960-2016					X
Soy Beans	0.87	5	0.48	WB PinkSheet	1960-2016		X			
Bananas	0.80	3	0.35	WB PinkSheet	1960-2016	X				
Coffee (Arabica)	0.77	2.6	0.31	WB PinkSheet	1960-2016				X	
Sugar	0.74	2.3	0.28	IFS/BJS(2013)	-				X	

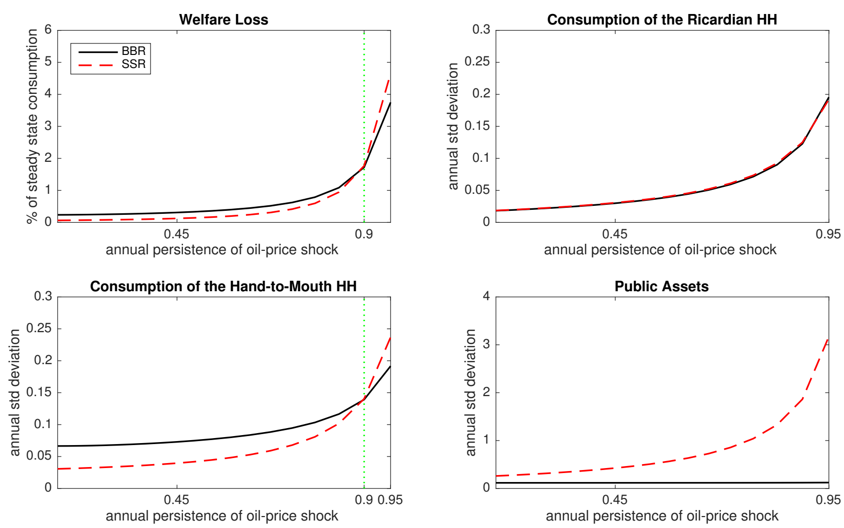

Figure 7.1: **Welfare loss and shock persistence with BBR and SSR without adjustment of σ_p** : This figure is a version of figure 3.2 in the main text without adjusting σ_p to keep the total commodity shock variance constant.

Table 7.2: Additional Parameters in the RBC model

Param.	**Value**	**Description**	**Target/Source**
η	1.45	Labor supply elasticity	Schmitt-Grohe & Uribe (2003) (Frisch= $(\eta-1)^{-1} = 2.22$)
γ	2	Coefficient of risk aversion	Common value in literature
α	1/3	Capital Share GDP	Schmitt-Grohe & Uribe (2003)
δ	0.1	Depreciation rate	SS I/Y; Schmitt-Grohe & Uribe (2003)
ϕ	0.028	Capital adjustment cost	SD of investment; Schmitt-Grohe & Uribe (2003)
ρ_z	0	Persistence of TFP	Consistency with exogenous income model
σ_z	0.013	SD resource GDP	Schmitt-Grohe & Uribe (2003)
Notes: Other parameters the same as in the exogenous income model (Table 3.1)			

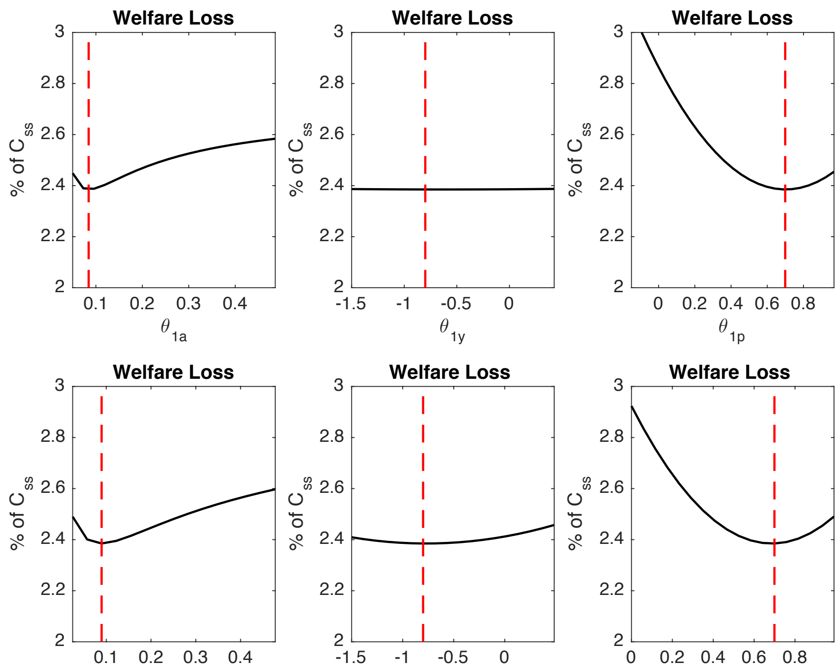

Figure 7.2: **How welfare loss changes with the Optimal Simple Rule coefficients (baseline calibration)**: the vertical line indicates OSR coefficients in table 3.2 which minimizes the welfare loss. Top row: fiscal rule coefficients for Ricardian HHs. Bottom row: fiscal rule coefficients for HtM HHs.

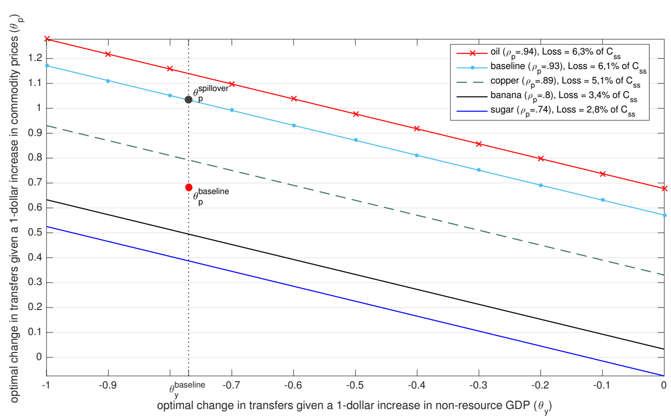

Figure 7.3: **Indifference Curves between θ_Y and θ_P, in the Exogenous Income Model with Spillovers** (baseline calibration$\beta_{YP} = 0.2$ and $\sigma_y = 0.04$ **):** combinations of θ_Y and θ_P along the line $\theta_P = intercept - 0.6 \times \theta_Y$ minimize the welfare loss. The optimal value of θ_A is 0.09.

7.4.3
IRF in the RBC model

Figure 7.4 shows how a commodity price shock in the RBC model increases in aggregate (public+private) assets and reduces interest rates (due to the debt-elastic spread) which then boosts capital accumulation and output non-resource sector.

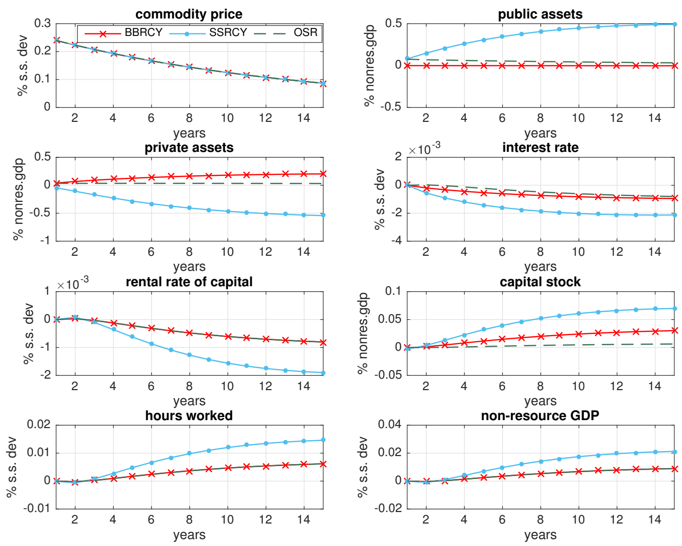

Figure 7.4: **Endogenous Feedback from Commodity Prices to Non-Resource GDP:** Impulse response function of key variables to an one standard-deviation (24%) shock of commodity prices in the baseline calibration of the RBC model.

7.4.4
The Importance of Persistence in the RBC model

Figure 7.5 plots the welfare loss of the SSR and BBR changing the persistence of commodity price shock (adjusting the variance of the shock as the persistence increases so the the variance of the commodity price ($V(\hat{p}_t) = \sigma_p^2/(1-\rho_p^2)$) is kept constant). Note that the cutoff 0.74 where BBR is better than SSR has moved to the left in comparison with the exogenous income model (0.9).

7.5
The Small Open Economy New-Keynesion Model

7.5.1
Model Description and Basic Defintions

Consider a small open economy model consisting of a small country (home) and the rest of the world (foreign, denoted with *). The small region has population n and the large region $1-n$. The home economy is inhabited by two types of households who have the same preferences and seek to maximize utility,

$$U_t^j = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{1}{1-\sigma} (C_t^j)^{1-\sigma} - \frac{1}{1+\varphi} (L_t^j)^{1+\varphi} \right] \qquad j \in \{\prime, \prime\prime\}$$

where the parameter σ is risk aversion and φ the Frisch elasticity. The superscript $\prime$ index the non-constrained (Ricardian) household and $\prime\prime$ the constrained (Hand-to-Mouth) household. A share $1-\omega$ of the housholds are Ricardians and ω are Hand-to-Mounth. L_t^j denotes hours of labour, and C_t^j is a composite consumption index defined by

$$C_t^j = \left[(1-\alpha)^{1/\eta} (C_{h,t}^j)^{\frac{\eta-1}{\eta}} + \alpha^{1/\eta} (C_{f,t}^j)^{\frac{\eta-1}{\eta}} \right]^{\frac{\eta}{1-\eta}}$$

where α measures the degree of oppeness of the economy, η is the elasticity of substitution between domestic and foreign goods, $C_{h,t}$ and $C_{f,t}$ are consumption indexes of domestic and foreign varieties that are aggregated according to the constant elasticity of substitution (CES) technology,

$$C_{h,t}^j = \left[\int_0^1 (C_{h,t}^j(i))^{\frac{\epsilon-1}{\epsilon}} di \right]^{\frac{\epsilon}{\epsilon-1}} \qquad C_{f,t}^j = \left[\int_0^1 (C_{f,t}^j(i))^{\frac{\epsilon-1}{\epsilon}} di \right]^{\frac{\epsilon}{\epsilon-1}}$$

where $i \in [0,1]$ denotes the good variety and ϵ the elasticity of substitution between varieties. Utility maximization is subject to a sequence of budget constraints of the form,

$$\int_0^1 P_{h,t}(i) C_{h,t}^j(i) di + \int_0^1 P_{f,t}(i) C_{f,t}^j(i) di = B_t^j - R_{t-1} B_{t-1}^j + \upsilon_t (B_t^{*,j} - R_{t-1}^* B_{t-1}^{*,j}) + \\ + W_t L_t^j + Tr_t^j + D_t^j$$

where $P_{f,t}(i)$ is the price of variety i imported from the foreign country and $P_{h,t}(i)$ the price of variety i produced in the home economy, both expressed in domestic currency. The Ricardian household can acquire two types of assets. A bond denominated in domestic currency that is issued by the Home government and pays nominal interest-rate R_t or a bond issued by the foreign country's government denominated in foreign currency with nominal interest rate R_t^*. υ_t is the nominal exchange rate, i.e., the price in dometic currency of 1 unit of the foreign currency. W_t denotes the nominal wage rate. Tr_t' and Π_t' denote nominal transfers and nominal profits. Domestic and Foreign producer price indexes (PPI) are defined as $P_{h,t} = \left[\int_0^1 (P_{h,t}(i))^{1-\epsilon} di\right]^{\frac{1}{1-\epsilon}}$ and $P_{f,t} = \left[\int_0^1 (P_{f,t}(i))^{1-\epsilon} di\right]^{\frac{1}{1-\epsilon}}$.

The hand-to-mouth household has no access to financial markets and doesn't own shares of the firms. We add these contraints,

$$B_t'' = B_t^{*,''} = D_t'' = 0 \tag{7-24}$$

The optimal allocation of any give expenditure within each category of goods yields the demand functions:

$$C_{h,t}^j(i) = (1-\alpha)\left(\frac{P_{h,t}(i)}{P_{h,t}}\right)^{-\epsilon} C_{h,t} \quad \text{and} \quad C_{f,t}^j(i) = \alpha\left(\frac{P_{f,t}(i)}{P_{f,t}}\right)^{-\epsilon} C_{f,t}$$

It follows from the last equation that $P_{h,t}C_{h,t^j} = \int_0^1 P_{h,t}(i)C_{h,t}^j(i)di$ and $P_{f,t}C_{f,t}^j = \int_0^1 P_{l,t}(i)C_{f,t}^j(i)di$. The optimal allocation of expenditures between domestic and imported goods is given by

$$C_{h,t}^j = (1-\alpha)\left(\frac{P_{h,t}}{P_t}\right)^{-\eta} C_t^j \quad \text{and} \quad C_{f,t}^j = \alpha\left(\frac{P_{f,t}}{P_t}\right)^{-\eta} C_t^j$$

where P_t is the consumer price index (CPI) defined as,

$$P_t = \left[(1-\alpha)P_{h,t}^{1-\eta} + \alpha P_{f,t}^{1-\eta}\right]^{\frac{1}{1-\eta}} \tag{7-25}$$

Accordingly, total consumption expenditures by domestic households is given by $P_{f,t}C_{f,t}^j + P_{h,t}C_{h,t}^j = P_tC_t^j$. Thus household's budget constraints can be writtern as

$$\upsilon_t B_t^{*,j} = \upsilon_t R_{t-1}^* B_{t-1}^{*,j} + (R_{t-1}B_{t-1} - B_t) + W_tL_t^j + Tr_t^j + (1-\omega)^{-1}D_t^j - P_tC_t^j$$
$$j \in \{\prime, \prime\prime\}$$

we want to express all variables in real terms. Divinding the last equation on both sides by $P_t^* \upsilon_t$ yields,

$$b_t^{*,j} = \frac{R_{t-1}^*}{1+\pi_t^*} b_{t-1}^{*,j} + Q_t^{-1}\left[\left(\frac{R_{t-1}}{1+\pi_t} b_{t-1} - b_t\right) + w_t L_t^j + tr_t^j + (1-\omega)^{-1} d_t^j - C_t^j\right]$$
$$j \in \{\prime, \prime\prime\} \tag{7-26}$$

where $x_t \equiv X_t/P_t$, $x_t^* \equiv X_t^*/P_t^*$ and $Q_t \equiv \frac{\upsilon_t P_t^*}{P_t}$ is the real exchange rate. The bilateral terms of trade between the domestic economy and the foreign country is defined as $S_t \equiv \frac{P_{f,t}}{P_{h,t}}$. Furthermore, we assume that the Law of One Price (LOP) (or Producer Currency Pricing (PCP)) holds so that at all times, the price of a given variety in different countries is identical once expressed in the same currency.

$$P_{f,t} = \upsilon_t P_t^* \tag{7-27}$$
$$S_t = \frac{\upsilon_t P_t^*}{P_{h,t}} \tag{7-28}$$
$$Q_t = \left[(1-\alpha) S_t^{\eta-1} + \alpha\right]^{\frac{1}{\eta-1}} \tag{7-29}$$

The last expressions are valid because the LOP also holds at the foreign PPI level ($P_{f,t} = \upsilon_t P_{f,t}^*$) and also because the small home economy has negligible impact on foreign PPI ($P_{f,t}^* = P_t^*$).

7.5.2
The Demand Side of the Economy

Investors arbitrage with domestic and foreign bonds until their expected discounted returns are equalized across the domestic and foreign households.

$$R_t \mathbb{E}_t \left(\frac{C'_{t+1}}{C'_t}\right)^{-\sigma} \frac{P_t}{P_{t+1}} = R_t^* \mathbb{E}_t \left(\frac{C'_{t+1}}{C'_t}\right)^{-\sigma} \frac{P_t}{P_{t+1}} \frac{\upsilon_{t+1}}{\upsilon_t} = R_t^* \mathbb{E}_t \left(\frac{C^*_{t+1}}{C^*_t}\right)^{-\sigma} \frac{P_t^*}{P_{t+1}^*} = 1$$

We can combine the last equations to yield three important equilibrium conditions,

$$\Theta_t \equiv \frac{\mathbb{E}_t C_{t+1}^{*-\sigma}/P_{t+1}^*}{\mathbb{E}_t C_{t+1}'^{-\sigma} \upsilon_{t+1}/P_{t+1}} \tag{7-30}$$

$$C_t' = \Theta_t^{1/\sigma} Q^{1/\sigma} C_t^* \tag{7-31}$$

$$R_t \mathbb{E}_t \left(\frac{C_{t+1}'}{C_t'} \right)^{-\sigma} \frac{P_t}{P_{t+1}} = 1 \tag{7-32}$$

$$\mathbb{E}_t \left(\frac{C_{t+1}'^{-\sigma}/P_{t+1}}{C_t'^{-\sigma}/P_t} \right) \left(R_t - R_t^* \frac{\upsilon_{t+1}}{\upsilon_t} \right) = 0 \tag{7-33}$$

where Θ_t measures the wedge of the value of an extra dolar between domestic and foreign households. Equation (7-31) relates domestic and foreign consumption. Under complete markets $\Theta_t = 1$ across all states and domestic consumption depends only on foreign consumption and the real exchange rate. The last equation is the uncoverd interest rate parity (UIP) and simply rules out excess expected returns.

We define aggregate domestic demand as

$$C_t = \omega C_t' + (1-\omega) C_t'' \tag{7-34}$$

7.5.3 The Supply-Side of the Economy

Price Setting. A firm in the home economy produces a differentiated good witha linear technology represented by the production function,

$$Y_t(j) = Z_t L_t(j) \qquad i \in [0,1]$$

Real marginal costs deflated by Home PPI is given by,

$$MC_{h,t} = (1-\tau)\frac{W_t}{P_{h,t} Z_t} = (1-\tau)\frac{w_t}{Z_t}\frac{S_t}{Q_t} \tag{7-35}$$

where τ is an employment subsidy to eliminate distortions caused by monopoly power. Note that the firm's markup is given by $\mu_t = \frac{P_{h,t}}{MC_t} = \frac{1}{MC_{h,t}}$. First-order appro.x of the marginal cost yields,

We consider a Calvo price setting, where in every period, a randomly selected fraction $1-\delta$ of firms can reset their prices. When setting a new price in period t a firm j seeks to maximize the current value of its dividend stream, conditional on that price being effective. the first-order condition,

$$\sum_{k=0}^{\infty}(\beta\delta)^k \mathbb{E}_t \left\{ C_{t+k}^{-\sigma} Y_{t+k}^h \frac{P_{h,t-1}}{P_{t+k}} \left(\frac{\bar{P}_{h,t}}{P_{h,t-1}} - \frac{\epsilon}{\epsilon-1} \Pi_{t-1,t+k}^h MC_{h,t+k} \right) \right\} = 0 \quad (7\text{-}36)$$

where $\bar{P}_{h,t}$ is the optimal price and $\Pi_{t-1,t+k} \equiv \frac{P_{h,t+k}}{P_{h,t-1}}$.

The dynamic of the domestic price index is described by the equation

$$P_{h,t} = \left[\delta P_{h,t-1}^{1-\epsilon} + (1-\delta)\bar{P}_{h,t}^{1-\epsilon} \right]^{\frac{1}{1-\epsilon}} \quad (7\text{-}37)$$

Labor Market. Labor is demand by firms according to the following equation,

$$L_t \equiv \int_0^1 L_t(j)dj = \frac{Y_t^h \Delta_t}{Z_t} \quad (7\text{-}38)$$

where $Y_{h,t} \equiv \left[\int_0^1 Y_{h,t}(j)^{\frac{\epsilon-1}{\epsilon}} \right]^{\frac{\epsilon}{\epsilon-1}}$ and $\Delta_t \equiv \int_0^1 \left(\frac{P_{h,t}(j)}{P_{h,t}} \right)^{-\epsilon}$ is a measure of price dispertion. Labor is supply by households according to the following condition,

$$L_t^{j\varphi} C_t^{j\sigma} = w_t \qquad j \in \{\prime, \prime\prime\} \quad (7\text{-}39)$$

Aggregate labor supply is given by

$$L_t = \omega L_t' + (1-\omega) L_t'' \quad (7\text{-}40)$$

Profits.

$$d_t' = \frac{Q_t}{S_t} Y_{h,t} - (1-\tau) w_t L_t \quad (7\text{-}41)$$

7.5.4 Market Clear

Market clearing for good i produced in the home country requires,

$$
\begin{aligned}
Y_{h,t}(i) &= C_{h,t}(i) + \left(\frac{1-n}{n}\right) C^*_{h,t}(i) \\
&= \left(\frac{P_{h,t}(i)}{P_{h,t}}\right)^{-\epsilon} \left[(1-\alpha)\left(\frac{P_{h,t}}{P_t}\right)^{-\eta} C_t + \alpha^* \left(\frac{1-n}{n}\right)\left(\frac{P_{h,t}}{\upsilon_t P^*_t}\right)^{-\eta} C^*_t\right]
\end{aligned}
$$

where n is the population of the small country relative to the population of the rest of the world and α^* is the degree of openess of the foreign country (subject to $\alpha^* = \frac{n}{n-1}\alpha$), $C^*_{h,t}(i)$ denotes foreign's country demand for variety i produced in the home country. C^*_t is the aggregate consumption of the foreign country which is assumed to follow an AR process in logs with zero mean, persistence ρ_{c^*} and standard deviation σ_{c^*}, and P^*_t the consumer-price index of the foreign country. Substituting the last expression in the agregation technology $Y_{h,t} = \left[\int_0^1 Y_{h,t}(i)^{\frac{\epsilon-1}{\epsilon}}\right]^{\frac{\epsilon}{\epsilon-1}}$ and the definition of $P_{h,t}$, yields,

$$
\begin{aligned}
Y_{h,t} &= (1-\alpha)\left(\frac{P_{h,t}}{P_t}\right)^{-\eta} C_t + \alpha^* \left(\frac{1-n}{n}\right)\left(\frac{P_{h,t}}{\upsilon_t P^*_t}\right)^{-\eta} C^*_t \\
&= (1-\alpha)\left(\frac{Q_t}{S_t}\right)^{-\eta} C_t + \alpha^* \left(\frac{1-n}{n}\right) S^\eta_t C^*_t
\end{aligned}
$$

substituting (7-31) into the last expression yields,

$$
Y_{h,t} = S^\eta_t C^*_t \left[(1-\alpha)\Theta_t Q_t^{\frac{1}{\sigma}-\eta} + \alpha^* \left(\frac{1-n}{n}\right)\right] \tag{7-42}
$$

The last expression equates aggregate demand (RHS) and aggregate supply (LHS).

7.5.5 The Public Sector

The government receive exogenous resource income in foreign currency, $P^R_t Y^R$, issue nominal bonds in domestic currency ,B_t, trades with foregin bonds, A_t, and makes transfers to househodls,

$$
A^*_t = R^*_{t-1}A^*_{t-1} + P^{R*}_t Y^R - \upsilon_t^{-1}\left[(1-\omega)Tr'_t + \omega Tr''_t - \tau W_t L_t + (R_{t-1}B_{t-1} - B_t)\right]
$$

Divinding both sides of the the last expression by P^*_t and rearranging yields,

$$a_t^* = \frac{R_{t-1}^*}{1+\pi_t^*} a_{t-1}^* + p_t^{R*} Y^R - Q_t^{-1} \left[(1-\omega) tr_t' + \omega tr_t'' + \tau w_t L_t - \left(\frac{R_{t-1}}{1+\pi_t} b_{t-1} - b_t \right) \right] \tag{7-43}$$

where $p_t^{R*} \equiv \frac{P_t^R}{P_t^*}$ is the price of the commodty good deflated by the foreign CPI.

7.5.6 Monetary Policy

We assume that the monetary policy follows a domestic inflation targeting regime (DIT) and hence sets the nominal interest rate according to the Taylor rule,

$$R_t = \left(\frac{P_{h,t}}{P_{h,t-1}} \right)^{\phi_\pi} exp(s_t^r) \tag{7-44}$$

where s^r is a shock of monetary policy. Following Schimitt-Grohe & M. Uribe (2003) we induce stationary in this model by assuming that Ricardian households and the government face and debt-elastic interest rate spread,

$$R_t^* = R_{ss} + \psi e^{-(A_{t-1}+(1-\omega)B_{t-1}-(A_{ss}+(1-\omega)B_{ss}))\chi_t} - \psi \tag{7-45}$$

where χ_t is an spread shock with persistence ρ_χ and variance σ_χ^2.

7.5.7 Fiscal Policy

The fiscal authority makes transfers to households according to a fixed rule. We assume that the transfers are function of (1) the value of public assets in terms of the home coog, (2) real non-resource GDP and (3) the value of commodity revenue in terms of home good.

$$tr_t' = Tr_{ss}' + \theta_a' Q_t (a_{t-1}^* - A_{ss}) + \theta_y' \frac{Q_t}{S_t} (Y_{h,t} - Y_{h,ss}) + \theta_p' Q_t Y^R (p_t^{R*} - P_{ss}^R) \tag{7-46}$$

$$tr_t'' = Tr_{ss}'' + \theta_a'' Q_t (a_{t-1}^* - A_{ss}) + \theta_y'' \frac{Q_t}{S_t} (Y_{h,t} - Y_{h,ss}) + \theta_p'' Q_t Y^R (p_t^{R*} - P_{ss}^R) \tag{7-47}$$

7.5.8
Other Definitions

Accounting Indentities

$$ps_t^* = p_t^{R*}Y^R - Q_t^{-1}\left[(1-\omega)tr_t' + \omega tr_t'' + \tau w_t L_t\right] \quad \text{[Primary Surplus of the Public Sector]} \tag{7-48}$$

$$tb_t^* = S_t^{-1}Y_{h,t} + p_t^{R*}Y^R - Q_t^{-1}C_t \quad \text{[Trade Balance]} \tag{7-49}$$

$$ca_t^* = tb_t^* + (R_t^* - 1)(a_{t-1}^* + (1-\omega)b_t^*) \quad \text{[Current Account]} \tag{7-50}$$

Exogenous Process

$$P_t^{R*} = (P_{t-1}^{R*})^{\rho_p} exp(\epsilon_t^p) \qquad \epsilon_t^p \sim N(0,\sigma_p) \tag{7-51}$$

$$Z_t = Z_{t-1}^{\rho_z} exp(\epsilon_t^z) \qquad \epsilon_t^z \sim N(0,\sigma_z) \tag{7-52}$$

$$\chi_t = \chi_{t-1}^{\rho_\chi} exp(\epsilon_t^\chi) \qquad \epsilon_t^\chi \sim N(0,\sigma_\chi) \tag{7-53}$$

$$s_t^r = s_{t-1}^{r\rho_r} exp(\epsilon_t^r) \qquad \epsilon_t^\chi \sim N(0,\sigma_r) \tag{7-54}$$

Specifc Assumptions. We will assume that the foreign country remains in steady state at all periods. Also, we don't want to deal with borrowing at domestic currency for now. We then add the following equilibrium conditions,

$$P_t^* = 1 \tag{7-55}$$

$$C_t^* = C^* \tag{7-56}$$

$$b_t' = 0 \tag{7-57}$$

7.5.9
Equilibrium

We may now define a rational expectations equilibirum as a collection of 39 stochastic processes $\{b_t'', b_t', b_t^{*\prime\prime}, b_t^{*\prime}, d_t^{*\prime\prime}, d_t^{*\prime}, C_t'', C_t', C_t, C_t^*, Y_{h,t}, R_t^*, R_t, Q_t, S_t\}$ $\{w_t, L_t', L_t'', L_t, tr_t', tr_t'', P_t, P_{h,t}, \bar{P}_{h,t}, P_{f,t}\}$, $\{P_t^*, \Theta_t, \upsilon_t, MC_{h,t}, \bar{P}_{h,t}, \Delta_t, a_t^*, tb_t^*, ps_t^*, ca_t^*, Z_t, p_t^{R*}, s_t^r, \chi_t\}$, with each endogenous variable specified as a function of the history of exogenous disturbances $\{\epsilon_t^p, \epsilon_t,^z \epsilon_t^\chi, \epsilon_t^r\}$ to that date, that satisfy each of the 39 conditions specified in the set of equations (7-24) to (7-57).

7.6 The First Best Allocation

The ricardian and the hand-to-mouth households are the equal from the point of view of the central planner so our heterogenous-agent set up boils down to a standard representative-agent utility maximization problem. The planner has to choose $\{C_t, L_t, Q_t, S_t, Y_t\}$ to maximize the houshold's itertemporal utility function in the economy without shocks.

$$\text{maximize} \quad \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{1}{1-\sigma} C_t^{1-\sigma} - \frac{1}{1+\varphi} L_t^{1+\varphi} \right]$$

$$\text{subject to} \quad C_t = \Theta_t C^* Q_t^{\frac{1}{\sigma}} \tag{7-58}$$

$$Y_t^h = \left[(1-\alpha) Q^{\frac{1}{\sigma}-\eta} \Theta_t + \alpha^* \left(\frac{1-n}{n} \right) \right] C^* S_t^{\eta} \tag{7-59}$$

$$Q_t = \left[(1-\alpha) S_t^{\eta-1} + \alpha \right]^{\frac{1}{\eta-1}} \tag{7-60}$$

$$Y_t^h = Z_{ss} L_t \tag{7-61}$$

$$A_{ss} + (1-\omega) B_{ss}^* = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^*}{C_0^*} \right)^{-\sigma} \left(Q_t^{-1} C_t - S_t^{-1} Y_t^h - Y^R P_{ss}^R \right) \tag{7-62}$$

Because the problem is convex and there is no shocks, optimality implies that the optimal path of the control variables are constant over time (so we can drop the time index from all variables). We can solve (7-58) - (7-62) for C and L as fucntions of S and C^*,

$$C(S, C^*) = \left[(1-\alpha) S^{\eta-1} + \alpha \right]^{\frac{\eta}{\eta-1}} \left\{ \alpha^{-1} W_0 S^{-1} + \left(\frac{\alpha^*}{\alpha} \right) \left(\frac{1-n}{n} \right) C^* S^{\eta-1} \right\}$$

$$L(S, C^*) = \frac{S}{ZQ} C(S, C^*) - \frac{S W_0}{Z}$$

where $W_0 \equiv Y^R P^R + (1-\beta)(A_{ss} + B_{ss})$.

Efficient Steady-State with $S = 1$. We want to find a steady state in which the planner has no incentine to manipulate the terms of trade in favor of the domestic household. We want to find values of C^* such that $S = 1$ maximizes the houshold's period utility. The first-order condition of the planner's problem evaluated at $S = 1$ is given by,

$$C(1, C^*)^{-\sigma} C_1(1, C^*) - Z^{-(1+\varphi)} \left[\alpha C(1, C^*) + C_1(1, C^*) - W_0 \right] \left(C(1, C^*) - W_0 \right)^{\varphi} = 0$$

where

$$C(1, C^*) = \alpha^{-1} W_0 + \left(\frac{\alpha^*}{\alpha}\right)\left(\frac{1-n}{n}\right) C^*$$
$$C_1(1, C^*) = [\eta(1-\alpha) + (\eta - 1)]\, C(1, C^*) - \eta \alpha^{-1} W_0$$

We can solve numerically the last expression for C^*. We can solve analytically for the special case in which W_0. In this case,

$$C^* = Z^{\frac{1+\varphi}{\varphi+\sigma}} \left(\frac{(2-\alpha)\eta - 1}{(2-\alpha)\eta - 1 + \alpha}\right)^{\frac{1}{\varphi+\sigma}} \left(\frac{\alpha}{\alpha^*}\right)\left(\frac{n}{1-n}\right)$$

we can now compute all steady state variables,

$$C^{\varphi} Z^{-\varphi} = AC^{\sigma} \tag{7-63}$$

$$S^{opt} = Q^{opt} = 1$$
$$\Theta^{opt} = \alpha^{-1} W_0 C^{*-1} + \left(\frac{\alpha^*}{\alpha}\right)\left(\frac{1-n}{n}\right)$$
$$C^{opt} = \alpha^{-1} W_0 + \left(\frac{\alpha^*}{\alpha}\right)\left(\frac{1-n}{n}\right) C^*$$
$$Y_h^{opt} = C^{opt} - W_0$$
$$L^{opt} = Z^{-1} Y_h^{opt}$$

The closed economy case ($\alpha = 0$) satisfies the usual condition for efficiency $C^{\sigma} L^{\varphi} = Z$ with $Y_h^{closed} = Z^{\frac{1+\varphi}{\varphi+\sigma}}$. Note however, that this is not the case in the open economy ($\alpha > 0$). The open economy output is lower than the one that equates the marginal rate of substituition between leisure and consumption with the marginal product of labor because one extra unit of consumption requires more than one unit of production because of the correlation of domestic and foreign consumption.

7.6.0.1
The Flexible Price Economy and the Optimal Subisdy Rate

In the flex-price economy the following expressing holds every period,

$$P^{h,flex} = (1-\tau)\left(\frac{\epsilon}{\epsilon-1}\right)\frac{W^{flex}}{Z}$$

divinding both sides by P^{flex}

$$\frac{Q^{flex}}{S^{flex}} = (1-\tau)\left(\frac{\epsilon}{\epsilon-1}\right)\frac{L^{\varphi,flex}C^{\sigma,flex}}{Z}$$

we want to find τ such that $Q^{flex} = Q^{opt}$, $S^{flex} = S^{opt}$, $L^{flex} = L^{opt}$ nad $C^{flex} = C^{opt}$ satisfy the last equation. Substituting these values in the last expression and solving for τ,

$$\tau = 1 - \left(\frac{\epsilon-1}{\epsilon}\right)\left(\frac{(2-\alpha)\eta - 1 + \alpha}{(2-\alpha)\eta - 1}\right) \leq \frac{1}{\epsilon} = \tau^{closed} \tag{7-64}$$

Note that the optimal τ does not quite converge to Werning & Farhi (2012) as $\gamma \Rightarrow \eta$. I think they have an algebraic mistake. Why the optimal subsidy is lower in the SOE case relative to the closed economy case if the SOE has monopoly power over the Home good? The reason is that we chose the exact value of C^* tha offsets the incentive that the Home country has to exploit its monopoly power (i.e. such that $S = 1$ is optimal). Moreover, we saw that the efficient output in the SOE is lower than in the closed economy case, and hence the subsidy that renders the flexible economy efficient in the open economy must be lower than the usual $\tau = 1/\epsilon$ of the closed economy.

We choose a symmetric steady state where consumption and hours worked is the same across households. We assume that $B_{ss} = B^*_{ss} = 0$ and that the firms profits are evenly shared by households. We impose that $C_{ss} = C_{opt}$ and,

$$W_{ss} = Z_{ss}\left(\frac{\epsilon-1}{\epsilon}\right)\left(\frac{1}{1-\tau}\right)$$
$$L_{ss} = (W_{ss}C_{ss}^{-\sigma})^{\frac{1}{\varphi}} = L^{opt}$$
$$Y^h_{ss} = Z_{ss}L_{ss} = Y^h_{opt}$$

We choose Z_{ss} so that $Y^h_{ss} = 1$. The following transfers support this allocation in equilibrium.

$$Tr'_{ss} = (R^*_{ss} - 1)A_{ss} - (R_{ss} - 1)B_{ss} + Y^R P^R_{ss} - \tau W_{ss} L_{ss} - \omega(1-\omega)^{-1} D_{ss}$$
$$Tr''_{ss} = (R^*_{ss} - 1)A_{ss} - (R_{ss} - 1)B_{ss} + Y^R P^R_{ss} - \tau W_{ss} L_{ss} + D_{ss}$$

7.6.1 Welfare Function

$$U_t = (1-\omega_U)\left[\frac{C_t'^{1-\sigma}}{1-\sigma} - \frac{L_t'^{1+\varphi}}{1+\varphi}\right] + \omega_U\left[\frac{C_t''^{1-\sigma}}{1-\sigma} - \frac{L_t''^{1+\varphi}}{1+\varphi}\right]$$

Second-order Taylor Expansion of each household's utility function yields,

$$U_t - U = C_{ss}^{1-\sigma}\left(\hat{c}_t - \frac{\sigma}{2}\hat{c}_t^2\right) - L_{ss}^{1+\varphi}\left(\hat{l}_t + \frac{\varphi}{2}\hat{l}_t^2\right) \tag{7-65}$$

$$\gamma = (1-\beta)E_0\sum_{t=0}^{\infty}\beta^t\left\{\frac{(1-\omega_U)\left[U'_t - U'_{SS}\right] + \omega_U\left[U''_t - U''_{SS}\right]}{(1-\omega_U)U'_{C'}[C'] + \omega_U U''_{C''}C''}\right\}$$

$$\frac{\gamma}{(1-\beta)} = \tag{7-66}$$
$$E_0\sum_{t=0}^{\infty}\beta^t\left\{\frac{(1-\omega_U)\left[C_{ss}'^{1-\sigma}\left(\hat{c}'_t - \frac{\sigma}{2}\hat{c}_t'^2\right) - L_{ss}'^{1+\varphi}\left(\hat{l}'_t + \frac{\varphi}{2}\hat{l}_t'^2\right)\right] + \omega_U\left[C_{ss}''^{1-\sigma}\left(\hat{c}''_t - \frac{\sigma}{2}\hat{c}_t''^2\right) - L_{ss}''^{1+\varphi}\left(\hat{l}''_t + \frac{\varphi}{2}\hat{l}_t''^2\right)\right]}{(1-\omega_U)C_{ss}'^{1-\sigma} + \omega_U C_{ss}''^{1-\sigma}}\right\} \tag{7-67}$$

Define,

$$\Phi'_c \equiv \frac{(1-\omega_U)C_{ss}'^{1-\sigma}}{(1-\omega_U)C_{ss}'^{1-\sigma} + \omega_U C_{ss}''^{1-\sigma}}$$
$$\Phi''_c \equiv \frac{\omega_U C_{ss}''^{1-\sigma}}{(1-\omega_U)C_{ss}'^{1-\sigma} + \omega_U C_{ss}''^{1-\sigma}}$$
$$\Phi'_l \equiv \frac{(1-\omega_U)L_{ss}'^{1+\varphi}}{(1-\omega_U)C_{ss}'^{1-\sigma} + \omega_U C_{ss}''^{1-\sigma}}$$
$$\Phi''_l \equiv \frac{\omega_U L_{ss}''^{1+\varphi}}{(1-\omega_U)C_{ss}'^{1-\sigma} + \omega_U C_{ss}''^{1-\sigma}}$$

Note that,

$$\hat{l}_t = \hat{y}^h_t - \hat{z}_t + \frac{\epsilon}{2}\mathbb{V}_i(P_{h,t}(i)) \qquad \text{(proof in Gali \& Monacelli (2005))}$$
$$\hat{l}_t = \omega\left(\frac{L'_{ss}}{L_{ss}}\right)\hat{l}'_t + (1-\omega)\left(\frac{L''_{ss}}{L_{ss}}\right)\hat{l}''_t$$

hence,

$$\hat{l}'_t = \left(\frac{L_{ss}}{L'_{ss}}\right)\left(\frac{1}{\omega}\right)\left[\hat{y}^h_t - \hat{z}_t + \frac{\epsilon}{2}\mathbb{V}_i(P_{h,t}(i))\right] - \left(\frac{L''_{ss}}{L'_{ss}}\right)\left(\frac{1-\omega}{\omega}\right)\hat{l}''_t \tag{7-68}$$
$$\hat{l}''_t = \left(\frac{L_{ss}}{L''_{ss}}\right)\left(\frac{1}{1-\omega}\right)\left[\hat{y}^h_t - \hat{z}_t + \frac{\epsilon}{2}\mathbb{V}_i(P_{h,t}(i))\right] - \left(\frac{L'_{ss}}{L''_{ss}}\right)\left(\frac{\omega}{1-\omega}\right)\hat{l}'_t \tag{7-69}$$

substituting the two last expression into (7-66) and using that *up to a first order* the uncondicional expectation of all endogenous variables is zero when expressed as percentage deviation from their steady state ($\mathbb{E}_0\hat{c}'_t = \mathbb{E}_0\hat{c}''_t = \mathbb{E}_0\hat{l}'_t = \mathbb{E}_0\hat{l}''_t = \mathbb{E}_0\hat{y}^h_t = \mathbb{E}_0\hat{z}_t = 0$), hence,

$$\frac{\gamma}{(1-\beta)} =$$
$$= \frac{1}{2}E_0\sum_{t=0}^{\infty}\beta^t\left\{\sigma(\Phi'_c\hat{c}'^2_t + \Phi''_c\hat{c}''^2_t) + \varphi(\Phi'_l\hat{l}'^2_t + \Phi''_l\hat{l}''^2_t) + \frac{\epsilon}{2}\left(\Phi'_l\left(\frac{L_{ss}}{L'_{ss}}\right)\left(\frac{1}{\omega}\right) + \Phi''_l\left(\frac{L_{ss}}{L''_{ss}}\right)\left(\frac{1}{1-\omega}\right)\right)\mathbb{V}_i(P_{h,t}(i))\right\}$$

Woodford (2003, chapter 6) shows that the discounted infinite sum of cross-sectional variation in domestic prices is proporcional to the discounted infinite sum of squared domestic inflation rate. That is,

$$\sum_0^{\infty}\beta^t\mathbb{V}_i(P_{h,t}(i)) = \frac{1}{\lambda}\sum_0^{\infty}\beta^t\pi^2_{h,t}$$

where $\lambda = \frac{(1-\delta)(1-\beta\delta)}{\delta}$ and $1-\delta$ is the randomly selected fraction of firms that can reset their price each period. We can then rewrite the cost of the business cycle as percentage of steady state consumption as

$$\gamma =$$
$$-\frac{1}{2}\left\{\Phi'_c\sigma\mathbb{V}(\hat{c}'_t) + \Phi'_l\varphi\mathbb{V}(\hat{l}'_t) + \Phi''_c\sigma\mathbb{V}(\hat{c}''_t) + \Phi''_l\varphi\mathbb{V}(\hat{l}''^2_t) + \frac{\epsilon}{\lambda}\left(\Phi'_l\left(\frac{L_{ss}}{L'_{ss}}\right)\left(\frac{1}{\omega}\right) + \Phi''_l\left(\frac{L_{ss}}{L''_{ss}}\right)\left(\frac{1}{1-\omega}\right)\right)\mathbb{V}(\pi_{h,t})\right\}$$

7.6.2 Special Case: $\omega = 1/2$ and $L'_{ss} = L''_{ss} \Rightarrow .C'_{ss} = C''_{ss}$

$$\frac{\gamma}{(1-\beta)} = \frac{1}{2}E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \left(\hat{c}'_t - \frac{\sigma}{2}\hat{c}'^2_t \right) + \left(\hat{c}''_t - \frac{\sigma}{2}\hat{c}''^2_t \right) - \frac{L'^{1+\varphi}_{ss}}{C'^{1-\sigma}_{ss}} \left(\hat{l}'_t + \frac{\varphi}{2}\hat{l}'^2_t + \hat{l}''_t + \frac{\varphi}{2}\hat{l}''^2_t \right) \right\}$$

using that $\hat{l}' + \hat{l}''_t = 2\hat{l}_t = 2(\hat{y}^h_t - \hat{z}_t + \frac{\epsilon}{2}\mathbb{V}_i(P_{h,t}(i)))$,

$$\frac{\gamma}{(1-\beta)} = \frac{1}{2}E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \left(\hat{c}'_t - \frac{\sigma}{2}\hat{c}'^2_t \right) + \left(\hat{c}''_t - \frac{\sigma}{2}\hat{c}''^2_t \right) - \frac{L'^{1+\varphi}_{ss}}{C'^{1-\sigma}_{ss}} \left(\frac{\varphi}{2}\hat{l}'^2_t + \frac{\varphi}{2}\hat{l}''^2_t + 2(\hat{y}^h_t - \hat{z}_t + \frac{\epsilon}{2}\mathbb{V}_i(P_{h,t}(i)) \right) \right\}$$

because $\mathbb{E}_0\hat{c}'_t = \mathbb{E}_0\hat{c}''_t = \mathbb{E}_0\hat{y}^h_t = \mathbb{E}_0\hat{z}_t = 0$,

$$\frac{\gamma}{(1-\beta)} = -\frac{\sigma}{4(1-\beta)} \left(\mathbb{V}(\hat{c}'_t) + \mathbb{V}(\hat{c}''_t) \right) - \frac{\varphi L'^{1+\varphi}_{ss}}{4(1-\beta)C'^{1-\sigma}_{ss}} \left(\mathbb{V}(\hat{l}'_t) + \mathbb{V}(\hat{l}''_t) \right) + E_0 \sum_{t=0}^{\infty} \beta^t \frac{\epsilon}{2}\mathbb{V}_i(P_{h,t}(i))$$

Woodford (2003, chapter 6) shows that the discounted infinite sum of cross-sectional variation in domestic prices is proporcional to the discounted infinite sum of squared domestic inflation rate. That is,

$$\sum_{0}^{\infty} \beta^t \mathbb{V}_i(P_{h,t}(i)) = \frac{1}{\lambda} \sum_{0}^{\infty} \beta^t \pi^2_{h,t}$$

where $\lambda = \frac{(1-\delta)(1-\beta\delta)}{\delta}$ and $1-\delta$ is the randomly selected fraction of firms that can reset their price each period. We can then rewrite the cost of the business cycle as percentage of steady state consumption as

$$\gamma = -\frac{1}{2}\left[\frac{\sigma}{2}\left(\mathbb{V}(\hat{c}'_t) + \mathbb{V}(\hat{c}''_t) \right) + \frac{\varphi}{2}\frac{L'^{1+\varphi}_{ss}}{C'^{1-\sigma}_{ss}} \left(\mathbb{V}(\hat{l}'_t) + \mathbb{V}(\hat{l}''_t) + \frac{2\epsilon}{\lambda}\mathbb{V}(\pi_{h,t}) \right) \right]$$

Note that $\lambda \to \infty$ in the flexible prices case.

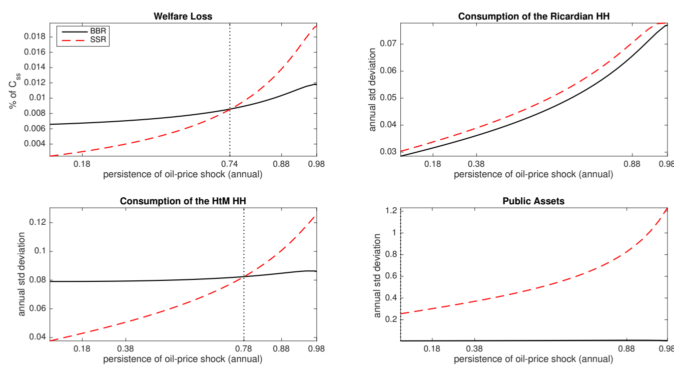

Figure 7.5: **Welfare loss and shock persistence with BBR and SSR**: Benchmark calibration of the RBC model and keeping the total variance of the commodity price shock constant $(V(\hat{p}_t) = \sigma_p^2/(1-\rho_p^2))$.