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Gabrielle Saller Nornberg

Methods of the Regularity Theory in the Study of Partial Differential Equations with Natural Growth in the Gradient

Tese de Doutorado

Thesis presented to the Programa de Pós–graduação em Matemática of PUC-Rio in partial fulfillment of the requirements for the degree of Doutor em Matemática.

Advisor: Prof. Boyan Slavchev Sirakov

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To my grandmother Ceni (in memoriam), with love.

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I wish one day my master colleagues and I can mitigate Leandro's loss, one of the greatest talents that our country has ever seen, a great person too. I hope that our theses can honor his memory.

Abstract

Nornberg, Gabrielle Saller; Sirakov, Boyan (Advisor). **Methods of the Regularity Theory in the Study of Partial Differential Equations with Natural Growth in the Gradient**. Rio de Janeiro, 2018. 136p. Tese de doutorado – Departamento de Matemática, Pontifícia Universidade Católica do Rio de Janeiro.

In this Ph.D. thesis we study a class of uniformly elliptic partial differential equations of second order in fully nonlinear nondivergence form with superlinear growth in the gradient and measurable coefficients. For equations with quadratic growth, we prove that multiplicity of solutions occurs when the operator is not coercive. We investigate the qualitative behavior of the continuums of solutions obtained for a parameterized family of problems. For this, we extend the Caffarelli-Święch-Winter $C^{1,\alpha}$ regularity estimates to equations with at most quadratic gradient growth, showing that the solutions are continuously differentiable up to the boundary. Furthermore, we show a priori estimates in the uniform norm using purely nonlinear techniques in the nondivergence form, such as Harnack type inequalities and a Vázquez's strong maximum principle for equations of our type.

Keywords

Regularity;   Multiplicity of solutions;   Existence;   A priori estimates; Viscosity.

Resumo

Nornberg, Gabrielle Saller; Sirakov, Boyan. **Métodos da Teoria de Regularidade no Estudo de Equações Diferenciais Parciais com Crescimento Natural no Gradiente**. Rio de Janeiro, 2018. 136p. Tese de Doutorado – Departamento de Matemática, Pontifícia Universidade Católica do Rio de Janeiro.

Nesta tese de Doutorado estudamos uma classe de equações diferenciais parciais de segunda ordem, uniformemente elípticas, completamente não-lineares na forma não-divergência, com crescimento superlinear no gradiente e coeficientes mensuráveis. Para equações com crescimento quadrático, provamos que ocorre multiplicidade de soluções quando o operador não é coercivo e investigamos o comportamento qualitativo dos contínuos de soluções obtidos para uma família parametrizada de problemas. Para isso, estendemos a regularidade e as estimativas $C^{1,\alpha}$ de Caffarelli-Święch-Winter para equações com crescimento, no máximo quadrático, no gradiente, mostrando que as soluções são continuamente diferenciáveis até o bordo. Além disso, mostramos estimativas a priori na norma uniforme via técnicas puramente não-lineares na forma não-divergência, entre elas desigualdades do tipo Harnack e o princípio do máximo forte de Vázquez para equações de nosso tipo.

Palavras-chave

Regularidade; Multiplicidade de soluções; Existência; Estimativas a priori; Viscosidade.

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1
Introduction

This thesis is devoted to the study of regularity, existence and multiplicity of viscosity solutions for a class of fully nonlinear uniformly elliptic equations with quadratic growth in the gradient.

The notion of viscosity solution began to be developed within the differential equations universe in the 1980s, with the works of Crandall and Lions [1], [2]. Since then it has been an enormous influence on the study of fully nonlinear elliptic and parabolic PDEs. This is so because its definition – weaker, where only continuity of solutions is required – is purely based upon the maximum principle, which is advantageous when we have an equation in nondivergence form. The first notion of viscosity solution, namely C-viscosity [3], had to be improved to cover the case when the coefficients or/and the right hand side of the equation are no longer continuous but merely measurable, which lead to the introduction of the L^p-viscosity notion [4]. In general, without the continuity hypothesis on the nonlinearity, the uniqueness is lost even for the standard Dirichlet problem for linear equations with bounded coefficients [5]. Therefore, we can expect *multiplicity of solutions*.

On the other hand, the study of quasilinear elliptic equations with quadratic dependence in the gradient also had its beginning in the '80s, essentially with the works of Boccardo, Murat and Puel [6], [7] and became a relevant research topic which still develops. This type of nonlinearity often appears in risk-sensitive stochastic and large deviation problems, hence their practical importance. Moreover, the set of equations under the form $Lu = g(x, u, Du)$, where L is a second order general operator and g has quadratic growth in the gradient, is invariant with respect to smooth changes of the function u and variable x, what makes this class also theoretically important. This fact was first observed by Kazdan and Kramer in [8] and it is responsible for this class to be usually referred as having *natural* growth in the gradient.

To acquaint ourselves with the notation, we consider the following model equation which features a second order uniformly elliptic operator with quadratic growth in the gradient

$$Lu + c(x)u + \langle M(x)Du, Du \rangle = h(x) \qquad \text{(ME)}$$

in a bounded $C^{1,1}$ domain Ω in $\mathbb{R}^n$ with a Dirichlet boundary condition $u = 0$ on $\partial\Omega$. Here, $c, h \in L^p(\Omega)$, $p > n \geq 1$, $c \not\equiv 0$ and the matrix M satisfies the nondegeneracy condition $0 < \mu_1 I \leq M \leq \mu_2 I$; with solutions of this Dirichlet problem understood as being continuous up to the boundary, so bounded. In [6], [7] the authors carried out a rather complete study of solvability of strictly coercive equations in divergence form, that is, when L is the divergence of an expression of x, u, and Du. Strictly coercive for (ME) means that $c(x) \leq c_0 < 0$, and then uniqueness of solutions is to be expected, see [9], [10]. For weakly coercive equations (when $c \leq 0$), existence and uniqueness can be proved only under a smallness assumption on c and M, as was first observed by Ferone and Murat [11]. All these pioneering works use the weak integral formulation of PDEs in divergence form.

In [12] Sirakov showed that the same type of existence and uniqueness results extends to general fully nonlinear coercive (i.e. proper) equations in nondivergence form, with superlinear growth in the gradient and unbounded coefficients, by using techniques based on the maximum principle. Moreover, C^α regularity results were developed there for fully nonlinear uniformly elliptic equations with at most quadratic growth in the gradient, in the most general setting of measurable coefficients for L^p-viscosity solutions.

In that paper it was also observed, for the first time and with a simple example with the Laplacian (specifically for $L = \Delta$, $c = 1$, $M = I$, $h = 0$), that the solution set may be very different in the nonproper case $c > 0$, and in particular more than one solution may appear. This example leveraged the study of such equations over a new research strand, whose main objective is the qualitative analysis of the set of solutions. Subsequent results were established in [13], [14], [15], [16], [17], [18], still using energy methods.

This Ph.D. thesis can be seen as a natural continuation of the regularity, existence and multiplicity results initiated in [12]. In particular, our work attempts to contribute to the theory of nondivergence fully nonlinear equations with superlinear gradient growth and unbounded coefficients, in the setting of L^p-viscosity solutions. This thesis covers and develops the results of the following research articles

[19] Nornberg, G.; Sirakov, B. *A priori bounds and multiplicity for fully nonlinear equations with quadratic growth in the gradient.* arXiv:1802.01661;

[20] Nornberg, G. S. $C^{1,\alpha}$ *regularity for fully nonlinear elliptic equations with superlinear growth in the gradient.* arXiv:1802.01643.

We stress that nondivergence (fully nonlinear) equations with natural growth are particularly relevant for applications, since problems with such dependence in the gradient are abundant in control and game theory [21], and more recently in mean-field problems [22], where Hamilton-Jacobi-Bellman and Isaacs operators appear as infinitesimal generators of the underlying stochastic processes.

The greatest challenge in these studies is to obtain regularity and a priori estimates for the solutions. We establish complete results of this nature for fully nonlinear uniformly elliptic equations in nondivergence form. In chapter 3 we compile the general $C^{1,\alpha}$ regularity and estimates for fully nonlinear equations with superlinear growth in the gradient and unbounded coefficients.

Our results are valid when L in (ME) is a general Hamilton-Jacobi-Bellman operator. However, to accommodate the reader and to elucidate the main ideas, let us give the statement of some of our results for L^p-viscosity solutions of (ME), in the very particular case when L is a second order uniformly elliptic linear operator in nondivergence form; namely

$$Lu \; = \; \mathrm{tr}\,(A(x)D^2u) + b(x) \cdot Du\,, \tag{1.1}$$

where A is a matrix function which is continuous up to the boundary and whose eigenvalues belong to some finite and closed interval in $(0,+\infty)$ and b is some bounded vector function. Consider Ω, c, h and M under the assumptions above (actually, for the regularity results, M can be any bounded matrix).

Theorem 1.1 ($C^{1,\alpha}$ *Regularity*) *Let u be an L^p-viscosity solution of* (ME)*, with $u \in C(\overline{\Omega})$ and $\|u\|_{L^\infty(\Omega)} + \|h\|_{L^p(\Omega)} \leq C_0$. Then there exists $\alpha \in (0,1)$ such that $u \in C^{1,\alpha}(\overline{\Omega})$ and satisfies the estimate*

$$\|u\|_{C^{1,\alpha}(\overline{\Omega})} \leq C\,\{\|u\|_{L^\infty(\Omega)} + \|h\|_{L^p(\Omega)}\},$$

where C and α depend only on the L^p-norm of the coefficients of the equation, on the ellipticity constants of L and on C_0.

As an application, $W^{2,p}$ regularity results and a generalized Nagumo's lemma are established at the end of chapter 3. In particular, for the linear operator L, this regularity implies that (bounded) solutions of (ME) are strong,

in the sense that they belong to the Sobolev space $W^{2,p}(\Omega)$ and satisfy the equation at almost every point.

In chapter 4 we present miscellaneous results on first eigenvalues of nonlinear operators, in the sense of weighted eigenvalues with unbounded coefficients – they play an important role in the proof of the multiplicity and nonexistence results.

We consider the following parameterized version of (ME)

$$
\begin{cases}
-Lu &= \lambda c(x)u + \langle M(x)Du, Du \rangle + h(x) & \text{in} \quad \Omega \\
u &= 0 & \text{on} \quad \partial\Omega
\end{cases}
\tag{P_λ}
$$

indexed by $\lambda \in \mathbb{R}$, with coefficients $c, h \in L^\infty(\Omega)$ and $c \gneqq 0$. For a priori estimates and multiplicity results related to (P_λ), we also need to assume that the problem (P_0) has a solution u_0.

In section 5.2 we prove that solutions of (P_λ) are bounded in the uniform norm by a new method, as suggested in [12]. Upper bounds are based on some standard estimates from regularity theory, such as half-Harnack inequalities, and their recent boundary extensions in [23]. On the other hand, lower bounds are shown to be equivalent, somewhat surprisingly, to a Vázquez type strong maximum principle for our equations, which we also establish. Our methods are (necessarily) very different from those in the preceding works, since for us, even in the linear scenario, no integral formulation of the equation is available.

Theorem 1.2 (*A priori uniform bounds*) *Let* Λ_1, Λ_2 *with* $0 < \Lambda_1 < \Lambda_2$. *Then every* L^p-*viscosity solution* u *of* (P_λ) *satisfies*

$$
\|u^-\|_{L^\infty(\Omega)} \le C, \quad \text{for all } \lambda \in [0, \Lambda_2], \quad \|u^+\|_{L^\infty(\Omega)} \le C, \quad \text{for all } \lambda \in [\Lambda_1, \Lambda_2],
$$

where C *depends on* n, p, μ_1, μ_2, Ω, Λ_1, Λ_2, $\|b\|_{L^\infty(\Omega)}$, $\|c\|_{L^\infty(\Omega)}$, $\|h\|_{L^\infty(\Omega)}$, $\|u_0\|_{L^\infty(\Omega)}$, *the ellipticity constants of* L, *the* $C^{1,1}$ *character of the boundary, and the set where* $c > 0$.

As a consequence, we obtain the following multiplicity result.

Theorem 1.3 (*Multiplicity of solutions*) *For* $\lambda \le 0$, *the problem* (P_λ) *has a unique* L^p-*viscosity solution* u_λ. *Moreover, the set*

$$
\Sigma = \left\{ (\lambda, u) \in \mathbb{R} \times C^1(\overline{\Omega}) \, ; \, (\lambda, u) \text{ solves } (P_\lambda) \right\}
$$

possesses an unbounded component $\mathcal{C}^+ \subset [0, +\infty] \times C^1(\overline{\Omega})$ *such that* $\mathcal{C}^+ \cap (\{0\} \times C^1(\overline{\Omega})) = \{u_0\}$ *and satisfies*

(i) either it bifurcates from infinity to the right of the axis $\lambda = 0$ with the corresponding solutions having a positive part blowing up to infinity in $C(\overline{\Omega})$ as $\lambda \to 0^+$;

(ii) or its projection on the λ axis is $[0, +\infty)$.

Moreover, there exists $\bar{\lambda} \in (0, +\infty]$ such that, for every $\lambda \in (0, \bar{\lambda})$, the problem (P_λ) has at least two L^p-viscosity solutions, $u_{\lambda,1}$ and $u_{\lambda,2}$, satisfying

$$u_{\lambda,1} \xrightarrow[\lambda \to 0^+]{} u_0 \;\; in \; C^1(\overline{\Omega}), \quad \max_{\overline{\Omega}} u_{\lambda,2} \xrightarrow[\lambda \to 0^+]{} +\infty,$$

and, if $\bar{\lambda} < +\infty$, the problem $(P_{\bar{\lambda}})$ has a unique L^p-viscosity solution.

See section 5.1 for the main a priori and multiplicity theorems in the general form; section 5.2 and chapter 6 are devoted to their proofs. In the latter we construct and study an auxiliary fixed point problem in order to obtain the existence and nonexistence results via degree theory.

At the beginning of each chapter, a historical overview and references concerning the problem is provided, as well as the hypotheses and the precise statements.

2
Preliminaries

In this chapter we recall some notation, definitions and known results (at least for specialists) that will be used throughout the thesis.

Let Ω be a bounded domain in $\mathbb{R}^n$ and $F(x, r, p, X) : \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n \to \mathbb{R}$ a measurable function. We consider the general structure condition

$$\mathcal{M}^-_{\lambda,\Lambda}(X - Y) - b(x)|p - q| - \mu|p - q|(|p| + |q|) - d(x)\,\omega(|r - s|)$$
$$\leq F(x, r, p, X) - F(x, s, q, Y) \quad (SC)^\mu$$
$$\leq \mathcal{M}^+_{\lambda,\Lambda}(X - Y) + b(x)|p - q| + \mu|p - q|(|p| + |q|) + d(x)\,\omega(|r - s|)$$

for a.e. $x \in \Omega$, where the operator F satisfies $F(\cdot, 0, 0, 0) \equiv 0$ with positive ellipticity coefficients $\lambda \leq \Lambda$, μ is a nonnegative constant, $b \in L^q(\Omega)$, $d \in L^p(\Omega)$, $b, d \geq 0$, for exponents p and q satisfying

$$q \geq p \geq n, \quad q > n \tag{2.1}$$

and ω is a modulus of continuity. As in [24], [25], hypothesis (2.1) is equivalent to the following two cases: (i) $p = q > n$ or (ii) $q > p = n$, that we are going to consider along the text. We use the same notation on exponents and on vectors in $(SC)^\mu$, as usual in the theory.

Notice that the assumption over the highest order term X, for $p = q$ and $r = s$, implies that F is a uniformly elliptic operator; F is uniformly continuous in (r, p, X). In $(SC)^\mu$,

$$\mathcal{M}^+_{\lambda,\Lambda}(X) := \sup_{\lambda I \leq A \leq \Lambda I} \operatorname{tr}(AX) \quad \text{and} \quad \mathcal{M}^-_{\lambda,\Lambda}(X) := \inf_{\lambda I \leq A \leq \Lambda I} \operatorname{tr}(AX)$$

are the Pucci's extremal operators, which satisfy the following properties.

Lemma 2.1 (*Lemma 2.1 in [12]*) *Let $X, Y \in \mathbb{S}^n$. Then,*

(i) $\mathcal{M}^-_{\lambda,\Lambda}(X) = -\mathcal{M}^+_{\lambda,\Lambda}(-X)$;

(ii) $\mathcal{M}^-_{\lambda,\Lambda}(X) = \lambda \sum_{\{\nu_i > 0\}} \nu_i + \Lambda \sum_{\{\nu_i < 0\}} \nu_i$, $\mathcal{M}^+_{\lambda,\Lambda}(X) = \Lambda \sum_{\{\nu_i > 0\}} \nu_i + \lambda \sum_{\{\nu_i < 0\}} \nu_i$, *where* $\{\nu_i\}_{1 \leq i \leq n} = \operatorname{spec}(X)$;

(iii) $\mathcal{M}^-_{\lambda,\Lambda}(X) + \mathcal{M}^-_{\lambda,\Lambda}(Y) \leq \mathcal{M}^-_{\lambda,\Lambda}(X + Y) \leq \mathcal{M}^-_{\lambda,\Lambda}(X) + \mathcal{M}^+_{\lambda,\Lambda}(X)$;

(iv) $\mathcal{M}^-_{\lambda,\Lambda}(X) + \mathcal{M}^+_{\lambda,\Lambda}(Y) \leq \mathcal{M}^+_{\lambda,\Lambda}(X+Y) \leq \mathcal{M}^+_{\lambda,\Lambda}(X) + \mathcal{M}^+_{\lambda,\Lambda}(X);$

(v) $\mathcal{M}^-_{\lambda a,\Lambda A}(X) \leq \mathcal{M}^-_{\lambda,\Lambda}(\phi X) \leq \mathcal{M}^-_{\lambda A,\Lambda a}(X),$ *for* $\phi(x) \in C(\overline{\Omega})$ *such that* $0 < a \leq \phi(x) \leq A.$

For a proof see [26, p. 15], [27, p. 4] and lemma 4.2 in [28].

By *modulus* we mean a function $\omega : [0,+\infty] \to [0,+\infty]$ continuous at 0 with $\omega(0) = 0$. We may consider ω increasing and continuous, up to replacing it by a larger function. We can also suppose ω subadditive, from where it holds the property $\omega(k) \leq (k+1)\,\omega(1)$ for all $k \geq 0$. More strongly, we will say that ω is a *Lipschitz modulus* if $\omega(k) \leq k\,\omega(1)$, for all $k \geq 0$.

In some cases, we are going to use the following linear structure condition

$$\mathcal{M}^-_{\lambda,\Lambda}(X-Y) - b(x)|p-q| - d(x)\,\omega((r-s)^+)$$
$$\leq F(x,r,p,X) - F(x,s,q,Y) \qquad \text{(SC)}$$
$$\leq \mathcal{M}^+_{\lambda,\Lambda}(X-Y) + b(x)|p-q| + d(x)\,\omega((s-r)^+) \quad \text{a.e. } x \in \Omega$$

with $F(\cdot,0,0,0) \equiv 0$, where $0 < \lambda \leq \Lambda$, $b \in L^p(\Omega)$, $d \in L^p(\Omega)$, $b,d \geq 0$, for p, q satisfying (2.1) and ω a modulus. Here, differently from $(SC)^\mu$, the condition over the zero order term in (SC) means that F is proper (or coercive), i.e. decreasing in r.

Remark 2.2 *Notice that equation* (ME)*, with* $L = F$ *satisfying* (SC)*, gives an example of an operator satisfying* $(SC)^\mu$*, for* $\mu = \|M\|_{L^\infty(\Omega)}$*. Indeed,*

$$\langle M(x)p,p \rangle - \langle M(x)q,q \rangle = \langle M(x)p,p \rangle + \langle M(x)p,q \rangle - \langle M(x)p,q \rangle - \langle M(x)q,q \rangle$$
$$= \langle M(x)p, p-q \rangle + \langle M(x)(p-q), q \rangle \leq \mu|p-q|(|p|+|q|).$$

Here, $\|M(x)\|_{L^\infty(\Omega)} = \sup_{x \in \Omega} \|M(x)\|_{\mathbb{M}_{n \times n}(\mathbb{R})}$*, for the matrix norm[1]*

$$\|M(x)\|_{\mathbb{M}_{n \times n}(\mathbb{R})} = \sup_{y \in \mathbb{R}^n,\,|y|=1} |M(x) \cdot y|, \quad \text{for each } x \in \Omega.$$

Next we recall the definition of L^p-viscosity solution. Here, solutions are real functions $u : \Omega \to \mathbb{R}$.

Definition 2.3 *Let* $f \in L^p_{loc}(\Omega)$*. We say that* $u \in C(\Omega)$ *is an* L^p*-viscosity subsolution (respectively, supersolution) of* $F = f$ *in* Ω *if for every* $\phi \in W^{2,p}_{loc}(\Omega)$

---

[1]Observe that both $\langle p,q \rangle$ and $p \cdot q$ denote the inner product of $p,q \in \mathbb{R}^n$.

and $x_0 \in \Omega$ such that $u - \phi$ has a local maximum (minimum) at x_0, one has

$$\text{ess } \overline{\lim_{x \to x_0}} \{F(x, u(x), D\phi(x), D^2\phi(x)) - f(x)\} \geq 0$$

$$(\text{ess } \underline{\lim_{x \to x_0}} \{F(x, u(x), D\phi(x), D^2\phi(x)) - f(x)\} \leq 0)$$

in the sense that for every $\varepsilon, r > 0$ there is a set $A \subset B_r(x_0)$ of positive measure such that $F(x, u(x), D\phi(x), D^2\phi(x)) - f(x) \geq -\varepsilon$ ($F(x, u(x), D\phi(x), D^2\phi(x)) - f(x) \leq \varepsilon$) for all $x \in A$.

In other words, whenever $\phi \in W^{2,p}_{\text{loc}}(\Omega)$, $\varepsilon > 0$ and $\mathcal{O} \subset \Omega$ open are such that $F(x, u(x), D\phi(x), D^2\phi(x)) - f(x) \leq -\varepsilon$ ($\geq \varepsilon$) for a.e. $x \in \mathcal{O}$, then $u - \phi$ cannot have a local maximum (minimum) in $\mathcal{O}$.

We can think about L^p-viscosity solutions for any $p > \frac{n}{2}$, since this restriction makes all test functions $\phi \in W^{2,p}_{\text{loc}}(\Omega)$ continuous [29, p. 284] and having a second order Taylor expansion [4]. We are going to deal mostly with the case $p > n$. In particular, for Ω bounded with $\partial\Omega \in C^{1,1}$, this implies that the continuous injection $W^{2,p}(\Omega) \subset C^1(\overline{\Omega})$ is compact, for all $n \geq 1$ [30, p. 213, 285].

If F and f are continuous in x, we can use the more usual notion of *C-viscosity* sub and supersolutions. Precisely, we have the following definition.

Definition 2.4 *Let $F(x, r, p, X) : \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n \to \mathbb{R}$ continuous and $f \in C(\Omega)$. We say that $u \in C(\Omega)$ is a C-viscosity subsolution (supersolution) of $F = f$ in Ω if for every $x_0 \in \Omega$ and $\phi \in C^2(B_r(x_0))$, $r > 0$, such that $u - \phi$ has a local maximum (minimum) at x_0, we have $F(x_0, u(x_0), D\phi(x_0), D^2\phi(x_0)) \geq f(x_0)$ ($F(x_0, u(x_0), D\phi(x_0), D^2\phi(x_0)) \leq f(x_0)$).*

Employing Taylor expansions of second order, we can replace the set of "test functions" in C^2 by polynomials of degree two [26, p. 13], or by an intrinsic approach on semi-jets [3, p. 9, 10]. We also refer to [31], [32], [33] for more properties and results concerning C-viscosity solutions.

Definitions 2.3 and 2.4 are equivalent when F satisfies (SC) with $b \in L^\infty(\Omega)$ and $p \geq n$, by theorem 2.1(ii) in [34] (which in turn goes back to proposition 2.9 in [4]) – we will be using them interchangeably in this case.

In both L^p or C cases, we only need to define a subsolution as an upper semicontinuous function, and a supersolution being lower semicontinuous, which is sufficient to ensure the attainment of a maximum or minimum, respectively, over compact sets. Moreover, if the function u is not semicontinuous, we say that u is a viscosity subsolution (supersolution) provided u^* (u_*) is. Here, u^* and u_* are the upper and lower semicontinuous

envelopes of u respectively, namely $u^*(x) := \overline{\lim}_{y \to x} u(y)$, $u_*(x) := \underline{\lim}_{y \to x} u(y)$, see [3, p. 22], [33, p. 40]. Note that u^* and u_* are always semicontinuous functions with $u^* \in USC(\Omega)$ and $u_* \in LSC(\Omega)$. Nevertheless, sub and supersolutions will be assumed continuous along the text, unless specifically noted otherwise.

A *strong* sub or subsolution belongs to $W^{2,p}_{\text{loc}}(\Omega)$ and satisfies the inequality at almost every point. These notions are related, up to quadratic growth, as shows the next proposition.

Proposition 2.5 *Assume F satisfies $(SC)^\mu$, p, q as in (2.1), and $f \in L^p(\Omega)$. Then, $u \in W^{2,p}_{\text{loc}}(\Omega)$ is a strong subsolution (supersolution) of $F = f$ in Ω if and only if it is an L^p-viscosity sub(super)solution of this equation.*

See theorem 3.1 and proposition 9.1 in [25] for a proof, even for more general conditions on μ and the exponents p, q. Is is also well known that the pointwise maximum of subsolutions, or supremum over any set (for instance, if this supremum is locally bounded), is still a subsolution; see theorem 4.2 in [33] for C-viscosity solutions, and also proposition 2 in [35] for a version for L^p-viscosity solutions related to quadratic growth and bounded coefficients.

When we refer to solutions of the Dirichlet problem in $C^{1,1}$ domains, we will assume that *strong solutions* belong to $W^{2,p}(\Omega)$ – despite some cases in which we prefer to keep on the "local" sense in order to preserve the maximal generality. Remember that a *solution* is always both a sub and supersolution of the equation.

Denote $\mathcal{L}^\pm[u] := \mathcal{M}^\pm_{\lambda,\Lambda}(D^2u) \pm b(x)|Du|$, where $b \in L^q_+(\Omega)$, $q > n$. We make the convention that $b \in L^p_+(\Omega)$ when $p > n$ and no other information on b or q is provided.

Next we recall Alexandrov-Bakelman-Pucci type results with unbounded ingredients and quadratic growth, which will be referred simply by ABP.

Proposition 2.6 *Let Ω bounded, $\mu \geq 0$, $b \in L^q_+(\Omega)$ and $f \in L^p(\Omega)$, for p, q as in (2.1). Then there exist $\delta = \delta(n, p, \lambda, \Lambda, \text{diam}(\Omega), \|b\|_{L^q(\Omega)}) > 0$ such that*

$$\mu\|f^-\|_{L^p(\Omega)} (\text{diam}(\Omega))^{\frac{n}{p}} \leq \delta \qquad (\mu\|f^+\|_{L^p(\Omega)} (\text{diam}(\Omega))^{\frac{n}{p}} \leq \delta)$$

implies that every $u \in C(\overline{\Omega})$ which is an L^p-viscosity sub(super)solution of

$$\mathcal{L}^+[u] + \mu|Du|^2 \geq f(x) \text{ in } \Omega^+ \quad \left(\mathcal{L}^-[u] - \mu|Du|^2 \leq f(x) \text{ in } \Omega^- \right),$$

where $\Omega^{\pm} = \Omega \cap \{\pm u > 0\}$, satisfies the estimate

$$\max_{\overline{\Omega}} u \le \max_{\partial\Omega} u + C_A \|f^-\|_{L^p(\Omega)} \quad \left(\min_{\overline{\Omega}} u \ge \min_{\partial\Omega} u - C_A \|f^+\|_{L^p(\Omega)}\right),$$

for a constant C_A depending on $n, p, \lambda, \Lambda, \|b\|_{L^q(\Omega)}, \mathrm{diam}(\Omega)$, which remains bounded if these quantities are bounded. Moreover, if $\mu = 0$, the norm of f on the right hand side can be taken in the L^n space instead of L^p.

As a matter of fact, ABP is valid under more general conditions, even for unbounded μ. We refer to proposition 2.8 in [36] for a complete proof in the case $\mu = 0$. See theorem 2.6 and lemma 9.3 in [25], and theorem 3.4 in [37], for a precise dependence on constants (see also [24] and [36]). For a simplified proof for $\mu > 0$ and $p > n$ (which is the only superlinear case that we need along the text), see appendix B.

A consequence of ABP in its quadratic form is the comparison principle for equations in the form (ME) (with $L = F$ there), concerning L^p-viscosity solutions and coercive operators. We make the convention, mainly when dealing with multiplicity results, that α and β will always denote a pair of sub and supersolutions, in a sense to be specified.

Lemma 2.7 (*Comparison Principle*) *Assume F satisfies* (SC), *$f \in L^p(\Omega)$ and $M \in L^\infty(\Omega)$ in the bounded domain Ω. Suppose that u is an L^p-viscosity supersolution of*

$$\begin{cases} -F[u] &= \langle M(x)Du, Du \rangle + f(x) & in & \Omega \\ u &= 0 & on & \partial\Omega \end{cases} \tag{2.2}$$

Then, for any $\alpha \in C(\overline{\Omega}) \cap W^{2,p}_{\mathrm{loc}}(\Omega)$ strong subsolution of (2.2), we have

$$\alpha \le u \ in \ \Omega.$$

Analogously, if u is an L^p-viscosity subsolution of (2.2), for any $\beta \in C(\overline{\Omega}) \cap W^{2,p}_{\mathrm{loc}}(\Omega)$ strong supersolution of (2.2), we have $u \le \beta$ in Ω.

Proof. Set $v := u - \alpha$ in Ω. By contradiction, assume $\min_{\overline{\Omega}} v = v(x_0) < 0$. As $v \ge 0$ on $\partial\Omega$, thus $x_0 \in \Omega$. Set $\widetilde{\Omega} := \{v < 0 \ in \ \Omega\}$, which is an open nonempty set since $x_0 \in \widetilde{\Omega}$. Let $\varphi \in W^{2,p}_{\mathrm{loc}}(\widetilde{\Omega})$ and $\tilde{x} \in \widetilde{\Omega}$ be such that $v - \varphi$ has a minimum at $\tilde{x}$. But then $u - (\alpha + \varphi)$ has a minimum at $\tilde{x}$, and by $\alpha + \varphi \in W^{2,p}_{\mathrm{loc}}(\widetilde{\Omega})$ together with the definition of u being an L^p-viscosity supersolution, we know that for every $\varepsilon > 0$, there exists $r > 0$ such that, for a.e. $x \in B_r(\tilde{x}) \cap \widetilde{\Omega}$,

$$-F(x, u, D\alpha + D\varphi, D^2\alpha + D^2\varphi) - \langle M(x)D(\alpha + \varphi), D(\alpha + \varphi) \rangle - f(x) \ge -\varepsilon$$

and $-F(x, \alpha, D\alpha, D^2\alpha) - \langle M(x)D\alpha, D\alpha \rangle - f(x) \leq 0$ from the definition of α as strong subsolution. By (SC),

$$
\begin{aligned}
\varepsilon &\geq F(x, u, D\alpha + D\varphi, D^2\alpha + D^2\varphi) - F(x, \alpha, D\alpha, D^2\alpha) \\
&\quad + \langle M(x)D(\alpha + \varphi), D(\alpha + \varphi) \rangle - \langle M(x)D\alpha, D\alpha \rangle \\
&\geq \mathcal{M}^-(D^2\varphi) - b(x)|D\varphi| - \mu\left(|D\alpha + D\varphi| + |D\alpha|\right)|D\varphi| - d(x)\,\omega((u - \alpha)^+) \\
&\geq \mathcal{M}^-(D^2\varphi) - b(x)|D\varphi| - \mu|D\varphi|^2 - 2\mu|D\alpha|\,|D\varphi|.
\end{aligned}
$$

Then, for $\widetilde{b} = b + 2\mu|D\alpha| \in L^p_+(\Omega)$, v is an L^p-viscosity supersolution of

$$
\begin{cases}
\mathcal{M}^-(D^2v) - \widetilde{b}(x)|Dv| - \mu|Dv|^2 &\leq\ 0 \quad \text{in}\quad \widetilde{\Omega} \\
v &\geq\ 0 \quad \text{on}\quad \partial\widetilde{\Omega} \subset \partial\Omega \cup \{v = 0\}.
\end{cases}
$$

Thus, ABP with $f = 0$ gives us $v \geq 0$ in $\widetilde{\Omega}$, contradicting the definition of $\widetilde{\Omega}$.

■

Remark 2.8 *The same result holds if $\alpha = \max_{1 \leq i \leq m} \alpha_i$ and $\beta = \min_{1 \leq j \leq l} \beta_j$, where α_i and β_j are continuous strong sub and supersolutions of (2.2) respectively. Indeed, in the proof above, we only need to note that $\min_{\overline{\Omega}}(u - \alpha) = (u - \alpha_i)(x_0)$ for some $i \in \{1, \ldots, m\}$, and consider $v := u - \alpha_i$.*

For equations with linear growth in the gradient, it is also a consequence of ABP that the notions of L^p and L^n viscosity solutions coincide for $p \geq n$. This is the content of the next proposition, which is a version of theorem 2.1(iii) in [34] for unbounded coefficients.

Proposition 2.9 *Suppose that F satisfies (SC) for $b, d \in L^q(\Omega)$, $q > n$, and let $f \in L^q_{\mathrm{loc}}(\Omega)$. If $q \geq p \geq n$, then u is an L^p-viscosity solution of $F \geq f$ in Ω if and only if u is an L^q-viscosity solution of $F \geq f$ in Ω.*

For this, we need the following lemma, which is a version of proposition 1 in [35] for unbounded coefficients. The proof relies on an argument due to Prof. Boyan Sirakov.

Lemma 2.10 *If F satisfies (SC) and (2.1), the maximum (minimum) in the definition of L^p-viscosity subsolution (supersolution) can be replaced by a strict maximum (minimum).*

Proof. Suppose, in order to obtain a contradiction, that there exists $\varepsilon, r > 0$ and $\phi \in W^{2,p}_{\mathrm{loc}}(\Omega)$ such that $u - \phi$ attains a local maximum at x_0 but

$$
F(x, u(x), D\phi(x), D^2\phi(x)) \leq f(x) - \varepsilon \quad \text{a.e. in } B_r(x_0). \tag{2.3}
$$

For simplicity of notation, suppose $x_0 = 0$ and $u - \phi \leq (u - \phi)(0)$ in $B_{2r}(0)$.

Set $\phi_\delta := \phi + \delta|x|^4 + \psi_\delta$, where ψ_δ is the strong solution of

$$\begin{cases} \mathcal{L}^+[\psi_\delta] & = & -4\delta b(x)|x|^3 + \varepsilon/8 & \text{in} & B_r \\ \psi_\delta & = & 0 & \text{on} & \partial B_r, \end{cases} \tag{2.4}$$

given, for instance, by proposition 2.4 in [24]. Also, consider small values of $\delta \in (0,1)$ such that $4\delta r^2(2 + n\Lambda) \leq \varepsilon/8$.

Claim 2.11 $u - \phi_\delta$ *attains a strict local maximum in B_r for small $\delta > 0$.*

Proof of Claim 2.11. Since $\mathcal{L}^+[\psi_\delta/\delta] = -4b(x)|x|^3 + \varepsilon/(4\delta) =: g(x)$ a.e. in B_r, by ABP and Chebyshev's inequality $|\{|h| > t\}| \leq t^{-q}\|h\|_{L^q}^q$ for $h \in L^q(\Omega)$, it follows that

$$\begin{aligned} \frac{\psi_\delta}{\delta} &\leq C_A \, \|g^-\|_{L^n(B_r)} \leq C_A \, |B_r \cap \{g < 0\}|^{\frac{1}{n} - \frac{1}{q}} \, \|g^-\|_{L^q(B_r)} \\ &\leq C_A \, |B_r \cap \{4b(x)|x|^3 > \varepsilon/(8\delta)\,\}|^{\frac{1}{n} - \frac{1}{q}} \, \|g^-\|_{L^q(B_r)} \\ &\leq C \, (\delta/\varepsilon)^{\frac{q}{n} - 1} \, \|b\|_{L^q(\Omega)}^{\frac{q}{n} - 1} \, (\varepsilon + \|b\|_{L^q(\Omega)}) \xrightarrow[\delta \to 0^+]{} 0 \end{aligned}$$

for all $x \in B_r$, since $q > n$. In particular, $\psi_\delta(0) < \delta r^4$ for small $\delta > 0$, so we have on ∂B_r that $u - \phi_\delta = u - \phi - \delta r^4 < (u - \phi)(0) - \psi_\delta(0) = (u - \phi_\delta)(0)$. Thus, $u - \phi_\delta$ must attain a strict local maximum at some point in B_r, say in the ball $B_s(x_0) \subset B_r(0)$, and the claim is proved. ∎

Therefore, by applying the hypothesis over ϕ_δ,

$$F(x, u(x), D\phi_\delta(x), D^2\phi_\delta(x)) \geq f(x) - \varepsilon/2 \quad \text{a.e. } x \in B_s(x_0).$$

So, using the latter and (2.3), we obtain

$$\begin{aligned} \varepsilon/2 &\leq F(x, u(x), D\phi_\delta(x), D^2\phi_\delta(x)) - F(x, u(x), D\phi(x), D^2\phi(x)) \\ &\leq \mathcal{L}^+[\delta|x|^4 + \psi_\delta] \leq 4\delta|x|^2(2 + n\Lambda) + 4\delta b(x)|x|^3 + \mathcal{L}^+[\psi_\delta] \leq \varepsilon/4 \end{aligned}$$

a.e. $x \in B_s(x_0)$, by (2.4) and the choice of δ, which yields a contradiction. ∎

Proof of Proposition 2.9. It is obvious by definition 2.3 that L^p-viscosity solutions are L^q-viscosity, since $f \in L^q_{\text{loc}}(\Omega)$. So, in order to obtain a contradiction, suppose that u is an L^q-viscosity but not L^p-viscosity subsolution of $F \geq f$ in Ω. Then, there exists $\varphi \in W^{2,p}_{\text{loc}}(\Omega)$ and $B_{2r}(x_0) \subset \Omega$ such that $u - \varphi$ has a local maximum at x_0 in $B_r(x_0)$ and

$$F(x, u, D\varphi, D^2\varphi) \leq f(x) - \varepsilon \quad \text{a.e. in } B_r(x_0). \tag{2.5}$$

We may assume that there exists $\delta > 0$ such that $(u - \varphi)(x_0) = 0$ and $u - \varphi \leq -\delta$ on $\partial B_r(x_0)$ by lemma 2.10.

Let $\varphi_k \in C^2(B_{2r}(x_0))$ with $\varphi_k \to \varphi$ in $W_{\text{loc}}^{2,p} \subset W_{\text{loc}}^{2,n}$. Hence, $w_k := u - \varphi_k$ is an L^q-viscosity subsolution of

$$\mathcal{L}^+[w_k] \geq F(x, u, Du, D^2u) - F(x, u, D\varphi_k, D^2\varphi_k) \geq -g_k^-(x)$$

in $B_r(x_0)$, for $g_k(x) := f(x) - F(x, u, D\varphi_k, D^2\varphi_k) \in L^q(B_r)$. Since by (2.5) we have $g_k^-(x) = -g_k(x) \leq F(x, u, D\varphi_k, D^2\varphi_k) - F(x, u, D\varphi, D^2\varphi) \leq \mathcal{L}^+[\varphi_k - \varphi]$ a.e. in $\{g_k \leq 0\} \cap B_r(x_0)$, then $\|g_k^-\|_{L^n(B_r(x_0))}$ is bounded by

$$(\lambda + \Lambda)\|D^2(\varphi_k - \varphi)\|_{L^n(B_r(x_0))} + \|b\|_{L^n(B_r(x_0))}\|D(\varphi_k - \varphi)\|_{L^\infty(B_r(x_0))}$$

which converges to zero when $k \to +\infty$, by the Sobolev continuous inclusion $W^{1,q} \subset L^\infty$ for $q > n$. Now it is just a question of applying ABP to obtain that $w_k \leq \sup_{\partial B_r(x_0)} w_k + \|g_k^-\|_{L^n(B_r(x_0))}$ in $B_r(x_0)$ for all $k \in \mathbb{N}$. Thus, the uniform convergence of φ_k to φ in this last inequality is responsible for producing a contradiction with the definition of δ. ∎

The next proposition follows from theorem 4 in [12] in the case $p = n$. We refer to proposition 9.4 in [25] for a more general version.

Proposition 2.12 (*Stability*) *Let* F, F_k *operators satisfying* $(SC)^\mu$ *with* p, q *as in* (2.1), *and* $f, f_k \in L^p(\Omega)$. *Let* $u_k \in C(\Omega)$ *be an* L^p-*viscosity subsolution* (*supersolution*) *of*

$$F_k(x, u_k, Du_k, D^2u_k) \geq f_k(x) \quad in \quad \Omega \quad (\leq) \quad for\ all\ \ k \in \mathbb{N}.$$

Suppose $u_k \to u$ *in* $L_{\text{loc}}^\infty(\Omega)$ *as* $k \to \infty$ *and for each ball* $B \subset\subset \Omega$ *and* $\varphi \in W^{2,p}(B)$, *setting*

$$g_k(x) := F_k(x, u_k, D\varphi, D^2\varphi) - f_k(x); \quad g(x) := F(x, u, D\varphi, D^2\varphi) - f(x),$$

we have $\|(g_k - g)^+\|_{L^p(B)}$ $(\|(g_k - g)^-\|_{L^p(B)}) \to 0$ *as* $k \to \infty$. *Then* u *is an* L^p-*viscosity subsolution* (*supersolution*) *of* $F(x, u, Du, D^2u) \geq f(x)$ $(\leq)$ *in* Ω.

If F *and* f *are continuous in* x, *then it is enough that the above holds for every* $\varphi \in C^2(B)$, *in which case* u *is a* C-*viscosity subsolution* (*supersolution*) *of* $F = f$ *in* Ω.

Remark 2.13 *Proposition 2.12 is valid if we have* $f_k \in L^p(\Omega_k)$, $u_k \in C(\Omega_k)$, *for an increasing sequence of domains* $\Omega_k \subset \Omega_{k+1}$ *such that* $\Omega := \bigcup_{k \in \mathbb{N}} \Omega_k$, *see proposition 1.5 in [38].*

The following proposition follows from the C^β superlinear regularity established in [12]. We give the statement in terms of more general exponents p, q as in (2.1), indicating the changes from the proof of theorem 2 there.

Proposition 2.14 (C^β *Regularity*) *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded domain. Assume* F *satisfies* $(SC)^\mu$ *for* $N = 0$, $q = 0$, $s = 0$ *and* $b \in L^q_+(\Omega)$, *for* p, q *satisfying* (2.1)*. Let* $u \in C(\Omega)$ *be an* L^p*-viscosity solution of* (3.2) *with* $f \in L^p(\Omega)$*. Then there exists* $\beta \in (0,1)$ *depending on* n, p, λ, Λ *and* $\|b\|_{L^q(\Omega)}$ *such that* $u \in C^\beta_{loc}(\Omega)$ *and for any subdomain* $\Omega' \subset\subset \Omega$ *we have*

$$\|u\|_{C^\beta(\Omega')} \leq K_1 \left\{ \|u\|_{L^\infty(\Omega)} + \|f\|_{L^p(\Omega)} + \|d\|_{L^p(\Omega)} \, \omega(\|u\|_{L^\infty(\Omega)}) \right\}$$

where K_1 *depends only on* $n, p, \lambda, \Lambda, \mu, \|b\|_{L^q(\Omega)}, \|u\|_{L^\infty(\Omega')}, \mathrm{dist}(\Omega', \partial\Omega)$*.*

If, in addition, $u \in C(\overline{\Omega}) \cap C^\tau(\partial\Omega)$ *and* Ω *satisfies a uniform exterior cone condition with size* L*, then there exists* $\beta_0 = \beta_0(n, p, \lambda, \Lambda, L, \|b\|_{L^q(\Omega)}) \in (0,1)$ *and* $\beta = \min(\beta_0, \frac{\tau}{2})$ *such that*

$$\|u\|_{C^\beta(\overline{\Omega})} \leq K_1 \left\{ \|u\|_{L^\infty(\Omega)} + \|f\|_{L^p(\Omega)} + \|u\|_{C^\tau(\partial\Omega)} + \|d\|_{L^p(\Omega)} \, \omega(\|u\|_{L^\infty(\Omega)}) \right\}$$

where K_1 *depends on* $n, p, \lambda, \Lambda, \mu, L, \|b\|_{L^q(\Omega)}, \omega(1)\|d\|_{L^p(\Omega)}, \mathrm{diam}(\Omega), \|u\|_{L^\infty(\Omega)}$*. In both cases,* K_1 *remains bounded if these quantities are bounded.*

The same result holds if, instead of a solution of (3.2)*,* u *is only an* L^p*-viscosity solution of the inequalities* $\mathcal{L}^-[u] - \mu|Du|^2 \leq g(x)$ *and* $\mathcal{L}^+[u] + \mu|Du|^2 \geq -g(x)$ *in* Ω*.*

If $\mu = 0$*, then* K_1 *does not depend on a bound from above on* $\|u\|_{L^\infty(\Omega)}$*.*

Proof. To obtain the statement in terms of p, it is only a question of reading L^n-viscosity sense in [12] as L^p-viscosity, changing $b \in L^p$, $d, f \in L^n$ there by $b \in L^q$, $d, f \in L^p$. The corresponding growth lemmas and exponents concerning ρ must be replaced by $\rho^{1-\frac{n}{p}}$, which appear by using proposition 2.6 (for $\mu = 0$) instead of theorem 3 there.

The zero order term is handled as part of the right hand side, since the whole proof is valid if we only have u as an L^p-viscosity solution of inequalities $\mathcal{L}^+[u] \geq -g(x)$ and $\mathcal{L}^-[u] \leq g(x)$ in the case $\mu = 0$ (see the final remark in the end of the proof of theorem 2 in [12]). ∎

Next we recall two boundary versions of the quantitative strong maximum principle and the weak Harnack inequality, which follow by theorems 1.1 and 1.2 in [23] respectively.

We denote with $B_r^+ = B_r \cap \{x_n > 0\}$ a half ball with a flat portion of the boundary included in $\{x_n = 0\}$, for $r > 0$. Further, $\mathbb{T}_r := B_r \cap \{x_n = 0\}$.

Theorem 2.15 (*BQSMP*) *Let* $d \in L^p(B_2^+)$, $f \in L^q(B_2^+)$, p, q *as in* (2.1). *Assume that* u *is an* L^p-*viscosity supersolution of* $\mathcal{L}^-[u] - du \leq f$, $u \geq 0$ *in* B_2^+. *Then there exist constants* $\varepsilon, c, C > 0$ *depending on* $n, \lambda, \Lambda, p, q$, $\|b\|_{L^q(B_2^+)}$ *and* $\|d\|_{L^p(B_2^+)}$ *such that*

$$\inf_{B_1^+} \frac{u}{x_n} \geq c \left(\fint_{B_{3/2}^+} (f^-)^\varepsilon \right)^{1/\varepsilon} - C\|f^+\|_{L^q(B_2^+)}.$$

Theorem 2.16 (*BWHI*) *Let* $d \in L^p(B_2^+)$, $f \in L^q(B_2^+)$, p, q *as in* (2.1). *Assume that* u *is an* L^p-*viscosity supersolution of* $\mathcal{L}^-[u] - du \leq f$, $u \geq 0$ *in* B_2^+. *Then there exist constants* $\varepsilon, c, C > 0$ *depending on* $n, \lambda, \Lambda, p, q$, $\|b\|_{L^q(B_2^+)}$ *and* $\|d\|_{L^p(B_2^+)}$ *such that*

$$\inf_{B_1^+} \frac{u}{x_n} \geq c \left(\fint_{B_{3/2}^+} \left(\frac{u}{x_n} \right)^\varepsilon \right)^{1/\varepsilon} - C\|f^+\|_{L^q(B_2^+)}.$$

In particular, theorem 2.16 implies the strong maximum principle when $f = 0$, i.e. for an L^p-viscosity solution u of $\mathcal{L}^-[u] - du \leq 0$, $u \geq 0$ in Ω, we have either $u \equiv 0$ in Ω or $u > 0$ in Ω and if $u(x_0) = 0$ at $x_0 \in \partial\Omega$, then $\partial_\nu u(x_0) > 0$. Here, ∂_ν is the derivative in the direction of the interior unit normal. We will refer to these consequences simply by SMP and Hopf, as below.

Proposition 2.17 (*SMP*) *Let* Ω *be a* $C^{1,1}$ *domain and* u *an* L^p-*viscosity solution of* $\mathcal{L}^-[u] - du \leq 0$, $u \geq 0$ *in* Ω, *where* $d \in L^p(\Omega)$ *and* p, q *are as in* (2.1). *Then either* $u > 0$ *in* Ω *or* $u \equiv 0$ *in* Ω.

Proposition 2.18 (*Hopf*) *Let* Ω *be a* $C^{1,1}$ *domain and* u *an* L^p-*viscosity solution of* $\mathcal{L}^-[u] - du \leq 0$, $u > 0$ *in* Ω, *where* $d \in L^p(\Omega)$ *and* p, q *are as in* (2.1). *If* $u(x_0) = 0$ *for some* $x_0 \in \partial\Omega$, *then* $\partial_\nu u(x_0) > 0$.

Notice that the results above generalize SMP and Hopf for C-viscosity solutions from [39]. In [23], theorems 2.15 and 2.16 (and consequently propositions 2.17 and 2.18) are proved for $d \equiv 0$, but exactly the same proofs there work for any $d \geq 0$. Moreover, since the function u has a sign, they are also valid for nonproper operators, by splitting the positive and negative parts of d and using $d^- u \geq 0$. We also refer to [40] for another proof of Hopf maximum principle.

The Local Maximum Principle (LMP) is well known in the literature, see for example [26, p. 36], [41, p. 244], [33, p. 85] and [42]. Its boundary version (BLMP) is given in theorem 1.3 in [23], without zero order term.

Theorem 2.1 (*BLMP*) *Assume that u is a locally bounded L^p-viscosity subsolution of*

$$\begin{cases} \mathcal{L}^+[u] + \nu(x)u & \geq & -f(x) & in & B_2^+ \\ u & \leq & 0 & on & \mathbb{T}_2 \end{cases}$$

with $f \in L^p(B_2^+)$, $b \in L_+^q(B_2^+)$, $\nu \in L^{p_1}(B_2^+) \cap L^p(B_2^+)$, for some $p_1 > n$ and p, q as in (2.1). Then, for each $r > 0$,

$$\sup_{B_1^+} u^+ \leq C \left(\left(\int_{B_{3/2}^+} (u^+)^r \right)^{1/r} + \|f^+\|_{L^p(B_2^+)} \right),$$

where C depends only on n, p, p_1, λ, Λ, r, $\|b\|_{L^q(B_2^+)}$ and $\|\nu\|_{L^{p_1}(B_2^+)}$.

Proof. As the proof of theorem 1.3 in [23], $v = u^+$ can be extended as 0 in $B_2 \setminus B_2^+$, since $v = 0$ on $\mathbb{T}_2$ and satisfies (2.6) in B_2, with f extended by 0, which remains in $L^p(B_2)$. Then we apply theorem 2.2 below to obtain the boundary result. ∎

Theorem 2.2 (*LMP*) *Let u be a locally bounded L^p-viscosity subsolution of $\mathcal{L}^+[u] + \nu(x)u \geq -f(x)$ in B_3, with $f \in L^p(B_3)$, $\nu \in L^{p_1}(B_3) \cap L^p(B_3)$, for $p_1 > n$ and p, q as in (2.1). Then, for each $r > 0$,*

$$\sup_{B_1} u^+ \leq C \left(\left(\int_{B_2} (u^+)^r \right)^{1/r} + \|f^+\|_{L^p(B_3)} \right)$$

where C depends only on n, p, p_1, λ, Λ, r, $\|b\|_{L^q(B_3)}$ and $\|\nu\|_{L^{p_1}(B_3)}$.

Remark 2.19 *Of course the same conclusion holds if we have a domain Ω and $\Omega' \subset\subset \Omega'' \subset\subset \Omega$ by a local covering argument. The same for any radii $\rho_1 < \rho_2 < \rho_3$ by rescaling.*

The only difference between theorem 2.2 and LMP given in theorem 3.1 (a) in [42] comes from the need to put the zero order term on the right hand side. For the sake of completeness, details are provided in the sequel.

Proof. Observe that u is an L^n-viscosity subsolution of the initial equation by proposition 2.9, as well as $v := u^+ = \max(u, 0)$ of

$$\mathcal{L}^+[v] + \nu(x)v \geq -f^+(x) \quad \text{in } B_3. \tag{2.6}$$

and so $\mathcal{L}^+[v] \geq -g(x)$, where $g := f^+ + \nu^+ v \in L^p_+(B_2)$. Applying theorem 3.1(a) of [42] in the ball B_2, we obtain in particular that

$$\sup_{B_1} v \leq C \left(\left(\int_{B_{3/2}} v^s \right)^{1/s} + \|g\|_{L^n(B_2)} \right)$$

for $s > n$ defined by the relation $\frac{1}{n} = \frac{1}{s} + \frac{1}{p_1}$. Then, Holder's inequality yields

$$\|\nu^+ v\|_{L^n(B_2)} \leq \|\nu^+\|_{L^{p_1}(B_2)} \|v\|_{L^s(B_2)}$$

and for a constant depending also on $\|\nu^+\|_{L^{p_1}(B_2)}$, it follows that

$$\sup_{B_1} v \leq C\{ \|v\|_{L^s(B_2)} + \|f^+\|_{L^n(B_2)} \}. \tag{2.7}$$

Now we perform a Moser type argument, as in [32, p. 75], to extend the validity of (2.7) for all $s > 0$. More precisely, we start rescaling (2.7). For $R \leq 2$, we define $w(x) := v(Rx/2)$ for $x \in B_2$ and apply the result just proved for w to obtain, in terms of v,

$$\sup_{B_{R/2}} v \leq C \{ R^{-\frac{n}{s}} \|v\|_{L^s(B_R)} + R \|f^+\|_{L^n(B_R)} \}. \tag{2.8}$$

Next, using (2.8) in $B = B_{(1-\theta)R}(y)$, for $R \in (0,2]$ and $\theta \in (0,1)$, we have

$$\sup_{B_{(1-\theta)R/2}(y)} v \leq C \{ ((1-\theta)R)^{-\frac{n}{s}} \|v\|_{L^s(B)} + (1-\theta)R \|f^+\|_{L^n(B)} \}. \tag{2.9}$$

In particular, (2.9) for all $y \in B_{\theta R}$ yields, for $R \in (0,2]$ and $\theta \in (0,1)$,

$$\|v\|_{L^\infty(B_{\theta R})} \leq C \{ ((1-\theta)R)^{-\frac{n}{s}} \|v\|_{L^s(B_R)} + \|f^+\|_{L^n(B_R)} \}. \tag{2.10}$$

Let $r \in (0,s)$ (notice that the validity of BLMP to $r \geq s$ is ensured by (2.7) and Holder's inequality), then,

$$\left(\int v^s \right)^{1/s} = \left(\int v^{s-r} v^r \right)^{1/s} \leq \|v\|_\infty^{1-\frac{r}{s}} \left(\int v^r \right)^{1/s}.$$

By Young's inequality with $\widetilde{p} = 1/(1-r/s)$ and $\widetilde{q} = s/r$,

$$\frac{C}{((1-\theta)R)^{n/s}} \|v\|_{L^s(B_R)} \leq \|v\|_{L^\infty(B_R)}^{1-\frac{r}{s}} \left(\frac{C}{((1-\theta)R)^n} \int_{B_R} v^r \right)^{1/s}$$

$$\leq \frac{1}{2} \|v\|_{L^\infty(B_R)} + \left(\frac{C}{((1-\theta)R)^n} \int_{B_R} v^r \right)^{1/r}.$$

Thus, applying this estimate in (2.10),

$$\|v\|_{L^\infty(B_{\theta R})} \le \frac{1}{2}\|v\|_{L^\infty(B_R)} + \frac{C}{((1-\theta)R)^{n/r}}\left(\int_{B_R} v^r\right)^{1/r} + C\|f^+\|_{L^n(B_R)}.$$

Next, by defining $\psi(t) := \|v\|_{L^\infty(B_t)}$, $t \in (0,2]$, we have

$$\psi(t) \le \frac{1}{2}\psi(R) + \frac{C}{(R-t)^{n/r}}\left(\int_{B_R} v^r\right)^{1/r} + C\|f^+\|_{L^n(B_1)}.$$

Now, lemma 4.3 in [32] implies that $\psi(t) \le \frac{C}{(R-t)^{n/r}}\left(\int_{B_R} v^r\right)^{1/r} + C\|f^+\|_{L^n(B_1)}$.
Taking $R \to 2^-$ we obtain, for instance for $\theta = 1/2$,

$$\|v\|_{L^\infty(B_1)} \le C\left(\left(\int_{B_2} v^r\right)^{1/r} + \|f^+\|_{L^n(B_2)}\right).$$

∎

We provide another proof of theorem 2.2 for C-viscosity solutions in appendix D, which is constructive and follows the classical proof in [41].

The following result, which follows from lemma 2.3 in [12], is a useful tool to deal with quadratic dependence in the gradient.

Lemma 2.20 (*Exponential change*) *Let $p \ge n$ and $u \in W^{2,p}_{\text{loc}}(\Omega)$. For $m > 0$ set*

$$v = \frac{e^{mu} - 1}{m}, \quad w = \frac{1 - e^{-mu}}{m}.$$

Then, a.e. in Ω we have $Dv = (1 + mv)Du$, $Dw = (1 - mw)Du$ and

$$\mathcal{M}^\pm(D^2u) + m\lambda|Du|^2 \le \frac{\mathcal{M}^\pm(D^2v)}{1+mv} \le \mathcal{M}^\pm(D^2u) + m\Lambda|Du|^2, \qquad (2.11)$$

$$\mathcal{M}^\pm(D^2u) - m\Lambda|Du|^2 \le \frac{\mathcal{M}^\pm(D^2w)}{1-mw} \le \mathcal{M}^\pm(D^2u) - m\lambda|Du|^2. \qquad (2.12)$$

and clearly $\{u = 0\} = \{v = 0\} = \{w = 0\}$ and $\{u > 0\} = \{v > 0\} = \{w > 0\}$.

Moreover, the same inequalities hold in the L^p-viscosity sense if u is merely continuous; for example, if $u \in C(\Omega)$ is an L^p-viscosity solution of

$$\mathcal{M}^+(D^2u) + b(x)|Du| + \mu|Du|^2 + c(x)u \ge f(x) \text{ in } \Omega \qquad (2.13)$$

where $b \in L^q_+(\Omega)$, $c, f \in L^p(\Omega)$, for p, q as in (2.1), then $v = \frac{1}{m}(e^{mu} - 1)$, for $m = \frac{\mu}{\lambda}$, is an L^p-viscosity solution of

$$\mathcal{M}^+(D^2v) + b(x)|Dv| + \frac{c(x)}{m}\ln(1+mv)(1+mv) - mf(x)v \ge f(x) \text{ in } \Omega$$

and analogously for the other inequalities.

The proof follows the original idea of [12] for viscosity solutions, with the slight improvements that can be found in theorem 6.9 in [33]. See also the proof of lemma 1.6 in [43].

Proof. Inequalities (2.11) and (2.12) follow by a simple computation, by using $\operatorname{spec}(\xi \otimes \zeta) = \{0, \ldots, 0, \xi \cdot \zeta\}$, where $\xi \otimes \zeta \in \mathbb{M}_{n \times n}(\mathbb{R})$, $(\xi \otimes \zeta)_{ij} := \xi_i \zeta_j$ for all[2] $\xi, \zeta \in \mathbb{R}^n$. Suppose, then, that u is an L^p-viscosity solution of (2.13).

Let $\psi \in W^{2,p}_{\text{loc}}(\Omega)$ such that $v - \psi$ attains a local maximum at x_0, namely $v \leq \psi$ and $v(x_0) = \psi(x_0)$. Let $\varepsilon > 0$, take some Ω' with $x_0 \in \Omega' \subset\subset \Omega$ and set $a = \|u\|_{L^\infty(\Omega')}$.

Define $\varphi = \frac{1}{m}\{\ln(1 + m\psi)\} \in W^{2,p}_{\text{loc}}(\Omega)$, i.e. $\psi = \frac{1}{m}\{e^{m\varphi} - 1\}$. Then, since $u - \varphi$ has a maximum at x_0, by using the definition of u being an L^p-viscosity subsolution of (2.13) and also (2.11) for the pair φ, ψ, we get

$$\frac{\mathcal{M}^+(D^2\psi)}{1 + m\psi} \geq \mathcal{M}^+(D^2\varphi) + \mu|Du|^2 \geq f(x) - b(x)|D\varphi| - c(x)u - \tilde{\varepsilon} \quad \text{a.e. in } \mathcal{O}$$

where $\tilde{\varepsilon} = \frac{\varepsilon}{e^{am}+1}$ and $\mathcal{O} \subset \Omega'$ is some subset with positive measure.

Let $\delta \in (0,1)$. Notice that, since $(\psi - u)(x_0) = 0$ and $\varphi \in W^{2,p}(\mathcal{O}) \subset C(\overline{\mathcal{O}})$ for $p \geq n > n/2$, there exists $\mathcal{O}_\delta \subset \mathcal{O}$ such that $v \leq \psi \leq v + \frac{\delta}{m}$ in $\mathcal{O}_\delta$. Thus $1 + m\psi \leq 1 + mv + \delta = e^{mu} + \delta$ and

$$\mathcal{M}^+(D^2\psi) + b(x)|D\psi| \geq f^+(x)e^{mu} - f^-(x)(e^{mu} + \delta)$$
$$+ (c^+u^- + c^-u^+)e^{mu} - (c^+u^+ + c^-u^-)(e^{mu} + \delta) - \varepsilon \quad \text{a.e. in } \mathcal{O}_\delta$$

i.e. we have shown that v is an L^p-viscosity subsolution of

$$\mathcal{M}^+(D^2v) + b(x)|D\psi| \geq f(x)e^{mu} - c(x)ue^{mu} - (f^- + c^+u^+ + c^-u^-)\delta \quad \text{in } \Omega$$

for any $\delta \in (0,1)$. The desired conclusion follows by letting $\delta \to 0$, since $\|f_\delta\|_{L^p(B)} \to 0$ for any $B \subset\subset \Omega$, where $f_\delta := (f^- + c^+u^+ + c^-u^-)\delta$, by proposition 2.12. The proof of the remaining inequalities in the L^p-viscosity sense are similar. ∎

A direct consequence of lemma 2.20 is the following extension of proposition 3.5 in [4]. Recall we say that u is twice super(sub)differentiable

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[2]Indeed, observe that if $A := \xi \otimes \zeta = \xi \zeta^T$ and $\mathbb{R}^n$ is the direct sum of V and W, for $V = \operatorname{span} \zeta$, $W = \operatorname{span}\{w_i\}_{i=1}^{n-1}$, then $A\xi = \xi(\zeta^T \xi) = (\xi \cdot \zeta)\xi$ and $Aw_i = 0$. Thus, $\xi \cdot \zeta$ is an eigenvalue with eigenvector ξ, while 0 is an eigenvalue with multiplicity $n-1$.

at $x \in \Omega$ if there exists $(p, X) \in \mathbb{R}^n \times \mathbb{S}^n$ such that

$$u(y) \leq (\geq) \, u(x) + p \cdot (y - x) + \frac{1}{2}\langle X(y - x), y - x\rangle + o(|y - x|^2) \ \text{ as } y \to x.$$

Corollary 2.21 *Let* $f \in L^p_{\mathrm{loc}}(\Omega)$ *for* $p \geq n$, $\mu \geq 0$, $b \in L^\infty_+(\Omega)$ *and* u *a locally bounded* L^p*-viscosity solution of*

$$\mathcal{L}^+[u] + \mu|Du|^2 \geq f(x) \quad in \ \Omega \qquad \left(\mathcal{L}^-[u] - \mu|Du|^2 \leq f(x) \quad in \ \Omega \right).$$

Then u *is twice superdifferentiable (subdifferentiable) a.e. in* Ω.

Proof. Set $v = \frac{1}{m}(e^{mu} - 1)$ for $m = \frac{\mu}{\lambda}$. Say $b(x) \leq \gamma$. Hence, by lemma 2.20 and $u \in L^\infty_{\mathrm{loc}}(\Omega)$, v is an L^p-viscosity solution of

$$\mathcal{M}^+(D^2 v) + \gamma|Dv| \geq f(x)(1 + mv) \in L^p_{\mathrm{loc}}(\Omega).$$

By proposition 3.5 in [4], v is twice superdifferentiable a.e. in Ω, and so is u.

∎

It follows from the argument in [4], that locally bounded L^p-viscosity solutions of $F = f$, with F satisfying the structure condition $(SC)^\mu$ for bounded coefficients, are twice differentiable a.e. and satisfy the equation at almost all points. Moreover, corollary 2.21 is true for $p > n - \varepsilon_0 > n/2$ as in [4], since lemma 2.20 also holds in this case – as well as most of the results stated here thanks to GMP; but not necessary for our purposes in this work.

We finish the chapter recalling some results about pure second order operators $F(D^2 u)$, that is, uniformly elliptic operators F depending only on X (so Lipschitz continuous in X) and satisfying $F(0) = 0$. These operators will play the role of $F(0, 0, 0, X)$ in the approximation lemmas. The next proposition is corollary 5.7 in [26], which deals with $C^{1,\bar{\alpha}}$ interior regularity.

Proposition 2.22 *Let* u *be a* C*-viscosity solution of* $F(D^2 u) = 0$ *in* B_1. *Then* $u \in C^{1,\bar{\alpha}}(\overline{B}_{1/2})$ *for some universal* $\bar{\alpha} \in (0, 1)$ *and there exists a constant* K_2, *depending on* n, λ *and* Λ, *such that*

$$\|u\|_{C^{1,\bar{\alpha}}(\overline{B}_{1/2})} \leq K_2 \, \|u\|_{L^\infty(B_1)}.$$

We also need the following solvability result about the Dirichlet problem.

Proposition 2.23 *Let Ω bounded with a uniform exterior cone condition, $\psi \in C(\partial\Omega)$. Then there exists a unique C-viscosity solution $u \in C(\overline{\Omega})$ of*

$$\begin{cases} F(D^2 u) & = & 0 & in & \Omega \\ u & = & \psi & on & \partial\Omega. \end{cases}$$

Proof. Uniqueness is corollary 5.4 in [26]. Let us recall how to obtain existence via Perron's Method, as in proposition II.1 in [44] (see also [45]). Indeed, comparison principle holds for $F(D^2 u)$ by theorem 5.3 and corollary 3.7 in [26]. Further, we obtain a pair of strong sub and supersolutions $\underline{u}, \overline{u} \in W^{2,p}_{\text{loc}}(\Omega) \cap C(\overline{\Omega})$ of Pucci's equations $\mathcal{M}^+(D^2 \overline{u}) \leq 0 \leq \mathcal{M}^-(D^2 \underline{u})$ in Ω with $\underline{u} = \overline{u} = \psi$ on $\partial\Omega$ by lemma 3.1 of [4]. They are L^p-viscosity sub and supersolutions of $F(D^2 u) = 0$, from where we obtain the pair of C-viscosity sub and supersolutions desired. ∎

We use the following notation from [38] and [46]. For $r, \nu > 0$,

$$B_r^\nu(x_0) := B_r(x_0) \cap \{x_n > -\nu\}, \quad \mathbb{T}_r^\nu(x_0) := B_r(x_0) \cap \{x_n = -\nu\}.$$

Proposition 2.24 *Let $u \in C(\overline{B_1^\nu})$ be a C-viscosity solution of*

$$\begin{cases} F(D^2 u) & = & 0 & in & B_1^\nu \\ u & = & \psi & on & \mathbb{T}_1^\nu \end{cases}$$

such that $\psi \in C(\partial B_1^\nu) \cap C^{1,\tau}(\mathbb{T}_1^\nu)$ for some $\tau > 0$. Then $u \in C^{1,\bar\alpha}(\overline{B_{1/2}^\nu})$, where $\bar\alpha = \min(\tau, \alpha_0)$ for a universal α_0. Moreover, for a constant K_3, depending only on n, λ, Λ and τ, we have

$$\|u\|_{C^{1,\bar\alpha}(\overline{B_{1/2}^\nu})} \leq K_3 \left\{ \|u\|_{L^\infty(B_1^\nu)} + \|\psi\|_{C^{1,\tau}(\mathbb{T}_1^\nu)} \right\}.$$

For a proof see proposition 2.2 in [46]; see also remark 3.3 in [38].

3
Hölder Regularity

The seminal work of Caffarelli [47] in 1989 brought an innovative approach of looking at Schauder type results via iterations from the differential quotients that are perturbations of solutions of the respective autonomous equations. The techniques in [47], which contains in particular $C^{1,\alpha}$ estimates for L^p-viscosity solutions of uniformly elliptic equations $F(x, D^2u) = f(x)$, allowed Świech [48] to extend them to more general operators $F(x, u, Du, D^2u)$ and later Winter [38] to boundary and global bounds. However, everything that is available in the literature, to our knowledge, for L^p-viscosity solutions in the fully nonlinear framework, concerns only structures with either linear gradient growth or bounded coefficients, except for some particular cases of extremal equations with small coefficients, see [24]. It is our goal here to obtain $C^{1,\alpha}$ regularity and estimates for general fully nonlinear uniformly elliptic equations, with at most quadratic growth in the gradient and unbounded coefficients.

We note that Trudinger, independently from [47], in [49] proved $C^{1,\alpha}$ regularity in a less general scenario than Świech and Winter, under a continuity hypothesis for F, dealing with C-viscosity solutions and approximations under supconvolutions. In that paper, it was stated that a priori estimates for solutions in $C^{1,\alpha}$ of superlinear equations could be derived from the arguments in [49] and [50]. However, the question of regularity is more complicated (for a discussion on differences between a priori bounds and regularity results we refer to [51]).

We also quote some other papers on $C^{1,\alpha}$ regularity, the classical works [52], [53], [54] for linear equations; [46] for Neumann boundary conditions; [55] for asymptotically convex operators; [56] (local) and [57] (global) for degenerate elliptic operators; [58] and [59] for parabolic equations possibly with VMO coefficients. Furthermore, Wang [43] has made an important contribution to $C^{1,\alpha}$ regularity for the parabolic equation $u_t + F(x, D^2u) = g(t, x, Du)$, where $|g(t, x, p)| \leq A |p|^2 + g(t, x)$, for bounded coefficients, see lemma 1.6 in [43] (which uses theorem 4.19 in [60]). Sharp regularity results for general parabolic equations with linear gradient growth can be found in [61], and very complete $C^{1,\alpha}$ estimates on the boundary for solutions in the so called S^*-class for equations with linear gradient growth and unbounded coefficients in [62].

It is also essential to mention an important series of papers due to Koike and Święch [25], [36], [24], [63], in which they proved ABP and weak Harnack inequalities for L^p-viscosity solutions of equations with superlinear growth in the gradient, together with several theorems about existence, uniqueness and $W^{2,p}$ estimates for solutions of extremal equations involving Pucci's operators with unbounded coefficients, see in particular theorem 3.1 in [24]. Many of our arguments depend on the machinery in these works.

3.1
Main Regularity Results

In this section we present the hypotheses and statements of our results.

First, in order to measure the oscillation of F in the x entry, we define $\beta(x, x_0) = \beta_F(x, x_0)$, as in [47], [38],

$$\beta(x, x_0) := \sup_{X \in \mathbb{S}^n \setminus \{0\}} \frac{|F(x, 0, 0, X) - F(x_0, 0, 0, X)|}{\|X\|} \quad \text{a.e. } x, x_0. \qquad (3.1)$$

Notice that β is a bounded function by $(SC)^\mu$ and lemma 2.1(ii). Next consider the usual hypothesis, as in [47], [38]: given $\theta > 0$, there exists $r_0 = r_0(\theta) > 0$ such that

$$\left(\frac{1}{r^n} \int_{B_r(x_0) \cap \Omega} \beta(x, x_0)^p \, \mathrm{d}x \right)^{\frac{1}{p}} \leq \theta, \quad \text{for all } r \leq r_0 \text{ a.e. } x_0. \qquad (H_\theta)$$

Notice that (H_θ) is satisfied for Pucci's extremal operators. Indeed, if for instance $F(x, r, p, X) = \mathcal{M}^+(X) + b(x)|p| + d(x)\omega(r^-)$, then $\beta_F(x, x_0) \equiv 0$. In particular this holds if F is the linear operator $\mathrm{tr}(A(x)X) + b(x) \cdot p + d(x)r$, with A a continuous matrix up to the boundary, as in chapter 1.

The following is our main regularity result. To simplify its statement, here we assume that ω is a Lipschitz modulus, i.e. $\omega(r) \leq \omega(1)r$, for all $r \geq 0$.

Theorem 3.1 (*$C^{1,\alpha}$ Regularity Estimates*) *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Assume F satisfies $(SC)^\mu$, $f \in L^p(\Omega)$, where $p > n$. Let u be an L^p-viscosity solution of*

$$F(x, u, Du, D^2u) = f(x) \quad \text{in } \Omega \qquad (3.2)$$

with $\|u\|_{L^\infty(\Omega)} + \|f\|_{L^p(\Omega)} \leq C_0$. Then, there exists $\alpha \in (0, 1)$ and $\theta = \theta(\alpha)$, depending on n, p, λ, Λ, $\|b\|_{L^p(\Omega)}$, such that if (H_θ) holds for all $r \leq \min\{r_0, \mathrm{dist}(x_0, \partial\Omega)\}$, for some $r_0 > 0$ and for all $x_0 \in \Omega$, this implies that

$u \in C^{1,\alpha}_{\text{loc}}(\Omega)$ *and for any subdomain* $\Omega' \subset\subset \Omega$,

$$\|u\|_{C^{1,\alpha}(\overline{\Omega'})} \leq C \left\{ \|u\|_{L^\infty(\Omega)} + \|f\|_{L^p(\Omega)} \right\} \tag{3.3}$$

where C *depends only on* $r_0, n, p, \lambda, \Lambda, \alpha, \mu, \|b\|_{L^p(\Omega)}, \omega(1)\|d\|_{L^p(\Omega)}, \operatorname{diam}(\Omega),$ $\operatorname{dist}(\Omega', \partial\Omega)$ *and on the bound* C_0.

If, in addition, $\partial\Omega \in C^{1,1}$ *and* $u \in C(\overline{\Omega}) \cap C^{1,\tau}(\partial\Omega)$ *is such that and* $\|u\|_{L^\infty(\Omega)} + \|f\|_{L^p(\Omega)} + \|u\|_{C^{1,\tau}(\partial\Omega)} \leq C_1$, *then there exists* $\alpha \in (0,\tau)$ *and* $\theta = \theta(\alpha)$, *depending on* $n, p, \lambda, \Lambda, \|b\|_{L^p(\Omega)}$, *so that if* (H_θ) *holds for some* $r_0 > 0$ *and for all* $x_0 \in \overline{\Omega}$, *this implies that* $u \in C^{1,\alpha}(\overline{\Omega})$ *and satisfies the estimate*

$$\|u\|_{C^{1,\alpha}(\overline{\Omega})} \leq C \left\{ \|u\|_{L^\infty(\Omega)} + \|f\|_{L^p(\Omega)} + \|u\|_{C^{1,\tau}(\partial\Omega)} \right\} \tag{3.4}$$

where C *depends on* $r_0, n, p, \lambda, \Lambda, \alpha, \mu, \|b\|_{L^p(\Omega)}, \omega(1)\|d\|_{L^p(\Omega)}, \operatorname{diam}(\Omega), C_1$ *and on the* $C^{1,1}$ *character of the boundary.*

Moreover, if $\mu = 0$, *then the constant* C *does not depend on* C_0, C_1.

We also consider a slightly different (smaller) version of β, as in [48] and chapter 8 in [26]; $\bar{\beta}(x, x_0) = \bar{\beta}_F(x, x_0)$ defined as

$$\bar{\beta}(x, x_0) := \sup_{X \in \mathbb{S}^n} \frac{|F(x, 0, 0, X) - F(x_0, 0, 0, X)|}{\|X\| + 1} \quad \text{for all } x, x_0. \tag{3.5}$$

Consider the hypothesis $(\overline{H})_\theta$, which is (H_θ) with β replaced by $\bar{\beta}$. This hypothesis is trivially satisfied if $F(x, 0, 0, X)$ is uniformly continuous in x, in the sense that there exists some modulus of continuity $\bar{\omega}$ such that

$$|F(x, 0, 0, X) - F(x_0, 0, 0, X)| \leq \bar{\omega}(|x - x_0|)\left(\|X\| + 1\right),$$

for all $x, x_0 \in \overline{\Omega}$ and $X \in \mathbb{S}^n$ (for instance, see [64]). See also remark 7.1 in [58] for a discussion over this hypothesis.

Remark 3.2 *If* $\mu = 0$ *we can replace the smallness condition* (H_θ) *by* $(\overline{H})_\theta$ *in the statement of Theorem 3.1, by adding 1 on the right hand side of* (3.3) *and* (3.4), *see remark 3.10 for details. For instance, in the global case,*

$$\|u\|_{C^{1,\alpha}(\overline{\Omega})} \leq C \left\{ \|u\|_{L^\infty(\Omega)} + \|f\|_{L^p(\Omega)} + \|u\|_{C^{1,\tau}(\partial\Omega)} + 1 \right\}. \tag{3.6}$$

Remark 3.3 *If* ω *is an arbitrary modulus, we still have regularity and estimates, with the same dependence on constants as before, by adding 1 on the right hand side of* (3.3) *and* (3.4), *as in* (3.6). *In this case, we can also obtain Theorem 3.1 in terms of* $(\overline{H})_\theta$, *see remark 3.11.*

Of course, explicit zero order terms that only depend on u and x, can always be handled as being part of the right hand side $f(x)$.

The proof of Theorem 3.1, given in section 3.2, is based on Caffarelli's iteration method. Compared to [48], [38], we use a simplified rescaling of variable which allows us to carry out the proof, without needing to use a twice differentiability property of viscosity solutions (whose validity is unknown for unbounded coefficients). We also use ideas of Wang to deal with superlinear terms.

The first application of the $C^{1,\alpha}$ theory is $W^{2,p}$ regularity for solutions of fully nonlinear equations with superlinear growth in the gradient, which are convex or concave in the variable X. This extends the results in [38] to superlinear growth in the gradient in the case $p > n$.

Theorem 3.4 ($W^{2,p}$ *Regularity*) *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $u \in C(\Omega)$ an L^p-viscosity solution of*

$$F(x, u, Du, D^2u) + g(x, Du) = f(x) \quad in \quad \Omega \tag{3.7}$$

where $f \in L^p(\Omega)$, $p > n$, g is a measurable function such that $g(x, 0) = 0$ and $|g(x,p) - g(x,q)| \leq \gamma|p - q| + \mu|p - q|(|p| + |q|)$, F is convex or concave in X satisfying $(SC)^0$, for $b, d \in L_+^\infty(\Omega)$ and ω a Lipschitz modulus. Also, suppose $\|u\|_{L^\infty(\Omega)} + \|f\|_{L^p(\Omega)} \leq C_0$. Then, there exists $\theta = \theta(n, p, \lambda, \Lambda, \|b\|_{L^p(\Omega)})$ such that, if (H_θ) holds for all $r \leq \min\{r_0, \text{dist}(x_0, \partial\Omega)\}$, for some $r_0 > 0$ and for all $x_0 \in \Omega$, this implies that $u \in W_{\text{loc}}^{2,p}(\Omega)$ and for every $\Omega' \subset\subset \Omega$,

$$\|u\|_{W^{2,p}(\Omega')} \leq C \left\{ \|u\|_{L^\infty(\Omega)} + \|f\|_{L^p(\Omega)} \right\}$$

where C depends on $r_0, n, p, \lambda, \Lambda, \mu, \|b\|_{L^p(\Omega)}, \omega(1)\|d\|_{L^p(\Omega)}, \text{dist}(\Omega', \partial\Omega), \text{diam}(\Omega)$ and on the bound C_0.

If, moreover, $\partial\Omega \in C^{1,1}$, $u \in C(\overline{\Omega})$ and $u = \psi$ on $\partial\Omega$ for some $\psi \in W^{2,p}(\Omega)$ with $\|u\|_{L^\infty(\Omega)} + \|f\|_{L^p(\Omega)} + \|\psi\|_{W^{2,p}(\Omega)} \leq C_1$ then, there exists $\theta = \theta(n, p, \lambda, \Lambda, \|b\|_{L^p(\Omega)})$ such that, if (H_θ) holds for some $r_0 > 0$ and for all $x_0 \in \overline{\Omega}$, this implies that $u \in W^{2,p}(\Omega)$ and satisfies the estimate

$$\|u\|_{W^{2,p}(\Omega)} \leq C \left\{ \|u\|_{L^\infty(\Omega)} + \|f\|_{L^p(\Omega)} + \|\psi\|_{W^{2,p}(\Omega)} \right\}$$

where C depends on $r_0, n, p, \lambda, \Lambda, \mu, \|b\|_{L^p(\Omega)}, \omega(1)\|d\|_{L^p(\Omega)}, \text{diam}(\Omega), C_1$ and on the $C^{1,1}$ character of the boundary.

Furthermore, if $\mu = 0$, the constant C does not depend on C_0, C_1.

From the regularity and estimates related to $\mu = 0$, we can give an alternative proof of proposition 2.4 in [24], about existence and uniqueness for the Pucci's extremal operators with unbounded coefficients in the case $p > n$.

Proposition 3.5 (*Solvability of the Dirichlet problem*) *Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{1,1}$ domain. Let $b, d \in L^p_+(\Omega)$, $p > n$ and ω a Lipschitz modulus. Let $f \in L^p(\Omega)$ and $\psi \in W^{2,p}(\Omega)$. Then, there exists $u_\pm \in C(\overline{\Omega})$ which are the unique L^p-viscosity solutions of the problems*

$$
\begin{cases}
\mathcal{M}^\pm_{\lambda,\Lambda}(D^2 u_\pm) \pm b(x)|Du_\pm| \pm d(x)w((\mp u_\pm)^+) & = & f(x) & in & \Omega \\
u_\pm & = & \psi & on & \partial\Omega.
\end{cases}
$$

Moreover, $u_\pm \in W^{2,p}(\Omega)$ and satisfies the estimate

$$
\|u_\pm\|_{W^{2,p}(\Omega)} \leq C \left\{ \|u_\pm\|_{L^\infty(\Omega)} + \|f\|_{L^p(\Omega)} + \|\psi\|_{W^{2,p}(\Omega)} \right\}
$$

where C depends only on $n, p, \lambda, \Lambda, \|b\|_{L^p(\Omega)}, \omega(1)\|d\|_{L^p(\Omega)}, \mathrm{diam}(\Omega)$ and on the $C^{1,1}$ character of the boundary.

On the other hand, as far as a priori bounds are concerned, we obtain the following version of Nagumo's lemma (for instance, a version of lemma 5.10 in [65]).

Lemma 3.6 (*Generalized Nagumo's lemma*) *Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{1,1}$ domain. Let F be a convex or concave operator in the X entry, satisfying $(SC)^\mu$, with $b, d \in L^p_+(\Omega)$ for $p > n$ and ω an arbitrary modulus. Suppose that there exists $\theta > 0$ such that (H_θ) holds for some $r_0 > 0$ and for all $x_0 \in \overline{\Omega}$. Let $f \in L^p(\Omega)$, $\psi \in W^{2,p}(\Omega)$ and let $u \in W^{2,p}(\Omega)$ be a strong solution of*

$$
\begin{cases}
F(x, u, Du, D^2 u) & = & f(x) & in & \Omega \\
u & = & \psi & on & \partial\Omega
\end{cases}
$$

such that $\|u\|_{L^\infty(\Omega)} + \|f\|_{L^p(\Omega)} + \|\psi\|_{W^{2,p}(\Omega)} \leq C_1$. Then we have

$$
\|u\|_{W^{2,p}(\Omega)} \leq C \left\{ \|u\|_{L^\infty(\Omega)} + \|f\|_{L^p(\Omega)} + \|\psi\|_{W^{2,p}(\Omega)} + \|d\|_{L^p(\Omega)} \, \omega(\|u\|_{L^\infty(\Omega)}) \right\}
$$

where C depends on r_0, n, p, λ, Λ, μ, $\|b\|_{L^p(\Omega)}$, $\mathrm{diam}(\Omega)$, C_1 and on the $C^{1,1}$ character of the boundary. The local case is analogous.

Moreover, if $\mu = 0$, the final constant C does not depend on C_1.

Remark 3.7 *Analogously to remark 3.2, we can replace (H_θ) by $(\overline{H})_\theta$ in Theorem 3.4 and Lemma 3.6 above, by adding 1 on the right hand side of the estimates, similar to (3.6).*

3.2
Proof of theorem 3.1.

In this section we give a detailed proof of theorem 3.1, splitting it into local, boundary and global parts, developed in subsections 3.2.1, 3.2.2 and 3.2.3, respectively.

3.2.1
Local Regularity

Fix a domain $\Omega' \subset\subset \Omega$. Consider K_1 and β the pair given by the C^β local superlinear estimate (proposition 2.14) for Ω', related to the initial $n, p, \lambda, \Lambda, \mu, \|b\|_{L^p(\Omega)}$, $\text{dist}(\Omega', \partial\Omega)$ and C_0 such that

$$\|u\|_{C^\beta(\Omega')} \leq K_1 \{\|u\|_{L^\infty(\Omega)} + \|f\|_{L^p(\Omega)} + \|d\|_{L^p(\Omega)} \, \omega(\|u\|_{L^\infty(\Omega)})\}.$$

Also, let K_2 (which we can suppose greater than 1) and $\bar\alpha$ be the constants of $C^{1,\bar\alpha}$ local estimate (proposition 2.22) associated to n, λ, Λ in the ball $B_1(0)$.

By taking K_1 larger and β smaller, we can suppose $K_1 \geq \widetilde{K_1}$ and $\beta \leq \widetilde{\beta}$, where $\widetilde{K_1}, \widetilde{\beta}$ is the pair of C^β local estimate in the ball B_1 (or $B_{1/2}$), with respect to an equation with given constants n, p, λ, Λ and bounds for the coefficients $\mu \leq 1$, $\|b\|_{L^p(B_2)} \leq 1 + 2K_2|B_1|^{1/p}$ and $\omega(1)\|d\|_{L^p(B_2)} \leq 1$, for all solutions in the ball B_2 with $\|u\|_{L^\infty(B_2)} \leq 1$ (or for all solutions in the ball B_1 with bounds on the coefficients in B_1).

The first step is to approximate our equation with one which already has the corresponding regularity and estimates that we are interested in.

Lemma 3.8 *Assume F satisfies $(SC)^\mu$ in B_1, $f \in L^p(B_1)$, where $p > n$. Let $\psi \in C^\tau(\partial B_1)$ with $\|\psi\|_{C^\tau(\partial B_1)} \leq K_0$. Then, for all $\varepsilon > 0$, there exists $\delta \in (0,1)$, $\delta = \delta(\varepsilon, n, p, \lambda, \Lambda, \tau, K_0)$, such that*

$$\|\bar\beta_F(\cdot, 0)\|_{L^p(B_1)} \leq \delta, \ \|f\|_{L^p(B_1)} \leq \delta, \ \mu \leq \delta, \ \|b\|_{L^p(B_1)} \leq \delta, \ \omega(1)\|d\|_{L^p(B_1)} \leq \delta$$

implies that any two L^p-viscosity solutions v and h of

$$\begin{cases} F(x, v, Dv, D^2v) &= f(x) \quad in \quad B_1 \\ v &= \psi \quad\;\; on \quad \partial B_1 \end{cases}$$

and

$$\begin{cases} F(0, 0, 0, D^2h) &= 0 \quad in \quad B_1 \\ h &= \psi \quad on \quad \partial B_1 \end{cases}$$

respectively, satisfy $\|v - h\|_{L^\infty(B_1)} \leq \varepsilon$.

Proof. We are going to prove that for all $\varepsilon > 0$, there exists a $\delta \in (0,1)$ satisfying the above, with $\delta \leq 2^{-\frac{n}{2p}} \widetilde{\delta}^{1/2}$, where $\widetilde{\delta}$ is the constant from proposition 2.6. Assume the conclusion is not satisfied, then there exists some $\varepsilon_0 > 0$ and a sequence of operators F_k satisfying $(SC)^{\mu_k}$ for $b_k, d_k \in L^p_+(B_1)$, $\mu_k \geq 0$, ω_k modulus, $f_k \in L^p(B_1)$ and $\delta_k \in (0,1)$ such that $\delta_k \leq 2^{-\frac{n}{2p}} \widetilde{\delta}_k^{1/2}$ for all $k \in \mathbb{N}$, where $\widetilde{\delta}_k$ is the number from ABP related to b_k, in addition to

$$\|\bar{\beta}_{F_k}(\cdot, 0)\|_{L^p(B_1)}, \ \|f_k\|_{L^p(B_1)}, \ \mu_k, \ \|b_k\|_{L^p(B_1)}, \ \omega_k(1)\|d_k\|_{L^p(B_1)} \leq \delta_k \xrightarrow[k \to \infty]{} 0$$

with $v_k, h_k \in C(\overline{B}_1)$ L^p-viscosity solutions of

$$\begin{cases} F_k(x, v_k, Dv_k, D^2 v_k) &= f_k(x) \quad \text{in} \quad B_1 \\ v_k &= \psi_k \quad\quad \text{on} \quad \partial B_1 \end{cases}$$

and

$$\begin{cases} F_k(0, 0, 0, D^2 h_k) &= 0 \quad \text{in} \quad B_1 \\ h_k &= \psi_k \quad \text{on} \quad \partial B_1 \end{cases}$$

where $\|\psi_k\|_{C^\tau(\partial B_1)} \leq K_0$, but $\|v_k - h_k\|_{L^\infty(B_1)} > \varepsilon_0$. We first claim that

$$\|v_k\|_{L^\infty(B_1)}, \ \|h_k\|_{L^\infty(B_1)} \leq C_0 \tag{3.8}$$

for large k, where $C_0 = C_0(n, p, \lambda, \Lambda, K_0)$. Indeed, in the first place, since we have $\mathcal{M}^-(D^2 h_k) \leq 0 \leq \mathcal{M}^+(D^2 h_k)$ in the viscosity sense, we obtain directly that $\|h_k\|_{L^\infty(B_1)} \leq \|\psi_k\|_{L^\infty(\partial B_1)} \leq K_0$. For v_k, we initially observe that

$$2^{\frac{n}{p}} \mu_k \delta_k \leq 2^{\frac{n}{p}} \delta_k^2 \leq \widetilde{\delta}_k, \quad \text{for all } k \in \mathbb{N}.$$

Further, v_k is an L^p-viscosity solution of

$$\mathcal{L}_k^+[v_k] + \mu_k |Dv_k|^2 + d_k(x)\omega_k(|v_k|) \geq f_k(x) \geq \mathcal{L}_k^-[v_k] - \mu_k |Dv_k|^2 - d_k(x)\omega_k(|v_k|).$$

Then, applying ABP in its quadratic form in B_1, we obtain, as in [38], that

$$\|v_k\|_{L^\infty(B_1)} \leq \|v_k\|_{L^\infty(\partial B_1)} + C_A^k \{\|f_k\|_{L^p(B_1)} + \|d_k\|_{L^p} \omega_k(1)(\|v_k\|_{L^\infty(B_1)} + 1)\}.$$

Since $\|b_k\|_{L^n(B_1)} \leq |B_1|^{\frac{p-n}{np}}$ for large k, then the constant in ABP is uniformly bounded, say $C_A^k \leq C_A$. Using also that $\|f_k\|_{L^p(B_1)} \leq 1$ and $C_A \omega_k(1)\|d_k\|_{L^p(\Omega)} \leq 1/2$ for large k, we obtain that $\|v_k\|_{L^\infty(B_1)} \leq C_0$, with $C_0 = C_0(n, p, \lambda, \Lambda, K_0)$, proving the claim (3.8).

Then, by the C^β global estimate (proposition 2.14), there exists some

$\beta \in (0, 1)$ such that

$$\|v_k\|_{C^\beta(\overline{B}_1)}, \ \|h_k\|_{C^\beta(\overline{B}_1)} \le C,$$

where $\beta = \min\left(\beta_0, \frac{\tau}{2}\right)$ for some $\beta_0 = \beta_0(n, p, \lambda, \Lambda)$, $C = C(n, p, \lambda, \Lambda, C_0)$. Here, β and C do not depend on k, since μ_k, $\|b_k\|_{L^p(B_1)}$, $\omega_k(1)\|d_k\|_{L^p(B_1)}$, $\|f_k\|_{L^p(B_1)} \le 1$ for all $k \in \mathbb{N}$. Then, by the compact inclusion $C^\beta(\overline{B}_1) \subset C(\overline{B}_1)$ we have, up to subsequences, that

$$v_k \longrightarrow v_\infty, \quad h_k \longrightarrow h_\infty \quad \text{in } C(\overline{B}_1) \quad \text{as } k \to \infty,$$

for some v_∞, $h_\infty \in C(\overline{B}_1)$ with $v_\infty = h_\infty = \psi_\infty$ on ∂B_1. Moreover, by Arzelà-Ascoli theorem, a subsequence of $F_k(0, 0, 0, X)$ converges uniformly on compact sets of $\mathbb{S}^n$ to some uniformly elliptic operator $F_\infty(X)$, since $\mathcal{M}_{\lambda,\Lambda}^-(X - Y) \le 0 = F_k(0, 0, 0, X) - F_k(0, 0, 0, Y) \le \mathcal{M}_{\lambda,\Lambda}^+(X - Y)$.

We claim that both v_∞ and h_∞ are viscosity solutions of

$$\begin{cases} F_\infty(D^2 u) &= 0 \quad \text{in} \quad B_1 \\ u &= \psi_\infty \quad \text{on} \quad \partial B_1 . \end{cases}$$

This implies that they are equal, by proposition 2.23, which contradicts $\|v_\infty - h_\infty\|_{L^\infty(B_1)} \ge \varepsilon_0$.

The claim for h_∞ follows by passing to uniform limits in the equation satisfied by h_k. On the other hand, for v_∞ we apply stability (proposition 2.12) by noticing that, for $\varphi \in C^2(B_1)$,

$$\begin{aligned} F_k(x, v_k, D\varphi, D^2\varphi) &- f_k(x) - F_\infty(D^2\varphi) \\ &= \{F_k(x, v_k, D\varphi, D^2\varphi) - F_k(x, 0, 0, D^2\varphi)\} \\ &\quad + \{F_k(x, 0, 0, D^2\varphi) - F_k(0, 0, 0, D^2\varphi)\} \\ &\quad + \{F_k(0, 0, 0, D^2\varphi) - F_\infty(D^2\varphi)\} - f_k(x) \end{aligned}$$

and that each one of the addends in braces tends to zero in $L^p(B_1)$ as $k \to \infty$. Indeed, the first one in modulus is less or equal than $\mu_k|D\varphi(x)|^2 + b_k(x)|D\varphi(x)| + \omega_k(\|v_k\|_{L^\infty(B_1)}) d_k(x)$, so its L^p-norm is bounded by

$$\mu_k\|D\varphi\|_{L^\infty(B_1)}^2 + \|b_k\|_{L^p(B_1)}\|D\varphi\|_{L^\infty(B_1)} + (C_0 + 1)\,\omega_k(1)\,\|d_k\|_{L^p(B_1)};$$

while the L^p-norm of the second and third are bounded by

$$\|\bar{\beta}_{F_k}(\cdot, 0)\|_{L^p(B_1)}(\|D^2\varphi\|_{L^\infty(B_1)} + 1)$$

and

$$|B_1|^{1/p}\,\|F_k(0,0,0,D^2\varphi) - F_\infty(D^2\varphi)\|_{L^\infty(B_1)}$$

respectively, what concludes the proof. ∎

Proof of Local Regularity Estimates in the set Ω'. The main difference from the case $\mu = 0$, in the present proof, consists of defining a slightly different scaling on the function, which allows us to have μ small in order to obtain the conditions of the approximation lemma 3.8. For this, we will bring forward an argument due to Wang [43], that uses the C^β regularity of u.

Set $W := \|u\|_{L^\infty(\Omega)} + \|f\|_{L^p(\Omega)} + \|d\|_{L^p(\Omega)}\,\omega(\|u\|_{L^\infty(\Omega)})$, which is less or equal than W_0, a constant that depends on C_0 and $\omega(1)\|d\|_{L^p(\Omega)}$.

For ease of notation, assume $0 \in \Omega'$ and set $s_0 := \min(r_0, \mathrm{dist}(0, \partial\Omega'))$. Recall that this $r_0 = r_0(\theta)$ is such (H_θ) holds for all $r \le \min\{r_0, \mathrm{dist}(x_0, \partial\Omega)\}$, for all $x_0 \in \Omega$. We will see, in the sequel, how the choice of θ is done.

We start assigning some constants. Fix an $\alpha \in (0, \bar\alpha)$ with $\alpha \le \min(\beta, 1 - n/p)$. Then, choose $\gamma = \gamma(\alpha, \bar\alpha, K_2) \in (0, \frac{1}{4}]$ such that

$$2^{2+\bar\alpha}K_2\,\gamma^{\bar\alpha} \le \gamma^\alpha \tag{3.9}$$

and define

$$\varepsilon = \varepsilon(\gamma) := K_2\,(2\gamma)^{1+\bar\alpha}. \tag{3.10}$$

This ε provides a $\delta = \delta(\varepsilon) \in (0,1)$, the constant of the approximation lemma 3.8 that, up to diminishing, can be supposed to satisfy

$$(5 + 2K_2)\,\delta \le \gamma^\alpha. \tag{3.11}$$

Now let $\sigma = \sigma(s_0, n, p, \alpha, \bar\alpha, \beta, \delta, \mu, \|b\|_{L^p(\Omega)}, \omega(1)\|d\|_{L^p(\Omega)}, K_1, K_2, C_0) \le \frac{s_0}{2}$ such that

$$\sigma^{\min\left(1-\frac{n}{p},\beta\right)}m \le \delta\,\{32K^2(K_2 + K + 1)|B_1|^{1/p}\}^{-1} \tag{3.12}$$

where $m := \max\{1, \|b\|_{L^p(\Omega)}, \omega(1)\|d\|_{L^p(\Omega)}, \mu(1 + 2^\beta K_1)W_0\}$. Consider the constant $K(\gamma, \alpha, K_2)$ defined as $K = K_2\,\gamma^{-\alpha}(1 - \gamma^\alpha)^{-1} + K_2\,\gamma^{-1-\alpha}(1 - \gamma^{1+\alpha})^{-1}$ which is greater than $K_2 \ge 1$. In particular, $\overline{B}_{2\sigma}(0) \subset \Omega'$ and we can define

$$N = N_\sigma(0) := \sigma W + \sup_{x \in B_2}|u(\sigma x) - u(0)|.$$

By construction and C^β local quadratic estimate, N is uniformly bounded by

$$\sigma W \le N \le (\sigma + 2^\beta K_1 \sigma^\beta) W \le (1 + 2^\beta K_1) W_0 \, \sigma^\beta. \qquad (3.13)$$

Claim 3.9 $\widetilde{u}(x) := \frac{1}{N}\{u(\sigma x) - u(0)\}$ *is an* L^p*-viscosity solution of* $\widetilde{F}[\widetilde{u}] = \widetilde{f}(x)$ *in* B_2, *where*

$$\widetilde{F}(x, r, p, X) := \frac{\sigma^2}{N} F\left(\sigma x, Nr + u(0), \frac{N}{\sigma}p, \frac{N}{\sigma^2}X\right) - \frac{\sigma^2}{N} F\left(\sigma x, u(0), 0, 0\right)$$

and $\widetilde{f} := \widetilde{f}_1 + \widetilde{f}_2$ *for*

$$\widetilde{f}_1(x) := \sigma^2 f(\sigma x)/N, \quad \widetilde{f}_2(x) := -\sigma^2 F\left(\sigma x, u(0), 0, 0\right)/N,$$

with $\widetilde{F}$ *satisfying* $(\widetilde{SC})^{\widetilde{\mu}}$ *for* $\widetilde{b}(x) := \sigma b(\sigma x)$, $\widetilde{\mu} := N\mu$, $\widetilde{d}(x) := \sigma^2 d(\sigma x)$ *and* $\widetilde{\omega}(r) := \omega(Nr)/N$.

  Proof of Claim 3.9. Let $\varepsilon > 0$ and $\widetilde{\varphi} \in W^{2,p}_{\text{loc}}(B_2)$ such that $\widetilde{u} - \widetilde{\varphi}$ has a minimum (maximum) at $x_0 \in B_2$. Define $\varphi(x) := N\widetilde{\varphi}(x/\sigma) + u(0)$ in $B_{2\sigma}(0)$ and notice that $u - \varphi$ has a minimum (maximum) at $\sigma x_0 \in B_{2\sigma}$. Since u is an L^p-viscosity solution on $B_{2\sigma}$, for this $\varepsilon > 0$ there exists $r > 0$ such that

$$F(\sigma x, u(\sigma x), D\varphi(\sigma x), D^2\varphi(\sigma x)) \le (\ge) f(\sigma x) + (-) N\varepsilon/\sigma^2 \quad \text{a.e. in } B_r(x_0),$$

which is equivalent to

$$\frac{\sigma^2}{N} F\left(\sigma x, N\widetilde{u}(x) + u(0), \frac{N}{\sigma}D\widetilde{\varphi}(x), \frac{N}{\sigma^2}D^2\widetilde{\varphi}(x)\right) \le (\ge) \frac{\sigma^2}{N} f(\sigma x) + (-) \varepsilon$$

a.e. in $B_r(x_0)$. Adding $-\sigma^2 F\left(\sigma x, u(0), 0, 0\right)/N$ in both sides, we have

$$\widetilde{F}(x, \widetilde{u}(x), D\widetilde{\varphi}(x), D^2\widetilde{\varphi}(x)) \le (\ge) \widetilde{f}(x) + (-) \varepsilon \quad \text{a.e. in } B_r(x_0).$$

Further, $\widetilde{F}(x, 0, 0, 0) = 0$ a.e. $x \in B_2$ and for all $r \in \mathbb{R}$, $p \in \mathbb{R}^n$, $X \in \mathbb{S}^n$,

$$\begin{aligned}
\widetilde{F}&(x, r, p, X) - \widetilde{F}(x, s, q, Y) \\
&= \frac{\sigma^2}{N}\left\{F\left(\sigma x, Nr + u(0), \frac{N}{\sigma}p, \frac{N}{\sigma^2}X\right) - F\left(\sigma x, Ns + u(0), \frac{N}{\sigma}q, \frac{N}{\sigma^2}Y\right)\right\} \\
&\le \mathcal{M}^+_{\lambda,\Lambda}(X - Y) + \sigma b(\sigma x)|p - q| + N\mu|p - q|(|p| + |q|) \\
&\qquad + \sigma^2 d(\sigma x)\omega(N|r - s|)/N \\
&= \mathcal{M}^+_{\lambda,\Lambda}(X - Y) + \widetilde{b}(x)|p - q| + \widetilde{\mu}|p - q|(|p| + |q|) + \widetilde{d}(x)\widetilde{\omega}(|r - s|).
\end{aligned}$$

The estimate from below in $(\widetilde{SC})^{\widetilde{\mu}}$ is analogous. ∎

Notice that, with this definition and the choice of σ in (3.12), we have

- $\|\widetilde{u}\|_{L^\infty(B_2)} \leq 1$ since $N \geq \sup_{B_2} |u(\sigma x) - u(0)|$;

- $\|\widetilde{f_1}\|_{L^p(B_2)} = \frac{\sigma^{2-\frac{n}{p}}}{N} \|f\|_{L^p(B_{2\sigma})} \leq \sigma^{1-\frac{n}{p}} \frac{\|f\|_{L^p(\Omega)}}{W} \leq \frac{\delta}{16}$;

- $\|\widetilde{f_2}\|_{L^p(B_2)} \leq \frac{\sigma^{2-\frac{n}{p}}}{N} \omega(|u(0)|) \|d\|_{L^p(B_{2\sigma})} \leq \sigma^{1-\frac{n}{p}} \frac{\omega(\|u\|_\infty)\|d\|_{L^p(\Omega)}}{W} \leq \frac{\delta}{16}$; thus $\|\widetilde{f}\|_{L^p(B_2)} \leq \frac{\delta}{8}$;

- $\widetilde{\mu} = N\mu \leq (1 + 2^\beta K_1) W_0\, \mu\, \sigma^\beta \leq \frac{\delta}{8 K^2 |B_1|^{1/p}}$;

- $\|\widetilde{b}\|_{L^p(B_2)} = \sigma^{1-\frac{n}{p}} \|b\|_{L^p(B_{2\sigma})} \leq \frac{\delta}{16K}$;

- $\widetilde{\omega}(1)\|\widetilde{d}\|_{L^p(B_2)} = \sigma^{2-\frac{n}{p}} \frac{\omega(N)}{N} \|d\|_{L^p(B_{2\sigma})} \leq \sigma^{2-\frac{n}{p}}\omega(1)\|d\|_{L^p(\Omega)} \leq \frac{\delta}{32(K_2+K+1)}$ from the hypothesis $\omega(r) \leq \omega(1)r$ for all $r \geq 0$;

- $\|\bar{\beta}_{\widetilde{F}}(\cdot, 0)\|_{L^p(B_1)} \leq \delta/4$, by choosing $\theta = \delta/8$. Indeed,

$$
\begin{aligned}
\bar{\beta}_{\widetilde{F}}(x, x_0) &\leq \frac{\sigma^2}{N} \sup_{X \in \mathbb{S}^n} \frac{|F(\sigma x, u(0), 0, \frac{N}{\sigma^2}X) - F(\sigma x, 0, 0, \frac{N}{\sigma^2}X)|}{\|X\| + 1} \\
&+ \sup_{X \in \mathbb{S}^n} \frac{|F(\sigma x, 0, 0, \frac{N}{\sigma^2}X) - F(\sigma x_0, 0, 0, \frac{N}{\sigma^2}X)|}{\frac{N}{\sigma^2}(\|X\| + 1)} \\
&+ \frac{\sigma^2}{N} \sup_{X \in \mathbb{S}^n} \frac{|F(\sigma x_0, 0, 0, \frac{N}{\sigma^2}X) - F(\sigma x_0, u(0), 0, \frac{N}{\sigma^2}X)|}{\|X\| + 1} \\
&+ \frac{\sigma^2}{N} \sup_{X \in \mathbb{S}^n} \frac{|F(\sigma x, u(0), 0, 0)| + |F(\sigma x_0, u(0), 0, 0)|}{\|X\| + 1} \\
&\leq \frac{2\sigma^2}{N}\{d(\sigma x) + d(\sigma x_0)\}\, \omega(|u(0)|) \sup_{X \in \mathbb{S}^n}(\|X\| + 1)^{-1} + \beta_F(\sigma x, \sigma x_0) \quad (3.14)
\end{aligned}
$$

and therefore,

$$
\begin{aligned}
\|\bar{\beta}_{\widetilde{F}}(\cdot, 0)\|_{L^p(B_1)} &\leq 4\sigma^{1-\frac{n}{p}} \frac{\omega(\|u\|_{L^\infty(\Omega)})\|d\|_{L^p(\Omega)}}{W} + \left(\frac{1}{\sigma^n} \int_{B_\sigma(0)} \beta_F(y, 0)^p \mathrm{d}y\right)^{\frac{1}{p}} \\
&\leq \delta/8 + \theta = \delta/4.
\end{aligned}
$$

Notice that the only place we had to use the dependence on the bound C_0 is to measure the smallness of μ. Thus, if $\mu = 0$, the final constant does not depend on W_0, neither on C_0.

Remark 3.10 *Still for $\mu = 0$, if we split our analysis in two cases (as usual for linear growth in the gradient, see for instance [27]), then we can obtain the conditions in terms of $(\overline{H}_\theta)$. Indeed, in this case, we consider $N := W$. If $N \geq 1$, then $\widetilde{u}$ is as in claim 3.9. In this case, using $N/\sigma^2 \geq 1$ we can replace β by $\bar{\beta}$ in (3.14); for the estimate in $\widetilde{\omega}(1)\|\widetilde{d}\|_{L^p(B_2)}$ we only need $\omega(r) \leq \omega(1)r$ for $r \geq 1$. On the other hand, if $N \leq 1$, we just define $\widetilde{u} = u(\sigma x)$ and use that each of the addends in W is less or equal than 1 and also*

$$\widetilde{\omega}(1)\|\widetilde{d}\|_{L^p(B_2)} = \sigma^{2-\frac{n}{p}}\omega(N)\|d\|_{L^p(B_{2\sigma})} \leq \sigma^{2-\frac{n}{p}}\omega(1)\|d\|_{L^p(\Omega)}.$$

Notice that, in this case, the final estimate we obtain for our original function u is that $\|u\|_{C^{1,\alpha}(\Omega)} \leq C \leq C(W+1)$, instead of $\|u\|_{C^{1,\alpha}(\Omega)} \leq CW$.

We refer to remarks 6.4 and 6.5 in [58] concerning β_F and $\bar\beta_F$; also theorem 7.3 there for an improvement of this estimate in the parabolic case in which $\|u\|_{C^{1,\alpha}(\Omega)}$ goes to zero when W does.

In particular $\widetilde{F}, \widetilde{u}, \widetilde{\mu}, \widetilde{b}, \widetilde{d}, \widetilde{\omega}$ satisfy the hypotheses of lemma 3.8. Thus, if we show that $\|\widetilde{u}\|_{C^{1,\alpha}(\overline{B}_1)} \leq C$, we will obtain that

$$\|u(\sigma x) - u(0)\|_{C^{1,\alpha}(\overline{B}_1)} \leq CN \leq (1 + 2^\beta K_1)\,CW$$

by (3.13), then $\|u\|_{C^{1,\alpha}(\overline{B}_\sigma)} \leq C\{\|u\|_{L^\infty(\Omega)} + \|f\|_{L^p(\Omega)}\}$, where the constant depends on σ; the local estimate following by a covering argument.

Remark 3.11 *In the case we have an arbitrary modulus of continuity, we define $N = \sigma \max\{W, 1\} + \sup_{x \in B_2} |u(\sigma x) - u(0)|$, which by construction and C^β local superlinear estimate,*

$$\sigma \leq N \leq (\sigma + 2^\beta K_1 \sigma^\beta)\max\{W, 1\} \leq (1 + 2^\beta K_1)W_0\,\sigma^\beta \leq 1.$$

Then we have $\widetilde{\omega}(1)\|\widetilde{d}\|_{L^p(B_2)} = \frac{\sigma^{2-\frac{n}{p}}}{N}\omega(N)\|d\|_{L^p(B_{2\sigma})} \leq \sigma^{1-\frac{n}{p}}\omega(1)\|d\|_{L^p(\Omega)}.$

Moreover, we can consider the smallness assumption in terms of $(\overline{H})_\theta$, with $\bar\beta$ instead of β in (3.14). In fact, in this case we use $N/\sigma^2 \geq 1$. In the end, we obtain that the original function u is such that $\|u\|_{C^{1,\alpha}(\overline{\Omega})}$ is bounded by $C\max\{W, 1\} \leq C(W+1)$, in place of CW.

With these rescalings in mind, we write $F, u, M, \mu, b, d, \omega$ instead of $\widetilde{F}, \widetilde{u}, \widetilde{\mu}, \widetilde{b}, \widetilde{d}, \widetilde{\omega}$, in order to ease of notation. Now we can proceed with Caffarelli's iterations as in [47], [26], [48], which consists of finding a sequence of linear functions $l_k(x) := a_k + b_k \cdot x$ such that

$(i)_k$ $\|u - l_k\|_{L^\infty(B_{r_k})} \leq r_k^{1+\alpha}$

$(ii)_k$ $|a_k - a_{k-1}| \leq K_2\, r_{k-1}^{1+\alpha}$, $|b_k - b_{k-1}| \leq K_2\, r_{k-1}^\alpha$

$(iii)_k$ $|(u-l_k)(r_k x) - (u-l_k)(r_k y)| \leq (1+3K_1)\, r_k^{1+\alpha}|x-y|^\beta$ for $x, y \in B_1$,

where $r_k = \gamma^k$ for some $\gamma \in (0,1)$, for all $k \geq 0$, with the convention $l_{-1} \equiv 0$.

Observe that this proves the result. Indeed, $b_k = b_0 + (b_1 - b_0) + \ldots + (b_k - b_{k-1})$ converges to some b, since $\sum_{k=0}^\infty |b_k - b_{k-1}| \leq K_2 \sum_{k=0}^\infty (\gamma^\alpha)^{k-1} < \infty$; also $|b_k - b| \leq \sum_{l=k}^\infty |b_{l+1} - b_l| \leq K_2 \sum_{l=k}^\infty \gamma^{\alpha l} = K_2 \frac{\gamma^{\alpha k}}{1-\gamma^\alpha}$. Similarly, $|a_k - a| \leq K_2 \frac{\gamma^{k(1+\alpha)}}{1-\gamma^{1+\alpha}}$ and a_k converges to some a.

Next, for each $x \in B_1$, there exists $k \geq 0$ such that $r_{k+1} < |x| \leq r_k$. Then, $|u(x) - a_k - b_k \cdot x| = |u(x) - l_k(x)| \leq r_k^{1+\alpha}$, since $x \in \overline{B}_{r_k}$, thus

$$|u(x) - a - b \cdot x| \leq |u(x) - a_k - b_k \cdot x| + |a_k - a| + |b_k - b||x|$$

$$\leq r_k^{1+\alpha} + K_2 \frac{r_k^{1+\alpha}}{1 - \gamma^{1+\alpha}} + K_2 \frac{r_k^\alpha}{1 - \gamma^\alpha} r_k$$

$$= \left\{ 1 + \frac{K_2}{1 - \gamma^{1+\alpha}} + \frac{K_2}{1 - \gamma^\alpha} \right\} \frac{1}{\gamma^{1+\alpha}} r_{k+1}^{1+\alpha} \leq C_\gamma |x|^{1+\alpha}.$$

By definition of a differentiable function, $a = u(0)$, $b = Du(0)$ and we will have obtained $|u(x) - u(0) - Du(0) \cdot x| \leq C|x|^{1+\alpha}$ and $|Du(0)| \leq C$.

Notice that there was nothing special in doing the initial argument around 0, which we had supposed in the beginning of the proof, belonging to Ω'. Actually, by replacing it by any $x_0 \in \Omega$ and setting the corresponding $s_0 = \min\{r_0, \text{dist}(x_0, \partial\Omega')\}$, we define $N = N_\sigma(x_0)$ by changing 0 by x_0 in there. Then, our initial function u is differentiable at x_0 with

$$|u(x) - u(x_0) - Du(x_0) \cdot (x - x_0)| \leq CW|x - x_0|^{1+\alpha}, \quad |Du(x_0)| \leq CW$$

which implies, by appendix C, that $Du \in C^\alpha(\overline{B}_\sigma)$ and $\|u\|_{C^{1,\alpha}(\overline{B}_\sigma)} \leq CW$. Thus, for the complete local estimate, we just take finitely many such points in order to cover Ω'.

We stress that $(i)_k$ and $(ii)_k$ are completely enough to imply the result, as above, while $(iii)_k$ is an auxiliary tool to get them. So, let us prove $(i)_k - (iii)_k$ by induction on k.

For $k = 0$ we set $a_0 = b_0 = 0$. Recall that β and K_1 are the constants from the C^β superlinear local estimate in B_1 such that $\|u\|_{C^\beta(\overline{B}_1)} \leq \widetilde{K}_1(1 + \delta + 1) \leq 3K_1$, which implies $(iii)_0$. Obviously $(i)_0$ and $(ii)_0$ are satisfied too.

Notice that $|b_k| \leq \sum_{l=0}^k |b_l - b_{l-1}| \leq \frac{K_2}{\gamma^\alpha} \sum_{k=0}^\infty \gamma^{\alpha k} = \frac{K_2}{\gamma^\alpha(1 - \gamma^\alpha)} \leq K$ and also, for all $x \in B_1$, $|l_k(x)| \leq |a_k| + |b_k||x| \leq \frac{K_2}{\gamma^{(1+\alpha)}} \sum_{k=0}^\infty \gamma^{(1+\alpha)k} + \frac{K_2}{\gamma^\alpha} \sum_{k=0}^\infty \gamma^{\alpha k} = K$.

As the induction step, we suppose the items $(i)_k - (iii)_k$ valid in order to construct a_{k+1} and b_{k+1} for which $(i)_{k+1} - (iii)_{k+1}$ hold. Define

$$v(x) = v_k(x) := \frac{(u - l_k)(r_k x)}{r_k^{1+\alpha}} = \frac{u(r_k x) - a_k - b_k \cdot x r_k}{r_k^{1+\alpha}}, \quad \text{for all } x \in B_2.$$

Note that $(i)_k$ says precisely that $|v(x)| \leq 1$ for all $x \in B_1$. In addition, from

this and $(iii)_k$ we get

$$\|v\|_{C^\beta(\overline{B}_1)} = \|v\|_{L^\infty(B_1)} + \sup_{\substack{x,y \in B_1 \\ x \neq y}} \frac{|v(x) - v(y)|}{|x - y|^\beta} \leq 2 + 3K_1 =: K_0.$$

Claim 3.12 *v is an L^p-viscosity solution of $F_k[v] = f_k(x)$ in B_2, for $f_k :=$
$f_k^1 + f_k^2$ with $f_k^1(x) := r_k^{1-\alpha} f(r_k x)$; $f_k^2(x) := -r_k^{1-\alpha} F(r_k x, l_k(r_k x), b_k, 0)$ and F_k
satisfying $(SC)_{F_k}^{\mu_{F_k}}$, where $F_k(x, s, p, X)$ is defined as*

$$r_k^{1-\alpha} F(r_k x, r_k^{1+\alpha} s + l_k(r_k x), r_k^\alpha p + b_k, r_k^{\alpha-1} X) - r_k^{1-\alpha} F(r_k x, l_k(r_k x), b_k, 0),$$

*and the coefficients as $b_{F_k}(x) := r_k b(r_k x) + 2 r_k \mu K$, $\mu_{F_k} := r_k^{1+\alpha} \mu$, $d_{F_k}(x) :=$
$r_k^2 d(r_k x)$ and $\omega_{F_k}(s) := r_k^{-1-\alpha} \omega(r_k^{1+\alpha} s)$.*

Proof of Claim 3.12. Let $\varepsilon > 0$ and $\psi \in W_{\text{loc}}^{2,p}(B_2)$ such that $v - \psi$
has a minimum (maximum) at x_0. Define $\varphi(x) := r_k^{1+\alpha} \psi(x/r_k) + l_k(x)$ for
all $x \in B_{2r_k}$; then $u - \psi$ has a minimum (maximum) at $r_k x_0$. Since u is an
L^p-viscosity solution in $B_{2r_k}(0)$, there exists $r \in (0, 2)$ such that

$$F(r_k x, u(r_k x), D\varphi(r_k x), D^2\varphi(r_k x)) \leq (\geq) f(r_k x) + (-) r_k^{\alpha-1} \varepsilon \quad \text{a.e. in } B_r(x_0).$$

Using $D\psi(x) = r_k^{-\alpha} \{D\varphi(r_k x) - b_k\}$ and $D^2\psi(x) = r_k^{1-\alpha} D^2\varphi(r_k x)$ a.e., we get

$$r_k^{1-\alpha} F(r_k x, r_k^{1+\alpha} v(x) + l_k(r_k x), r_k^\alpha D\psi(x) + b_k, r_k^{\alpha-1} D^2\psi(x))$$
$$\leq (\geq) r_k^{1-\alpha} f(r_k x) + (-) \varepsilon$$

a.e. in $B_r(x_0)$. Adding $-r_k^{1-\alpha} F(r_k x, l_k(r_k x), b_k, 0)$ in both sides we obtain

$$F_k(x, v(x), D\psi, D^2\psi) \leq (\geq) f_k(x) + (-) \varepsilon \quad \text{a.e. in } B_r(x_0).$$

Moreover, F_k satisfies $(SC)_k^{\mu_{F_k}}$, since $F_k(x, 0, 0, 0) = 0$ a.e. $x \in B_2$ and

$$F_k(x, r, p, X) - F_k(x, s, q, Y) = r_k^{1-\alpha} \{F(r_k x, r_k^{1+\alpha} r + l_k(r_k x), r_k^\alpha p + b_k, r_k^{\alpha-1} X)$$
$$- F(r_k x, r_k^{1+\alpha} s + l_k(r_k x), r_k^\alpha q + b_k, r_k^{\alpha-1} Y)\}$$
$$\leq \mathcal{M}_{\lambda,\Lambda}^+(X - Y) + r_k b(r_k x) |p - q|$$
$$+ r_k \mu |p - q| \{r_k^\alpha(|p| + |q|) + b_k\} + r_k^{1-\alpha} d(r_k x) \omega(r_k^{1+\alpha} |r - s|)$$
$$= \mathcal{M}_{\lambda,\Lambda}^+(X - Y) + b_{F_k}(x) |p - q| + \mu_{F_k} |p - q|(|p| + |q|) + d_{F_k}(x) \omega_{F_k}(|r - s|)$$

and the left hand side is completely analogous. ∎

Note that $F_k, v, \mu_{F_k}, b_{F_k}, d_{F_k}, \omega_{F_k}$ also satisfy the hypotheses of lemma 3.8,

$$\|b_{F_k}\|_{L^p(B_1)} \leq r_k^{1-\frac{n}{p}}\|b\|_{L^p(B_{r_k})} + 2\mu K|B_1|^{1/p} \leq \delta;$$

$$\|f_k^1\|_{L^p(B_1)} \leq r_k^{1-\frac{n}{p}-\alpha}\|f\|_{L^p(B_{r_k})} \leq \frac{\delta}{2};$$

$$\|f_k^2\|_{L^p(B_1)} \leq r_k^{1-\frac{n}{p}-\alpha}\left\{ \|b\|_{L^p(B_{r_k})}|b_k| + (K+1)\omega(1)\|d\|_{L^p(B_{r_k})} \right\}$$
$$+ r_k^{1-\alpha}\mu|b_k|^2|B_1|^{\frac{1}{p}} \leq \frac{\delta}{2};$$

$$\omega_{F_k}(1)\|d_{F_k}\|_{L^p(B_1)} = r_k^{1-\frac{n}{p}-\alpha}\omega(r_k^{1+\alpha})\|d\|_{L^p(B_{r_k})} \leq r_k^{1-\frac{n}{p}-\alpha}\omega(1)\|d\|_{L^p(B_1)} \leq \delta$$

and

$$\bar{\beta}_{F_k}(x, x_0) \leq r_k^{1-\alpha}\sup_{X\in\mathbb{S}^n}\frac{|F(r_k x, l_k(r_k x), b_k, r_k^{\alpha-1}X) - F(r_k x, 0, 0, r_k^{\alpha-1}X)|}{\|X\|+1}$$
$$+ \sup_{X\in\mathbb{S}^n}\frac{|F(r_k x, 0, 0, r_k^{\alpha-1}X) - F(r_k x_0, 0, 0, r_k^{\alpha-1}X)|}{r_k^{\alpha-1}(\|X\|+1)}$$
$$+ r_k^{1-\alpha}\sup_{X\in\mathbb{S}^n}\frac{|F(r_k x_0, 0, 0, r_k^{\alpha-1}X) - F(r_k x_0, l_k(r_k x_0), b_k, r_k^{\alpha-1}X)|}{\|X\|+1}$$
$$+ r_k^{1-\alpha}\sup_{X\in\mathbb{S}^n}\frac{|F(r_k x, l_k(r_k x), b_k, 0)| + |F(r_k x_0, l_k(r_k x_0), b_k, 0)|}{\|X\|+1}$$
$$\leq 2r_k^{1-\alpha}\{(d(r_k x) + d(r_k x_0))\,\omega(\|l_k(r_k x)\|_{L^\infty(\Omega)}) + (b(r_k x) + b(r_k x_0))$$
$$|b_k| + \mu|b_k|^2\}\sup_{X\in\mathbb{S}^n}(\|X\|+1)^{-1} + \bar{\beta}_F(r_k x, r_k x_0)$$

since $r_k^{\alpha-1} \geq 1$, then

$$\|\bar{\beta}_{F_k}(\cdot, 0)\|_{L^p(B_1)} \leq 4r_k^{1-\frac{n}{p}-\alpha}(K+1)\,\omega(1)\|d\|_{L^p(B_{r_k})} + 4Kr_k^{1-\frac{n}{p}-\alpha}\|b\|_{L^p(B_{r_k})}$$
$$+ 2\mu K^2|B_1|^{1/p} + \|\bar{\beta}_F(\cdot, 0)\|_{L^p(B_{r_k})} \leq \delta.$$

Let $h = h_k \in C(\overline{B}_1)$ be the C-viscosity solution of

$$\begin{cases} F_k(0, 0, 0, D^2 h) = 0 & \text{in} \quad B_1 \\ h = v & \text{on} \quad \partial B_1. \end{cases}$$

By ABP we have $\|h\|_{L^\infty(B_1)} \leq \|h\|_{L^\infty(\partial B_1)} \leq 1$ and by the $C^{1,\bar{\alpha}}$ local estimate (proposition 2.22), $\|h\|_{C^{1,\bar{\alpha}}(\overline{B}_{1/2})} \leq K_2\|h\|_{L^\infty(B_1)} \leq K_2$. Hence, by lemma 3.8 applied to $F_k, v, \mu_{F_k}, b_{F_k}, d_{F_k}, \omega_{F_k}, \psi := v\mid_{\partial B_1}, \tau := \beta, K_0$ and h we obtain, for ε given in (3.10), that $\|v - h\|_{L^\infty(B_1)} \leq \varepsilon$.

Define $\bar{l}(x) = \bar{l}_k(x) := h(0) + Dh(0) \cdot x$ in B_1, then,

$$\|v - \bar{l}\|_{L^\infty(B_{2\gamma})} \leq \gamma^{1+\alpha}. \tag{3.15}$$

In fact, by the choice of $\gamma \leq \frac{1}{4}$ in (3.9), we have for all $x \in B_{2\gamma}(0)$ that

$$|v(x) - \bar{l}(x)| \leq |v(x) - h(x)| + |h(x) - h(0) - Dh(0) \cdot x|$$
$$\leq K_2 (2\gamma)^{1+\bar{\alpha}} + K_2 |x|^{1+\bar{\alpha}} \leq 2K_2 (2\gamma)^{1+\bar{\alpha}} \leq \gamma^{1+\alpha}.$$

However, inequality (3.15) and the definition of v imply

$$|u(r_k x) - l_k(r_k x) - r_k^{1+\alpha} h(0) - r_k^{1+\alpha} Dh(0) \cdot x| \leq r_k^{1+\alpha} \gamma^{1+\alpha} = r_{k+1}^{1+\alpha} \text{ for } x \in B_{2\gamma},$$

which is equivalent to

$$|u(y) - l_{k+1}(y)| \leq r_k^{1+\alpha} \gamma^{1+\alpha} = r_{k+1}^{1+\alpha} \quad \text{for all } y = r_k x \in B_{2\gamma r_k} = B_{2r_{k+1}},$$

where $l_{k+1}(y) := l_k(y) + r_k^{1+\alpha} h(0) + r_k^\alpha Dh(0) \cdot y$. Then, we define

$$a_{k+1} := a_k + h(0) \, r_k^{1+\alpha}, \quad b_{k+1} := b_k + Dh(0) \, r_k^\alpha$$

obtaining $(i)_{k+1}$. Further, $|a_{k+1} - a_k| \leq K_2 \, r_k^{1+\alpha}$, $|b_{k+1} - b_k| \leq K_2 \, r_k^\alpha$, which is $(ii)_{k+1}$. To finish we note that, in order to prove $(iii)_{k+1}$, it is enough to show

$$\|v - \bar{l}\|_{C^\beta(\overline{B}_\gamma)} \leq (1 + 2K_1) \, \gamma^{1+\alpha-\beta}. \tag{3.16}$$

Indeed, if $x, y \in B_1$ and (3.16) is true, then

$$|(v - \bar{l})(\gamma x) - (v - \bar{l})(\gamma y)| \leq (1 + 2K_1)\gamma^{1+\alpha-\beta}|\gamma x - \gamma y|^\beta$$
$$\Leftrightarrow |(u - l_k)(\gamma r_k x) - (u - l_k)(\gamma r_k y) - r_k^\alpha Dh(0) \cdot (x - y)\gamma r_k|$$
$$\leq (1 + 2K_1)\gamma^{1+\alpha} r_k^{1+\alpha}|x - y|^\beta$$
$$\Leftrightarrow |(u - l_{k+1})(r_{k+1}x) - (u - l_{k+1})(r_{k+1}y)| \leq (1 + 2K_1) \, r_{k+1}^{1+\alpha}|x - y|^\beta.$$

Now, we obtain (3.16) applying the local quadratic C^β estimate (proposition 2.14) to the function $w := v - \bar{l}$, which is an L^p-viscosity solution in B_2 of the inequalities

$$\mathcal{L}_k^-[w] - \mu_{F_k}|Dw|^2 \leq g_k(x), \quad \mathcal{L}_k^+[w] + \mu_{F_k}|Dw|^2 \geq -g_k(x), \tag{3.17}$$

where $g_k := g_k^1 + g_k^2$, for $g_k^1(x) := |f_k(x) - F_k(x, \bar{l}(x), Dh(0), 0)|$ and $g_k^2(x) := d_{F_k}(x)\omega_{F_k}(|w|)$, with $\mathcal{L}_k^\pm[u] := \mathcal{M}_{\lambda,\Lambda}^\pm(D^2 u) \pm (b_{F_k} + 2K_2 \, \mu_{F_k})|Du|$. Surely, this finishes the proof of (3.16), since

$$|g_k^1(x)| \leq |f_k(x)| + b_{F_k}(x)|Dh(0)| + \omega_{F_k}(|\bar{l}(x)|) \, d_{F_k}(x) + \mu_{F_k}|Dh(0)|^2,$$

then using that $|\bar{l}(x)| \le |h(0)| + |Dh(0)|\,|x| \le \|h\|_{C^{1,\bar{\alpha}}(\overline{B}_{1/2})} \le K_2$ for all $x \in B_1$, we have

$$\| g_k\|_{L^p(B_1)} \le \|f_k\|_{L^p(B_1)} + \|b_{F_k}\|_{L^p(B_1)}K_2 + (K_2 + 1)\,\omega_{F_k}(1)\|d_{F_k}\|_{L^p(B_1)}$$
$$+ \mu K_2^2\,|B_1|^{\frac{1}{p}} + (1 + \|w\|_{L^\infty(B_1)})\,\omega_{F_k}(1)\|d_{F_k}\|_{L^p(B_1)} \le (5 + 2K_2)\,\delta \le \gamma^\alpha$$

from the definition of δ in (3.11). Thus, using the estimate above and (3.15) in the C^β local estimate, properly scaled to the ball of radius γ, we obtain in particular that

$$[w]_{\beta,\overline{B_\gamma}} \le \gamma^{-\beta}\widetilde{K}_1\left\{\,\|w\|_{L^\infty(B_{2\gamma})} + \gamma^{2-\frac{n}{p}}\|g_k\|_{L^p(B_{2\gamma})}\,\right\}$$
$$\le \gamma^{-\beta}K_1\left\{\,\gamma^{1+\alpha} + \gamma^{2-\frac{n}{p}}\gamma^\alpha\,\right\} \le 2K_1\,\gamma^{1+\alpha-\beta}$$

and so

$$\|w\|_{C^\beta(\overline{B_\gamma})} = \|w\|_{L^\infty(B_\gamma)} + [w]_{\beta,\overline{B_\gamma}}$$
$$\le \gamma^{1+\alpha} + 2K_1\,\gamma^{1+\alpha-\beta} \le (1 + 2K_1)\,\gamma^{1+\alpha-\beta},$$

which is (3.16), as desired. ∎

Remark 3.13 *By the proof above we see that, under $\mu, \|b\|_{L^p(\Omega)}, \omega(1)\|d\|_{L^p(\Omega)}$ $\le C_1$, both σ and the final constant C depend on $n, p, \lambda, \Lambda, \alpha, \beta, K_1, K_2, C_0$ and C_1. This is very useful in applications, when we have, for example, a sequence of solutions u_k with their respective coefficients uniformly bounded; with $\|u_k\|_{L^\infty}$ and the L^p norm of the right hand side a priori bounded. Then we can uniformly bound the $C^{1,\alpha}$ norm of u_k.*

3.2.2
Boundary Regularity

Since our equation is invariant under diffeomorphisms and $\partial\Omega \in C^{1,1}$, we only need to prove regularity and estimates for some half ball, say $B_1^+(0)$. Precisely, we need to prove the following theorem.

Theorem 3.14 *Suppose F satisfies $(SC)^\mu$, $f \in L^p(B_1^+)$, for $p > n$ and $\psi \in C^{1,\tau}(\mathbb{T})$. Let $u \in C(\overline{\Omega})$ be an L^p-viscosity solution of*

$$\begin{cases} F(x, u, Du, D^2u) & = & f(x) & in & B_1^+ \\ u & = & \psi & on & \mathbb{T}_1 \end{cases}$$

with $\|u\|_{L^\infty(B_1^+)} + \|f\|_{L^p(B_1^+)} + \|\psi\|_{C^{1,\tau}(\mathbb{T}_1)} \le C_0$. Let $\alpha \in (0, \bar{\alpha})$ such that $\alpha \le \min(\beta, 1 - n/p, \tau, \bar{\alpha}(1 - \tau))$. Then, there exists $\theta = \theta(n, p, \lambda, \Lambda, \alpha, \bar{\alpha}, K_3)$

such that if (H_θ) holds for some $r_0 > 0$ and for all $x_0 \in \overline{B}_1^+$, this implies that $u \in C^{1,\alpha}(\overline{B_{1/2}^+})$ and

$$\|u\|_{C^{1,\alpha}(\overline{B_{1/2}^+})} \leq C\left\{\|u\|_{L^\infty(B_1^+)} + \|f\|_{L^p(B_1^+)} + \|\psi\|_{C^{1,\tau}(\mathbb{T}_1)}\right\}$$

where C depends on $r_0, n, p, \lambda, \Lambda, \alpha, \bar{\alpha}, \beta, \mu, \|b\|_{L^p(B_1^+)}, \omega(1)\|d\|_{L^p(B_1^+)}, K_1, K_3$ and the bound C_0.

Here, K_1, β is the pair of C^β global superlinear estimate (proposition 2.14) in B_1^+, related to the initial $n, p, \lambda, \Lambda, \mu, \|b\|_{L^p(\Omega)}, \tau$ and C_1, such that

$$\|u\|_{C^\beta(\overline{B}_1^+)} \leq K_1\left\{\|u\|_{L^\infty(B_1^+)} + \|f\|_{L^p(B_1^+)} + \|\psi\|_{C^\tau(\mathbb{T}_1)} + \|d\|_{L^p(B_1^+)}\omega(\|u\|_{L^\infty(B_1^+)})\right\}.$$

As in [38], we start proving a boundary version of the approximation lemma in B_1^ν. For this set, let $K_3 \geq 1$ and $\bar{\alpha}$ be the pair of $C^{1,\bar{\alpha}}$ boundary estimate (proposition 2.24) associated to n, λ, Λ and τ, independently of $\nu > 0$.

We can suppose that $K_1 \geq \widetilde{K}_1$ and $\beta \leq \widetilde{\beta}$, where $\widetilde{K}_1, \widetilde{\beta}$ is the pair of C^β global estimate for the set B_1^ν (or $B_{1/2}^\nu$), independently of $\nu > 0$, with respect to an equation with given constants n, p, λ, Λ and bounds for the coefficients $\mu \leq 1$, $\|b\|_{L^p(B_2^\nu)} \leq 1 + 2K_3(3 + 2C_n)|B_1|^{1/p}$ (for a constant C_n, depending only on n, from lemma 6.35 of [41] for $\epsilon = 1/2$, that will appear in the sequel) and $\omega(1)\|d\|_{L^p(B_2^\nu)} \leq 1$, for any solution in B_2^ν satisfying $\|u\|_{L^\infty(B_2^\nu)} \leq 1$ and $\|\psi\|_{C^{1,\tau}(\mathbb{T}_2^\nu)} \leq 2$ (or for any solution in B_1^ν with coefficients in B_1^ν).

Lemma 3.15 *Assume F satisfies $(SC)^\mu$ in B_1^ν for some $\nu \in [0,1]$ and $f \in L^p(B_1^\nu)$, where $p > n$. Let $\psi \in C^\tau(\partial B_1^\nu)$ with $\|\psi\|_{C^\tau(\partial B_1^\nu)} \leq K_0$. Then, for all $\varepsilon > 0$, there exists $\delta \in (0,1)$, $\delta = \delta(\varepsilon, n, p, \lambda, \Lambda, \tau, K_0)$, such that if*

$$\|\bar{\beta}_F(\cdot, 0)\|_{L^p(B_1^\nu)} \leq \delta, \ \|f\|_{L^p(B_1^\nu)} \leq \delta, \ \mu \leq \delta, \ \|b\|_{L^p(B_1^\nu)} \leq \delta, \ \omega(1)\|d\|_{L^p(B_1^\nu)} \leq \delta$$

then any two L^p-viscosity solutions v and h of

$$\begin{cases} F(x, v, Dv, D^2v) &= f(x) & in & B_1^\nu \\ v &= \psi & on & \partial B_1^\nu \end{cases}$$

and

$$\begin{cases} F(0,0,0,D^2h) &= 0 & in & B_1^\nu \\ h &= \psi & on & \partial B_1^\nu \end{cases}$$

respectively, satisfy $\|v - h\|_{L^\infty(B_1^\nu)} \leq \varepsilon$.

Proof. For $\varepsilon > 0$, we will prove the existence of $\delta \in (0,1)$ as above with $\delta \leq 2^{-\frac{n}{2p}} \widetilde{\delta}^{1/2}$, where $\widetilde{\delta}$ is the constant from proposition 2.6. Suppose the contrary, then there exist $\varepsilon_0 > 0$ and sequences $\nu_k \in [0,1]$, F_k satisfying $(SC)^{\mu_k}$ for $b_k, d_k \in L^p_+(B_1^{\nu_k})$, $\mu_k \geq 0$, ω_k modulus, $\delta_k \in (0,1)$ with $\delta_k \leq 2^{-\frac{n}{2p}} \widetilde{\delta}_k^{1/2}$ and $f_k \in L^p(B_1^{\nu_k})$ such that

$$\|\bar{\beta}_{F_k}(\cdot, 0)\|_{L^p(B_1^{\nu_k})}, \ \|f_k\|_{L^p(B_1^{\nu_k})}, \ \mu_k, \ \|b_k\|_{L^p(B_1^{\nu_k})}, \ \omega_k(1)\|d_k\|_{L^p(B_1^{\nu_k})} \leq \delta_k \xrightarrow[k\to\infty]{} 0$$

with $v_k, h_k \in C(\overline{B_1^{\nu_k}})$ L^p-viscosity solutions of

$$\begin{cases} F_k(x, v_k, Dv_k, D^2 v_k) &=& f_k(x) & \text{in} & B_1^{\nu_k} \\ v_k &=& \psi_k & \text{on} & \partial B_1^{\nu_k} \end{cases}$$

and

$$\begin{cases} F_k(0, 0, 0, D^2 h_k) &=& 0 & \text{in} & B_1^{\nu_k} \\ h_k &=& \psi_k & \text{on} & \partial B_1^{\nu_k} \end{cases}$$

where $\|\psi_k\|_{C^\tau(\partial B_1^{\nu_k})} \leq K_0$ but $\|v_k - h_k\|_{L^\infty(B_1^{\nu_k})} > \varepsilon_0$.

Analogously to the proof of lemma 3.8, ABP implies that $\|v_k\|_{L^\infty(B_1^{\nu_k})}$, $\|h_k\|_{L^\infty(B_1^{\nu_k})} \leq C_0$ for large k, where C_0 is a constant that depends only on n, p, λ, Λ and K_0.

Notice that $B_1^{\nu_k}$ has the exterior cone property, then by the C^β global quadratic estimate (proposition 2.14) we obtain $\beta \in (0,1)$ such that

$$\|v_k\|_{C^\beta(\overline{B_1^{\nu_k}})}, \ \|h_k\|_{C^\beta(\overline{B_1^{\nu_k}})} \leq C, \quad \text{for all } k \in \mathbb{N}, \tag{3.18}$$

where $\beta = \min(\beta_0, \tau/2)$ for some $\beta_0 = \beta_0(n, p, \lambda, \Lambda)$ and C depending on n, p, λ, Λ and C_0. Observe that β and C do not depend on k, since μ_k, $\|b_k\|_{L^p(B_1^{\nu_k})}$, $\omega_k(1)\|d_k\|_{L^p(B_1^{\nu_k})}$, $\|f_k\|_{L^p(B_1^{\nu_k})} \leq 1$ and $\text{diam}(B_1^{\nu_k}) \leq 2$, for all $k \in \mathbb{N}$. Here we have different domains, what prevents us from directly using the compact inclusion C^β into the set of continuous functions, in order to produce convergent subsequences. But this is just a technicality, as in [38], by taking a subsequence of ν_k that converges to some $\nu_\infty \in [0,1]$, which we can suppose monotonous. Hence we consider two cases: $B_1^{\nu_\infty} \subset B_1^{\nu_k} \subset B_1^{\nu_{k+1}} \subset ...$ or $... \subset B_1^{\nu_{k+1}} \subset B_1^{\nu_k} \subset B_1^{\nu_\infty}$, for all $k \in \mathbb{N}$. In the first one, we use the compact inclusion on $\overline{B_1^{\nu_\infty}}$. In the second, we make a trivial extension of our functions to the larger domain $\overline{B_1^{\nu_\infty}}$, i.e. by defining ψ_k in $\widetilde{B}_k = B_1 \cap \{-\nu_\infty \leq x_n \leq -\nu_k\}$ in such a way that $\|\psi_k\|_{C^\tau(\widetilde{B}_k)} \leq C_0$, from where we can suppose that (3.18) holds on $\overline{B_1^{\nu_\infty}}$ for the extended v_k and h_k. In both cases, we obtain convergent subsequences $v_k \longrightarrow v_\infty$, $h_k \longrightarrow h_\infty$ in $C(\overline{B_1^{\nu_\infty}})$ as $k \to \infty$, for some continuous

functions v_∞, h_∞ in $\overline{B_1^{\nu\infty}}$, with $v_\infty = h_\infty = \psi_\infty$ on $\partial B_1^{\nu\infty}$.

Finally, we claim that v_∞ and h_∞ are viscosity solutions of

$$\begin{cases} F_\infty(D^2 u) = 0 & \text{in} \quad B_1^{\nu\infty} \\ u = \psi_\infty & \text{on} \quad \partial B_1^{\nu\infty} \end{cases},$$

therefore equal by proposition 2.23, which contradicts $\|v_\infty - h_\infty\|_{L^\infty(B_1)} \geq \varepsilon_0$.

For h_∞, it follows by passing to uniform limits in the inequalities satisfied by h_k. For v_∞, we apply proposition 2.12 together with observation 2.13, since for each $\varphi \in C^2(B)$, where $B \subset B_1^{\nu\infty}$, we have that $F_k(x, v_k, D\varphi, D^2\varphi) - f_k(x) - F_\infty(D^2\varphi) \to 0$ as $k \to \infty$ in $L^p(B)$, analogously to the end of the proof of lemma 3.8. ■

Proof of theorem 3.14. We proceed as in the local case, introducing the corresponding changes that come from the boundary context. Our approach is similar to [38]. Now we set $W := \|u\|_{L^\infty(B_1^+)} + \|f\|_{L^p(B_1^+)} + \|\psi\|_{C^{1,\tau}(\mathbb{T}_1)} + \|d\|_{L^p(B_1^+)} \omega(\|u\|_{L^\infty(B_1^+)}) \leq W_0$ and $s_0 := \min(r_0, \frac{1}{2})$.

Fix $\alpha \in (0, \bar\alpha)$ with $\alpha \leq \min(\beta, 1 - \frac{n}{p}, \tau, \bar\alpha(1-\tau))$ and choose $\gamma = \gamma(n, \alpha, \bar\alpha, K_3) \in (0, \frac{1}{4}]$ such that $2^{2+\bar\alpha} K_4 \gamma^{\bar\alpha} \leq \gamma^\alpha$, where $K_4 = K_4(K_3, n) \geq 1$ will be specified later. Thus, define $\varepsilon = \varepsilon(\gamma)$ by $K_4(2\gamma)^{1+\bar\alpha}$. This ε provides a $\delta = \delta(\varepsilon) \in (0,1)$, the constant of the approximation lemma 3.15 which, up to diminishing, can be supposed to satisfy $(5 + 2K_4)\delta \leq \gamma^\alpha$. Next we chose $\sigma = \sigma(s_0, n, p, \alpha, \bar\alpha, \beta, \delta, \mu, \|b\|_{L^p(B_1^+)}, \omega(1)\|d\|_{L^p(B_1^+)}, K_1, K_3, C_0) \leq \frac{s_0}{2}$ such that

$$\sigma^{\min(1-\frac{n}{p}, \beta)} m \leq \delta \{32 K^2 (K_4 + K + 1)|B_1|^{1/p}\}^{-1}$$

where $m := \max\{1, \|b\|_{L^p(B_1^+)}, \omega(1)\|d\|_{L^p(B_1^+)}, \mu(1+2^\beta K_1)W_0\}$ and $K := K_4 \gamma^{-\alpha}(1-\gamma^\alpha)^{-1} + K_4 \gamma^{-1-\alpha}(1-\gamma^{1+\alpha})^{-1} \geq K_4 \geq 1$.

Fix $z = (z', z_n) \in B_{1/2}^+(0)$. We split our analysis in two cases, depending on the distance of the point z to the bottom boundary:

1) $z_n < \frac{\sigma}{2} \Leftrightarrow \nu < \frac{1}{2}$ and 2) $z_n \geq \frac{\sigma}{2} \Leftrightarrow \nu \geq \frac{1}{2}$, for $\nu := \frac{z_n}{\sigma}$.

Suppose the first one. In this case we proceed as in [38] by translating the problem to the set B_2^ν, in order to use the approximation lemma in its boundary version 3.15. Notice that, for $B_{2\sigma}^+(z) = B_{2\sigma}(z) \cap \{x_n > 0\}$, which is a subset of $B_1^+(0)$, and for $B_2^\nu(0) = B_2(0) \cap \{x_n > -\nu\}$, we have

$$x \in B_2^\nu(0) \Leftrightarrow \sigma x + z \in B_{2\sigma}^+(z) \subset B_1^+(0).$$
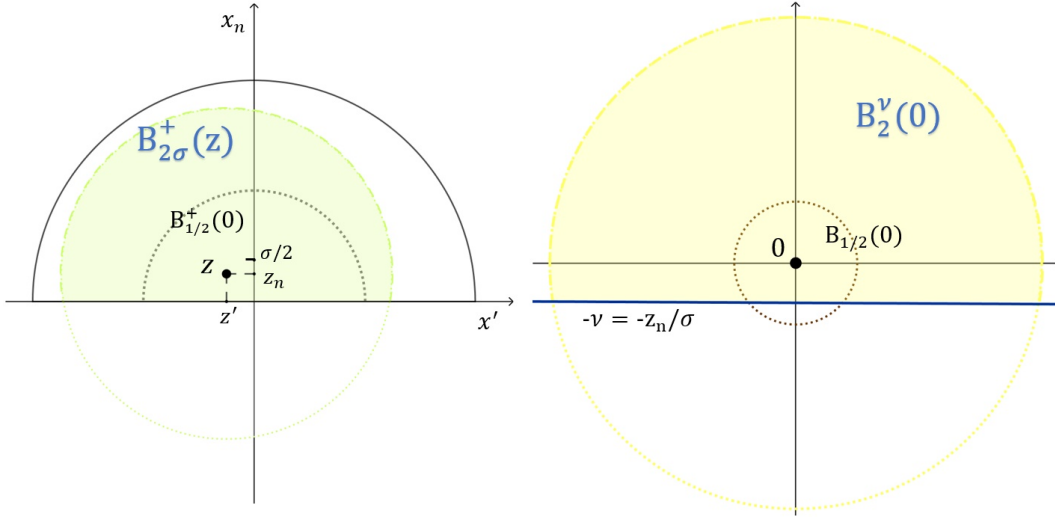
Figure 3.1: Illustration of the variable change, from $B_{2\sigma}^+(z)$ to $B_2^\nu(0)$.

Then we define $N = N_\sigma(z) := \sigma W + \sup_{x \in B_2^\nu(0)} |u(\sigma x + z) - u(z)|$. The C^β estimate, this time the global one, restricted to the set $B_{2\sigma}^+(z)$, yields

$$\sigma W \leq N \leq (\sigma + 2^\beta K_1 \sigma^\beta) W \leq (1 + 2^\beta K_1) \sigma^\beta W_0. \qquad (3.19)$$

Next we set $\widetilde{u}(x) := \frac{1}{N}\{u(\sigma x + z) - u(z)\}$. As in claim 3.9, $\widetilde{u}$ is an L^p-viscosity solution of

$$\begin{cases} \widetilde{F}(x, \widetilde{u}, D\widetilde{u}, D^2\widetilde{u}) &=& \widetilde{f}(x) \quad \text{in} \quad B_2^\nu \\ \widetilde{u} &=& \widetilde{\psi} \quad \text{on} \quad \mathbb{T}_2^\nu \end{cases}$$

for

$$\widetilde{F}(x, r, p, X) := \frac{\sigma^2}{N} F\left(\sigma x + z, Nr + u(z), \frac{N}{\sigma} p, \frac{N}{\sigma^2} X\right) - \frac{\sigma^2}{N} F(\sigma x + z, u(z), 0, 0),$$

$\widetilde{\psi}(x) := \frac{1}{N}\{\psi(\sigma x + z) - u(z)\}$ and $\widetilde{f} := \widetilde{f}_1 + \widetilde{f}_2$ where

$$\widetilde{f}_1(x) := \sigma^2 f(\sigma x + z)/N; \quad \widetilde{f}_2(x) := -\sigma^2 F(\sigma x + z, u(z), 0, 0)/N,$$

$\widetilde{F}$ satisfying $(\widetilde{SC})^{\widetilde{\mu}}$ for $\widetilde{b}(x) = \sigma b(\sigma x + z)$, $\widetilde{\mu} = N\mu$, $\widetilde{d}(x) = \sigma^2 d(\sigma x + z)$ and $\widetilde{\omega}(r) = \omega(Nr)/N$.

With this definition and the choice of σ in (3.12), we get $\|\widetilde{u}\|_{L^\infty(B_2^\nu)} \leq 1$, $\|\widetilde{f}\|_{L^p(B_2^\nu)} \leq \frac{\delta}{8}$, $\widetilde{\mu} \leq \frac{\delta}{8K^2|B_1|^{1/p}}$, $\|\widetilde{b}\|_{L^p(B_2^\nu)} \leq \frac{\delta}{16K}$, $\widetilde{\omega}(1)\|\widetilde{d}\|_{L^p(B_2^\nu)} \leq \frac{\delta}{32(K_4+K+1)}$ and $\|\bar{\beta}_{\widetilde{F}}(0, \cdot)\|_{L^p(B_1^\nu)} \leq \delta/4$ by choosing $\theta = \delta/8$, as in the local case.

Furthermore, we have $\|\widetilde{\psi}\|_{L^\infty(\mathbb{T}_2^\nu)} \leq \|\widetilde{u}\|_{L^\infty(B_2^\nu)} \leq 1$ and then

$$\|D\widetilde{\psi}\|_{C^\tau(\mathbb{T}_2^\nu)} \leq \frac{\sigma}{N}\|D\psi\|_{L^\infty(B_{2\sigma}(z)\cap\mathbb{T})} + \frac{\sigma}{N}\sup_{x\neq y\in\mathbb{T}_2^\nu}\frac{|D\psi(\sigma x + z) - D\psi(\sigma y + z)|}{|\sigma x - \sigma y|^\tau}\sigma^\tau$$

$$\leq \frac{\|\psi\|_{C^{1,\tau}(\mathbb{T})}}{W} \leq 1$$

since $N \geq \sigma W$. Therefore, we obtain

$$\|\widetilde{\psi}\|_{C^{1,\tau}(\mathbb{T}_2^\nu)} = \|\widetilde{\psi}\|_{L^\infty(\mathbb{T}_2^\nu)} + \|D\widetilde{\psi}(x)\|_{C^\tau(\mathbb{T}_2^\nu)} \leq 2.$$

We can suppose, up to this rescaling, that F, u, μ, b, d, ω satisfy the former hypotheses related to $\widetilde{F}, \widetilde{u}, \widetilde{\mu}, \widetilde{b}, \widetilde{d}, \widetilde{\omega}$. Thus, we move to the construction of $l_k(x) := a_k + b_k \cdot x$ such that

$(i)_k$ $\|u - l_k\|_{L^\infty(B_{r_k}^\nu)} \leq r_k^{1+\alpha}$

$(ii)_k$ $|a_k - a_{k-1}| \leq K_4\, r_{k-1}^{1+\alpha}$, $|b_k - b_{k-1}| \leq K_4\, r_{k-1}^\alpha$

$(iii)_k$ $|(u-l_k)(r_k x) - (u-l_k)(r_k y)| \leq C_{1,4}\, r_k^{1+\alpha}|x-y|^\beta$ for all $x, y \in B_1^{\nu_k}$

where $C_{1,4} = C_{1,4}(K_1, K_4)$ and $\nu_k := \frac{\nu}{r_k}$, $r_k = \gamma^k$ for some $\gamma \in (0,1)$, for all $k \geq 0$; again $l_{-1} \equiv 0$.

We emphasize that these iterations will prove that the function u (which plays the role of $\widetilde{u}$) is differentiable at 0 and provide

$$|u(x) - u(0) - Du(0) \cdot x| \leq C|x|^{1+\alpha}, \;\; |Du(0)| \leq C$$

for every $x \in B_1^\nu$. In terms of our original function defined on B_1^+, it means that u will be differentiable at z, for all z with $z_n < \frac{\sigma}{2}$. On the other hand, the second case $z_n \geq \frac{\sigma}{2}$ is covered by the local part, section 3.2.1, since in this situation we are far away from the bottom boundary. Consequently, boundary superlinear regularity and estimates on B_1^+ will follow by a covering argument.

For the proof of $(i)_k - (iii)_k$, we use induction on k. For $k = 0$ we set $a_0 = b_0 = 0$. Recall that β and K_1 are the constants from C^β quadratic global estimate in the set B_1^ν, then we have $\|u\|_{C^\beta(\overline{B_1^\nu})} \leq \widetilde{K}_1(1+\delta+2+1) \leq 5K_1$ and so $(iii)_0$ for $\nu_0 = \nu$, $(i)_0$ and $(ii)_0$ are valid.

Analogously to the the local case, we have $|b_k|, \|l_k\|_{L^\infty(B_{r_k}^\nu)} \leq K$. For the induction's step we suppose $(i)_k - (iii)_k$ and construct a_{k+1}, b_{k+1} such that $(i)_{k+1} - (iii)_{k+1}$ are valid. Define

$$v(x) = v_k(x) := \frac{(u - l_k)(r_k x)}{r_k^{1+\alpha}} = \frac{u(r_k x) - a_k - b_k \cdot x r_k}{r_k^{1+\alpha}}, \;\; \text{for all } x \in B_2^{\nu_k}.$$

Since $r_k x \in B_{r_k}^\nu \Leftrightarrow x \in B_1^{\nu_k}$, $(i)_k$ says that $|v| \le 1$ in $B_1^{\nu_k}$. From this and $(iii)_k$,

$$\|v\|_{C^\beta(\overline{B_1^{\nu_k}})} = \|v\|_{L^\infty(B_1^{\nu_k})} + \sup_{\substack{x,y \in B_1^{\nu_k} \\ x \neq y}} \frac{|v(x) - v(y)|}{|x - y|^\beta} \le 1 + C_{1,4} =: K_0.$$

Notice that, as in claim 3.12, v is an L^p-viscosity solution of

$$\begin{cases} F_k(x, v, Dv, D^2 v) &=& f_k(x) & \text{in} & B_2^{\nu_k} \\ v &=& \psi_k & \text{on} & \mathbb{T}_2^{\nu_k} \end{cases}$$

for $f_k := f_k^1 + f_k^2$; $f_k^1(x) := r_k^{1-\alpha} f(r_k x)$, $f_k^2(x) := -r_k^{1-\alpha} F(r_k x, l_k(r_k x), b_k, 0)$ and $F_k(x, s, p, X)$, defined as

$$r_k^{1-\alpha} F(r_k x, r_k^{1+\alpha} s + l_k(r_k x), r_k^\alpha p + b_k, r_k^{\alpha-1} X) - r_k^{1-\alpha} F(r_k x, l_k(r_k x), b_k, 0),$$

satisfying $(SC)_{F_k}^{\mu_{F_k}}$ with coefficients $b_{F_k}(x) = r_k b(r_k x) + 2 r_k \mu K$, $\mu_{F_k} = r_k^{1+\alpha} \mu$, $d_{F_k}(x) = r_k^2 d(r_k x)$ and $\omega_{F_k}(s) = r_k^{-1-\alpha} \omega(r_k^{1+\alpha} s)$.

These $F_k, v, \mu_{F_k}, b_{F_k}, d_{F_k}, \omega_{F_k}$ still satisfy the hypotheses of lemma 3.15, since $\|b_{F_k}\|_{L^p(B_1^{\nu_k})} \le \delta$, $\omega_{F_k}(1)\|d_{F_k}\|_{L^p(B_1^{\nu_k})} \le \delta$, $\|f_k\|_{L^p(B_1^{\nu_k})} \le \delta$ and $\|\bar{\beta}_{F_k}(\cdot, 0)\|_{L^p(B_1^{\nu_k})} \le \delta$, see section 3.2.1.

Let $h = h_k \in C(\overline{B_1^{\nu_k}})$ be the C-viscosity solution of

$$\begin{cases} F_k(0, 0, 0, D^2 h) &=& 0 & \text{in} & B_1^{\nu_k} \\ h &=& v & \text{on} & \partial B_1^{\nu_k} \end{cases}$$

given by proposition 2.23, since $B_1^{\nu_k}$ has the uniform exterior cone condition. From ABP we get $\|h\|_{L^\infty(B_1^{\nu_k})} \le \|h\|_{L^\infty(\partial B_1^{\nu_k})} \le 1$. Further, $h = v = \psi_k \in C^{1,\tau}(B_1 \cap \{x_n = -\nu_k\})$ and we can find a uniform bound for the $C^{1,\tau}$ norm of ψ_k. Indeed, $\|\psi_k\|_{L^\infty(\mathbb{T}_1^{\nu_k})} \le \|v\|_{L^\infty(\overline{B_1^{\nu_k}})} \le 1$ and

$$[D\psi_k]_{\tau, \mathbb{T}_1^{\nu_k}} = \sup_{\substack{x,y \in \mathbb{T}_1^{\nu_k} \\ x \neq y}} \frac{|D\psi_k(x) - D\psi_k(y)|}{|x - y|^\tau} = \sup_{\substack{\tilde{x}, \tilde{y} \in \mathbb{T}_{r_k}^\nu \\ \tilde{x}=r_k x, \tilde{y}=r_k y}} \frac{|D\psi(\tilde{x}) - D\psi(\tilde{y})|}{|\tilde{x} - \tilde{y}|^\tau} r_k^{\tau-\alpha} \le 1$$

since $\|D\psi\|_{C^\tau(\mathbb{T}_1^\nu)} \le 1$ and $\alpha \le \tau$. Moreover, using the global Holder interpolation in smooth domains, lemma 6.35 of [41], for $\epsilon = \frac{1}{2}$, there exists a constant C_n, which does not depend[1] on k, such that

$$\|\psi_k\|_{C^1(\mathbb{T}_1^{\nu_k})} \le C_n \|\psi_k\|_{C(\mathbb{T}_1^{\nu_k})} + \frac{1}{2} \|\psi_k\|_{C^{1,\tau}(\mathbb{T}_1^{\nu_k})}$$

[1] The proof of lemma 6.35 in [41] is based on an interpolation inequality (6.89) for adimensional Holder norms (that does not depend on the domain); followed by a partition of unity that straightens the boundary (not necessary in our case $\mathbb{T}_1^{\nu_k} \subset \mathbb{R}^{n-1}$).

hence

$$\|\psi_k\|_{C^{1,\tau}(\mathbb{T}_1^{\nu_k})} = \|\psi_k\|_{C^1(\mathbb{T}_1^{\nu_k})} + [D\psi_k]_{\tau,\mathbb{T}_1^{\nu_k}} \leq C_n + \frac{1}{2}\|\psi_k\|_{C^{1,\tau}(\mathbb{T}_1^{\nu_k})} + 1$$

i.e. $\|\psi_k\|_{C^{1,\tau}(\mathbb{T}_1^{\nu_k})} \leq 2(C_n+1)$. Thus, the $C^{1,\bar{\alpha}}$ global estimate (proposition 2.24) yields

$$\|h\|_{C^{1,\bar{\alpha}}(\overline{B_{1/2}^{\nu_k}})} \leq K_3\left\{\|h\|_{L^\infty(B_1^{\nu_k})} + \|\psi_k\|_{C^{1,\tau}(\mathbb{T}_1^{\nu_k})}\right\} \leq K_3(3+2C_n) =: K_4.$$

Now, the approximation boundary lemma 3.8 applied to F_k, v, h, ν_k, μ_{F_k}, b_{F_k}, d_{F_k}, ω_{F_k}, ψ_k, β, K_0 gives us that $\|v-h\|_{L^\infty(B_1^{\nu_k})} \leq \varepsilon$.

Therefore, defining $\bar{l}(x) = \bar{l}_k(x) := h(0) + Dh(0)\cdot x$ in $B_1^{\nu_k}$, it follows that

$$\|v-\bar{l}\|_{L^\infty(B_{2\gamma}^{\nu_k})} \leq \gamma^{1+\alpha}. \tag{3.20}$$

In fact, by the choice of γ we have, for all $x \in B_{2\gamma}^{\nu_k}(0)$,

$$|v(x)-\bar{l}(x)| \leq |v(x)-h(x)| + |h(x)-h(0)-Dh(0)\cdot x| \leq 2K_4(2\gamma)^{1+\bar{\alpha}} \leq \gamma^{1+\alpha}.$$

Next, (3.20) and the definition of v imply

$$|u(r_k x)-l_k(r_k x)-r_k^{1+\alpha}h(0)-r_k^{1+\alpha}Dh(0)\cdot x| \leq r_k^{1+\alpha}\gamma^{1+\alpha} = r_{k+1}^{1+\alpha} \text{ for } x \in B_{2\gamma}^{\nu_k},$$

which is equivalent to

$$|u(y)-l_{k+1}(y)| \leq r_k^{1+\alpha}\gamma^{1+\alpha} = r_{k+1}^{1+\alpha} \text{ for all } y = r_k x \in B_{2\gamma r_k}^\nu = B_{2r_{k+1}}^\nu,$$

where $l_{k+1}(y) := l_k(y) + r_k^{1+\alpha}h(0) + r_k^\alpha Dh(0)\cdot y$. Then, we define $a_{k+1} := a_k + h(0)r_k^{1+\alpha}$, $b_{k+1} := b_k + Dh(0)r_k^\alpha$, obtaining $(i)_{k+1}$. Also, $|a_{k+1}-a_k| \leq K_4 r_k^{1+\alpha}$, $|b_{k+1}-b_k| \leq K_4 r_k^\alpha$, which is $(ii)_{k+1}$. As in the local case, to finish the proof of $(iii)_{k+1}$, it is enough to show that

$$\|v-\bar{l}\|_{C^\beta(\overline{B_\gamma^{\nu_k}})} \leq C_{1,4}\gamma^{1+\alpha-\beta}.$$

Let us see that this is obtained by applying the global superlinear C^β estimate (proposition 2.14) to the function $w := v - \bar{l}$.

Analogously to the local case, w is an L^p-viscosity solution in $B_2^{\nu_k}$ of (3.17) (see notations and coefficients there), in addition to $w = \psi_k - \bar{l}$ on $\mathbb{T}_2^{\nu_k}$. The definition of δ gives us $\|g_k\|_{L^p(B_1^{\nu_k})} \leq (5+2K_4)\delta \leq \gamma^\alpha$. Further, using that $\psi_k = h$ on $\mathbb{T}_{2\gamma}^{\nu_k}$, we obtain $\|\psi_k-\bar{l}\|_{L^\infty(\mathbb{T}_{2\gamma}^{\nu_k})} \leq \gamma^{1+\alpha}$.

Now, since $\psi_k - \bar{l} \in C^1(\mathbb{T}_{2\gamma}^{\nu_k})$, it is a Lipschitz function with constant less

or equal than $\|D\psi_k - D\bar{l}\|_{C(\mathbb{T}_{2\gamma}^{\nu_k})} \le 2(C_n + 1) + K_4 \le 2K_4$ and thus

$$
\begin{aligned}
|(\psi_k - \bar{l})(x) - (\psi_k - \bar{l})(y)| \\
&= |(\psi_k - \bar{l})(x) - (\psi_k - \bar{l})(y)|^\tau |(\psi_k - \bar{l})(x) - (\psi_k - \bar{l})(y)|^{1-\tau} \\
&\le (2K_4)^\tau (2K_4)^{1-\tau} |x - y|^\tau \gamma^{(1+\bar{\alpha})(1-\tau)} \\
&= 2K_4 |x - y|^\tau \gamma^{1-\tau+\bar{\alpha}(1-\tau)}.
\end{aligned}
$$

Then, the choice of α implies that $[\psi_k - \bar{l}]_{\tau, \mathbb{T}_{2\gamma}^{\nu_k}} \le 4K_4 \gamma^{1-\tau+\alpha}$. Hence, from this, (3.20) and C^β global estimate, properly scaled for the radius γ, we obtain

$$
\begin{aligned}
{[w]}_{\beta, \overline{B_\gamma^{\nu_k}}} &\le \gamma^{-\beta} \widetilde{K_1} \big\{ \|w\|_{L^\infty(B_{2\gamma}^{\nu_k})} + \gamma^{2-\frac{n}{p}} \|g_k\|_{L^p(B_{2\gamma}^{\nu_k})} \\
&\qquad\qquad + \|\psi_k - \bar{l}\|_{L^\infty(\mathbb{T}_{2\gamma}^{\nu_k})} + \gamma^\tau [\psi_k - \bar{l}]_{\tau, \mathbb{T}_{2\gamma}^{\nu_k}} \big\} \\
&\le \gamma^{-\beta} K_1 \big\{ 2\gamma^{1+\alpha} + \gamma^{2-\frac{n}{p}} \gamma^\alpha + 4K_4 \gamma^{1+\alpha} \big\} \le K_1 (3 + 4K_4) \gamma^{1+\alpha-\beta}
\end{aligned}
$$

and finally, for $C_{1,4} := 1 + (3 + 4K_4)K_1 = C_{1,4}(K_1, K_4)$, we conclude

$$
\begin{aligned}
\|w\|_{C^\beta(\overline{B_\gamma^{\nu_k}})} &= \|w\|_{L^\infty(B_\gamma^{\nu_k})} + [w]_{\beta, \overline{B_\gamma^{\nu_k}}} \\
&\le \gamma^{1+\alpha} + (3 + 4K_4)K_1 \gamma^{1+\alpha-\beta} \le C_{1,4} \gamma^{1+\alpha-\beta}.
\end{aligned}
$$

∎

3.2.3
The Global Blend

Our goal in this section is to prove the global statement of theorem 3.1, by combining the local regularity estimates derived in section 3.2.1 with theorem 3.14 in section 3.2.2. Even though the argument is easily assembled from the previous constructions, we compile the details for completeness since, on the one hand, some technical difficulties arise due to the presence of quadratic growth in the gradient; on the other hand, it permits to clarify some points in the linear growth context in [38].

Let us deal with the model equation (ME) for $L = F$. More precisely, assume that $u \in C(\overline{\Omega})$ is an L^p-viscosity solution of

$$
\begin{cases}
F(x, u, Du, D^2u) + c(x)u + \langle M(x)Du, Du \rangle = f(x) & \text{in} \quad \Omega \\
u = \psi & \text{on} \quad \partial\Omega
\end{cases}
\tag{3.21}
$$

for $\psi \in C^{1,\tau}(\partial\Omega)$ and F satisfying (SC). Meanwhile we quickly discuss the changes when we have an arbitrary superlinear growth (since the latter is easier to handle when we are interested in diffeomorphic invariance property).

From section 3.2.1 we obtain local regularity estimates for any $\Omega' \subset\subset \Omega$, with a fixed distance from the boundary. So, as in [38], we only need to make a careful analysis on the neighborhoods of boundary points. In fact, in order to use theorem 3.14, near a boundary point we will make a diffeomorphic change of independent variable, which takes a neighborhood of $\partial\Omega$ into B_1^+. This change only depends on the coefficients of the equation and on the $C^{1,1}$ character of the boundary.

Let $x_0 \in \partial\Omega$. Since $\partial\Omega \in C^{1,1}$, there exists an open set $\mathcal{U}_{x_0}$ containing x_0 and a $C^{1,1}$ diffeomorphism

$$
\begin{aligned}
\Phi \; : \quad & \mathcal{U}_{x_0} \subset \mathbb{R}^n \; \longrightarrow \; B_1(0) \subset \mathbb{R}^n \\
& x = (x_1, \ldots, x_n) \longmapsto (\Phi_1(x), \ldots, \Phi_n(x))
\end{aligned}
$$

such that $\Phi(x_0) = 0$ and $\Phi(\mathcal{U}_{x_0} \cap \Omega) = B_1^+$, $\Phi(\overline{\mathcal{U}}_{x_0} \cap \partial\Omega) = \mathbb{T}_1 = B_1 \cap \{x_n = 0\}$.

Note that $u \in C(\overline{\Omega})$ is, in particular, an L^p-viscosity solution of the first equation in (3.21) in the smaller domain $\mathcal{U}_{x_0} \cap \Omega$, with $u = \psi$ on $\partial\Omega \cap \overline{\mathcal{U}}_{x_0}$. Also, the function $\tilde{u} := u \circ \Phi^{-1}$ is continuous in $\overline{B_1^+}$.

We claim that $\tilde{u}$ is an L^p-viscosity solution of

$$
\begin{cases}
\widetilde{F}(x, \tilde{u}, D\tilde{u}, D^2\tilde{u}) + \tilde{c}(x)\tilde{u} + \langle \widetilde{M}(x)D\tilde{u}, D\tilde{u} \rangle & = \; \tilde{f}(x) \quad \text{in} \quad B_1^+ \\
\tilde{u} & = \; \tilde{\psi} \quad \text{on} \quad \mathbb{T}_1
\end{cases} \tag{3.22}
$$

where $\tilde{c} := c \circ \Phi^{-1}$, $\tilde{f} := f \circ \Phi^{-1}$, $\widetilde{M} := (D\Phi^T \circ \Phi^{-1})M \circ \Phi^{-1}(D\Phi \circ \Phi^{-1})$, $\tilde{\psi} := \psi \circ \Phi^{-1}$ and $\widetilde{F}(x, r, p, X)$, defined as

$$
F\big(\Phi^{-1}(x), r, (D\Phi \circ \Phi^{-1})p, (D\Phi^T \circ \Phi^{-1})X(D\Phi \circ \Phi^{-1}) + \big((\partial_{ij}\Phi \circ \Phi^{-1})\,p \big)_{i,j} \big) \big),
$$

satisfies $(\widetilde{SC})$, with ellipticity coefficients $\tilde{\lambda} = \lambda c_\Phi^1$, $\tilde{\Lambda} = \Lambda c_\Phi^2$, for some positive constants c_Φ^1, c_Φ^2, $\tilde{b} := \|D\Phi \circ \Phi^{-1}\|_\infty b \circ \Phi^{-1} + n^2 \Lambda \max_{1 \le i,j \le n} |\partial_{ij}\Phi \circ \Phi^{-1}|$, $\tilde{d} := d \circ \Phi^{-1}$ and $\tilde{\omega} := \omega$. Moreover,

$$
\beta_{\widetilde{F}}(x, x_0) \le C_\Phi^1 \, \beta_F(\Phi^{-1}(x), \Phi^{-1}(x_0)) + C_\Phi^2 \, |x - x_0| \tag{3.23}
$$

and

$$
\bar{\beta}_{\widetilde{F}}(x, x_0) \le C_\Phi^3 \, \bar{\beta}_F(\Phi^{-1}(x), \Phi^{-1}(x_0)) + C_\Phi^2 \, |x - x_0|, \tag{3.24}
$$

for some positive constants C_Φ^1, C_Φ^2 and C_Φ^3. Furthermore, $\widetilde{F}$ is convex or concave provided F is (notice that this last statement is important to deal with the $W^{2,p}$ results).

Let us prove the claim. Of course $\tilde{c}, \tilde{d} \; \tilde{f} \in L^p(B_1^+)$, $\tilde{\psi} \in C^{1,\tau}(\mathbb{T})$ and

$\widetilde{u} = \widetilde{\psi}$ on $\mathbb{T}$. Observe that $\|M\|_{L^\infty(B_1^+)} \leq C(\Phi)\|M\|_{L^\infty(\Omega)}$.

Let $\widetilde{\varphi} \in W^{2,p}_{\text{loc}}(B_1^+)$ a test function and set $\varphi := \widetilde{\varphi} \circ \Phi \in W^{2,p}_{\text{loc}}(\mathcal{U}_{x_0} \cap \Omega)$. Since $\Phi \in C^{1,1}$, it follows that $D\Phi$ is Lipschitz and so differentiable a.e., then the second order partial derivatives of Φ exist at almost every point. We know (for instance, theorem 4 in section 5.8 of [29]) that $D\Phi \in W^{1,\infty}(\Omega)$, so $\Phi \in W^{2,q}(\Omega)$ for all $q \leq +\infty$. Thus $\widetilde{b} \in L^p(\Omega)$ and we can calculate

$$\partial_i\,\varphi\,(x) = \sum_{k=1}^{n} \partial_k\,\widetilde{\varphi}\,(\Phi(x))\,\frac{\partial \Phi_k}{\partial x_i}(x)$$

$$\partial_{ij}\,\varphi\,(x) = \sum_{k=1}^{n}\sum_{l=1}^{n} \partial_{kl}\,\widetilde{\varphi}\,(\Phi(x))\,\frac{\partial \Phi_l}{\partial x_j}\frac{\partial \Phi_k}{\partial x_i} + \sum_{k=1}^{n} \partial_k\,\widetilde{\varphi}\,(\Phi(x))\,\frac{\partial^2 \Phi_k}{\partial x_i x_j}$$

and then

$$D\varphi = D\Phi\,(D\widetilde{\varphi} \circ \Phi) \in \mathbb{M}_{n\times 1}(\mathbb{R})$$

$$D^2\varphi = D\Phi^T(D^2\widetilde{\varphi} \circ \Phi)D\Phi + (\partial_{ij}\,\Phi\,(D\widetilde{\varphi} \circ \Phi))_{1\leq i,j\leq n} \in \mathbb{M}_{n\times n}(\mathbb{R})$$

under the convention that $\partial_{ij}\,\Phi$ is the line matrix in $\mathbb{M}_{1\times n}(\mathbb{R})$ given by the second order derivatives in direction i,j of the coordinate functions Φ_k, i.e. $\partial_{ij}\Phi = \left(\frac{\partial^2 \Phi_1}{\partial x_i x_j} \cdots \frac{\partial^2 \Phi_n}{\partial x_i x_j}\right)$. In order to give meaning to the quadratic form $\langle M(x)p, p\rangle$, for a matrix $M(x) \in \mathbb{M}_{n\times n}(\mathbb{R})$, we make the convention that vector derivatives of functions from $\mathbb{R}^n$ into $\mathbb{R}$ are column matrices in $\mathbb{M}_{n\times 1}(\mathbb{R})$.

Observe that the definition on the coefficients of the equation satisfied by $\widetilde{u}$ in (3.22) is the only one that makes sense when we evaluate our original equation (3.21) with u, on the derivatives of the test function φ, and apply Φ^{-1} to obtain the desired equation in B_1^+, i.e.

$$F(x, u, D\varphi, D^2\varphi) \circ \Phi^{-1} = F(\Phi^{-1}(x), u \circ \Phi^{-1}, (D\Phi \circ \Phi^{-1})D\widetilde{\varphi},$$
$$(D\Phi^T \circ \Phi^{-1})D^2\widetilde{\varphi}\,(D\Phi \circ \Phi^{-1}) + \left((\partial_{ij}\,\Phi \circ \Phi^{-1})\,D\widetilde{\varphi}\right)_{i,j})$$
$$= \widetilde{F}(x, \widetilde{u}, D\widetilde{\varphi}, D^2\widetilde{\varphi})$$

and

$$\langle M(x)D\varphi, D\varphi\rangle \circ \Phi^{-1} = \langle\,(M \circ \Phi^{-1})(x)\,(D\Phi \circ \Phi^{-1})D\widetilde{\varphi}, (D\Phi \circ \Phi^{-1})D\widetilde{\varphi}\,\rangle$$
$$= \left((D\Phi \circ \Phi^{-1})D\widetilde{\varphi}\right)^T (M \circ \Phi^{-1})(x)\left((D\Phi \circ \Phi^{-1})D\widetilde{\varphi}\right)$$
$$= (D\widetilde{\varphi})^T\left((D\Phi \circ \Phi^{-1})^T(M \circ \Phi^{-1})(x)(D\Phi \circ \Phi^{-1})\right)(D\widetilde{\varphi})$$
$$= \langle\widetilde{M}(x)D\widetilde{\varphi}, D\widetilde{\varphi}\rangle.$$

Notice that

$$\sup_{\lambda I \leq A = (a_{ij})_{ij} \leq \Lambda I} \operatorname{tr}\left(A\left(\partial_{ij}\Phi \circ \Phi^{-1}\right)(p - q)\right)$$

$$= \sup_{\lambda I \leq A \leq \Lambda I} \sum_{i,j=1}^{n} |a_{ij}(\partial_{ij}\Phi \circ \Phi^{-1})(p - q)| \leq n^2\Lambda \max_{1 \leq i,j \leq n} |\partial_{ij}\Phi \circ \Phi^{-1}|\,|p - q|$$

and

$$\sup_{\lambda I \leq A \leq \Lambda I} \operatorname{tr}\left(A(D\Phi^T \circ \Phi^{-1})X(D\Phi \circ \Phi^{-1})\right)$$

$$= \sup_{\lambda I \leq A \leq \Lambda I} \operatorname{tr}\left((D\Phi \circ \Phi^{-1})A(D\Phi^T \circ \Phi^{-1})X\right) = \sup_{\tilde\lambda I \leq \tilde A \leq \tilde\Lambda I} \operatorname{tr}(\tilde A X),$$

since $\langle \tilde A \xi, \xi \rangle = \xi^T(D\Phi \circ \Phi^{-1})A(D\Phi^T \circ \Phi^{-1})\xi = (D\Phi^T \circ \Phi^{-1}\xi)^T A(D\Phi^T \circ \Phi^{-1}\xi) \in [\lambda, \Lambda]|\eta|^2 \subset [\tilde\lambda, \tilde\Lambda]|\xi|^2$, for $\eta = D\Phi^T \circ \Phi^{-1}\xi \Leftrightarrow \xi = D\Phi \circ \Phi^{-1}\eta$ which is such that $\|D\Phi \circ \Phi^{-1}\|_\infty^{-1}|\xi| \leq |\eta| \leq \|D\Phi^T \circ \Phi^{-1}\|_\infty|\xi|$. Then, define c_Φ^1 and c_Φ^2, respectively, as the minimum and maximum of the numbers $\|D\Phi \circ \Phi^{-1}\|_\infty^{-2}$ and $\|D\Phi^T \circ \Phi^{-1}\|_\infty^2$, both being positive since the determinant of the Jacobian matrix of Φ is equal to 1.

Therefore, $\tilde F$ satisfies $(\widetilde{SC})$, since

$$\tilde F(x, r, p, X) - \tilde F(x, s, q, Y)$$
$$= F(\Phi^{-1}(x), r, (D\Phi \circ \Phi^{-1})p, (D\Phi^T \circ \Phi^{-1})X(D\Phi \circ \Phi^{-1}) + \left((\partial_{ij}\Phi \circ \Phi^{-1})p\right)_{i,j}))$$
$$- F(\Phi^{-1}(x), s, (D\Phi \circ \Phi^{-1})q, (D\Phi^T \circ \Phi^{-1})Y(D\Phi \circ \Phi^{-1}) + \left((\partial_{ij}\Phi \circ \Phi^{-1})q\right)_{i,j}))$$
$$\leq \mathcal{M}_{\lambda,\Lambda}^+((D\Phi^T \circ \Phi^{-1})(X - Y)(D\Phi \circ \Phi^{-1}) + \left((\partial_{ij}\Phi \circ \Phi^{-1})(p - q)\right)_{i,j}))$$
$$\quad + b(\Phi^{-1}(x))\,|(D\Phi \circ \Phi^{-1})(p - q)| + d(\Phi^{-1}(x))\omega((s - r)^+)$$
$$\leq \mathcal{M}_{\lambda c_\Phi^1, \Lambda c_\Phi^2}^+(X - Y) + \mathcal{M}_{\lambda,\Lambda}^+\left((\partial_{ij}\Phi \circ \Phi^{-1})(p - q)\right)_{i,j})$$
$$\quad + \|D\Phi \circ \Phi^{-1}\|_\infty\, b(\Phi^{-1}(x))\,|p - q| + d(\Phi^{-1}(x))\omega((s - r)^+)$$
$$\leq \mathcal{M}_{\tilde\lambda,\tilde\Lambda}^+(X - Y) + \tilde b(x)|p - q| + \tilde d(x)\tilde\omega((s - r)^+)$$

Of course the bounds on the left hand side of $(\widetilde{SC})$ are completely analogous. On the other hand, if we compute F with $(SC)^\mu$, we directly obtain $\tilde F$ satisfying $(SC)^{\tilde\mu}$ for $\tilde\mu := \|D\Phi \circ \Phi^{-1}\|_\infty^2 \mu$.

Now we prove that the oscillations of β and $\bar\beta$ remain controlled after the change of variable. In this case we need extra care to deal with Φ at different points x and x_0. This was first observed by Prof. Diego Moreira, who suggested the following argument.

Set $Q_x := D\Phi\left(\Phi^{-1}(x)\right)$, then $\widetilde{F}(x, 0, 0, X) = F(\Phi^{-1}(x), 0, 0, Q_x^T X Q_x)$, so

$$|\widetilde{F}(x, 0, 0, X) - \widetilde{F}(x_0, 0, 0, X)|$$
$$= |F(\Phi^{-1}(x), 0, 0, Q_x^T X Q_x) - F(\Phi^{-1}(x_0), 0, 0, Q_{x_0}^T X Q_{x_0})|$$
$$\leq |F(\Phi^{-1}(x), 0, 0, Q_x^T X Q_x) - F(\Phi^{-1}(x), 0, 0, Q_{x_0}^T X Q_{x_0})|$$
$$+ |F(\Phi^{-1}(x), 0, 0, Q_{x_0}^T X Q_{x_0}) - F(\Phi^{-1}(x_0), 0, 0, Q_{x_0}^T X Q_{x_0})|$$

The second addend above is bounded by

$$\beta_F(\Phi^{-1}(x), \Phi^{-1}(x_0)) \, \|Q_{x_0}^T X Q_{x_0}\| \leq C_\Phi^1 \, \beta_F(\Phi^{-1}(x), \Phi^{-1}(x_0)) \|X\|, \qquad (3.25)$$

where $C_\Phi^1 := \|D\Phi \circ \Phi^{-1}\|_\infty \|D\Phi^T \circ \Phi^{-1}\|_\infty$. For the first one, notice that $|F(x, 0, 0, X) - F(x, 0, 0, X_0)| \leq \max\{\mathcal{M}^+(X - X_0), \mathcal{M}^+(X_0 - X)\}$, and

$$\mathcal{M}^+(Q_x^T X Q_x - Q_{x_0}^T X Q_{x_0}) \leq \mathcal{M}^+((Q_x^T - Q_{x_0}^T) X Q_x) + \mathcal{M}^+(Q_{x_0}^T X (Q_x - Q_{x_0}))$$
$$\leq C_\Phi^2 \, |x - x_0| \, \|X\|, \qquad (3.26)$$

where $C_\Phi^2 := n\Lambda \max\{\|D\Phi \circ \Phi^{-1}\|_\infty, \|D\Phi^T \circ \Phi^{-1}\|_\infty\} \{\text{Lip}(D\Phi \circ \Phi^{-1}) + \text{Lip}(D\Phi^T \circ \Phi^{-1})\}$, hence (3.23) is proved. On the other hand, using that (3.26) is less or equal than $C_\Phi^2 \, |x - x_0|(\|X\| + 1)$, and replacing (3.25) by

$$\bar\beta_F(\Phi^{-1}(x), \Phi^{-1}(x_0)) \, (\|Q_{x_0}^T X Q_{x_0}\| + 1) \leq C_\Phi^3 \, \bar\beta_F(\Phi^{-1}(x), \Phi^{-1}(x_0))(\|X\| + 1),$$

where $C_\Phi^3 := \max\{C_\Phi^1, 1\}$, we obtain (3.24).

Finally, consider $\widetilde{F}_{x,r,p} = F_{\Phi^{-1}(x), r, (D\Phi \circ \Phi^{-1})p}$ for fixed (x, r, p). Then

$$\widetilde{F}(tX + (1-t)Y)$$
$$= F\left(t(D\Phi^T \circ \Phi^{-1})X(D\Phi \circ \Phi^{-1}) + (1-t)(D\Phi^T \circ \Phi^{-1})Y(D\Phi \circ \Phi^{-1})\right)$$
$$\lesseqqgtr tF\left((D\Phi^T \circ \Phi^{-1})X(D\Phi \circ \Phi^{-1})\right) + (1-t)F\left((D\Phi^T \circ \Phi^{-1})Y(D\Phi \circ \Phi^{-1})\right)$$
$$= t\widetilde{F}(X) + (1-t)\widetilde{F}(Y)$$

for all $X, Y \in \mathbb{S}^n$ and $t \in [0, 1]$, provided F is convex or concave in the X entry, and the claim is proved.

To finish we cover the boundary with a finitely many neighborhoods of this type to obtain regularity and an estimate near the boundary. In turn, the complete proof of regularity and estimates in the global case is derived by a covering argument over the domain Ω, using local and boundary results.

3.3
$W^{2,p}$ **Results**

In this section we prove the $W^{2,p}$ type results stated in section 3.1.

Proof of Theorem 3.4. We prove only the global case, since in the local one we just ignore the term with ψ, by considering it equal to zero in what follows. Notice that $\psi \in W^{2,p}(\Omega) \subset C^{1,\tau}(\overline{\Omega})$ for some $\tau \in (0,1)$ with continuous inclusion, then $\|u\|_{L^\infty(\Omega)} + \|f\|_{L^p(\Omega)} + \|\psi\|_{C^{1,\tau}(\partial\Omega)} \le C_2$.

Thus, by $C^{1,\alpha}$ regularity theorem, we have that $\bar{f}(x) := f(x) - g(x, Du) \in L^p(\Omega)$ and also

$$\|u\|_{C^{1,\alpha}(\overline{\Omega})} \le C_3 \left\{ \|u\|_{L^\infty(\Omega)} + \|\bar{f}\|_{L^p(\Omega)} + \|\psi\|_{C^{1,\tau}(\partial\Omega)} \right\}.$$

Claim 3.16 *u is an L^p-viscosity solution of $F(x, u, Du, D^2u) = \bar{f}(x)$ in Ω.*

Proof of Claim 3.16. Let us prove the subsolution case; for the supersolution it is analogous. Assuming the contrary, there exists some $\phi \in W^{2,p}_{loc}(\Omega)$, $x_0 \in \Omega$ and $\varepsilon > 0$ such that $u - \phi$ has a local maximum at x_0 and $F(x, u, D\phi, D^2\phi) - \bar{f}(x) \le -\varepsilon$ a.e. in $B_r(x_0)$.

In turn, by the definition of u being an L^p-viscosity subsolution of (3.7),

$$F(x, u, D\phi, D^2\phi) + g(x, D\phi) \ge f(x) - \varepsilon/2 \text{ a.e. in } B_r(x_0)$$

up to diminishing $r > 0$. By subtracting the last two inequalities, we obtain

$$-\{\gamma + \mu(|Du| + |D\phi|)\}|Du - D\phi| \le g(x, Du) - g(x, D\varphi) \le -\varepsilon/2 \quad (3.27)$$

a.e. in $B_r(x_0)$. Since $u - \phi \in C^1(B_r(x_0))$ has a local maximum at x_0, we have $D(u - \phi)(x_0) = 0$ and, moreover, $|D(u - \phi)(x)| < \varepsilon\{\gamma + \mu(\|Du\|_{L^\infty(B_r(x_0))} + \|D\phi\|_{L^\infty(B_r(x_0))}) + 1\}^{-1}/4$ for all $x \in B_r(x_0)$, possibly for a smaller r, which contradicts (3.27). ∎

Thus by Winter's result, theorem 4.3 in [38] (or Święch [48] in the local case), we have that $u \in W^{2,p}(\Omega)$ (respectively $u \in W^{2,p}_{loc}(\Omega)$) and

$$\begin{aligned}\|u\|_{W^{2,p}(\Omega)} &\le C\left\{\|u\|_{L^\infty(\Omega)} + \|\bar{f}\|_{L^p(\Omega)} + \|\psi\|_{W^{2,p}(\Omega)}\right\}\\ &\le C\left\{\|u\|_{L^\infty(\Omega)} + \|f\|_{L^p(\Omega)} + \mu\|u\|^2_{C^1(\overline{\Omega})} + \gamma\|u\|_{C^1(\overline{\Omega})} + \|\psi\|_{W^{2,p}(\Omega)}\right\}\\ &\le C\left\{\|u\|_{L^\infty(\Omega)} + \|f\|_{L^p(\Omega)} + \|\psi\|_{W^{2,p}(\Omega)}\right.\\ &\quad\left. + (\mu C_2 + \gamma)C_3\{\|u\|_{L^\infty(\Omega)} + \|f\|_{L^p(\Omega)} + \|\psi\|_{C^{1,\tau}(\partial\Omega)}\}\right\}\end{aligned}$$

which implies the estimate. ∎

In theorem 3.4, the final constants only depend on the L^p-norm of the coefficients, despite the boundedness hypothesis on b, d. The latter hypothesis

is needed to conclude that solutions are twice differentiable a.e. Observe that, in [38] (see theorem 4.3 there), $W^{2,p}$ results consist of two parts: (i) introducing a new equation $F(x, 0, 0, D^2 u) = \tilde{f}(x)$ (via corollary 1.6 in [48]), in which u remains a solution in the L^p-viscosity sense; (ii) obtaining $W^{2,p}$ estimates for solutions of $F(x, 0, 0, D^2 u) = \tilde{f}(x)$, which are independent of the zero and first order coefficients.

Proof of Proposition 3.5. It is enough to treat the upper extremal case. Let b_k, $d_k \in L^\infty_+(\Omega)$ be such that $b_k \to b$ and $d_k \to d$ in $L^p(\Omega)$. Let $u_k \in W^{2,p}(\Omega)$ be the unique L^p-viscosity solution of

$$\begin{cases} \mathcal{M}^+_{\lambda,\Lambda}(D^2 u_k) + b_k(x)|Du_k| + d_k(x)\,\omega(u_k^-) &= f(x) \quad \text{in} \quad \Omega \\ u_k &= \psi \quad \text{on} \quad \partial\Omega \end{cases}$$

given by theorem 4.6 of [38]. From the estimates in theorem 3.4, we have

$$\|u_k\|_{W^{2,p}(\Omega)} \le C_k \{\|u_k\|_{L^\infty(\Omega)} + \|f\|_{L^p(\Omega)} + \|\psi\|_{W^{2,p}(\Omega)}\}, \tag{3.28}$$

where C_k remains bounded, since b_k and d_k are bounded in $L^p(\Omega)$.

Now, by ABP we have that that $\|u_k\|_{L^\infty(\Omega)} \le \|\psi\|_{L^\infty(\partial\Omega)} + C\,\|f\|_{L^p(\Omega)}$. From this and (3.28) we get $\|u_k\|_{W^{2,p}(\Omega)} \le C$ and hence there exists $u \in C^1(\overline{\Omega})$ such that $u_k \to u$ in $C^1(\overline{\Omega})$.

Next, proposition 2.12 implies that u is an L^p-viscosity solution of

$$\begin{cases} \mathcal{M}^+_{\lambda,\Lambda}(D^2 u) + b(x)|Du| + d(x)\,\omega(u^-) &= f(x) \quad \text{in} \quad \Omega \\ u &= \psi \quad \text{on} \quad \partial\Omega. \end{cases} \tag{3.29}$$

Notice that $W^{2,p}(\Omega)$ is reflexive [30, p. 203, 264], so there exists $\tilde{u} \in W^{2,p}(\Omega)$ such that u_k converges weakly to $\tilde{u}$. By uniqueness of the limit, $\tilde{u} = u$ a.e. in Ω, and u is a strong solution of (3.29).

Finally, if there exists another L^p-viscosity solution of (3.29), say $v \in C(\overline{\Omega})$, then the function $w := u - v$ satisfies $w = 0$ on $\partial\Omega$ and it is an L^p-viscosity solution of $\mathcal{L}^+[w] \ge 0$ in $\Omega \cap \{w > 0\}$. Indeed, since u is strong, we can apply the definition of v as an L^p-viscosity supersolution with u as a test function; we also use that $u^- \le v^- + (u-v)^-$, monotonicity and subadditivity of the modulus. Then, by ABP we have that $w \le 0$ in Ω. Analogously, from the definition of subsolution of v, we obtain $w \ge 0$ in Ω, and so $w \equiv 0$ in Ω. ∎

The approximation procedure in the above proof cannot be used to extend theorem 3.4 for unbounded b and d, since in this case we do not have uniqueness results to infer that the limiting function is the same as the one we had started with. However, knowing a priori that the solution is strong, we

can obtain $W^{2,p}$ a priori estimates in the general case, as stated in lemma 3.6. We give a short proof for such Nagumo's lemma in the sequel.

Proof of Lemma 3.6. Note that, in particular, $u \in C^{1,\alpha}(\overline{\Omega})$ and satisfies $F(x, 0, 0, D^2u) = g(x)$ a.e. in Ω, where

$$g(x) := f(x) - F(x, u, Du, D^2u) + F(x, 0, 0, D^2u) \in L^p(\Omega),$$

since $|F(x, u, Du, D^2u) - F(x, 0, 0, D^2u)| \leq b(x)|Du| + \mu|Du^2| + d(x)\,\omega(|u|) \in L^p(\Omega)$. Now, by theorem 3.4 (for $b, d, \mu, \gamma = 0$) and the proof there dealing with $C^{1,\alpha}$ estimates,

$$
\begin{aligned}
\|u\|_{W^{2,p}(\Omega)} \leq\ & C\left\{\|u\|_{L^\infty(\Omega)} + \|g\|_{L^p(\Omega)} + \|\psi\|_{W^{2,p}(\Omega)}\right\} \\
\leq\ & C\left\{\|u\|_{L^\infty(\Omega)} + \|f\|_{L^p(\Omega)} + \mu\|u\|^2_{C^1(\overline{\Omega})}\right. \\
& \left. + \|\psi\|_{W^{2,p}(\Omega)} + \|b\|_{L^p(\Omega)}\|u\|_{C^1(\overline{\Omega})} + \|d\|_{L^p(\Omega)}\,\omega(\|u\|_\infty)\right\}
\end{aligned}
$$

from where the estimate follows. ∎

4
The Eigenvalue Problem

This chapter is related to existence of eigenvalues for general operators with nonnegative unbounded weight. So, we leave aside for a moment the superlinear growth and "go back to the roots" of the theory, focusing on the linear growth case. As we mentioned in chapter 1, results of this type will play an important role in the multiplicity and nonexistence arguments of chapter 6. Indeed, the lack of energy methods require techniques based on the maximum principle to overcome these limitations in the nondivergence scenario, see [66].

Throughout the chapter we are going to consider all solutions in the L^n-viscosity sense. There is no particular reason to deal with n instead of p; we just prefer to keep the usual notation from [28]. We stress that all L^n-viscosity results are valid for L^p-viscosity ones by proposition 2.9; the only important assumption relies on the weight – we ask for $c \in L^p(\Omega)$, $c \gneq 0$, with $p > n$, as well as $b \in L^p_+(\Omega)$.

We start recalling some notations. A subset $K \subset E$ of a Banach space is an *order cone* if it is closed, convex, $\lambda K \subset K$ for all $\lambda \geq 0$ and $K \cap (-K) = \{0\}$. This cone induces a partial order on E, namely for $u, v \in E$, $u \leq v \Leftrightarrow v - u \in K$. We say that K is solid if $\text{int} K \neq \emptyset$. Further, a *completely continuous* operator, defined in E, is continuous and takes bounded sets into precompact ones.

Following the construction in [66], [67], we have the next Krein-Rutman theorem for general nonlinear operators; a proof is provided in section 4.1.

Theorem 4.1 (*Generalized Krein-Rutman*) *Let $K \subset E$ be an order solid cone and let $T : K \to K$ be a completely continuous operator that is also*

(i) positively 1-homogeneous, i.e. $T(\lambda u) = \lambda Tu$, for all $\lambda \geq 0, u \in K$;

(ii) monotone increasing, i.e. for all $u, v \in K$, $u \leq v$ we have $Tu \leq Tv$;

(iii) strongly positive with respect to the cone, in the sense $T(K \setminus \{0\}) \subset \text{int} K$.

Then T has a positive eigenvalue $\alpha_1 > 0$ associated to a positive eigenfunction $w_1 \in \text{int} K$.

Consider $\Omega \subset \mathbb{R}^n$ a bounded $C^{1,1}$ domain along this chapter. The application of Krein-Rutman is very standard for positive weights [66], [68]. Let us recall its use when we have a fully nonlinear operator with unbounded coefficients. About structure, we suppose (SC) with ω a Lipschitz modulus.

Consider $E := C_0^1(\overline{\Omega})$ and the usual order solid cone of nonnegative functions over this space, i.e. $K =: \{u \in E;\ u \geq 0 \text{ in } \Omega\}$.

Let $c(x) \in L_+^p(\Omega)$ with $c > 0$ in $\overline{\Omega}$, $p > n$. As the operator on K, we take $T = -F^{-1} \circ c$ in the sense that $U = Tu$ iff U is the unique L^n-viscosity solution of the Dirichlet problem

$$\begin{cases} F(x, U, DU, D^2U) & = & -c(x)u & \text{in} & \Omega \\ U & = & 0 & \text{on} & \partial\Omega \end{cases} \qquad (T_u)$$

where F satisfies the following hypotheses

$$\begin{cases} \text{there exists } \theta > 0 \text{ such that } (\overline{H})_\theta \text{ holds for a.e. } x_0 \in \overline{\Omega}, \\ \text{(SC) and (S) hold,}\quad F(x, tr, tp, tX) = tF(x, r, p, X) \text{ for all } t \geq 0. \end{cases} \qquad (H)$$

Here, hypothesis (S) means the solvability in L^n-viscosity sense with data in L^p, i.e. for any $f \in L^p(\Omega)$, there exists a unique

$$u \in C(\overline{\Omega})\ L^n\text{-viscosity solution of } F[u] = f(x) \text{ in } \Omega;\ u = 0 \text{ on } \partial\Omega. \qquad (S)$$

Of course, Pucci's extremal operators

$$\mathcal{L}^\pm[u] := \mathcal{M}^\pm(D^2u) \pm b(x)|Du| \pm d(x)\omega(u^\mp),\quad b,\, d \in L_+^p(\Omega),$$

where ω is a Lipschitz modulus, are particular examples of F satisfying (H). Indeed, remember that proposition 3.5 provides us with a strong solution $u \in W^{2,p}(\Omega) \subset W^{2,n}(\Omega)$, which is an L^n-viscosity solution by proposition 2.5. Furthermore, since it is unique among L^p-viscosity solutions, it is also unique among L^n-viscosity ones. Here all the coefficients can be unbounded. Observe that (S) and $(\overline{H})_\theta$ also holds when F is a uniformly continuous operator in x satisfying the growth conditions in [64] (see also [69]), in this case concerning C-viscosity notions of solutions.

On the other hand, $(\overline{H})_\theta$, (SC) and (S) are completely enough to ensure existence, uniqueness and $C^{1,\alpha}$ global regularity and estimates for the problem (T_u) from theorem 3.1, which in turn implies that the operator T is well defined and completely continuous.

Let us check the homogeneity of the operator from the homogeneity of F. For $\lambda = 0$ it is obvious, since the problem $F(x, u, Du, D^2u) = 0$ in Ω with

$u = 0$ on $\partial\Omega$ has only the trivial solution. Indeed, this is ensured by applying (SC) and ABP in both directions. Thus, if $\lambda > 0$ and $V = T(\lambda u)$, by the homogeneity and uniqueness of F, we have $V/\lambda = Tu$ i.e. $T(\lambda u) = \lambda Tu$.

Furthermore, $T = -F^{-1} \circ c$ is strictly positive with respect to the cone, thanks to SMP and Hopf. In general, without the strict positiveness of c in Ω there is no guarantee on this property, i.e., under $c \geq 0$ and $c \not\equiv 0$ in Ω, we only obtain that $T(K \setminus \{0\}) \subset K$.

Notice that $T = -F^{-1} \circ c$ has an eigenvalue $\alpha_1 > 0$ associated to the positive eigenfunction φ_1 if and only if φ_1 is an L^n-viscosity solution of

$$\begin{cases} F[\varphi_1] + 1/\alpha_1 \, c(x)\varphi_1 & = & 0 & \text{in} & \Omega \\ \varphi_1 & > & 0 & \text{in} & \Omega \\ \varphi_1 & = & 0 & \text{on} & \partial\Omega. \end{cases}$$

For any $c \in L^p(\Omega)$ with $p > n$ and F satisfying (H), we can define, similar to the foregoing works [70], [28], the *principal weighted eigenvalues*

$$\lambda_1^{\pm} = \lambda_1^{\pm}(F(c), \Omega) = \sup\left\{\lambda > 0; \ \Psi^{\pm}(F(c), \Omega, \lambda) \neq \emptyset\right\} \tag{4.1}$$

where

$$\Psi^{\pm}(F(c), \Omega, \lambda) := \left\{\psi \in C(\overline{\Omega}); \ \pm\psi > 0 \text{ in } \Omega, \ \pm(F[\psi] + \lambda c(x)\psi) \leq 0 \text{ in } \Omega\right\};$$

with inequalities holding in the L^n-viscosity sense. Notice that, by definition, $\lambda_1^{\pm}(G(c), \Omega) = \lambda_1^{\mp}(F(c), \Omega)$, where $G(x, r, p, X) := -F(x, r, p, X)$.

With a simple approximation result by positive weights given by Krein-Rutman theorem as above, for F satisfying (H), our main result in this chapter is the following existence of eigenvalues with nonnegative weight.

Theorem 4.2 *Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{1,1}$ domain, $c \in L^p(\Omega)$, $c \gneqq 0$ for $p > n$ and F satisfying (H) for $b, d \in L_+^{\infty}(\Omega)$. Then F has two positive weighted eigenvalues $\alpha_1^{\pm} > 0$ corresponding to normalized and signed eigenfunctions $\varphi_1^{\pm} \in C^{1,\alpha}(\overline{\Omega})$ that satisfies*

$$\begin{cases} F[\varphi_1^{\pm}] + \alpha_1^{\pm}c(x)\varphi_1^{\pm} & = & 0 & in & \Omega \\ \pm\varphi_1^{\pm} & > & 0 & in & \Omega \\ \varphi_1^{\pm} & = & 0 & on & \partial\Omega \end{cases} \tag{4.2}$$

in the L^p-viscosity sense, with $\max_{\overline{\Omega}}(\pm\varphi_1^{\pm}) = 1$.

If, moreover, the operator F has $W^{2,p}$ regularity of solutions (in the sense that every $u \in C(\overline{\Omega})$ which is an L^p-viscosity solution of $F[u] = f(x) \in L^p(\Omega)$,

$u = 0$ *on* $\partial\Omega$, *satisfies* $u \in W^{2,p}(\Omega)$), *then* $\alpha_1^{\pm} = \lambda_1^{\pm}$ *and the conclusion is valid also for* $b \in L_+^p(\Omega)$.

Significant contributions on eigenvalues of continuous operators in nondivergence form in bounded domains include the fundamental work [70] for linear operators; [28] for convex fully nonlinear operators; [71] for nonlocal operators; [72], [73], [74] and the recent [75] for degenerate elliptic operators. Theorem 4.2 is a slight improvement to the general existence theory about nonconvex operators possessing first eigenvalues in [64] (see also [69]), since we are not supposing that our nonlinearity is uniformly continuous in x.

If, in addition, we have $W^{2,p}$ regularity of solutions, we can extend theorem 4.2 even further, allowing an unbounded first order coefficient. Eigenvalues for fully nonlinear operators with such coefficients have been previously studied, to our knowledge, only for radial operators and eigenfunctions, in [76] and [77]. As a particular case of theorem 4.2, we obtain the existence of positive eigenvalues with a nonnegative unbounded weight for the extremal Pucci's operators with unbounded coefficients.

Proposition 4.3 *Let* $\Omega \subset \mathbb{R}^n$ *a bounded* $C^{1,1}$ *domain,* $b, c \in L_+^p(\Omega)$, $c \gneqq 0$, *for* $p > n$. *Then, there exists* $\varphi_1^{\pm} \in W^{2,p}(\Omega)$ *such that, for* $\lambda_1^{\pm}$ *defined in (4.1), we have* $\lambda_1^{\pm} > 0$ *and*

$$\begin{cases} \mathcal{M}_{\lambda,\Lambda}^{\pm}(D^2\varphi_1^{\pm}) \pm b(x)|D\varphi_1^{\pm}| + \lambda_1^{\pm}c(x)\varphi_1^{\pm} &=& 0 & in & \Omega \\ \varphi_1^{\pm} &>& 0 & in & \Omega \\ \varphi_1^{\pm} &=& 0 & on & \partial\Omega. \end{cases} \quad (4.3)$$

Notice that we obtain positive eigenvalues because F is proper. For general existence related to nonproper operators see the script in [28] for bounded coefficients. We also stress that, without regularity assumptions on the domain, it is still possible to obtain the existence of an eigenpair, as in [28] and [70]; in such cases the eigenfunction belongs to $C_{\text{loc}}^{1,\alpha}(\Omega) \cap C(\overline{\Omega})$ by using $C^{1,\alpha}$ local regularity instead of the global one.

We start proving some auxiliary results which take into account the unboundedness of c.

Proposition 4.4 *Let* $u, v \in C(\overline{\Omega})$ *be* L^n-*viscosity solutions of*

$$\begin{cases} F[u] + c(x)u &\geq& 0 & in & \Omega \\ u &<& 0 & in & \Omega \end{cases}, \quad \begin{cases} F[v] + c(x)v &\leq& 0 & in & \Omega \\ v &\geq& 0 & on & \partial\Omega \\ v(x_0) &<& 0 & x_0 \in \Omega \end{cases} \quad (4.4)$$

with F satisfying (H), $c \in L^p(\Omega)$, $p > n$. Suppose one, u or v, is a strong solution. Then, $u = tv$ for some $t > 0$. The conclusion is the same if

$$
\begin{cases}
F[u] + c(x)u \leq 0 & in \;\; \Omega \\
u > 0 & in \;\; \Omega
\end{cases}
, \qquad
\begin{cases}
F[v] + c(x)v \geq 0 & in \;\; \Omega \\
v \leq 0 & on \;\; \partial\Omega \\
v(x_0) > 0 & x_0 \in \Omega.
\end{cases}
\qquad (4.5)
$$

For the proof of proposition 4.4, as in [64], [70], [28], we need the following consequence of ABP, which is a maximum principle for small domains.

Lemma 4.5 *Assume F satisfies* (SC) *and $c \in L^p(\Omega)$, $p > n$. Then there exists $\varepsilon_0 > 0$, depending on n, p, λ, Λ, $\|b\|_{L^p(\Omega)}$, $\|c^+\|_{L^p(\Omega)}$ and $\mathrm{diam}(\Omega)$, such that if $|\Omega| \leq \varepsilon_0$ then any $u \in C(\overline{\Omega})$ which is an L^n-viscosity solution of*

$$
\begin{cases}
F[u] + c(x)u \geq 0 & in \;\; \Omega \\
u \leq 0 & on \;\; \partial\Omega
\end{cases}
\qquad (4.6)
$$

satisfies $u \leq 0$ in Ω. Analogously, if $v \in C(\overline{\Omega})$ is an L^n-viscosity solution of

$$
\begin{cases}
F[v] + c(x)v \leq 0 & in \;\; \Omega \\
v \geq 0 & on \;\; \partial\Omega
\end{cases}
$$

then we have that $v \geq 0$ in Ω provided $|\Omega| \leq \varepsilon_0$.

Proof. Assume u satisfies (4.6). In order to obtain a contradiction, suppose that $\Omega^+ := \{u > 0\}$ is not an empty set. By (SC), we have that u is an L^n-viscosity solution of $\mathcal{L}^+[u] \geq \mathcal{L}^+[u] - c^-(x)u \geq -c^+(x)u$ in Ω^+. So, ABP gives us that

$$
\sup_{\Omega^+} u \leq C_1 \, \mathrm{diam}(\Omega) \, \|c^+\|_{L^n(\Omega)} \sup_{\Omega^+} u \leq C_1 \, \mathrm{diam}(\Omega) \, |\Omega|^{1 - \frac{n}{p}} \|c^+\|_{L^p(\Omega)} \sup_{\Omega^+} u.
$$

Then we choose $\epsilon_0 > 0$ such that $C_1 \, \mathrm{diam}(\Omega) \, \varepsilon_0^{1 - \frac{n}{p}} \|c^+\|_{L^p(\Omega)} \leq 1/2$ to obtain a contradiction. If v is a supersolution the proof is similar, by using ABP in the opposite direction. ∎

Remark 4.6 *We emphasize once more that, for equations satisfying* (SC), *the notions of L^p and L^n viscosity solutions are equivalent, by proposition 2.9. Notice that the addition of the term $c(x)u$ does not change the proof there, since the behavior of the zero order term is irrelevant to that proof.*

However, we can give a more direct proof of proposition 4.4 for L^p-viscosity solutions. As a matter of fact, it is just a question of using ABP

(for L^p-visc.) with the L^n-norm of f in the proof above (lemma 4.5); the proof of proposition 4.4 below is also unchangeable, just reading L^p instead of L^n.

We are going to use this result in chapter 6 with the operator F being $\mathcal{L}^-[u] + c(x)u$, in the case of inequalities (4.5).

Proof of Proposition 4.4. We are going to prove the first case, since the second is analogous. Let u, v be L^n-viscosity solutions of (4.4). Say both are strong, otherwise just use test functions for one of them and read all inequalities below in the L^n-viscosity sense. Set $z_t := tu - v$ for $t > 0$. Then, using 1-homogeneity and (SC), we have that z_t is a solution of

$$\mathcal{L}^+[z_t] + d(x)\omega((-z_t)^+) + c(x)z_t \geq F[tu] - F[v] + c(x)z_t$$
$$= t\left\{F[u] + c(x)u\right\} - \left\{F[v] + c(x)v\right\} \geq 0 \ \text{ in } \Omega. \qquad (4.7)$$

Let K be a compact subset of Ω such that $x_0 \in K$ and MP lemma 4.5 holds for $\Omega \setminus K$. Further, let $t_0 > 0$ be large enough such that $z_{t_0} \leq 0$ in K. In fact, this t_0 can be taken as $\min_K v / \max_K u > 0$, since $u < 0$ in K and $\min_K v \leq v(x_0) < 0$. Then, since $z_{t_0} \leq 0$ in $\partial(\Omega \setminus K) \subset \partial\Omega \cup \partial K$, we obtain from lemma 4.5 that $z_{t_0} \leq 0$ in $\Omega \setminus K$ and so in Ω.

Define

$$\tau := \inf\{\, t > 0; \ z_t \leq 0 \ \text{ in } \Omega \,\} \geq t_0 > 0.$$

By (SC), z_τ is a solution of $\mathcal{L}^-[-z_\tau] + \{c(x) - d(x)\,\omega(1)\}(-z_\tau) \leq 0$ in Ω, with $z_\tau \leq 0$ in Ω. Hence, by SMP, we have either $z_\tau \equiv 0$ or $z_\tau < 0$ in Ω. In the first case we are done. Suppose, then, $z_\tau < 0$ in Ω in order to obtain a contradiction.

Next we choose some $\varepsilon > 0$ such that $z_{\tau-\varepsilon} < 0$ in K. Indeed, we can take, for example, $\varepsilon = \min\{-\min_K z_\tau / (2\|u\|_{L^\infty(K)}), \tau/2\}$, which implies

$$z_{\tau-\varepsilon} = z_\tau - \varepsilon u \leq \min_K z_\tau + \varepsilon\|u\|_{L^\infty(K)} < 0 \ \text{ in } K,$$

as in [78]. In particular, z_t satisfies (4.7) for $t = \tau - \varepsilon > 0$. Thus, $z_{\tau-\varepsilon} \leq 0$ by MP in the set $\Omega \setminus K$. By SMP, $z_{\tau-\varepsilon} < 0$ in Ω, which contradicts the definition of τ being an infimum. ∎

The next result was first introduced in [70] and extended in [28] to nonlinear operators. We show below that, when we add an unbounded weight c, all we need is its positiveness on a subset of positive measure in order to obtain a bound from above on λ_1; we also make some remarks about the constant.

Lemma 4.7 *Suppose (H) with $b, d \in L^\infty_+(\Omega)$. If $c \geq \delta > 0$ a.e. in $B_R \subset\subset \Omega$, for $R \leq 1$, then*

$$\lambda_1^\pm(F(c), \Omega) \leq \frac{C_0}{\delta R^2}$$

where C_0 depends on n, λ, Λ, R, $\|b\|_{L^\infty(\Omega)}$ and $\omega(1)\|d\|_{L^\infty(\Omega)}$.

If, moreover, F has no term of order zero (i.e. d or ω is equal to zero), then R can be any positive number. On the other hand, if $b \equiv 0$, then the constant C_0 does not depend on R.

Proof. Observe that $\lambda_1^\pm(F(c), \Omega) \leq \lambda_1^\pm(F(c), B_R)$ by definition.

As in [70], [28], consider the radial function $\sigma(x) = -(R^2 - |x|^2)^2 < 0$ in B_R. Let us treat the λ_1^- case; for λ_1^+ it is just a question of looking at $-\sigma$.

Suppose, in order to obtain a contradiction, that there exists some $\lambda > \frac{C_0}{\delta R^2}$ such that $\Psi^-(F(c), \Omega, \lambda) \neq \emptyset$, i.e. let $\psi \in C(\overline{\Omega})$ be a negative L^n-viscosity solution of $F[\psi] + \lambda c(x)\psi \geq 0$ in Ω. Hence ψ is also an L^n-viscosity solution of $F[\psi] + \frac{C_0}{\delta R^2}c(x)\psi \geq 0$ in B_R.

Claim 4.8 *We have $F[\sigma] + \frac{C_0}{\delta R^2}c(x)\,\sigma \leq 0$ a.e. in B_R.*

Proof of Claim 4.8. Notice that

$$D\sigma(x) = 4(R^2 - |x|^2)\,x\,, \quad D^2\sigma(x) = 4(R^2 - |x|^2)I - 8x \otimes x.$$

Say, for example, $0 \leq b(x) \leq \gamma$ and $0 \leq d(x) \leq \eta$ a.e., thus

$$F[\sigma] \leq \mathcal{M}^+(D^2\sigma) + \gamma|D\sigma| - \theta A\,\sigma$$
$$\leq 4(R^2 - |x|^2)\,\mathcal{M}^+(I) - 8\mathcal{M}^-(x \otimes x) + 4\gamma(R^2 - |x|^2)\,|x| - \eta\,\omega(1)\,\sigma(x)$$

and therefore

$$\frac{F[\sigma]}{\sigma} \geq \frac{8\lambda\,|x|^2}{(R^2 - |x|^2)^2} - \frac{4n\Lambda}{R^2 - |x|^2} - \frac{4\gamma R}{R^2 - |x|^2} - \eta\,\omega(1) \quad \text{a.e. in } B_R\,.$$

The first term is always nonnegative, but it is extremely important to control the middle terms when x is near from R. Explicitly, we have that

$$\frac{8\lambda\,|x|^2}{(R^2 - |x|^2)^2} \geq \frac{4n\Lambda}{R^2 - |x|^2} + \frac{4\gamma R}{R^2 - |x|^2} \quad \Leftrightarrow \quad |x|^2 \geq \alpha R^2$$

for $\alpha = \frac{n\Lambda + \gamma R}{2\lambda + n\Lambda + \gamma R} \in (0, 1)$. Then we separate the analysis in two cases.

(a) $|x|^2 \geq \alpha R^2$: It follows that $F[\sigma]/\sigma \geq -\eta\,\omega(1) \geq -\eta\,\omega(1)\,c(x)/(\delta R^2)$ from construction;

(b) $|x|^2 \leq \alpha R^2$: In this case we use that the first term is nonnegative and so $F[\sigma]/\sigma \geq -4(n\Lambda + \gamma R)/((1 - \alpha)R^2) - \eta\,\omega(1) \geq -C_0\,c(x)/(\delta R^2)$.

If η or ω is equal to zero, then we do not need to use $R \leq 1$ in the above. Furthermore, if $\gamma = 0$, C_0 does not depend on R. ∎

Now we apply proposition 4.4, since $\sigma \in C^2(\overline{B}_R)$, obtaining that $\psi = t\sigma$, for some $t > 0$. However, this is not possible, since $\psi < 0$ on $\partial B_R \subset \Omega$ while $\sigma = 0$ on ∂B_R. ∎

Moving to the last statement in theorem 4.2, we first prove an eigenvalue bound that takes into account an unbounded b, when the weight is a continuous and positive function in $\overline{\Omega}$. Note that, in this case, theorem 4.1 gives us a pair $\alpha_1 > 0$ and $\varphi_1 \in C^1(\overline{\Omega})$ such that

$$
\begin{cases}
G[\varphi_1] + \alpha_1\, c(x)\varphi_1 &=\ 0 \quad \text{in} \quad \Omega \\
\varphi_1 &>\ 0 \quad \text{in} \quad \Omega \\
\varphi_1 &=\ 0 \quad \text{on} \quad \partial\Omega
\end{cases}
\tag{4.8}
$$

in the L^n-viscosity sense, with $\max_{\overline{\Omega}} \varphi_1 = 1$ and

$$
0 < \alpha_1 \le \lambda_1^+(G(c), \Omega) = \lambda_1^-(F(c), \Omega).
$$

The following proposition is a delicate point in our construction of an eigenpair. It states that α_1 in (4.8) is bounded, and this does not seem to be a consequence of the usual methods for bounding a first eigenvalue, such as the one in lemma 4.7. Instead, we use the classical *blow-up method* [79] of Gidas and Spruck.

Proposition 4.9 *Assume $c \in C(\overline{\Omega})$ with $c > 0$ in $\overline{\Omega}$, and G satisfying (H) for $b \in L^p_+(\Omega)$ and $d \in L^\infty_+(\Omega)$. Let α_1 and φ_1 as in (4.8). Then $\alpha_1 \le C$, for a constant $C = C(n, \lambda, \Lambda, \Omega, \|b\|_{L^p(\Omega)}, \omega(1)\|d\|_{L^\infty(\Omega)})$.*

Proof. If the conclusion is not true, then exists a sequence $b_k \in L^\infty_+(\Omega)$, with $\|b_k\|_{L^p(\Omega)} \le C$, $\|b_k\|_{L^\infty(\Omega)} \to +\infty$ and the respective eigenvalue problem

$$
\begin{cases}
G_k\,[\varphi_k] + \alpha_1^k\, c(x)\varphi_k &=\ 0 \quad \text{in} \quad \Omega \\
\varphi_k &>\ 0 \quad \text{in} \quad \Omega \\
\varphi_k &=\ 0 \quad \text{on} \quad \partial\Omega
\end{cases}
\tag{4.9}
$$

in the L^n-viscosity sense, with $\max_{\overline{\Omega}} \varphi_k = 1$ for all $k \in \mathbb{N}$ and $\alpha_1^k \to +\infty$ as $k \to +\infty$, where G_k is a fully nonlinear operator satisfying $(H)_k$, i.e. (H) for b_k and d_k. Say $d_k \le \eta$ and $\max_{\overline{\Omega}} \varphi_k = \varphi_k(x_0^k)$ for $x_0^k \in \Omega$. Then, $x_0^k \to x_0 \in \overline{\Omega}$ as $k \to +\infty$, up to a subsequence.

Case 1: $x_0 \in \Omega$. Let $2\rho = \text{dist}(x_0, \partial\Omega) > 0$ and notice that $x_0^k \in B_\rho(x_0)$ for all $k \ge k_0$. Set $r_k = (\alpha_1^k)^{-1/2}$ and define $\psi_k(x) = \varphi_k(x_0^k + r_k x)$. Thus, ψ_k is an L^n (so L^p) viscosity solution of

$$
\widetilde{G}_k(x, \psi_k, D\psi_k, D^2\psi_k) + c_k(x)\psi_k(x) = 0 \quad \text{in} \quad \widetilde{B}_k := B_{\rho/r_k}(0)
$$

where $c_k(x) := c(x_0^k + r_k x)$, $\widetilde{G}_k(x, r, p, X) := r_k^2 G_k(x_0^k + r_k x, r, p/r_k, X/r_k^2)$ satisfies $(\widetilde{H})_k$, i.e. (H) for $\widetilde{b}_k$ and η_k, where $\widetilde{b}_k(x) := r_k b_k(x_0^k + r_k x)$ and $\eta_k = r_k^2 \eta$. Notice that b_k and η_k converge locally to zero in $L^p(\widetilde{B}_k)$ as $k \to +\infty$, since $p > n$.

Furthermore, $\sup_{\widetilde{B}_k} \psi_k = \psi_k(0) = 1$ for all $k \in \mathbb{N}$ and $B_R(0) \subset\subset \widetilde{B}_k$ for large k, for any fixed $R > 0$. By theorem 3.1 we have that ψ_k is locally in $C^{1,\alpha}$ and satisfies the estimate

$$\|\psi_k\|_{C^{1,\alpha}(\overline{B}_R(0))} \leq C_k \|\psi_k\|_{L^\infty(\widetilde{B}_k)} \leq C,$$

since ψ_k attains its maximum at 0 and C_k only depends on the L^p-norm of the coefficients b_k and c_k, which are uniformly bounded in there. Hence, by compact inclusion we have that there exists $\psi \in C^1(\overline{B}_R(0))$ such that $\psi_k \to \psi$ as $k \to +\infty$, up to a subsequence. Performing the same argument for each ball $B_R(0)$, for every $R > 0$, we obtain in particular that $\psi_k \to \psi$ in $L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$, by using the uniqueness of the limit for ψ_k in the smaller balls.

Using stability (proposition 2.12 together with observation 2.13) and the continuity of c, we have that ψ is an L^p-viscosity solution of $J(x, D^2\psi) + c(x_0)\psi = 0$ in $\mathbb{R}^n$ for some measurable operator J still satisfying (H) with coefficients of zero and first order term, d and b, equal to zero. Also, $\psi(0) = 1$ and $\psi > 0$ in $\mathbb{R}^n$ by SMP. This implies that $1 \leq \lambda_1^+(J(c(x_0)), B_R) \leq \frac{C_0}{c(x_0)R^2}$ for all $R > 0$, which gives a contradiction when we take $R \to +\infty$.

Case 2: $x_0 \in \partial\Omega$. By passing to new coordinates, that come from the smoothness property of the domain $\partial\Omega \in C^{1,1}$, we can suppose that $\partial\Omega \subset \{x_n = 0\}$ and $\Omega \subset \{x_n > 0\}$.

Set $\rho_k = \mathrm{dist}(x_0^k, \partial\Omega) = x_0^k \cdot e_n = x_{0,n}^k$, where $e_n = (0, \ldots, 0, 1)$ and $x_0^k = (x_{0,1}^k, \ldots, x_{0,n}^k)$. Analogously, consider $\psi_k(y)$ in $y \in B_{\rho_k/r_k}(0)$ and the respective equation $\widetilde{G}_k$ as in case 1. Thus, for x, y satisfying $r_k y = x - x_0^k$, we have that the set $\{x_n > 0\}$ corresponds to $A_k := \{y_n = (x - x_0^k) \cdot e_n / r_k > -\rho_k/r_k\}$. So we need to analyze the behavior of the set A_k when we pass to limits as $k \to +\infty$.

We first claim that ρ_k/r_k is bounded below by a constant $C_1 > 0$, which means that A_k does not converge to $\{y_n > 0\}$. This is an easy consequence of our $C^{1,\alpha}$ boundary regularity and estimates in a half ball (theorem 3.14), applied to ψ_k and $\widetilde{G}_k$. Indeed, since $\|D\psi_k\|_{L^\infty(B_r^+(0))} \leq C$, $r > 0$ fixed, then

$$1 = |\varphi(x_0^k) - \varphi(\bar{x}_0^k)| = |\psi_k(0, 0) - \psi_k(0, -\rho_k/r_k)| \leq C\rho_k/r_k,$$

where $\bar{x}_0^k = (x_{0,1}^k, \ldots, x_{0,n-1}^k, 0) \in \partial\Omega$, from where we obtain the desired bound.

Next observe that we have two possibilities about the fraction ρ_k/r_k, either it converges to $+\infty$ or it is uniformly bounded. In the first one, $A_k \to \mathbb{R}^n$

and we finish as in case 1. In the second, $A_k \to \{y_n > \varrho\}$, $\varrho \in (0, +\infty)$, by taking a subsequence. The proof carries on as in the case 1, since we have a smooth domain which contains a ball with radius $R = (2\, C_0 / c(x_0)\,)^{1/2}$; this gives the final contradiction. ∎

Lemma 4.10 *Let $c \in L^p(\Omega)$, $c \geq \delta$ in B_R for some $B_R \subset\subset \Omega$ and F satisfying hypothesis (H), then*

$$\lambda_1^{\pm}(F(c), \Omega) \leq \frac{\lambda_1^{\pm}(F(1), B_R)}{\delta}.$$

Proof. Let us prove the λ_1^+ case; for λ_1^- we use G instead of F. We already know that both quantities are nonnegative, by the properness of the operator F. Hence, it is enough to verify that $\mathcal{A} \cap \{\lambda \geq 0\} \subset \mathcal{B}/\delta \cap \{\lambda \geq 0\}$, where

$$\lambda_1^+(F(c), \Omega) = \sup_{\mathcal{A}} \lambda = \sup_{\mathcal{A} \cap \{\lambda \geq 0\}} \lambda \,, \quad \lambda_1^+(F(1), B_R) = \sup_{\mathcal{B}} \lambda = \sup_{\mathcal{B} \cap \{\lambda \geq 0\}} \lambda$$

as defined before. Let $\lambda \in \mathcal{A} \cap \{\lambda \geq 0\}$, then there exists $\psi \in C(\overline{\Omega})$ a nonnegative L^n-viscosity solution of $F[\psi] + c(x)\lambda\psi \leq 0$ in Ω. Then, ψ is also a nonnegative L^n-viscosity solution of $F[\psi] + \delta\lambda\psi \leq 0$ in B_R, so $\delta\lambda \in \mathcal{B}$. ∎

Proof of Theorem 4.2. First, from the fact that $c > 0$ in a set of positive measure, there exists $\delta > 0$ such that $\{c \geq \delta\}$ is a nontrivial set. In fact, if this was not true, i.e. if $|\{c \geq \delta\}| = 0$ for all δ, then $\{c > 0\} = \bigcup_{\delta > 0}\{c \geq \delta\}$ would have measure zero, as the union of such sets, contradicting the hypothesis. Namely, then, $c \geq \delta > 0$ a.e. in some ball $B_R \subset\subset \Omega$.

Let us prove the λ_1^- case, applying Krein-Rutman results to G; for λ_1^+ replace G by F. Let $\varepsilon \in (0, 1)$ and define $c_\varepsilon := c + \varepsilon > 0$ in Ω, for all ε. From theorem 4.1, we obtain the existence of pairs $\alpha_1^\varepsilon > 0$ and $\varphi_1^\varepsilon \in C^1(\overline{\Omega})$ such that

$$\begin{cases} G[\varphi_1^\varepsilon] + \alpha_1^\varepsilon\, c_\varepsilon(x)\varphi_1^\varepsilon &=\ 0 \quad \text{in} \quad \Omega \\ \varphi_1^\varepsilon &>\ 0 \quad \text{in} \quad \Omega \\ \varphi_1^\varepsilon &=\ 0 \quad \text{on} \quad \partial\Omega \end{cases} \tag{4.10}$$

with $\max_{\overline{\Omega}} \varphi_1^\varepsilon = 1$ for all $\varepsilon \in (0, 1)$. Then,

$$0 < \alpha_1^\varepsilon \leq \lambda_1^+(G(c_\varepsilon), \Omega) = \lambda_1^-(F(c_\varepsilon), \Omega) \leq \frac{C_0}{\delta R^2} \quad \text{for all } \varepsilon \in (0, 1). \tag{4.11}$$

Next, $\alpha_1^\varepsilon \to \alpha_1 \in [0, C_0/\delta R^2]$ up to a subsequence. Then we apply $C^{1,\alpha}$ global regularity and estimates (theorem 3.1) in the case $\mu = 0$ (recall again that L^n-viscosity solutions are L^p-viscosity for $p > n$), by considering

$\alpha_1^\varepsilon c_\varepsilon(x) \varphi_1^\varepsilon \in L^p(\Omega)$ as the right hand side, from where

$$\|\varphi_1^\varepsilon\|_{C^{1,\alpha}(\overline{\Omega})} \le C\,\{\,\|\varphi_1^\varepsilon\|_{L^\infty(\Omega)} + \alpha_1^\varepsilon\,\|c_\varepsilon\|_{L^p(\Omega)}\,\|\varphi_1^\varepsilon\|_\infty\,\}$$
$$\le C\,C_1\,(\|c\|_{L^p(\Omega)} + 1)\,\}\le C.$$

Hence the compact inclusion $C^{1,\alpha}(\overline{\Omega}) \subset C^1(\overline{\Omega})$ yields $\varphi_1^\varepsilon \to \varphi_1 \in C^1(\overline{\Omega})$, up to a subsequence. Of course that this implies $\max_{\overline{\Omega}} \varphi_1 = 1$, $\varphi_1 \ge 0$ in Ω and $\varphi_1 = 0$ on $\partial\Omega$.

Since $c_\varepsilon \to c$ in $L^p(\Omega)$ as $\varepsilon \to 0$, by proposition 2.12 we have that φ_1 is an L^p-viscosity solution of $G[\varphi_1] + \alpha_1 c(x)\varphi_1 = 0$ in Ω, which allows us to apply $C^{1,\alpha}$ regularity again to obtain that $\varphi_1 \in C^{1,\alpha}(\overline{\Omega})$.

Using now that φ_1 is a nonnegative L^p-viscosity solution of

$$\mathcal{L}^-[\varphi_1] - (d(x)\omega(1) - \alpha_1 c(x))\,\varphi_1 \le 0 \text{ in } \Omega,$$

together with SMP, we have that $\varphi_1 > 0$ in Ω, since $\max_{\overline{\Omega}} \varphi_1 = 1$. Moreover, we must have $\alpha_1 > 0$, because the case $\alpha_1 = 0$ would imply that φ_1 is an L^p-viscosity solution of $\mathcal{L}^+[\varphi_1] \ge 0$ in $\Omega \cap \{\varphi_1 > 0\}$ (since F is proper, and so G) which, in turn, would give us $\varphi_1 \le 0$ in Ω, by ABP. Thus, the existence property is completed.

In order to conclude that, under $W^{2,p}$ regularity assumptions over F, the α_1 obtained is equal to $\lambda_1^- = \lambda_1^-(F(c), \Omega)$, related to the eigenfunction $\varphi_1^- = \varphi_1^-(F(c), \Omega) = -\varphi_1 < 0$ in Ω, we have to work a little bit more, as in proposition 4.7 in [28].

We already have $\alpha_1 \le \lambda_1^-$. Suppose by contradiction that $\alpha_1 < \lambda_1^-$. By definition of λ_1^- as a supremum, we know that α_1 cannot be an upper bound, that is, there exists $\lambda > 0$ such that $\Psi^-(F(c), \Omega, \lambda) \ne \emptyset$ and $\alpha_1 < \lambda \le \lambda_1^-$. Then we obtain $\psi \in C(\overline{\Omega})$ such that $F[\psi] + \lambda c(x)\psi \ge 0$ in Ω in the L^n-viscosity sense, with $\psi < 0$ in Ω. Now, since $c \gneqq 0$, we have $c(x)(\lambda - \alpha_1) \gneqq 0$. Therefore ψ is a negative L^n-viscosity solution of

$$F[\psi] + \alpha_1\,c(x)\psi \gneqq F[\psi] + \lambda\,c(x)\psi \ge 0 \text{ in } \Omega. \tag{4.12}$$

Then, under $W^{2,p}$ regularity, we have that $\varphi_1^- \in W^{2,p}(\Omega) \subset W^{2,n}(\Omega)$ is a strong solution of

$$\begin{cases} F[\varphi_1^-] + \alpha_1\,c(x)\varphi_1^- &= 0 \quad \text{in} \quad \Omega \\ \varphi_1^- &< 0 \quad \text{in} \quad \Omega \\ \varphi_1^- &= 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

Applying proposition 4.4 we obtain that $\psi = t\varphi_1^-$ for some $t > 0$; but this contradicts the strict inequality in (4.12). Thus, we must have $\alpha_1 = \lambda_1^-$. The case of λ_1^+ is completely analogous, by reversing the inequalities.

From this last paragraph, under $W^{2,p}$ regularity of the solutions, the only possibility for α_1 is to coincide with λ_1. Therefore, by using proposition 4.9 (with $c \equiv 1$) and lemma 4.10, we obtain that $\lambda_1^-(F(c_\varepsilon), \Omega) \leq C_1/\delta$, for all $\varepsilon \in (0, 1)$, where C_1 depends on $n, \lambda, \Lambda, R, \|b\|_{L^p(\Omega)}$ and $\omega(1)\|d\|_{L^\infty(\Omega)}$. Thus, we carry on this bound on λ_1, instead of (4.11), in the limiting procedure, in order to get the desired existence result for $b \in L_+^p(\Omega)$. ■

4.1
The Krein-Rutman Theorem

In this section we prove theorem 4.1. We start recalling the following "Rabinowitz type" result about existence of solutions via Leray-Schauder degree for *perturbations of the identity*, i.e. for operators in the form $I - T$, where T is completely continuous. We state it as in theorem 3.3 in [66] (see there for a proof); see also theorem 3.5.3 in [80] and theorem 4.11 in [81]. For this, consider the fixed point problem

$$x = T(\lambda, x). \tag{4.13}$$

In the following we say *continuum* to mean a closed and connected set.

Proposition 4.11 *Let $T : \mathbb{R} \times E \to E$ be such that $T_\lambda = T(\lambda, \cdot) : E \to E$ is completely continuous for all $\lambda \in \mathbb{R}$ and $T(\lambda, x)$ is continuous in λ uniformly with respect to x in balls of E. Let (λ_0, x_0) be a solution of (4.13).*

Suppose that $\mathcal{U} \subset E$ is an open bounded set such that $x_0 \in \mathcal{U}$ and there is no other solution x of $x = T(\lambda_0, x)$ in $\overline{\mathcal{U}}$, with $\deg(I - T_{\lambda_0}, \mathcal{U}, 0) \neq 0$.

Then there exist two continuums, $\mathcal{C}^- \subset (-\infty, \lambda_0] \times E$ and $\mathcal{C}^+ \subset [\lambda_0, +\infty) \times E$, of solutions of (4.13) with $(\lambda_0, x_0) \in \mathcal{C}^+ \cap \mathcal{C}^-$ and such that one of the following alternatives holds

(i) $\mathcal{C}^+$ is unbounded;

(ii) $\mathcal{C}^+ \cap (\{\lambda_0\} \times (E \setminus \overline{\mathcal{U}})) \neq \emptyset$, i.e. $\mathcal{C}^+$ bends back to λ_0.

The same alternatives hold for $\mathcal{C}^-$.

In particular, if $T(0, x) = 0$ for all $x \in E$, then there exist two unbounded continuums of solutions meeting each other at $(0, 0)$.

With this tool at hand, we give a proof for the generalized Krein-Rutman theorem in the sequel.

Proof of Theorem 4.1. Fix $u_0 \in K \setminus \{0\}$. We first claim that there exists $m > 0$ such that $mTu_0 \geq u_0$. If this was not true, i.e. if for all $m > 0$, $Tu_0 - u_0/m \notin K$, by taking the limit as $m \to +\infty$ we would have $Tu_0 \notin \text{int} K$, which contradicts item *(iii)*. Now, for each $\varepsilon > 0$, define the map

$$T_\varepsilon \; : \; [0, +\infty) \times K \to K$$

$$(\lambda, u) \mapsto \lambda Tu + \varepsilon \lambda Tu_0$$

which is completely continuous, continuous in λ and satisfies $T_\varepsilon(0, \cdot) \equiv 0$. By theorem 4.11, there exists an unbounded connected component C_ε of solutions of $T_\varepsilon(\lambda, u) = u$. Let us see that C_ε is bounded in the λ-direction.

Claim 4.12 $u \geq \varepsilon \left(\frac{\lambda}{m} \right)^k u_0$, *for all* $k \in \mathbb{N}$ *and all* $(\lambda, u) \in C_\varepsilon$.

Notice that this claim implies that $\lambda \leq m$. If not, if there would exists $(\lambda, u) \in C_\varepsilon$ with $\lambda > m$, we would have $u_0 \leq \frac{1}{\varepsilon} \left(\frac{m}{\lambda} \right)^k u \to 0$ as $k \to +\infty$ in the norm of E, yielding $u_0 \notin K$, a contradiction.

Proof. Let $(\lambda, u) \in C_\varepsilon$, then $u = \lambda Tu + \varepsilon \lambda Tu_0 \geq \varepsilon \lambda Tu_0 \geq \varepsilon \frac{\lambda}{m} u_0$ and the claim is proved for $k = 1$. Moreover, using that that T is increasing and positively 1-homogeneous, $Tu \geq T\left(\varepsilon \frac{\lambda}{m} u_0 \right) = \varepsilon \frac{\lambda}{M} Tu_0 \geq \varepsilon \frac{\lambda}{m^2} u_0$. Since $u \geq \lambda Tu$, it follows that $u \geq \varepsilon \left(\frac{\lambda}{m} \right)^2 u_0$. By iteration we obtain the conclusion for all $k \in \mathbb{N}$. ∎

Since C_ε is unbounded, it has to be unbounded in the K-direction, that is, there must exist a sequence of solutions $(\lambda, u) \in C_\varepsilon$ with $\|u\|_E \to +\infty$ for each $\varepsilon > 0$; in particular, there exists $(\lambda_\varepsilon, u_\varepsilon) \in C_\varepsilon$ with $\|u_\varepsilon\|_E = 1$. Then, by taking the limit as $\varepsilon \to 0$ we can obtain the desired pair (λ_1, u_1) with $\lambda_1 \in [0, M]$ and $\|u_1\|_E = 1$ such that $u_1 = \lambda_1 Tu_1$.

Indeed, since T takes bounded sets into precompact ones, there exists some $v \in K$ such that $Tu_\varepsilon \to v$ in E, for a subsequence of $\varepsilon \to 0$. Then there exists the limit of $u_\varepsilon = \lambda_\varepsilon Tu_\varepsilon + \varepsilon \lambda_\varepsilon Tu_\varepsilon \to \lambda_1 v =: u_1$ for some $\lambda_1 \in [0, m]$, up to a subsequence of $\varepsilon \to 0$. Finally, T continuous yields $v = Tu_1$.

We notice that λ_1 must be positive; otherwise we would have $u_1 \equiv 0$, contradicting $\|u_1\|_E = 1$. Furthermore, since $u_1 \in K \setminus \{0\}$, it follows that Tu_1 belongs to the interior of K, and so does $u_1 = \lambda_1 Tu_1$. ∎

5
A Priori Bounds and Multiplicity Results

In this chapter we consider a family of fully nonlinear uniformly elliptic problems of the following form

$$\begin{cases} -F(x, u, Du, D^2u) & = & \lambda c(x)u + \langle M(x)Du, Du \rangle + h(x) & \text{in} & \Omega \\ u & = & 0 & \text{on} & \partial\Omega \end{cases} \quad (P_\lambda)$$

where Ω is a bounded $C^{1,1}$ domain in $\mathbb{R}^n$, $\lambda \in \mathbb{R}$, $n \geq 1$, $c, h \in L^p(\Omega)$, M is a bounded matrix, and F is a fully nonlinear uniformly elliptic operator of Isaacs type. As it was already seen (chapter 1), a particular case, for which all our multiplicity results are new as well, is when F is a linear operator in nondivergence form (1.1).

In the recent years have appeared a series of papers which unveil the complex nature of the solution set for noncoercive equations ($\lambda c \gneqq 0$), in the particular case when F is the Laplacian. In 2013, Jeanjean and Sirakov [17] used a *mountain pass* argument related to Cerami sequences, when a classical exponential change of variables (as in lemma 5.11) reduces the equation to a semilinear one in linear and divergence form. They extended the multiplicity result in [12] under a smallness condition over c and μh (similar to the condition on quadratic ABP) in the case $M = const.I$.

Later Arcoya, de Coster, Jeanjean and Tanaka [13] (2015, see also [82]) developed a method based on degree theory which applies to general gradient terms. They stablished the existence of a *continuum of solutions* for the problem with $F = \Delta$, $M(x) = \mu(x)I$ for $0 < \mu_1 \leq \mu(x) \leq \mu_2$ and $h \gneqq 0$. Moreover, the authors proved the existence of a positive $\bar{\lambda}$ such that (P_λ) there is no nonnegative solutions for λ greater than $\bar{\lambda}$. Therefore it was realized that the sign of h definitely matters for the problem; this because few years earlier (2007) Abdellaoui, Peral and Primo [83] exhibited one solution for the problem with $F = \Delta$, $M(x) \equiv I$, for a specific radial c and $h \lneqq 0$, for all $\lambda > 0$.

The work [13] was extended and completed by Jeanjean and Quorin [16] and by Jeanjean and de Coster [15]. In the latter the authors gave a description of the solution set in terms of the parameter λ; the sign of h could be replaced by a more general condition over the sign of u_0, which is the solution of (P_0). In fact, they obtained a continuum when (P_0) has a solution and the existence

of at least two solutions, for every $\lambda < \bar{\lambda}$, with $\bar{\lambda}$ being finite for $u_0 \gneqq 0$ and infinite if $u_0 \lneqq 0$; still for the laplacian and $M(x) = \mu(x)I$. A result with the p-Laplacian is found in [14].

Souplet [18] showed that the study of (P_λ) is even more difficult if M is allowed to vanish somewhere (note that in any case a hypothesis which prevents $M \equiv 0$ is necessary). In all these works the crucial a priori bounds for u in the L^∞-norm rely on the fact that the second order operator is the Laplacian, or a divergence form operator.

Hence, it appears to be a natural question if this same kind of multiplicity result can be extended to a more general setting, namely for a nonlinearity F in the nondivergence form, (P_λ) given in the L^p-viscosity sense. In the sequel we see that the answer is affirmative. In other words, it is our goal here to perform a similar study for general operators in nondivergence form, and extend the results from [12] to noncoercive equations. Since $c \gneqq 0$, this means to give some description of the solution set of (P_λ) when the parameter λ is positive, similar to [15].

5.1
Main Results

We start this section by introducing our hypotheses.

From now on Ω is a bounded domain in $\mathbb{R}^n$ with $C^{1,1}$ boundary. We assume that the matrix M satisfies the nondegeneracy condition

$$\mu_1 I \leq M(x) \leq \mu_2 I \quad \text{a.e. in } \Omega \qquad (M)$$

for some $\mu_1, \mu_2 > 0$, and that (P_λ) has the (SC) structure, recalling

$$\mathcal{M}^-(X - Y) - b(x)|p - q| - d(x)\omega((r - s)^+)$$
$$\leq F(x, r, p, X) - F(x, s, q, Y) \qquad (\text{SC})$$
$$\leq \mathcal{M}^+(X - Y) + b(x)|p - q| + d(x)\omega((s - r)^+) \text{ a.e. } x \in \Omega$$
$$F(\cdot, 0, 0, 0) \equiv 0, \; b, d, c, h \in L^p(\Omega), p > n, b, d \geq 0, \; \omega \text{ a modulus.}$$

Here, $\mathcal{M}^\pm$ are the Pucci's operators with constants $0 < \lambda_P \leq \Lambda_P$[1]. We also consider $\mathcal{L}^\pm[u] := \mathcal{M}^\pm(D^2u) \pm b(x)|Du|, b \in L_+^p(\Omega)$, for the corresponding ellipticity coefficients λ_P, Λ_P.

Notice that, for ease of notation, we included in (SC) the unboundedness condition over the coefficients. On the other hand, we make the convention

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[1]We are denoting the ellipticity coefficients by λ_P and Λ_P instead of the usual λ and Λ in order to avoid any confusions with λ in the problem (P_λ).

that all of them are bounded functions when using the following stronger assumption

$$\mathcal{M}^-(X - Y) - b(x)|p - q| \leq F(x, r, p, X) - F(x, s, q, Y)$$

$$\leq \mathcal{M}^+(X - Y) + b(x)|p - q| \quad \text{a.e. } x \in \Omega \qquad (SC)_0$$

$$F(\cdot, 0, 0, 0) \equiv 0, \qquad b, c, h \in L^\infty(\Omega), \ b \geq 0,$$

needed for most of our results. Observe that a very particular case of the last hypothesis appears when F is the linear operator (1.1), but we can go much further, allowing F to be an arbitrary supremum or infimum of such linear operators, i.e. a Hamilton-Jacobi-Bellman (HJB) operator, and even to be a sup-inf of linear operators (Isaacs operator).

We will also assume that for some $\theta > 0$, $r_0 > 0$ and all $x_0 \in \overline{\Omega}$

$$\left(\frac{1}{r^n} \int_{B_r(x_0) \cap \Omega} \bar{\beta}_F(x, x_0)^p \right)^{\frac{1}{p}} \leq \theta, \quad \text{for all } r \leq r_0 \qquad (H_\beta)$$

where $\bar{\beta}$ is defined in (3.5). As we already mentioned, this is satisfied, for instance, if $F(x, 0, 0, X)$ is continuous in $x \in \overline{\Omega}$ (if F is linear this means $a_{ij}(x)$ are continuous, as in chapter 1). The conditions (M)-(SC)-(H_β) guarantee that the L^p-viscosity solutions of (P_λ) have global $C^{1,\alpha}$ regularity and estimates, by theorem 3.1 and remark 3.3.

Solutions of the Dirichlet problem (P_λ) are understood in the L^p-viscosity sense and belong to $C(\overline{\Omega})$. Thus, we study and prove multiplicity of *bounded solutions*. We note that multiple unbounded solutions can easily be found for simple equations with natural growth. For instance, in [84] it was observed that $\Delta u = |Du|^2$ admits infinitely many weak solutions in $W_0^{1,2}(B_1)$, namely $u_k = \ln((|x|^{2-n} - k)(1 - k)^{-1})$, $0 \leq k < 1$, in the case $n > 2$.

Remember that strong solutions of (P_λ), for $\partial\Omega \in C^{1,1}$ are functions in $W^{2,p}(\Omega)$ which satisfy the equation almost everywhere. Strong solutions are L^p-viscosity solutions (see proposition 2.5). Conversely, if F is for instance convex in the matrix X and satisfies $(SC)_0$ (such are the HJB operators), then L^p-viscosity solutions are strong from theorem 3.3; the convexity assumption can be removed in some cases but not in general, see [85]. For some of our results we will need to assume that L^p-viscosity solutions of (P_λ) are strong – see hypothesis (H_2) below.

Since we want to study the way the nature of the solution set changes when we go from negative to positive zero order term, we will naturally assume that the problem with $\lambda = 0$ has a solution. We also assume that the Dirichlet problem for F is uniquely solvable, so that we can concentrate on the way

the coefficients c and M influence the solvability. We now summarize these conditions on F. First, we assume that

$$\text{the problem } (P_0) \text{ has a strong solution } u_0. \tag{H_0}$$

Further, setting $F[u] := F(x, u, Du, D^2u)$, we assume that for each $f \in L^p(\Omega)$,

$$\text{there exists a unique } L^p\text{-viscosity solution of } \begin{cases} -F[u] = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{H_1}$$

Given c, h for which we study (P_λ), if $(\overline{P}_\lambda)$ denotes the problem (P_λ) with c and h replaced by $\overline{c}$ and $\overline{h}$, we sometimes require that L^p-viscosity solutions $\overline{u}_\lambda$ of $(\overline{P}_\lambda)$ are such that

$$\overline{u}_\lambda \in W^{2,p}(\Omega), \text{ for every } 0 \leq \overline{c} \leq c\,, \ |\overline{h}| \leq |h| + 1 + c. \tag{H_2}$$

We observe that, by Theorem 1(iii) of [12], the function u_0 is the unique L^p-viscosity solution of (P_0). Theorem 1(ii) of [12] shows that (H_0) holds for instance if Mh has small L^p-norm (examples showing that in general this hypothesis cannot be removed are also found in that paper). Moreover, recall that (H_1) and (H_2) are both true if F satisfies $(SC)_0$ and is convex or concave in X, by the results in [38] and chapter 3.

We now state our main multiplicity results. The following theorem contains a crucial uniform estimate for solutions of (P_λ), which is both important in itself and instrumental for the existence statements below.

Theorem 5.1 *Let $\Omega \in C^{1,1}$ be a bounded domain. Suppose $(SC)_0$, (H_0) hold and let Λ_1, Λ_2 with $0 < \Lambda_1 < \Lambda_2$. Then every L^p-viscosity solution u of (P_λ) satisfies*

$$\|u^-\|_{L^\infty(\Omega)} \leq C\,, \ \text{for all } \lambda \in [0, \Lambda_2], \quad \|u^+\|_{L^\infty(\Omega)} \leq C\,, \ \text{for all } \lambda \in [\Lambda_1, \Lambda_2],$$

where C depends on n, p, μ_1, Ω, Λ_1, Λ_2, $\|b\|_{L^\infty(\Omega)}$, $\|c\|_{L^\infty(\Omega)}$, $\|h\|_{L^\infty(\Omega)}$, $\|u_0\|_{L^\infty(\Omega)}$, λ_P, Λ_P, the $C^{1,1}$ character of the boundary, and the set where $c > 0$.

It will be clear from the next theorems that the restrictions on λ cannot be removed. As in the previous works, we use the following order in the space $E := C^1(\overline{\Omega})$.

Definition 5.2 *Let $u, v \in E$. We say that $u \ll v$ if for every $x \in \Omega$ we have $u(x) < v(x)$ and for $x_0 \in \partial\Omega$ we have either $u(x_0) < v(x_0)$, or $u(x_0) = v(x_0)$ and $\partial_\nu u(x_0) < \partial_\nu v(x_0)$ (recall that $\vec{\nu}$ is the interior unit normal to $\partial\Omega$).*

The hypotheses $\Omega \in C^{1,1}$ bounded domain, $c \gneq 0$ and (M) are implied attached to the problem (P_λ); so we will no longer mention it each time.

Theorem 5.3 *Assume* (SC), (H_β), (H_0), *and* (H_1).

1. Then, for $\lambda \leq 0$, the problem (P_λ) has an L^p-viscosity solution u_λ that converges to u_0 in E as $\lambda \to 0^-$. Moreover, the set

$$\Sigma = \{\,(\lambda, u) \in \mathbb{R} \times E \,;\, u \text{ solves } (P_\lambda)\,\}$$

possesses an unbounded component $\mathcal{C}^+ \subset [0, +\infty] \times E$ such that $\mathcal{C}^+ \cap (\{0\} \times E)$ reduces to $\{u_0\}$. From now on we assume $(SC)_0$.

2. The component from 1. is such that

(i) either it bifurcates from infinity to the right of the axis $\lambda = 0$ with the corresponding solutions having a positive part blowing up to infinity in $C(\overline{\Omega})$ as $\lambda \to 0^+$;

(ii) or its projection on the λ axis is $[0, +\infty)$.

3. There exists $\bar{\lambda} \in (0, +\infty]$ such that, for every $\lambda \in (0, \bar{\lambda})$, the problem (P_λ) has at least two L^p-viscosity solutions, $u_{\lambda,1}$ and $u_{\lambda,2}$, satisfying

$$u_{\lambda,1} \xrightarrow[\lambda \to 0^+]{} u_0 \text{ in } E, \quad \max_{\overline{\Omega}} u_{\lambda,2} \xrightarrow[\lambda \to 0^+]{} +\infty,$$

and, if $\bar{\lambda} < +\infty$, the problem $(P_{\bar{\lambda}})$ has at least one L^p-viscosity solution. The latter is unique if $F(x, r, p, X)$ is convex in (r, p, X).

4. If in addition (H_2) holds, the solutions u_λ for $\lambda \leq 0$ are unique among L^p-viscosity solutions; whereas the solutions from 3. for $\lambda > 0$ are ordered, $u_{\lambda,1} \ll u_{\lambda,2}$.

This theorem proves the multiplicity conjectures from [12], [86]. We recall that theorem 5.3 is new even when F is a linear operator in nondivergence form – in this case, it reduces to theorem 1.3.

The supplementary hypotheses for the uniqueness results in the above theorem are unavoidable – we recall that, in the universe of L^p-viscosity solutions, uniqueness is only available in the presence of a strong solution (see [12] and the references in that paper).

In the next two theorems, we show that it is possible to obtain a more precise description of the set Σ, provided we know the sign of u_0. Such results for the divergence case $F = \Delta$ were already proved in theorems 1.4 and 1.5 in [15]. Note that if h has a sign, then u_0 has the same sign, by the maximum principle (see remark 6.32).

Theorem 5.4 *Suppose* $(SC)_0$, (H_β), (H_1), (H_2) *and* (H_0) *with* $u_0 \leq 0$ *and* $cu_0 \lneq 0$. *Then every nonpositive solution of* (P_λ) *with* $\lambda > 0$ *satisfies* $u \ll u_0$. *Furthermore, for every* $\lambda > 0$, *the problem* (P_λ) *has at least two nontrivial* L^p-*viscosity solutions* $u_{\lambda,1} \ll u_{\lambda,2}$, *such that* $u_{\lambda_2,1} \ll u_{\lambda_1,1} \ll u_0$ *if* $0 < \lambda_1 < \lambda_2$, *and*

$$u_{\lambda,1} \xrightarrow[\lambda \to 0^+]{} u_0 \quad in \ E, \quad \max_{\overline{\Omega}} u_{\lambda,2} \xrightarrow[\lambda \to 0^+]{} +\infty.$$

If $F(x, r, p, X)$ *is convex in* (r, p, X) *then* $\max_{\overline{\Omega}} u_{\lambda,2} > 0$ *for all* $\lambda > 0$.
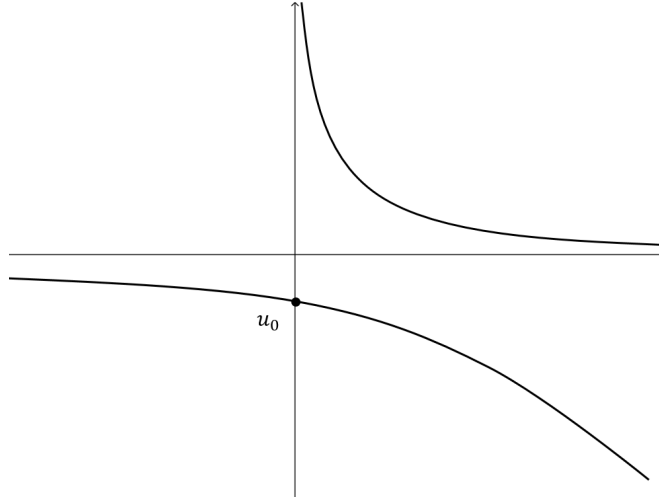

Figure 5.1: Illustration of theorem 5.4.

In this figure we put λ on the horizontal axis. On the negative side of the vertical axis we have $u_{\lambda,1}(x_0)$ for any fixed $x_0 \in \Omega$ (or $\min_{\overline{\Omega}} u_{\lambda,1}$), which is a negative quantity for $\lambda > 0$; whereas on the positive side of the vertical axis we find $\|u_{\lambda,2}\|_{L^\infty(\Omega)}$ (or $\max_{\overline{\Omega}} u_{\lambda,2}$ if F is convex).

Theorem 5.5 *Suppose* $(SC)_0$, (H_β), (H_1), (H_2) *and* (H_0) *with* $u_0 \geq 0$ *and* $cu_0 \gneq 0$. *Then every nonnegative solution of* (P_λ) *with* $\lambda > 0$ *satisfies* $u \gg u_0$. *Moreover, there exists* $\overline{\lambda} \in (0, +\infty)$ *such that*

(i) for every $\lambda \in (0, \overline{\lambda})$, *the problem* (P_λ) *has at least two nontrivial* L^p-*viscosity solutions with* $u_{\lambda,1} \ll u_{\lambda,2}$, *where* $u_0 \ll u_{\lambda_1,1} \ll u_{\lambda_2,1}$ *if* $0 < \lambda_1 < \lambda_2$ *and*

$$u_{\lambda,1} \xrightarrow[\lambda \to 0^+]{} u_0 \quad in \ E, \quad \max_{\overline{\Omega}} u_{\lambda,2} \xrightarrow[\lambda \to 0^+]{} +\infty;$$

(ii) the problem $(P_{\overline{\lambda}})$ *has at least one* L^p-*viscosity solution* $u_{\overline{\lambda}}$; *this solution is unique if* F *is convex in* (r, p, X);
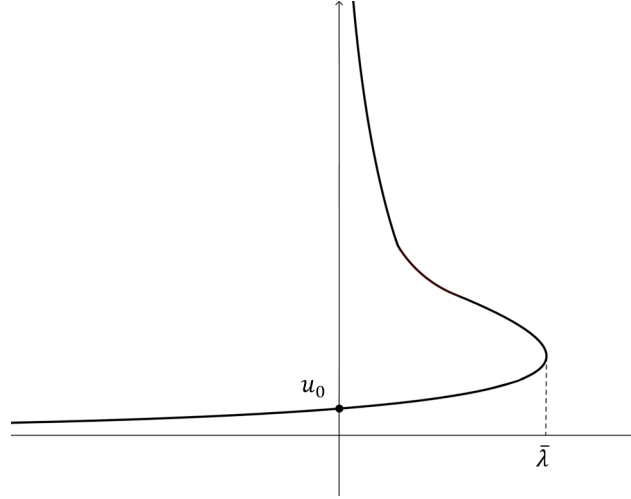
Figure 5.2: Illustration of theorem 5.5.

(iii) for $\lambda > \bar{\lambda}$, the problem (P_λ) has no nonnegative solution.

Notice that, up to replacing u for $-u$, we are also taking into account, indirectly, the case $-\mu_2 I \leq M(x) \leq -\mu_1 I$ for $\mu_1, \mu_2 > 0$. We only need to pay attention to the sign of u_0, which is reversed in this case.

5.2
A Priori Bounds

In this section we look at our family of problems (P_λ) for $\lambda > 0$, assuming $c \gneq 0$ and that the matrix M satisfies (M). Here we consider $(SC)_0$ i.e. we suppose that all coefficients of the problem (P_λ) are bounded and $d \equiv 0$. With the latter, the zero order term in (P_λ) is explicit, so we can obtain a clear behavior of the solutions with respect to λ.

Theorem 5.6 *Let $\Omega \in C^{1,1}$ be a bounded domain. Suppose $(SC)_0$, (H_0) hold and let Λ_1, Λ_2 with $0 < \Lambda_1 < \Lambda_2$. Then every L^p-viscosity solution u of (P_λ) satisfies*

$$\|u\|_\infty \leq C, \quad \text{for all } \lambda \in [\Lambda_1, \Lambda_2],$$

where C depends on $n, p, \mu_1, \Omega, \Lambda_1, \Lambda_2, \|b\|_{L^\infty(\Omega)}, \|c\|_{L^\infty(\Omega)}, \|h\|_{L^\infty(\Omega)}, \|u_0\|_{L^\infty(\Omega)},$ on the ellipticity coefficients λ_P, Λ_P, on the $C^{1,1}$ character of the boundary, and on the set where $c > 0$.

The proof of theorem 5.6 uses and develops the ideas sketched in [86], adding some improvements in order to remove restrictions on the size of c. We start by proving that all supersolutions stay uniformly bounded from below, even when λ is close to zero.

Proposition 5.7 *Let $\Omega \in C^{1,1}$ be a bounded domain. Suppose $(SC)_0$ and let $\Lambda_2 > 0$. Then every L^p-viscosity supersolution u of (P_λ) satisfies*

$$\|u^-\|_\infty \leq C, \quad \text{for all } \lambda \in [0, \Lambda_2]$$

where C depends only on $n, p, \mu_1, \Lambda_2, \|b\|_{L^\infty(\Omega)}, \|c\|_{L^\infty(\Omega)}, \|h^-\|_{L^\infty(\Omega)}, \lambda_P, \Lambda_P, |\Omega|$ and on the $C^{1,1}$ norm of $\partial\Omega$.

Proof. First observe that both $-u$ and 0 are L^p-viscosity subsolutions of

$$\widetilde{F}(x, U, DU, D^2U) \leq \lambda c(x)U - \mu_1|DU|^2 + h^-(x) \quad \text{in} \quad \Omega,$$

where $\widetilde{F}(x, r, p, X) = -F(x, -r, -p, -X)$. Then, using $(SC)_0$, these functions are also L^p-viscosity subsolutions of

$$\begin{cases} \mathcal{M}^+(D^2U) + b(x)|DU| - \mu_1|DU|^2 & \geq & -\lambda c(x)U - h^-(x) & \text{in} & \Omega \\ U & \leq & 0 & \text{on} & \partial\Omega \end{cases}$$

and so is $U := u^- = \max\{-u, 0\}$, as the maximum of subsolutions. Moreover, $U \geq 0$ in Ω and $U = 0$ on $\partial\Omega$. We make the following exponential change

$$w := \frac{1 - e^{-mU}}{m}, \quad \text{with} \quad m = \frac{\mu_1}{\Lambda_P}$$

where Λ_P is the constant from the definition of Pucci's operators. From lemma 2.20, we know that w is an L^p-viscosity solution of

$$\begin{cases} -\mathcal{L}_1^+[w] & \leq & h^-(x) + \frac{\lambda}{m}c(x)\left|\ln(1 - mw)\right|(1 - mw) & \text{in} & \Omega \\ w & = & 0 & \text{on} & \partial\Omega \end{cases} \quad (Q_\lambda)$$

where $\mathcal{L}_1^+[w] := \mathcal{L}^+[w] - mh^-(x)w$. Notice that the logarithm above is well defined, since

$$0 \leq w = \frac{1}{m}\{1 - e^{-mU}\} < \frac{1}{m} \quad \text{in} \quad \Omega.$$

Now set $w_1 := \frac{1}{m}(1 - e^{-mu_1^-})$, where u_1 is some fixed supersolution of (P_λ), $\lambda \geq 0$ (if there was not such supersolution, we would have nothing to prove). Then, by the above argument, $w_1 \in [0, 1/m)$ is an L^p-viscosity solution of (Q_λ). Define $\overline{w} := \sup \mathcal{A}$, where

$$\mathcal{A} := \{w : w \text{ is an } L^p\text{-viscosity solution of } (Q_\lambda); \ 0 \leq w < 1/m \ \text{in} \ \Omega\}.$$

Then $\mathcal{A} \neq \emptyset$, since $w_1 \in \mathcal{A}$, and $w_1 \leq \overline{w} \leq 1/m$ in Ω. Also, as a supremum of subsolutions (locally bounded, since it belongs to the interval $[0, 1/m]$), $\overline{w}$

is an L^p-viscosity solution[2] (possibly discontinuous) of the first inequality in (Q_λ). Clearly, $\overline{w} = 0$ on $\partial\Omega$.

Observe that $\mathcal{L}_1^+$ is a coercive operator and the function

$$f(x) := f_\lambda(x, \overline{w}(x)) = h^-(x) + \frac{\lambda}{m}c(x)\,|\ln(1 - m\overline{w})|(1 - m\overline{w}) \in L_+^p(\Omega)$$

with

$$\|f^+\|_{L^p(\Omega)} \le \|h^-\|_{L^p(\Omega)} + \frac{\Lambda_2}{m}\|c\|_{L^p(\Omega)}\,C_0$$

since $A(\overline{w}) := |\ln(1 - m\overline{w})|\,(1 - m\overline{w}) \le C_0$. Indeed, from $\lim_{t\to 0^+} t\ln t = 0$ there exists a $\delta \in (0,1)$ such that $t|\ln t| \le 1$, for all $0 < t < \delta$ i.e. $A(\overline{w}) \le 1$ when $x \in \left\{\overline{w} > \frac{1-\delta}{m}\right\}$. If $x \in \left\{\overline{w} \le \frac{1-\delta}{m}\right\}$ then $A(\overline{w}) \le |\ln\delta|$ (notice that $1 - m\overline{w} \le 1$) so take $C_0 = \max\{1, |\ln\delta|\}$.

Therefore, by the proof of the boundary Lipschitz bound (see theorem 2.3 in [23], which does not require any continuity assumption on $\overline{w}$),

$$\overline{w}(x) \le C\|f^+\|_{L^p(\Omega)}\,\mathrm{dist}(x, \partial\Omega) \to 0 \quad \text{as} \quad x \to \partial\Omega$$

and so $\overline{w} \not\equiv \frac{1}{m}$. Observe that the function $\overline{w}$ can be equal to $1/m$ at some interior points.

If there was a sequence of supersolutions u_k of (P_λ) in Ω with unbounded negative parts, then there would exist a subsequence such that

$$u_k^-(x_k) = \|u_k^-\|_\infty \xrightarrow[k\to\infty]{} +\infty, \quad x_k \in \overline{\Omega}, \quad x_k \xrightarrow[k]{} x_0 \in \overline{\Omega}$$

with $x_k \in \Omega$ for large k, since $u_k \ge 0$ on $\partial\Omega$. Then the respective sequence

$$w_k(x_k) = \frac{1}{m}\{1 - e^{-mu_k^-(x_k)}\} \xrightarrow[k\to\infty]{} \frac{1}{m}, \quad w_k \in \mathcal{A}$$

i.e. for every $\varepsilon > 0$, there exists some $k_0 \in \mathbb{N}$ such that

$$\frac{1}{m} \ge \overline{w}(x_k) \ge w_k(x_k) \ge \frac{1}{m} - \varepsilon, \quad \text{for all } k \ge k_0$$

thus there exists the limit $\lim_k \overline{w}(x_k) = \frac{1}{m}$ and also

$$\overline{w}(x_0) \ge \lim_{x_k \to x_0} \overline{w}(x_k) = \lim_{k\to\infty} \overline{w}(x_k) = \frac{1}{m}.$$

---

[2]If we need to use the fact that the supremum of C-viscosity subsolutions is still a C-viscosity subsolution, we apply theorem 4.2 in [33] for continuous equations. Indeed, in the problem (Q_λ), we consider γ, h_0 and c_0 (instead of $b(x)$, $h^-(x)$ and $c(x)$) such that $b(x) \le \gamma$, $h^- \le h_0$, $c(x) \le c_0$, and the logarithm term on the right hand side being defined as 0 at points where $\overline{w} = 1/m$; we can also take $\mathcal{L}^+$ instead of $\mathcal{L}_1^+$, since $mh^-w \ge 0$.

Hence $x_0 \in \Omega$, since $\overline{w} = 0$ on $\partial\Omega$, and $\overline{w}(x_0) = \frac{1}{m}$.

Finally, define $z := 1 - m\overline{w}$. Then z is an L^p-viscosity supersolution of

$$-\mathcal{L}_1^- z \geq -\lambda c(x) \left|\ln z\right| z \quad \text{in} \quad \Omega, \quad z \gneqq 0 \text{ in } \Omega, \ z(x_0) = 0,$$

where $\mathcal{L}_1^- := \mathcal{L}^- - mh^-$ is a coercive operator. But this contradicts the following nonlinear version of the SMP. $\blacksquare$

Lemma 5.8 *Set* $\mathcal{L}_1^-[u] := \mathcal{M}^-(D^2u) - \gamma\left|Du\right| - du$, *for a constant* $d \geq 0$. *Let* $f \in C[0, +\infty)$ *be defined by* $f(s) = a\,s\left|\ln s\right|$ *if* $s > 0$, $f(0) = 0$, *where* $a \geq 0$. *Then, the SMP holds for the operator* $\mathcal{L}_1^-[\cdot] - f(\cdot)$, *i.e. if* u *is a C-viscosity solution* (*possibly discontinuous*) *of*

$$\begin{cases} \mathcal{L}_1^-[u] & \leq & f(u) & in & \Omega \\ u & \geq & 0 & in & \Omega \end{cases}$$

then either $u > 0$ *in* Ω *or* $u \equiv 0$ *in* Ω.

This lemma can be seen as a form of the Vazquez's strong maximum principle [87] for our operators, since one over the square root of the primitive of $z\left|\ln z\right|$ is not integrable at 0. The proof of lemma 5.8 is given in section 5.2.1.

Note that we apply lemma 5.8 with $d = m\left\|h^-\right\|_\infty$ and $a = \Lambda_2\left\|c\right\|_\infty$.

Remark 5.9 *Notice that, by lemma 2.7, given any pair of* L^p-*viscosity sub and supersolutions* α, β *of* (P_0), *we have that* $\alpha \leq u \leq \beta$ *in* Ω, *for any strong solution* u *of* (P_0).

Before giving the proof of Theorem 5.6 we recall once more that the class of equations we study is invariant with respect to diffeomorphic changes of the spatial variable. In particular we can assume that the boundary of Ω is a hyperplane in a neighborhood of any given point of $\partial\Omega$. Indeed, straightening of the boundary leads to an equation of the same type, as showed in section 3.2.3, with bounds on the the coefficient norms depending on the $C^{1,1}$ norm of $\partial\Omega$.

Proof of Theorem 5.6. Fix Λ_1, Λ_2 with $0 < \Lambda_1 < \Lambda_2$. From proposition 5.7, there exists a constant $C_1 > 0$ such that

$$u^- \leq C_1, \quad \text{for every supersolution } u \text{ of } (P_\lambda), \text{ for all } \lambda \in [0, \Lambda_2]. \quad (5.1)$$

Suppose then, in order to obtain a contradiction, that solutions are not bounded from above in $[\Lambda_1, \Lambda_2]$, by picking out a sequence u_k of L^p-viscosity solutions of (P_λ) such that

$$u_k^+(x_k) \xrightarrow[k\to\infty]{} +\infty, \quad x_k \in \overline{\Omega}, \quad x_k \xrightarrow[k]{} x_0 \in \overline{\Omega}.$$

where x_k is the point of maximum of $|u_k|$ in $\overline{\Omega}$, i.e. $\|u_k\|_\infty = |u(x_k)|$. Here, $\|u_k\|_\infty = u_k^+(x_k) + u_k^-(x_k)$ and $u_k^-(x_k) \in [0, C_1]$.

We claim that, up to changing the blow-up limit point x_0, we can suppose that there is a ball around x_0 in which c is not identically zero.

To prove the claim, consider G, a maximal domain such that $c \equiv 0$ in G. Obviously there is no need of such argument if $|\{c = 0\}| = 0$ or even if $c \gneqq 0$ in a neighborhood of x_0. Suppose, hence, that x_0 is an interior point of G, and so $x_k \in G$ for large k (considering a half ball in G if $x_0 \in \partial\Omega$, after a diffeomorphic change of independent variable which straightens the boundary). Notice that both u_k and u_0 satisfy the same equation

$$-F(x, u, Du, D^2u) = \langle M(x)Du, Du \rangle + h(x) \quad \text{in } G \qquad (5.2)$$

in the L^p-viscosity sense, for each $k \in \mathbb{N}$ (recall $c = 0$ in G). Because of $(SC)_0$ we have $F(x, r, p, X) = F(x, r + a, p, X)$ for $a \in \mathbb{R}$, hence the functions $v_k := u_k - \inf_{\partial G} u_k$ and $v_0 := u_0 - \sup_\Omega u_0$ still satisfy equation (5.2). Also $v_k \leq 0 \leq v_0$ on ∂G, so v_k and v_0 are respectively L^p-viscosity sub and supersolution of (P_0), with v_0 strong. We apply lemma 2.7 (see remark 5.9) to obtain that $v_k \leq v_0$ in G and, in particular, for large k,

$$u_k^+ \geq u_k^+(x_k) - 2\|u_0\|_{L^\infty(\Omega)} - C_1 \xrightarrow[k\to\infty]{} +\infty \quad \text{on } \partial G.$$

This means that we have blow-up also at the boundary of G, in the sense that there exists a sequence $y_k \in \partial G$ with $u_k^+(y_k) \to +\infty$ and $y_k \to y_0 \in \partial G$, as $k \to \infty$. Next, since G is maximal, so $\partial G \subset \partial\Omega \cup \partial(\{c = 0\})$, and using $u_k = 0$ on $\partial\Omega$, we have $y_k \in \partial(\{c = 0\})$. Therefore, we can take a ball $B_r(y_0)$ centered at y_0 (or a half ball if $y_0 \in \partial\Omega$) which, by enlarging r if necessary, becomes a neighborhood which meets the set $\{c > 0\}$; in other words, such that $c \gneqq 0$ in $B_r(y_0)$. Hence, up to changing x_k and x_0 by y_k and y_0, we can suppose that $c \gneqq 0$ in $B_r(x_0)$, or in a half ball if $x_0 \in \partial\Omega$, after straightening the boundary around x_0.

Suppose we are in the more difficult case of a half ball. For simplicity, and up to rescaling, say $c \gneqq 0$ in $B_1^+ = B_1^+(x_0)$, with our equation being defined in $B_2^+(x_0) \subset \Omega$.

We make the convention of assuming that the constant C may change from line to line and depends on n, p, λ_P, Λ_P, Λ_1, Λ_2, μ_1, $\|b\|_{L^p(\Omega)}$, $\|h\|_{L^p(\Omega)}$, $\|c\|_{L^p(\Omega)}$ and C_1. The constant C_1 is fixed in (5.1) with its dependence described in the statement of proposition 5.7.

Notice that, from (5.1), for every L^p-viscosity solution u of (P_λ), the

function $v := u + C_1$ is a nonnegative L^p-viscosity solution of

$$\mathcal{M}^-(D^2 v) - b(x)|Dv| \leq F(x, v - C_1, Dv, D^2 v)$$
$$= -\lambda c(x)v - \langle M(x)Dv, Dv\rangle - h(x) + \lambda c(x)C_1$$
$$\leq -\lambda c(x)v - \mu_1|Dv|^2 + \tilde{h}(x)$$

where $\tilde{h}(x) := h^-(x) + \Lambda_2\, c(x)C_1 \geq 0$, by $(SC)_0$. Thus, by lemma 2.20, the function

$$v_1 := \frac{1}{m_1}\left\{e^{m_1 v} - 1\right\}, \quad \text{where} \ \ m_1 = \frac{\mu_1}{\Lambda_P} \tag{5.3}$$

is a nonnegative L^p-viscosity supersolution of

$$\mathcal{L}_1^-[v_1] \leq f_1(x) \quad \text{in} \ \ B_2^+ \tag{5.4}$$

where $\mathcal{L}_1^-[v_1] := \mathcal{M}^-(D^2 v_1) - b(x)|Dv_1| - m_1\,\tilde{h}(x)\,v_1$ and

$$f_1(x) := -\tfrac{\lambda}{m_1}c(x)(1 + m_1 v_1)\ln(1 + m_1 v_1) + \tilde{h}(x) \in L^p(\Omega)$$

since $v_1 \in L^\infty(\Omega)$. Notice that, in the set $B_2^+ \cap \{f_1 \geq 0\}$, we have

$$0 \leq \tfrac{\lambda}{m_1}c(x)(1 + m_1 v_1)\ln(1 + m_1 v_1) \leq \tilde{h}$$

and $f_1^+ = |f_1| \leq \tfrac{\lambda}{m_1}c(x)(1 + m_1 v_1)\ln(1 + m_1 v_1) + \tilde{h} \leq 2\,\tilde{h}$, so

$$\|f_1^+\|_{L^p(B_2^+)} \leq 2\|\tilde{h}\|_{L^p(B_2^+)} \leq 2\|h^-\|_{L^p(\Omega)} + 2\Lambda_2\|c\|_{L^p(\Omega)}C_1 \leq C. \tag{5.5}$$

Then, using proposition 2.15 ($BQSMP$, case $p = q > n$) applied to (5.4), we obtain positive constants c_0, C_0 and $\varepsilon \leq 1$, depending on n, λ, Λ_P, p and $\|b\|_{L^p(\Omega)}$, such that

$$I := \inf_{B_1^+} \frac{v_1}{x_n} \geq c_0 \left(\int_{B_{3/2}^+} (f_1^-)^\varepsilon\right)^{1/\varepsilon} - C_0 \|f_1^+\|_{L^p(B_2^+)}$$

$$= c_0 \left(\int_{B_{3/2}^+} \left\{\left(\frac{\lambda}{m_1}c(x)(1 + m_1 v_1)\ln(1 + m_1 v_1) - \tilde{h}(x)\right)^+\right\}^\varepsilon\right)^{1/\varepsilon} - C$$

$$\geq c_0 \inf_{B_1^+} \frac{v_1}{x_n} \left(\int_{B_1^+} \left(\left(\lambda c(x)\frac{1 + m_1 v_1}{m_1 v_1} x_n \ln(1 + m_1 v_1)\right.\right.\right.$$
$$\left.\left.\left. - \tilde{h}(x)\frac{1 + m_1 v_1}{v_1} x_n\right)^+\right)^\varepsilon\right)^{\frac{1}{\varepsilon}} - C$$

$$\geq c_0\, I \left(\int_{B_1^+} \left\{\left(\lambda c(x)\, x_n \ln(1 + I\, m_1 x_n) - m_1\,\tilde{h}(x)\, x_n\right)^+\right\}^\varepsilon\right)^{1/\varepsilon} - C$$

using (5.5) and that $v_1(x) \geq x_n I$ for all $x \in B_1^+$. Thus,

$$I \left\{ c_0 \left(\int_{B_1^+} x_n^\varepsilon \left(\left(\lambda c(x) \ln(1 + I\, m_1 x_n) - m_1 \widetilde{h}(x) \right)^+ \right)^\varepsilon \right)^{1/\varepsilon} - 1 \right\} \leq \bar{C}. \quad (5.6)$$

We claim that this is only possible if $I \leq C$, with a constant that does not depend on v_1 (and consequently on u), neither $\lambda \in [\Lambda_1, \Lambda_2]$. Indeed, if this was not the case, we would obtain a sequence of supersolutions v_1^k of $\mathcal{L}_1^-[v_1^k] \leq f_1^k(x)$ in B_2^+ such that $I_k := \inf_{B_1^+} \frac{v_1^k}{x_n} \to +\infty$ when $k \to +\infty$ and (5.6) holding with I replaced by I_k. So, up to a subsequence and renumbering, we can assume that $I_k \geq k^2$ and $\frac{\bar{C}}{k^2} \leq 1$ for all $k \geq k_0$, from where it follows

$$\int_{B_1^+} x_n^\varepsilon \left(\left(\lambda c(x) \ln(1 + I_k\, m_1 x_n) - m_1 \widetilde{h}(x) \right)^+ \right)^\varepsilon \leq c_0^{-\varepsilon} \left(1 + \frac{\bar{C}}{k^2} \right)^\varepsilon \leq C$$

and finally, using $\lambda \geq \Lambda_1$,

$$\int_{B_1^+ \cap \{x_n \geq 1/k\}} x_n^\varepsilon \left(\left(\Lambda_1 c(x) - m_1 \frac{\widetilde{h}(x)}{\ln(1 + m_1 k)} \right)^+ \right)^\varepsilon \leq \frac{C}{\ln(1 + m_1 k)}. \quad (5.7)$$

Passing to limits as $k \to +\infty$ we have $\int_{B_1^+} (x_n\, c(x))^\varepsilon\, \mathrm{d}x = 0$, since $\Lambda_1 > 0$, which contradicts $c(x) \gneqq 0$ in B_1^+. More precisely, for the limit in (5.7) we can use, for example, the dominated convergence theorem: for $\varepsilon = 1$ this is obvious; for $0 < \varepsilon < 1$ we use Young's inequality to estimate

$$\left(\left(\lambda c(x) - m_1 \widetilde{h}(x)/\ln(1 + m_1 k) \right)^+ \right)^\varepsilon \leq \left(\lambda c(x) - m_1 \widetilde{h}(x)/\ln(1 + m_1 k) \right)^+ + 1$$

ensuring the desired convergence. In this way we establish the claim

$$\inf_{B_1^+} \frac{v_1}{x_n} \leq C.$$

Thus, by theorem 2.16 (*BWHI*, again with $p = q > n$) applied to (5.4), we obtain that there exists other positive constants ε, c_0, C_0, depending on n, λ_P, Λ_P, p and $\|b\|_{L^p(\Omega)}$, such that

$$\left(\int_{B_{3/2}^+} v_1^\varepsilon \right)^{1/\varepsilon} \leq c_0 \left(\int_{B_{3/2}^+} \left(\frac{v_1}{x_n} \right)^\varepsilon \right)^{1/\varepsilon} \leq \inf_{B_1^+} \frac{v_1}{x_n} + C_0 \|f_1^+\|_{L^p(B_2^+)} \leq C. \quad (5.8)$$

Now we go back to u and define

$$v_2 := \frac{1}{m_2} \{ e^{m_2 u} - 1 \}, \quad \text{with } m_2 = \frac{\mu_2}{\lambda_P} \quad (5.9)$$

which by lemma 2.20 and $(SC)_0$ is an L^p-viscosity solution of

$$\begin{cases} \mathcal{M}^+(D^2v_2) + b(x)|Dv_2| + \nu(x)v_2 \geq -h^+(x) & \text{in} \quad B_2^+ \\ v_2 = 0 & \text{on} \quad B_2^0 \end{cases} \tag{5.10}$$

where $\nu(x) := \frac{\lambda}{m_2 v_2} c(x)(1+m_2 v_2)\ln(1+m_2 v_2) + m_2 h^+(x) \in L^p(B_2^+)$.

Notice that by the definitions (5.3) and (5.9),

$$v_2 = \frac{1}{m_2}\left\{ (1+m_1 v_1)^{\frac{m_2}{m_1}} e^{-m_2 C_1} - 1 \right\}. \tag{5.11}$$

As in [15], observe that

$$\lambda c(x)\left|\frac{(1+m_2 v_2)}{m_2 v_2}\ln(1+m_2 v_2)\right| \leq C_s\, c(x)\,(1+|v_2|^s) \tag{5.12}$$

for any $s > 0$. Indeed, $v_2 \geq R_1$, for $R_1 := \frac{1}{m_2}\{e^{-C_1 m_2}-1\} \in (-1/m_2, 0)$ by (5.1). Then, since $\lim_{t\to 0}\frac{\ln(1+t)}{t} = 1$, there exists a $\delta \in (0, -R_1)$ such that the expression inside the modulus sign in (5.12) is less than $1+m_2\delta$, for every $x \in \{|v_2| < \delta\}$. Further, from $\lim_{t\to +\infty}\frac{1+t}{t} = 1$ and $\lim_{t\to +\infty}\frac{\ln(1+t)}{t^s} = 0$, there exists a $R_2 > 0$ such that this expression is less than $2\,v_2{}^s$, for all $x \in \{v_2 \geq R_2\}$. Finally, the same term is bounded by a constant which depends on R_1, R_2 and δ if $v_2 \in [R_1, -\delta] \cup [\delta, R_2]$, and so (5.12) holds for any $s > 0$.

Now, if we take $s = \varepsilon\frac{m_1}{m_2}\frac{p-n}{p(p+n)}$ and $p_1 = \frac{p+n}{2} \in (n, p)$ then, by Holder's inequality, the right hand side in (5.12) belongs to $L^{p_1}(B_2^+)$ and

$$\| c\,|v_2|^s\|_{L^{p_1}(B_2^+)} \leq \|c\|_{L^p(B_2^+)}\|\,|v_2|^s\|_{L^{p_2}(B_2^+)}, \quad \text{with} \quad \frac{1}{p_1} = \frac{1}{p} + \frac{1}{p_2}$$

$$\leq \|c\|_{L^p(B_2^+)}\left(\int_{B_2^+}|v_2|^{\varepsilon\frac{m_1}{m_2}}\right)^{\frac{p-n}{p(p+n)}} \leq C \tag{5.13}$$

and then

$$\|\nu\|_{L^{p_1}(B_2^+)} \leq C_{n,p}\,\|c\|_{L^p(\Omega)} + \| c\,|v_2|^s\|_{L^{p_1}(B_2^+)} + m_2\|h^+\|_{L^p(\Omega)} \leq C. \tag{5.14}$$

Next, the uniform bound on (5.14) allows us to use theorem 2.1 (*BLMP*, case $p = q > n$) applied to (5.10), for v_2 and $r = \varepsilon\frac{m_1}{m_2}$ as in (5.13), to obtain

$$v_2{}^+ \leq C\left\{\left(\int_{B_{3/2}^+}|v_2|^r\right)^{1/r} + \|h^+\|_{L^p(\Omega)}\right\} \leq C \quad \text{in } B_1^+.$$

Hence, $u^+ = \frac{1}{m_2}\ln(1 + m_2 v_2{}^+)$ is uniformly bounded in B_1^+, for every L^p-viscosity solution of (P_λ), for all $\lambda \in [\Lambda_1, \Lambda_2]$. ∎

5.2.1
Proof of Lemma 5.8

We show that the result is valid for the most general notion of C-viscosity solutions (see [3], [33]), which can be discontinuous functions. We can suppose $u \in LSC(\Omega)$, just replacing u by its lower semicontinuous envelope u_*, see chapter 2. Remember that, in this case, we say that u is a C-viscosity supersolution provided u_* is.

In this section, for simplicity, we denote $\mathcal{L}_1^-[u] = \mathcal{M}_{\lambda,\Lambda}^-(D^2u) - \gamma|Du| - du$, with ellipticity coefficients λ and Λ.

By contradiction, let u be a nonnegative C-viscosity supersolution of

$$\mathcal{L}_1^-[u] \leq f(u) \quad \text{in} \ \Omega$$

with both $\Omega_0 := \{x \in \Omega; \ u(x) = 0\}$ and $\Omega^+ := \{x \in \Omega; \ u(x) > 0\}$ nonempty sets. Notice that Ω^+ is open, since $u \in LSC(\Omega)$. As in the usual proof of SMP (see, for instance, [39]), choose $\tilde{x} \in \Omega^+$ such that $\text{dist}(\tilde{x}, \Omega_0) < \text{dist}(\tilde{x}, \partial\Omega)$ and consider the ball $B_R = B_R(\tilde{x}) \subset \Omega^+$ such that $\partial B_R(\tilde{x}) \cap \partial\Omega_0 \neq \emptyset$.

Observe that f is a strictly increasing function on the interval $(0,\delta)$, for some $\delta < 1$.

Fix a $x_0 \in \partial B_R(\tilde{x}) \cap \partial\Omega_0$, so $u(x_0) = 0$ and $u(x) > 0$ in $B_R = B_R(\tilde{x})$. Note that, up to diminishing R, we can suppose also that $u < \delta$ in $B_R(\tilde{x})$. Indeed, since $u(x_0) = 0$ there exists a ball $B_{r_0}(x_0)$ such that $u < \delta$ in this ball; so by taking $R_1 < R$ with $B_{R_1}(\tilde{x}_1) \subset B_{r_0}(x_0)$ for a point $\tilde{x}_1 \in \Omega^+$, now just replace $\tilde{x}, R$ by $\tilde{x}_1, R_1$.

Consider the annulus $E_R = B_R \setminus \overline{B}_{R/2}$ centered in $\tilde{x}$ and set μ as the quantity $\mu = \min_{\partial B_{R/2}} u \in (0,\delta)$. We need to find a good barrier in E_R for our nonlinear problem. This is accomplished in the following claim.

Claim 5.10 *There exists a nonnegative classical subsolution $v \in C^2(\overline{E}_R)$ of*

$$\begin{cases} \mathcal{L}_1^-[v] &> \ f(v) & in & \overline{E}_R \\ v &= \ \mu & on & \partial B_{R/2} \\ v &= \ 0 & on & \partial B_R\,, \end{cases}$$

radially decreasing in $r = |x - \tilde{x}|$, convex and such that $\partial_\nu v > 0$ on ∂B_R.

Proof. We start choosing a large $\alpha > 1$ such that

$$\frac{1}{R^2}\left\{\alpha\left[\lambda(\alpha+1) + (n-1)\Lambda - \gamma R\right] - dR^2\right\} > C_0\,\alpha\,, \tag{5.15}$$

for $C_0 := a\left(|\ln\mu| + 2|\ln R| + |\ln 2| + |\ln(R/2)\,|\right) + m_0\,$, $m_0 := \max_{s \in [0,1]} f(s) > 0$.

Let $\varepsilon > 0$ be such that $\varepsilon = \varepsilon(\alpha) := \mu\,\{(R/2)^{-\alpha} - R^{-\alpha}\}^{-1}$, and set

$$v(x) := \varepsilon\,\{|x - \tilde{x}|^{-\alpha} - R^{-\alpha}\}. \tag{5.16}$$

Remark 5.11 (*For instance, lemma 2.2 in [12].*) *Let $u \in C^2(B)$ be a radial function, $u(x) = \varphi(|x - x_0|)$, defined in a ball B in $\mathbb{R}^n$ centered on x_0. Then*

$$\mathrm{spec}(D^2 u(x)\,) = \left\{ \frac{\varphi'(|x - x_0|)}{|x - x_0|}, \ldots, \frac{\varphi'(|x - x_0|)}{|x - x_0|}, \varphi''(|x - x_0|) \right\}.$$

With this choice of ε, of course $v = \mu$ on $\partial B_{R/2}$ and $v = 0$ on ∂B_R. Notice that $v(x) = \varphi(r)$ with $r = |x - \tilde{x}|$, thus $\varphi'(r) = -2\alpha\varepsilon\,r^{-\alpha-1} < 0$ and so

$$\partial_\nu v(x) = Dv(x) \cdot \vec{\nu} = -\varphi'(R)\frac{x - \tilde{x}}{|x - \tilde{x}|} \cdot \frac{x - \tilde{x}}{|x - \tilde{x}|} = -\varphi'(R) > 0$$

for every $x \in \partial B_R = \partial B_R(\tilde{x})$, where $\vec{\nu} = -\frac{x-\tilde{x}}{|x-\tilde{x}|}$ is the interior unit normal to the ball B_R. Further, $\varphi''(r) = \alpha(\alpha + 1)\varepsilon\,r^{-\alpha-2} > 0$, and by remark 5.11 we have, in $\overline{E}_R$,

$$
\begin{aligned}
\mathcal{L}_1^-[v] &= \mathcal{M}_{\lambda,\Lambda}^-(D^2 v) - \gamma\,|Dv| - d\,v(x) \\
&= \alpha\varepsilon|x - \tilde{x}|^{-\alpha-2}\,\{\lambda(\alpha + 1) + (n - 1)\Lambda\} - \gamma\alpha\varepsilon|x - \tilde{x}|^{-\alpha-1} \\
&\quad - d\varepsilon|x - \tilde{x}|^{-\alpha} + d\varepsilon R^{-\alpha} \\
&\geq \varepsilon|x - \tilde{x}|^{-\alpha}\,R^{-2}\,\{\,\alpha\,[\lambda(\alpha + 1) + (n - 1)\Lambda - \gamma R] - dR^2\,\} \\
&> \alpha\,\varepsilon\,C_0\,|x - \tilde{x}|^{-\alpha}
\end{aligned}
$$

by the choice of α in (5.15). Now we claim that

$$f(v) \leq \alpha\,\varepsilon\,C_0\,|x - \tilde{x}|^{-\alpha} \ \text{ in } \ \overline{E}_R \tag{5.17}$$

and this will finish the proof of the claim 5.10. For $r = R$ this is obvious. Note that, for $r \neq R$, (5.17) is equivalent to

$$a\left(1 - \frac{r^\alpha}{R^\alpha}\right)\left|\ln\varepsilon - \ln(r^\alpha) + \ln\left(1 - \frac{r^\alpha}{R^\alpha}\right)\right| \leq \alpha\,C_0. \tag{5.18}$$

Set $y := 1 - \left(\frac{r}{R}\right)^\alpha \in \left[0, 1 - \frac{1}{2^\alpha}\right]$ for $r \in \left[\frac{R}{2}, R\right]$. But now the left hand side of (5.18) is less or equal than

$$
\begin{aligned}
a\,|\ln\varepsilon| &+ \alpha\,a\,|\ln r| + ay\,|\ln y| \\
&\leq a\,\{\,|\ln\mu| + \alpha\,|\ln R| + |\ln(2^\alpha - 1)|\,\} + \alpha a\,\{\,|\ln R| + |\ln(R/2)|\,\} + ay|\ln y| \\
&\leq \alpha\,C_0
\end{aligned}
$$

by using $\alpha > 1$ and definitions of ε and m_0. ∎

By construction, $u \geq v$ on $\partial E_R = \partial B_R \cup \partial B_{R/2}$.

Claim 5.12 *$u \geq v$ in E_R.*

Proof. Suppose not, i.e. that the open set $\mathcal{O} := E_R \cap \{u < v\}$ is not empty.

Notice that since v is decreasing in r and $v = \mu < \delta$ on $r = R/2$, then both $u, v \in (0, \delta)$ in $E_R = B_R \setminus \overline{B}_{R/2}$, and we can use the monotonicity of f in the interval $(0, \delta)$ to obtain

$$\begin{cases} \mathcal{L}_1^-[v] - \mathcal{L}_1^-[u] > f(v) - f(u) \geq 0 & \text{in} \quad \mathcal{O} \\ v \leq u & \text{on} \quad \partial\mathcal{O} \subset \partial E_R \cup \{v = u\} \end{cases}$$

in the C-viscosity sense, and so the comparison principle (for example proposition 3.3 in [33]), gives us that $v \leq u$ in $\mathcal{O}$, which contradicts the definition of the set $\mathcal{O}$. ∎

To finish the proof of lemma 5.8, if $u \in C^1(\Omega)$, we can just use the fact that u has a minimum at the interior point x_0, so by claim 5.12 we have $0 = \partial_\nu u(x_0) \geq \partial_\nu v(x_0) > 0$, a contradiction.

Assume u is only in $LSC(\Omega)$. Let $\rho < R/2$. Observe that v is a C^2 function in $B_\rho(x_0)$ with $v < 0 \leq u$ in $B_\rho(x_0) \setminus E_R$ and, since $v \leq u$ in E_R with $u(x_0) = v(x_0) = 0$, then v touches u from below at the interior point x_0. Thus, the definition of u being a C-viscosity supersolution on Ω yields

$$0 < \mathcal{L}^-[v](x_0) - dv(x_0) - f(v(x_0)) = \mathcal{L}^-[v](x_0)$$
$$= \mathcal{L}^-[v](x_0) - du(x_0) - f(u(x_0)) \leq 0,$$

a contradiction.

6
The Degree Theory Approach

This chapter is devoted to the proof of theorems 5.3, 5.4 and 5.5, split in sections 6.3, 6.4 and 6.5, respectively. However, the first section in the sequel is completely independent from the others sections and only requires the usual bounds necessary to the application of $C^{1,\alpha}$ regularity and estimates from chapter 3.

6.1
Existence results through fixed point theorems

In this section we construct and study an auxiliary fixed point problem in order to obtain the existence statements in theorems 5.3–5.5.

Consider the problem (P_λ) without λ dependence, i.e.

$$\begin{cases} -F(x, u, Du, D^2u) &= c(x)u + \langle M(x)Du, Du\rangle + h(x) & \text{in} & \Omega \\ u &= 0 & \text{on} & \partial\Omega \end{cases} \quad (\text{P})$$

under (SC). In this section all results hold for functions b, c and h in $L^p(\Omega)$. About the matrix M, we only need to assume that $M \in L^\infty(\Omega)$. We set $\mu = \|M\|_{L^\infty(\Omega)}$. As for c, no sign condition is needed in this section.

We define, under hypothesis (H_1), the operator $\mathcal{T} : E \to E$ that takes a function $u \in E = C^1(\overline{\Omega})$ into $U = \mathcal{T}u$, the unique L^p-viscosity solution of the problem

$$\begin{cases} -F(x, U, DU, D^2U) &= c(x)u + \langle M(x)Du, Du\rangle + h(x) & \text{in} & \Omega \\ U &= 0 & \text{on} & \partial\Omega \end{cases} \quad (\mathcal{T}_u)$$

Claim 6.1 *The operator $\mathcal{T}$ is completely continuous.*

Proof. Let $u_k \in E$, $u_k \to u$ in E. Then $\|u_k\|_E \le C_0$, for all $k \in \mathbb{N}$. Set $f_k(x) := c(x)u_k + \langle M(x)Du_k, Du_k\rangle + h(x)$ so

$$\|f_k\|_{L^p(\Omega)} \le \|h\|_{L^p(\Omega)} + C_0\,\|c\|_{L^p(\Omega)} + \mu\,C_0^2 \le C.$$

Next, ABP on the sequence $U_k = \mathcal{T}u_k$ produces $\|U_k\|_{L^\infty} \le C\,\|f_k\|_{L^p(\Omega)} \le C$. Therefore the $C^{1,\alpha}$ global estimate (theorem 3.1 and remark 3.3, see also

remark 3.2) gives us $\|U_k\|_{C^{1,\alpha}(\overline{\Omega})} \leq C$. By the compact inclusion of $C^{1,\alpha}(\overline{\Omega})$ into E, there exists $U \in E$ and a subsequence such that $U_k \to U$ in E. This already shows that $\mathcal{T}$ takes bounded sets into precompact ones.

We need to see that $U = \mathcal{T}u$. This easily follows from stability (proposition 2.12), by defining, for each $\varphi \in W^{2,p}_{\text{loc}}(\Omega)$,

$$g_k(x) := -F(x, U_k, D\varphi, D^2\varphi) - f_k(x), \quad g(x) := -F(x, U, D\varphi, D^2\varphi) - f(x)$$

a.e. $x \in \Omega$, where f is the same as f_k with u_k replaced by u. Indeed, by (SC),

$$
\begin{aligned}
\|g_k - g\|_{L^p(\Omega)} &\leq \|d\|_{L^p(\Omega)}\,\omega(\|U_k - U\|_{L^\infty(\Omega)}) + \|c\|_{L^p(\Omega)}\|u_k - u\|_{L^\infty(\Omega)} \\
&\quad + \mu\left(\|Du_k\|_{L^\infty(\Omega)} + \|Du\|_{L^\infty(\Omega)}\right)\|Du_k - Du\|_{L^\infty(\Omega)}|\Omega|^{1/p} \\
&\leq \|d\|_{L^p(\Omega)}\,\omega(\|U_k - U\|_{L^\infty(\Omega)}) + \left\{\|c\|_{L^p(\Omega)} + 2C_0\,\mu\,|\Omega|^{1/p}\right\}\|u_k - u\|_E,
\end{aligned}
$$

which converges to zero as $k \to +\infty$, since ω is increasing, continuous in 0, with $\omega(0) = 0$. Since the problem $(\mathcal{T}_u)$ has a unique solution, $U = \mathcal{T}u$. On the other hand, since this argument can be made for any subsequence of the original $(U_k)_{k\in\mathbb{N}}$, the whole sequence converges to $U = \mathcal{T}u$. ∎

The next existence statement is a typical result about existence between sub and supersolutions, and it is a version of theorem 2.1 of [15] for fully nonlinear equations. We start with a definition.

Definition 6.2 *An L^p-viscosity subsolution $\alpha \in E$ (supersolution β) of* (P) *is said to be strict if every L^p-viscosity supersolution (subsolution) $u \in E$ of* (P) *such that $\alpha \leq u$ ($u \leq \beta$) in Ω, also satisfies $\alpha \ll u$ ($u \ll \beta$) in Ω.*

Set $\mathcal{B}_r = \mathcal{B}_r^E(0) := \{u \in E;\ \|u\|_E < r\}$, for any $r > 0$, $E = C^1(\overline{\Omega})$.

Theorem 6.3 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $\partial\Omega \in C^{1,1}$. Suppose* (SC), (H_β) *and* (H_1). *Let $\alpha = \max_{1\leq i\leq\kappa} \alpha_i$, $\beta = \min_{1\leq j\leq\iota} \beta_j$, where $\alpha_i, \beta_j \in W^{2,p}(\Omega)$ are strong sub and supersolutions of* (P) *respectively, with $\alpha \leq \beta$ in Ω. Then* (P) *has an L^p-viscosity solution satisfying $\alpha \leq u \leq \beta$ in Ω. Furthermore,*

(i) If α and β are strict in the sense of definition 6.2, then for large $R > 0$ we have

$$deg(I - \mathcal{T}, \mathcal{S}, 0) = 1$$

where $\mathcal{S} = \mathcal{O} \cap \mathcal{B}_R$, for $\mathcal{O} = \mathcal{O}_{\alpha,\beta} := \{u \in C_0^1(\overline{\Omega});\ \alpha \ll u \ll \beta \text{ in } \Omega\}$.

(ii) If (H_2) holds, there exists a minimal solution $\underline{u}$ and a maximal solution $\overline{u}$ of (P) *in the sense that every (strong) solution u of* (P) *in the order interval $[\alpha, \beta]$ (i.e. such that $\alpha(x) \leq u(x) \leq \beta(x)$ for all $x \in \Omega$) satisfies*

$$\alpha \leq \underline{u} \leq u \leq \overline{u} \leq \beta \quad \text{in} \quad \Omega.$$

Notice that, under the assumptions of theorem 6.3 and by the global $C^{1,\alpha}$ estimate (theorem 3.1 and remark 3.3), every solution u of (P) satisfying $\alpha \le u \le \beta$ in Ω is such that

$$\|u\|_{C^{1,\alpha}(\overline{\Omega})} \le C, \tag{6.1}$$

where C depends on r_0, n, p, μ, $\|b\|_{L^p(\Omega)}$, $\|c\|_{L^p(\Omega)}$, $\|h\|_{L^p(\Omega)}$, $\omega(1)\|d\|_{L^p(\Omega)}$, $\mathrm{diam}(\Omega)$, λ_P, Λ_P, on the $C^{1,1}$ character of the boundary and, of course, on the L^∞ uniform bounds on u given by $\|\alpha\|_\infty$ and $\|\beta\|_\infty$.

Proof. Consider any $R \ge \max\{C, \|\alpha\|_E, \|\beta\|_E\} + 1$, with C from (6.1).

Part 1. Existence of a solution in the order interval $[\alpha, \beta]$.

First of all, we construct a modified problem, similar to but a little bit simpler than the one given in [15]. In order to avoid technicalities, consider $\kappa = \iota = 1$; later we indicate the corresponding changes. We set

$$f(x, r, p) := h(x) + c(x)\, r + \langle M(x)\, p, p\rangle;$$
$$\overline{f}(x, r, p) := h(x) + c(x)\, r + \langle \overline{M}(x)\, p, p\rangle$$

for

$$\overline{M}(x) = \overline{M}(x, p) := \begin{cases} M(x), & \text{if } |p| < R \\ M(x)\dfrac{R^2}{|p|^2}, & \text{if } |p| \ge R, \end{cases}$$

and also

$$\widetilde{f}(x, r, p) := \begin{cases} \overline{f}(x, \alpha(x), D\alpha(x)), & \text{if } r < \alpha(x) \\ \overline{f}(x, r, p), & \text{if } \alpha \le r \le \beta(x) \\ \overline{f}(x, \beta(x), D\beta(x)), & \text{if } r > \beta(x). \end{cases}$$

Consider the problems

$$-F[u] = f(x, u, Du) \quad \text{in } \Omega, \tag{P}$$
$$-F[u] = \overline{f}(x, u, Du) \quad \text{in } \Omega, \tag{$\overline{\text{P}}$}$$
$$-F[u] = \widetilde{f}(x, u, Du) \quad \text{in } \Omega, \tag{$\widetilde{\text{P}}$}$$
$$\text{and } u = 0 \quad \text{on } \partial\Omega.$$

Notice that, by the construction of $\overline{f}$, $\widetilde{f}$ and $|D\alpha|$, $|D\beta| < R$, we have

$$\widetilde{f}(x, \alpha(x), D\alpha(x)) = \overline{f}(x, \alpha(x), D\alpha(x)) = f(x, \alpha(x), D\alpha(x)) \quad \text{a.e. in } \Omega \tag{6.2}$$

and the same for β. So, α, β are also a pair of strong sub and supersolutions for $(\overline{\mathrm{P}})$ and $(\widetilde{\mathrm{P}})$. Observe also that $\|\overline{M}\|_{L^\infty(\Omega)} \le \mu$.

Claim 6.4 *The operator $F(x, r, p, X) + \langle \overline{M}(x)p, p \rangle$ still satisfies $(SC)^\mu$.*

Proof. We need to estimate the difference $A_{p,q} := \langle \overline{M}(x, p)p, p \rangle - \langle \overline{M}(x, q)q, q \rangle$. First, if both $|p|, |q| < R$, then $\overline{M}(x, p) = \overline{M}(x, q) = M(x)$ and we have nothing to prove (see remark 2.2). In the second place, if both $|p|, |q| \ge R$ then

$$A_{p,q} = \frac{R^2}{|p|^2}\langle M(x)p, p \rangle - \frac{R^2}{|q|^2}\langle M(x)q, q \rangle \le \langle M(x)p, p \rangle - \frac{R^2}{|q|^2}\langle M(x)q, q \rangle.$$

So, reversing the roles of p, q, it is enough to consider $|p| < R$ and $|q| \ge R$. In this case, the modulus of the last expression above is equal to

$$\frac{1}{|q^2|}|\langle M(x)|q|\, p, |q|\, p \rangle - \langle M(x)Rq, Rq \rangle| \le \mu \frac{|\,|q|p - Rq\,|}{|q|} \frac{|q|\,|p| + R|q|}{|q|}$$

$$\le \mu\,|\,p - Rq/|q|\,|\,(|p| + R) \le \mu\,|p - q|\,(|p| + |q|)$$

since the distance between p and q is greater[1] or equal than the distance between p and the projection of q on the ball B_R. ∎

By claim 6.4, (6.1) and the definition of R, every solution u of $(\overline{\mathrm{P}})$ with $\alpha \le u \le \beta$ in Ω is $C^{1,\alpha}$ up to the boundary and satisfies

$$\|u\|_E < R, \tag{6.3}$$

so $\overline{M}(x) = \overline{M}(x, Du) = M(x)$, and u is a solution of the original problem (P).

Claim 6.5 *Every L^p-viscosity solution u of $(\widetilde{\mathrm{P}})$ satisfies $\alpha \le u \le \beta$ in Ω, hence is a solution of $(\overline{\mathrm{P}})$, and, by the above, a solution of (P).*

Proof. Let u be an L^p-viscosity solution of $(\widetilde{\mathrm{P}})$. As in the proof of lemma 2.7, assume $v := u - \alpha$ is such that $\min_{\overline{\Omega}} v = v(x_0) < 0$ in order to produce a contradiction. Since $v \ge 0$ on $\partial\Omega$, it follows that $x_0 \in \Omega$. Consider, then, the set $\widetilde{\Omega} := \{v < 0\} \ne \emptyset$.

We claim that v is an L^p-viscosity supersolution of

$$\mathcal{M}^-(D^2 v) - b(x)|Dv| \le 0 \ \text{ in } \widetilde{\Omega}. \tag{6.4}$$

Indeed, let $\varphi \in W^{2,p}_{\mathrm{loc}}(\widetilde{\Omega})$ and $\widehat{x} \in \widetilde{\Omega}$ such that $v - \varphi$ has a local maximum at $\widehat{x}$ in $\widetilde{\Omega}$. Then, $\alpha + \varphi \in W^{2,p}_{\mathrm{loc}}(\widetilde{\Omega})$ and $u - (\alpha + \varphi)$ has also a local maximum at

---

[1]Observe that $|p - q|$ is the length of the opposite side to the obtuse angle.

$\widehat{x}$. By definition of u as an L^p-viscosity supersolution of $(\widetilde{P})$, for every $\varepsilon > 0$, there exists an $r > 0$ such that for a.e. in $B_r(\widehat{x}) \cap \widetilde{\Omega}$,

$$-F(x, u, D(\alpha + \varphi, D^2(\alpha + \varphi))) - \overline{f}(x, \alpha, D\alpha) \geq -\varepsilon, \qquad (6.5)$$

since $u < \alpha$ in $\widetilde{\Omega}$. Further, from (6.2), $-F(x, \alpha, D\alpha, D^2\alpha) - \overline{f}(x, u, D\alpha) \leq 0$ a.e. in $\widetilde{\Omega}$. Subtracting this from (6.5) and using (SC), we obtain $\mathcal{L}^-[u] \leq \varepsilon$ a.e. in $B_r(\widehat{x}) \cap \widetilde{\Omega}$, since $d(x)\,\omega(v^+) \equiv 0$ in $\widetilde{\Omega}$. Hence, (6.4) is proved.

Now, (6.4) and ABP imply that $v \geq 0$ in $\widetilde{\Omega}$, contradicting the definition of $\widetilde{\Omega}$. So, $\min_{\overline{\Omega}} v \geq 0$ i.e. $u \geq \alpha$ in Ω. Analogously, $u \leq \beta$ in Ω. ∎

Next, we move to build a solution for the operator $\widetilde{\mathcal{T}} : E \to E$ that takes a function $u \in E$ into $U = \widetilde{\mathcal{T}}_u$, the unique L^p-viscosity solution of the problem

$$\begin{cases} -F[U] &= \widetilde{f}(x, u, Du) \quad \text{in} \quad \Omega \\ U &= 0 \qquad\qquad \text{on} \quad \partial\Omega. \end{cases} \qquad (\widetilde{\mathcal{T}}_u)$$

Solutions of $(\widetilde{P})$ are fixed points of $\widetilde{\mathcal{T}}$, and belong to the order interval $[\alpha, \beta]$, by claim 6.5. Moreover,

$$\|\widetilde{\mathcal{T}}u\|_E < R_0, \quad \text{for all } u \in E, \qquad (6.6)$$

for an appropriate $R_0 > R$. In fact, by observing that

$$|\langle \overline{M}(x)\,p, p\rangle| = \begin{cases} |\langle M(x)\,p, p\rangle|, & \text{if } |p| < R \\ \frac{R^2}{|p|^2}|\langle M(x)\,p, p\rangle|, & \text{if } |p| \geq R \end{cases} \leq \mu R^2, \text{ for all } p \in \mathbb{R}^n,$$

then $|\widetilde{f}(x, r, p)| \leq |h(x)| + \mu\,R^2 + |c(x)| \max\{\|\alpha\|_\infty, \|\beta\|_\infty\}$. In addition, for the function $\gamma(x) := \widetilde{f}(x, u(x), Du(x))$,

$$\|\gamma\|_{L^p(\Omega)} \leq \|h\|_{L^p(\Omega)} + \mu\,R^2 + \|c\|_{L^p(\Omega)} \max\{\|\alpha\|_\infty, \|\beta\|_\infty\},$$

for every $u \in E$. Thus, $C^{1,\alpha}$ estimates (theorem 3.1 and remark 3.3, see also remark 3.2) together with ABP, applied to the problem $(\widetilde{\mathcal{T}}_u)$, give us that $U = \widetilde{\mathcal{T}}u \in C^{1,\alpha}(\overline{\Omega})$ and $\|U\|_{C^{1,\alpha}(\overline{\Omega})} \leq C < R_0$. Thus, (6.6) follows.

Notice that $\widetilde{\mathcal{T}}$ takes bounded sets in E into precompact ones, by the above and the compact inclusion $C^{1,\alpha} \subset E$. Also, if $u_k \to u$ in E, then $U_k = \widetilde{\mathcal{T}}u_k \to U$ in E up to a subsequence; thus we can conclude that $\widetilde{\mathcal{T}}$ is completely continuous if we show that $U = \widetilde{\mathcal{T}}u$. To prove the latter, similarly to the argument with

$\mathcal{T}$ we set, for $\varphi \in W^{2,p}_{\mathrm{loc}}(\Omega)$,

$$g_k(x) := -F(x, U_k, D\varphi, D^2\varphi) - c(x)\,\widetilde{u}_k - \langle \overline{M}(x, Du_k)\, Du_k, Du_k \rangle - h(x)$$
$$g(x) := -F(x, U, D\varphi, D^2\varphi) - c(x)\,\widetilde{u} - \langle \overline{M}(x, Du)\, Du, Du \rangle - h(x),$$

where, for each fixed function v, the function $\widetilde{v}$ is the following truncation

$$\widetilde{v}(x) := \begin{cases} \alpha(x) & \text{if } v(x) < \alpha(x) \\ v(x) & \text{if } \alpha(x) \le v(x) \le \beta(x) \\ \beta(x) & \text{if } v(x) > \beta(x). \end{cases}$$

Observe that

$$\widetilde{u}_k(x) - \widetilde{u}(x) = \begin{cases} u_k(x) - u(x) & \text{if } x \in \{\alpha \le u_k \le \beta\} \cap \{\alpha \le u \le \beta\} \\ u_k(x) - \alpha(x) & \text{if } x \in \{\alpha \le u_k \le \beta\} \cap \{u < \alpha\} \\ u_k(x) - \beta(x) & \text{if } x \in \{\alpha \le u_k \le \beta\} \cap \{u > \beta\} \\ \alpha(x) - u(x) & \text{if } x \in \{u_k < \alpha\} \cap \{\alpha \le u \le \beta\} \\ \beta(x) - u(x) & \text{if } x \in \{u_k > \beta\} \cap \{\alpha \le u \le \beta\} \\ \beta(x) - \alpha(x) & \text{if } x \in \{u_k > \beta\} \cap \{u < \alpha\} \\ \alpha(x) - \beta(x) & \text{if } x \in \{u_k < \alpha\} \cap \{u > \beta\} \\ 0 & \text{otherwise} \end{cases}$$

thus $|\widetilde{u}_k - \widetilde{u}|$ is given by

$$\begin{cases} |u_k - u| & \text{in } \{\alpha \le u_k \le \beta\} \cap \{\alpha \le u \le \beta\} \\ u_k - \alpha < u_k - u & \text{in } \{\alpha \le u_k \le \beta\} \cap \{u < \alpha\} \\ \beta - u_k < u - u_k & \text{in } \{\alpha \le u_k \le \beta\} \cap \{u > \beta\} \\ u - \alpha < u - u_k & \text{in } \{u_k < \alpha\} \cap \{\alpha \le u \le \beta\} \\ \beta - u < u_k - u & \text{in } \{u_k > \beta\} \cap \{\alpha \le u \le \beta\} \\ \beta - \alpha < |u_k - u| & \text{in } (\{u_k > \beta\} \cap \{u < \alpha\}) \cup (\{u_k < \alpha\} \cap \{u > \beta\}) \\ 0 & \text{otherwise} \end{cases}$$

and then $\|\widetilde{u}_k - \widetilde{u}\|_{L^\infty(\Omega)} \le \|u_k - u\|_{L^\infty(\Omega)} \to 0$ as $k \to \infty$. By the estimates for $\mathcal{T}$ and using that the function $p \mapsto \langle \overline{M}(x, p)p, p \rangle$ is continuous in p for a.e. $x \in \Omega$, we get $\|g_k - g\|_{L^p(\Omega)} \to 0$ as $k \to \infty$ and so $U = \widetilde{\mathcal{T}}u$.

By complete continuity of $\mathcal{T}$ and (6.6), the degree $\deg(I - \widetilde{\mathcal{T}}, \mathcal{B}_{R_0}, 0)$ is well defined and is equal to one. Indeed, set $H_t(u) := t\,\widetilde{\mathcal{T}}u$ for all $t \in [0, 1]$ and notice that $(I - H_t)\,u = 0 \iff u = t\,\widetilde{\mathcal{T}}u \in \mathcal{B}_{R_0}$. Then $I - H_t$ does not vanish

on $\partial \mathcal{B}_{R_0}$ and

$$\deg(I - \widetilde{\mathcal{T}}, \mathcal{B}_{R_0}, 0) = \deg(I - H_1, \mathcal{B}_{R_0}, 0)$$
$$= \deg(I - H_0, \mathcal{B}_{R_0}, 0) = \deg(I, \mathcal{B}_{R_0}, 0) = 1.$$

Therefore, $\widetilde{\mathcal{T}}$ has a fixed point $u \in E$, which is a solution of $(\widetilde{P})$. By claim 6.5, the first existence statement in theorem 6.3 is proved in the case $\kappa = \iota = 1$.

If α and β are in the general case as the maximum and minimum of strong sub and supersolutions, respectively, we define $\widetilde{f}$ as

$$\widetilde{f}(x, r, p) := \begin{cases} \max_{1 \leq i \leq \kappa} \overline{f}(x, \alpha_i(x), D\alpha_i(x)), & \text{if } r < \alpha(x) \\ \overline{f}(x, r, p), & \text{if } \alpha \leq r \leq \beta(x) \\ \min_{1 \leq j \leq \iota} \overline{f}(x, \beta_j(x), D\beta_j(x)), & \text{if } r > \beta(x) \end{cases}$$

and consider $R > \max\{\|\alpha_i\|_E, \|\beta_j\|_E; \ 1 \leq i \leq \kappa, 1 \leq j \leq \iota\}$. In claim 6.5, choose $i \in \{1, \ldots, \kappa\}$ such that $\min_{\overline{\Omega}}(u - \alpha) = (u - \alpha_i)(x_0)$, define $v = u - \alpha_i$ and replace α by $\alpha_i \in W^{2,p}(\Omega)$ until the end of the proof, observing that $x \in \widetilde{\Omega} = \{v < 0\}$ implies $u(x) < \alpha(x)$. The rest of the proof is exactly the same.

Denote by $\mathcal{H}$ the set of fixed points of $\mathcal{T}$ belonging to the order interval $[\alpha, \beta]$. In claim 6.5 we saw that this set contains the set of fixed points of $\widetilde{\mathcal{T}}$. The converse is also true, since any solution of (P) in the order interval $[\alpha, \beta]$ satisfies $\overline{f}(x, u, Du) = f(x, u, Du)$ and (6.3), hence is a solution of $(\widetilde{P})$, i.e.

$$\mathcal{H} = \{u \in E; \ u = \mathcal{T}u, \ \alpha \leq u \leq \beta \ \text{in} \ \Omega\} = \{u \in E; \ u = \widetilde{\mathcal{T}}u\}. \qquad (6.7)$$

Notice that we just proved that this set is nonempty.

Part 2. Degree computation in $\mathcal{S}$ under strictness of α, β – proof of (i)

Suppose α, β are strict, and consider the set $\mathcal{S}$ as in the statement of theorem 6.3. Since there exists a solution $u \in C_0^1(\overline{\Omega})$ of (P) with $\alpha \leq u \leq \beta$ in Ω, by definition 6.2 we have $\alpha \ll u \ll \beta$ in Ω and so $\mathcal{S}$ is a nonempty open set in $C_0^1(\overline{\Omega})$. Further, from part 1, we see that all fixed points of $\widetilde{\mathcal{T}}$ are in $\mathcal{S} \subset \mathcal{B}_R \subset \mathcal{B}_{R_0}$, the degree over $\mathcal{B}_{R_0}$ is equal to 1 and solutions of (P) and $(\widetilde{P})$ in $\mathcal{S}$ coincide, leading to

$$\deg(I - \mathcal{T}, \mathcal{S}, 0) = \deg(I - \widetilde{\mathcal{T}}, \mathcal{S}, 0) = \deg(I - \widetilde{\mathcal{T}}, \mathcal{B}_{R_0}, 0) = 1.$$

Part 3. Existence of extremal solutions under (H_2) – proof of (ii).

From the proof of part 1, we know that for any sequence $(u_k)_{k\in\mathbb{N}}$ in E with $u_k \to u$ in E there exists $U \in E$ such that $\widetilde{\mathcal{T}}u_k \to U = \widetilde{\mathcal{T}}u$ in E. Then, if additionally $u_k = \widetilde{\mathcal{T}}u_k$, we obtain that $u = \widetilde{\mathcal{T}}u$ thus the set $\mathcal{H}$ in (6.7) is a (nonempty) compact set in E.

Now consider, for each $u \in \mathcal{H}$, the set $C_u := \{z \in \mathcal{H}\,;\ z \leq u \text{ in } \Omega\}$.

Claim 6.6 *The family $\{C_u\}_{u\in\mathcal{H}}$ has the finite intersection property, i.e. the intersection of any finite number of sets C_u is not empty.*

Proof. Let $u_1, ..., u_\kappa \in \mathcal{H}$. Observe that $\widetilde{\beta} := \min_{1\leq i\leq\kappa} u_i$ is an L^p-viscosity supersolution of (P), with $\alpha \leq \widetilde{\beta} \leq \beta$ in Ω. Furthermore, under hypothesis (H_2), such $\widetilde{\beta}$ is a minimum of strong solutions of (P), which is exactly what we need in order to use part 1 of the above proof, obtaining the existence of a solution v of (P) with $\alpha \leq v \leq \widetilde{\beta} \leq \beta$ in Ω, i.e. $v \in \mathcal{H}$ and $v \leq u_i$, for every $i \in \{1, ..., \kappa\}$, so $v \in \cap_{1\leq i\leq\kappa} C_{u_i} \neq \emptyset$. ∎

By the definition of compacity of $\mathcal{H}$ by open covers and claim 6.6, there exists $\underline{u} \in \cap_{u\in\mathcal{H}} C_u$ (see, for example, theorem 26.9 in [88]). But then there exists a solution $\underline{u}$ of (P) with $\alpha \leq \underline{u} \leq u$ in Ω, for all $u \in \mathcal{H}$. Analogously we prove the existence of $\overline{u}$, with $u \leq \overline{u} \leq \beta$ in Ω, for every $u \in \mathcal{H}$. ∎

Remark 6.7 *The conclusion of the theorem 6.3 is still true if, instead of $c(x)$, we have some $c(x,u)$ such that $c(x,u)u = c(x)T_a u$, where T_a is a truncation of u, i.e. $T_a(u) = u$ for $u \geq a$, $T_a(u) = a$ for $u < a$. In this case, similiar to the "tilde" truncation, $|T_a u_k - T_a u| \leq |u_k - u|$ and the rest of the proof carries on in the same way.*

6.2
Some auxiliary results

We start by constructing an auxiliary problem $(P_{\lambda,k})$, for which we can assure that there are no solutions for large k, and that $(P_{\lambda,0})$ reduces to the problem (P_λ). This is a typical but essential argument (see [66], [15]) that allows us to find a second solution via degree theory, by homotopy invariance in k.

Fix $\Lambda_2 > 0$. Recall that proposition 5.7 gives us an a priori lower uniform bound C_0, depending only on $n, p, \lambda_P, \Lambda_P, \mu_1, \Omega, \Lambda_2, \|b\|_{L^\infty(\Omega)}, \|c\|_{L^\infty(\Omega)}$ and $\|h^-\|_{L^\infty(\Omega)}$, such that

$$u \geq -C_0\,, \quad \text{for all } L^p\text{-visc. supersol. } u \text{ of } (P_\lambda),\ \text{for all } \lambda \in [0, \Lambda_2]. \qquad (6.8)$$

Consider, thus, the problem

$$\begin{cases} -F[u] = \lambda c(x)u + h(x) + \langle M(x)Du, Du\rangle + k\,\widetilde{c}(x) & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases} \quad (P_{\lambda,k})$$

for $k \geq 0$, $\lambda \in [0, \Lambda_2]$, F satisfing $(SC)_0$, (H_β) and (H_1), M satisfying (M), $c \gneqq 0$, $c, h \in L^\infty(\Omega)$ and $\widetilde{c}$ being defined as

$$\widetilde{c}(x) = \widetilde{c}_{\Lambda_2}(x) := Ac(x) + h^-(x) + \Lambda_2\,C_0\,c(x) \in L_+^\infty(\Omega), \qquad (6.9)$$

with $A := \lambda_1/m$, $m = \mu_1/\Lambda_P$, where $\lambda_1 = \lambda_1^+\,(\mathcal{L}^-(c), \Omega) > 0$ is the first eigenvalue with weight c of the proper operator $\mathcal{L}^-$, associated to the positive eigenfunction $\varphi_1 \in W^{2,p}(\Omega)$, given by proposition 4.3, that is,

$$\begin{cases} (\mathcal{L}^- + \lambda_1 c)\,[\varphi_1] = 0 & \text{in} \quad \Omega \\ \varphi_1 > 0 & \text{in} \quad \Omega \\ \varphi_1 = 0 & \text{on} \quad \partial\Omega. \end{cases} \qquad (6.10)$$

Note that every L^p-viscosity solution of $(P_{\lambda,k})$ is also an L^p-viscosity supersolution of (P_λ), since $k\,\widetilde{c} \geq 0$, and so satisfies (6.8). From this and (6.9) we have, for all $k \geq 1$ and for a.e. in Ω,

$$\lambda c(x)u + h(x) + k\,\widetilde{c}(x) \geq -\Lambda_2 C_0\,c(x) - h^-(x) + \widetilde{c}(x) = Ac(x) \gneqq 0. \quad (6.11)$$

Lemma 6.8 *For each fixed $\Lambda_2 > 0$, $(P_{\lambda,k})$ has no solutions for all $k \geq 1$ and $\lambda \in [0, \Lambda_2]$.*

Proof. First observe that every L^p-viscosity solution of $(P_{\lambda,k})$, for $\lambda \in [0, \Lambda_2]$, is positive in Ω. Indeed, from (6.11), $(SC)_0$ and $M \geq 0$, we have that u is an L^p-viscosity solution of

$$\begin{cases} \mathcal{L}^-[u] \lneqq 0 & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

and this implies that $u \geq 0$ in Ω by ABP. Then $u > 0$ in Ω by SMP.

Assume, in order to obtain a contradiction, that $(P_{\lambda,k})$ has a solution u. Again by (M), (6.11) and $(SC)_0$, u is also an L^p-viscosity solution of

$$\begin{cases} \mathcal{L}^-[u] \leq -\mu_1|Du|^2 - Ac(x) & \text{in} \quad \Omega \\ u > 0 & \text{in} \quad \Omega, \end{cases}$$

and from lemma 2.20,

$$\begin{cases} (\mathcal{L}^- + \lambda_1 c)[v] & \leq & -Ac(x) & \text{in} & \Omega \\ v & > & 0 & \text{in} & \Omega, \end{cases} \tag{6.12}$$

using $mA = \lambda_1$, where $v = \frac{1}{m}\{e^{mu} - 1\}$, for m and A given by (6.9). Then (6.12) and (6.10), together with proposition 4.4 (see also remark 4.6), yields $v = t\varphi_1$ for some $t > 0$. But this contradicts the first inequality in (6.12), since $Ac(x) \not\geq 0$ and $(\mathcal{L}^- + \lambda_1 c)[t\varphi_1] = t(\mathcal{L}^- + \lambda_1 c)[\varphi_1] = 0$ in Ω. ∎

When we are assuming hypothesis (H_2) we just say solutions to mean strong solutions of $(\overline{P}_\lambda)$. However, sub and supersolutions of such equations, in general, are not assumed strong (since we are considering the problem in the L^p-viscosity sense), unless specified otherwise. To avoid possible confusion, we always make explicit the notion of sub/supersolution we are referring to.

The next result is important in degree arguments, bearing in mind the set $\mathcal{S}$ in theorem 6.3.

Lemma 6.9 *Suppose* $(\text{SC})_0$, (H_β) *and* (H_2). *Then, for every* $\lambda > 0$, *there exists a strong strict subsolution* α_λ *of* (P_λ) *which is strong minimal, in the sense that every strong supersolution* β *of* (P_λ) *satisfies* $\alpha_\lambda \leq \beta$ *in* Ω.

When u_0 has a sign, we will see in the proofs of theorems 5.4 and 5.5 that u_0 can be taken as β and α, respectively, in theorem 6.3 for the problem (P_λ), for all $\lambda > 0$. In the second case, lemma 6.9 will not be necessary.

Proof. Let K be the positive constant from proposition 5.7 such that every L^p-viscosity supersolution β of

$$\begin{cases} -F[\beta] & \geq & \lambda c(x)\beta + \langle M(x)D\beta, D\beta \rangle - h^-(x) - 1 & \text{in} & \Omega \\ \beta & \geq & 0 & & \text{on} & \partial\Omega \end{cases} \tag{$\tilde{P}_\lambda$}$$

satisfies $\beta \geq -K$ in Ω. Let α_0 be the strong solution of the problem

$$\begin{cases} \mathcal{L}^-[\alpha_0] & = & \lambda K c(x) + h^-(x) + 1 & \text{in} & \Omega \\ \alpha_0 & = & 0 & & \text{on} & \partial\Omega, \end{cases} \tag{6.13}$$

given, for example, by proposition 3.5. Then, as the right hand side of (6.13) is positive, by ABP, SMP and Hopf, we have $\alpha_0 \ll 0$ in Ω.

Claim 6.10 *Every* L^p-*viscosity supersolution* β *of* (P_λ) *satisfies* $\beta \geq \alpha_0$ *in* Ω.

Proof. First notice that β is an L^p-viscosity supersolution of $(\tilde{P}_\lambda)$ and so satisfies $\beta \geq -K$. Second, by $(\text{SC})_0$ and $M \geq 0$, β is also an L^p-viscosity

supersolution of

$$-\mathcal{L}^-[\beta] \geq \lambda c(x)\beta + h(x) \geq -\lambda K c(x) - h^-(x) - 1 \quad \text{in} \ \Omega$$

and setting $v := \beta - \alpha_0$ in Ω, v is an L^p-viscosity solution of

$$\mathcal{L}^-[v] \leq \mathcal{M}^-(D^2\beta) + \mathcal{M}^+(-D^2\alpha_0) - b(x)|D\beta| + b(x)|D\alpha_0| = \mathcal{L}^-[\beta] - \mathcal{L}^-[\alpha_0] \leq 0$$

since α_0 is strong. Further, $v \geq 0$ on $\partial\Omega$ then, by ABP, $v \geq 0$ in Ω. ∎

Set

$$\bar{c}(x,t) = \begin{cases} c(x) & \text{if} \ t \geq -K \\ -K\, c(x)/t & \text{if} \ t < -K. \end{cases}$$

Note that $0 \leq \bar{c}(x,t) \leq c(x)$ a.e. in Ω and $\bar{c}(x,t)t \geq -Kc(x)$ for $t \in \mathbb{R}$. Then,

$$\begin{aligned} -F[\alpha_0] &\leq -\mathcal{L}^-[\alpha_0] = -\lambda K c(x) - h^-(x) - 1 \\ &\leq \lambda \bar{c}(x,\alpha_0)\alpha_0 + \langle M(x)D\alpha_0, D\alpha_0 \rangle - h^-(x) - 1 \end{aligned}$$

since $M \geq 0$ and α_0 is a strong subsolution of $(\overline{P}_\lambda)$.

Consider the problem $(\overline{P}_\lambda)$, which we define as the problem (P_λ) with c, h replaced by $\bar{c} = \bar{c}(x,u)$, $\bar{h} = -h^- - 1$. So, solutions of $(\overline{P}_\lambda)$ are strong, by hypothesis (H_2). Moreover, observe that we are in the situation of remark 6.7, since $\bar{c}(x,u)u = c(x)T_{-K}u$ as in there, which allows us to use theorem 6.3 in order to obtain solutions of $(\overline{P}_\lambda)$.

Let β_0 be some fixed strong supersolution of (P_λ) (if there were not strong supersolutions of (P_λ), the proof is finished). Then, by claim 6.10, we have $\alpha_0 \leq \beta_0$ in Ω. Also, in that proof we observed that $\beta_0 \geq -K$, then $\bar{c}(x,\beta_0) \equiv c(x)$ a.e. $x \in \Omega$, which means that

$$\begin{aligned} -F[\beta_0] &\geq \lambda c(x)\beta_0 + \langle M(x)D\beta_0, D\beta_0 \rangle + h(x) \\ &\geq \lambda \bar{c}(x)\beta_0 + \langle M(x)D\beta_0, D\beta_0 \rangle - h^-(x) - 1 \end{aligned}$$

and so β_0 is a strong supersolution of $(\overline{P}_\lambda)$. By theorem 6.3 and remark 6.7, we obtain an L^p-viscosity solution w of this problem, with $\alpha_0 \leq w \leq \beta_0$ in Ω, which is strong and can be chosen as the minimal solution in the order interval $[\alpha_0, \beta_0]$, by hypothesis (H_2).

Remark 6.11 *Notice that $\bar{c}(x,t)t \geq c(x)t$ a.e. $x \in \Omega$ for all $t \in \mathbb{R}$, so*

$$\begin{aligned}
-F[w] &= \lambda \bar{c}(x,w)w + \langle M(x)Dw, Dw \rangle - h^-(x) - 1 \\
&\geq \lambda c(x)w + \langle M(x)Dw, Dw \rangle - h^-(x) - 1
\end{aligned} \tag{6.14}$$

a.e. in Ω, that is, w is also a strong supersolution of $(\widetilde{P}_\lambda)$.

Claim 6.12 *For every β strong supersolution of (P_λ), $\beta \geq w$ in Ω.*

Proof. Let β be any strong supersolution of (P_λ). As in the argument above for β_0, we have that β is also a strong supersolution of $(\overline{P}_\lambda)$. Suppose that the conclusion is not verified, i.e. that there exists $x_0 \in \Omega$ such that $\beta(x_0) < w(x_0)$ and define

$$\beta_1 := \min\{w, \beta\} \not\equiv w.$$

Then β_1 is the minimum of strong supersolutions, hence itself is an L^p-viscosity supersolution of $(\overline{P}_\lambda)$ and of $(\widetilde{P}_\lambda)$, by remark 6.11. Following the same lines as in claim 6.10, $\beta_1 \geq \alpha_0$ in Ω. Thus, by theorem 6.3 and remark 6.7, there exists an L^p-viscosity solution w_1 of $(\overline{P}_\lambda)$, strong by (H_2), such that $\alpha_0 \leq w_1 \leq \beta_1 \lneqq w \leq \beta_0$ in Ω, which contradicts the minimality of w. ∎

Claim 6.13 *w is a strong strict subsolution of (P_λ).*

Proof. From remark 6.11, $w \geq -K$ and $\bar{c}(x,w) \equiv c(x)$, which implies that w actually satisfies (6.14) with equality, from where

$$-F[w] < \lambda c(x)w + \langle M(x)Dw, Dw \rangle + h(x) \quad \text{a.e. in } \Omega, \tag{6.15}$$

with $w = 0$ on $\partial\Omega$. Thus w is a strong subsolution of (P_λ). What remains to be proved is that w is strict, in order to choose α_λ as w. Therefore, in the sense of definition 6.2, let $u \in E$ be an L^p-viscosity supersolution of (P_λ) with $u \geq w$ in Ω. Then, since w is strong, $U := u - w \geq 0$ in Ω is an L^p-viscosity supersolution of

$$-\mathcal{L}^-[U] \geq \lambda c(x)U + \langle M(x)DU, DU \rangle + \langle M(x)DU, Dw \rangle + \langle M(x)Dw, DU \rangle$$

in Ω, using $(SC)_0$ and $0 \leq M(x) \leq \mu_2$. Hence, for $\widetilde{b} = b + 2\mu_2|Dw| \in L^p_+(\Omega)$, U satisfies

$$\begin{cases}
\mathcal{M}^-(D^2 U) - \widetilde{b}(x)|DU| &< \ 0 \quad \text{in } \ \Omega \\
U &\geq \ 0 \quad \text{in } \ \overline{\Omega}
\end{cases}$$

in the L^p-viscosity sense and by SMP, $U > 0$ in Ω. If there exists $x_0 \in \partial\Omega$ with $U(x_0) = 0$, Hopf lemma implies that $\partial_\nu U(x_0) > 0$. Then, $U \gg 0$ in Ω. ∎ ∎

6.3
Proof of Theorem 5.3

Suppose, at first, (SC), (H_β) and (H_1).

We start proving the first statement in theorem 5.3, about existence of solutions for $\lambda < 0$. Set $\alpha := u_0 - \|u_0\|_\infty$ and $\beta := u_0 + \|u_0\|_\infty$. Thus, α, β is a pair of strong sub and supersolutions of (P_λ), for each $\lambda < 0$, with $\alpha \leq \beta$ in Ω. Indeed, using (SC), $\alpha \leq u_0$ and $0 \leq \lambda c(x)\alpha$, we have

$$\begin{cases} -F[\alpha] \leq -F[u_0] \leq \lambda c(x)\alpha + h(x) + \langle M(x)D\alpha, D\alpha \rangle & \text{in} \quad \Omega \\ \alpha \leq u_0 = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{6.16}$$

and similarly for β, with $\beta \geq u_0$, $0 \geq \lambda c(x)\beta$ and reversed inequalities. Therefore, theorem 6.3 gives us a solution $u_\lambda \in [\alpha, \beta]$, for all $\lambda < 0$.

Observe that, since $\alpha \leq u_\lambda \leq \beta$, we can say that $\|u_\lambda\|_{C^{1,\alpha}(\overline{\Omega})} \leq C$ for all $\lambda \in [0,1]$, by the $C^{1,\alpha}$ estimates (theorem 3.1 and remark 3.3). Thus, take a sequence $\lambda_k \leq 0$ with $\lambda_k \to 0$ as $k \to \infty$. Next, the compact inclusion $C^{1,\alpha}(\overline{\Omega}) \subset E$ gives us some $u \in E$ such that $u_k \to u$ in E, up to a subsequence. Hence we can define, for each $\varphi \in W^{2,p}_{\text{loc}}(\Omega)$,

$$\begin{cases} g_k(x) := F(x, u_k, D\varphi, D^2\varphi) + h(x) + \lambda_k c(x)u_k, \\ g(x) := F(x, u, D\varphi, D^2\varphi) + h(x). \end{cases} \tag{6.17}$$

Therefore

$$\|g_k - g\|_{L^p(\Omega)} \leq |\lambda_k| \|c\|_{L^p(\Omega)} \|u_0\|_\infty + \|d\|_{L^p(\Omega)} \omega(\|u_k - u\|_{L^\infty(\Omega)}) \to 0$$

as $k \to \infty$. By proposition 2.12, we have that u is an L^p-viscosity solution of (P_0). From the uniqueness of the solution at $\lambda = 0$, u needs to be equal to u_0. Finally, since the sequence of λ converging to zero is arbitrary, we obtain $\|u_\lambda - u_0\|_E \to 0$ as $k \to \infty$.

Now we prove the existence of a continuum from u_0.

Fix an $\varepsilon > 0$ and consider another pair of sub and supersolutions given by $\alpha := u_0 - \varepsilon$ and $\beta := u_0 + \varepsilon$. Analogously to (6.16), we see that α, β is a pair of strong sub and supersolutions for (P_0). Notice that they are not a pair for (P_λ) with $\lambda < 0$, since they do not have a sign. However, $\alpha < u_0 < \beta$ in $\overline{\Omega}$, which implies that $\alpha \ll u_0 \ll \beta$ in Ω. Since u_0 is the unique L^p-viscosity solution of the problem (P_0), then α, β are strict in the sense of definition 6.2. Then, theorem 6.3 (i) gives us, for $\mathcal{S} = \mathcal{O} \cap \mathcal{B}_R$ defined there, that

$$\deg(I - \mathcal{T}_0, \mathcal{S}, 0) = 1. \tag{6.18}$$

Using again that u_0 is the unique L^p-viscosity solution of (P_0), we have further that $\text{ind}(I - \mathcal{T}_0, u_0) = 1$. Then, by proposition 4.11 (since $\mathcal{T}_\lambda$ is continuous in λ, uniformly with respect to u in balls of E, by stability), there exists a continuum $\mathcal{C} \subset \Sigma$ such that both components

$$\mathcal{C} \cap ([0, +\infty) \times E) \quad \text{and} \quad \mathcal{C} \cap ((-\infty, 0] \times E)$$

are unbounded in $\mathbb{R}^\pm \times E$. This proves item *1.*

From now on, we suppose $(SC)_0$.

Let us prove point *2.* in theorem 5.3. The continuum $\mathcal{C} \subset \Sigma$ is such that its projection on the λ-axis is either $\mathbb{R}$ (and we obtain (ii) in theorem 5.3) or it is $(-\infty, \bar{\lambda}]$, with $0 < \bar{\lambda} < +\infty$. In the second case, since we know that the component $\mathcal{C}^+$ is unbounded in $R^+ \times E$, its projection on the E axis must be unbounded in E.

Under $(SC)_0$, by theorem 5.6, for any $0 < \Lambda_1 < \Lambda_2$ there is an L^∞ a priori bound for the solutions of (P_λ), for all $\lambda \in [\Lambda_1, \Lambda_2]$. Then, by $C^{1,\alpha}$ global estimate (theorem 3.1), we have also a $C^{1,\alpha}$ a priori bound for these solutions i.e. the projection of $\mathcal{C}^+ \cap ([\Lambda_1, \Lambda_2] \times E)$ on E is bounded. So, $\mathcal{C}^+$ needs to be unbounded in E when we approach $\lambda = 0$ from the right.

Now, by proposition 5.7, there is a lower L^∞ bound for the solutions, for every $\lambda \leq \Lambda_2$. Therefore, $\mathcal{C}^+$ must emanate from plus infinity to the right of $\lambda = 0$, with the positive part of its solutions blowing up to infinity in $C(\overline{\Omega})$. Thus, (i) and (ii) in *2.* are proved.

Now we pass to the multiplicity results in item *3.* of theorem 5.3.

Observe that, up to taking a larger R in (6.18), by $C^{1,\alpha}$ estimates we can suppose that for every u, L^p-viscosity solution of (P_λ) in $[u_0 - \varepsilon, u_0 + \varepsilon]$,

$$\|u\|_{C^{1,\alpha}(\overline{\Omega})} < R, \quad \text{for all } \lambda \in [0, 1]. \tag{6.19}$$

Claim 6.14 *There exists a $\lambda_0 > 0$ such that*

$$\deg(I - \mathcal{T}_\lambda, \mathcal{S}, 0) = 1, \quad \text{for all } \lambda \in (0, \lambda_0).$$

Proof. Let us prove the existence of a $\lambda_0 > 0$ such that, for all $\lambda \in (0, \lambda_0)$, (P_λ) has no solution on $\partial \mathcal{S}$. Suppose not, i.e. that for all $\lambda_0 > 0$, there exists a $\lambda \in (0, \lambda_0)$ such that $\mathcal{T}_\lambda$ has a fixed point on $\partial \mathcal{S}$. Then for every $k \in \mathbb{N}$, there exists $\lambda_k \in \left(0, \frac{1}{k}\right)$ and $u_k \in \partial \mathcal{S}$ a solution of (P_{λ_k}). By (6.19), $u_k \notin \partial B_R$ and so $u_k \in \partial \mathcal{O}$, for all $k \in \mathbb{N}$. Note that

$$\partial \mathcal{O} = \{ u \in C_0^1(\overline{\Omega}) \, ; \, \alpha \leq u \leq \beta \text{ in } \overline{\Omega} \text{ and } u \text{ "touches" } \alpha \text{ or } \beta \}$$

where "touches" in the above, as in [66], has the following meaning.

Definition 6.15 *Let $u, v \in C_0^1(\overline{\Omega})$. We say that u touches v if there exists $x \in \Omega$ with $u(x) = v(x)$ or if there exists $x \in \partial\Omega$ with $\partial_\nu u(x) = \partial_\nu v(x)$. In any case, $u(x) = v(x)$ at a point $x \in \overline{\Omega}$.*

If u_k touches α, there exists a $x \in \overline{\Omega}$ such that $u_k(x) = u_0(x) - \varepsilon$, and since $u_k \geq u_0 - \varepsilon$, then $\max_{\overline{\Omega}}(u_0 - u_k) = \varepsilon$. If on the other side u_k touches β, there exists a $x \in \overline{\Omega}$; $u_k(x) = u_0(x) + \varepsilon$, and since $u_k \leq u_0 + \varepsilon$, then $\max_{\overline{\Omega}}(u_k - u_0) = \varepsilon$. Anyway,

$$\|u_k - u_0\|_\infty = \max_{\overline{\Omega}} |u_k - u_0| = \varepsilon, \quad \text{for all } k \in \mathbb{N}. \tag{6.20}$$

By (6.19) and compact inclusion $C^{1,\alpha}(\overline{\Omega}) \subset E$, there exists $u \in E$ such that $u_k \to u$ in E as $k \to \infty$, up to taking a subsequence. Hence, by stability (proposition 2.12), u is an L^p-viscosity solution of (P_0). Indeed, we define g and g_k as in (6.17), for each $\varphi \in W_{loc}^{2,p}(\Omega)$, from where

$$\|g_k - g\|_{L^p(\Omega)} \leq \lambda_k \|c\|_{L^p(\Omega)} (\|u_0\|_\infty + \varepsilon) \to 0 \text{ as } k \to \infty.$$

By uniqueness in $\lambda = 0$, $u = u_0$, which contradicts (6.20) by passing to limits.

Therefore, the following degree is well defined and by the homotopy invariance and (6.18) we obtain

$$\deg(I - \mathcal{T}_\lambda, \mathcal{S}, 0) = \deg(I - \mathcal{T}_0, \mathcal{S}, 0) = 1, \text{ for all } \lambda \in (0, \lambda_0).$$

∎

Claim 6.16 *(P_λ) has two solutions when $\lambda \in (0, \lambda_0/2]$.*

Proof. The existence of a first solution $u_{\lambda,1}$ with $u_0 - \varepsilon \ll u_{\lambda,1} \ll u_0 + \varepsilon$ is ensured by claim 6.14. Set $\Lambda_2 := \lambda_0/2$. Then, lemma 6.8 implies that $(P_{\lambda,k})$ has no solutions for $k \geq 1$ and $\lambda \in (0, \Lambda_2]$.

Fix a $\lambda \in (0, \Lambda_2]$. With h replaced by $h + k\tilde{c}$ (see (6.9)) we have, by theorem 5.6, an L^∞ a priori bound for solutions of $(P_{\lambda,k})$, for every $k \in [0, 1]$. Precisely, we get an L^∞ a priori bound for solutions of $(P_{\mu,k})$, for all $\mu \in [\lambda, \Lambda_2]$, depending on λ and Λ_2. This provides, by the $C^{1,\alpha}$ estimates (theorem 3.1), an a priori bound for solutions in E, namely

$$\|u\|_E < R_0, \text{ for every } u \text{ } L^p\text{-viscosity solution of } (P_{\lambda,k}), \text{ for all } k \in [0, 1],$$

and for some $R_0 > R$ that depends, in addition to the coefficients of the equation, also on λ and the L^p-norm of $\tilde{c}$. By the homotopy invariance of the

degree in $k \in [0,1]$ and the fact that, for $k = 1$, $(P_{\lambda,k})$ has no solution,

$$\deg(I - \mathcal{T}_\lambda, \mathcal{B}_{R_0}, 0) = \deg(I - \mathcal{T}_{\lambda,0}, \mathcal{B}_{R_0}, 0) = \deg(I - \mathcal{T}_{\lambda,1}, \mathcal{B}_{R_0}, 0) = 0$$

where $\mathcal{T}_{\lambda,k}$ is the operator $\mathcal{T}_\lambda$ in which we replace h by $h + k\widetilde{c}$ (of course $\mathcal{T}_{\lambda,k}$ is still completely continuous). But then, by the excision property of the degree,

$$\deg(I - \mathcal{T}_\lambda, \mathcal{B}_{R_0} \setminus \mathcal{S}, 0) = \deg(I - \mathcal{T}_\lambda, \mathcal{B}_{R_0}, 0) - \deg(I - \mathcal{T}_\lambda, \mathcal{S}, 0) = -1$$

which provides the second solution $u_{\lambda,2} \in \mathcal{B}_{R_0} \setminus \mathcal{S}$ that we were looking for. ∎

Next, by claim 6.16, the quantity

$$\bar{\lambda} := \sup\{\, \mu \,; \, \forall \lambda \in (0,\mu), \ (P_\lambda) \text{ has at least two solutions} \} \in (0,+\infty]$$

it is well defined and greater or equal than $\lambda_0/2$.

Claim 6.17 $u_{\lambda,1} \to u_0$ *in* E *and* $\max_{\overline{\Omega}} u_{\lambda,2} \to +\infty$ *as* $\lambda \to 0^+$.

Proof. Let $(\lambda_k)_{k \in \mathbb{N}} \subset (0,\bar{\lambda})$ be a decreasing sequence with $\lambda_k \to 0$. Say $\lambda_k \leq \lambda_0/2$ for $k \geq k_0$. Since $u_{\lambda_k,1} \in \mathcal{S}$, $u_0 - \varepsilon \leq u_{\lambda_k,1} \leq u_0 + \varepsilon$ in $\overline{\Omega}$, therefore it is bounded in $C^{1,\alpha}(\overline{\Omega})$ by theorem 3.1. Hence, exactly as in (6.17), we show by stability that $u_{\lambda_k,1} \to u$ in E, where u is a solution of (P_0). Then $u = u_0$.

If, in turn, the respective sequence $u_{\lambda_k,2}$ were uniformly bounded from above, it would be unifomly bounded in $C(\overline{\Omega})$ using proposition 5.7, so bounded in $C^{1,\alpha}(\overline{\Omega})$ and the paragraph above would imply that $u_{\lambda_k,2} \to u_0$ in E. Since $u_0 \in \mathcal{S}$ and $\mathcal{S}$ is open in E, then $u_{\lambda_k,2}$ should belong to $\mathcal{B}_r^E(u_0) \subset \mathcal{S}$ for large k, for some $r > 0$. But this contradicts the fact that $u_{\lambda_k,2} \notin \mathcal{S}$. ∎

Claim 6.18 *In case* $\bar{\lambda} < +\infty$, *the problem* $(P_{\bar{\lambda}})$ *has at least one solution.*

Proof. Let $\lambda_k \in (0,\bar{\lambda})$ be such that $\lambda_k \to \bar{\lambda}$ and let u_k be a sequence of solutions for (P_{λ_k}). Say $\lambda_k \in [\bar{\lambda}/2, \bar{\lambda}]$ for $k \geq k_0$. This provides an L^∞ a priori bound for u_k, by theorem 5.6, i.e. $\|u_k\|_\infty \leq C_1$, which implies that $\|u_k\|_{C^{1,\alpha}(\overline{\Omega})} \leq C_2$. Again, by compact inclusion and stability, we obtain $u_k \to u$ in E, where u is a solution of $(P_{\bar{\lambda}})$. Surely, for stability we need to consider, this time, $g(x) := F(x, u, D\varphi, D^2\varphi) + \bar{\lambda}c(x)u + h(x)$ and so $\|g_k - g\|_{L^p(\Omega)} \leq (\bar{\lambda} - \lambda_k)\|c\|_{L^p(\Omega)} C_2 + \bar{\lambda}\|c\|_{L^p(\Omega)}\|u_k - u\|_\infty \to 0$ as $k \to \infty$. ∎

To finish the proof of theorem 5.3, it remains to show the last statements in item *4.* concerning ordering and uniqueness considerations, in which we assume (H_2). Notice that this automatically implies that solutions $u_{\lambda,1}$ and $u_{\lambda,2}$ are strong, as well as every L^p-viscosity solution of (P_λ).

Further, note that (H_2) provided existence of minimal and maximal solutions on the order interval $[\alpha, \beta]$ in theorem 6.3. Such existence of minimal solution made it possible to find a minimal strong subsolution α_λ of (P_λ) in lemma 6.9.

Claim 6.19 $u_{\lambda,1} \ll u_{\lambda,2}$, *for all* $\lambda \in (0, \bar{\lambda})$.

Proof. Fix a $\lambda \in (0, \bar{\lambda})$ and consider the strict strong subsolution $\alpha = \alpha_\lambda$ given by lemma 6.9. Since in particular $\alpha \leq u$ for every (strong) solution of (P_λ), we can choose $u_{\lambda,1}$ as the minimal strong solution such that $u_{\lambda,1} \geq \alpha$ in Ω. This choice implies that

$$u_{\lambda,1} \lneqq u_{\lambda,2} \quad \text{in} \ \ \Omega. \tag{6.21}$$

Indeed, $u_{\lambda,1} \neq u_{\lambda,2}$ and, if there would exist a point $x_0 \in \Omega$ such that $u_{\lambda,1}(x_0) > u_{\lambda,2}(x_0)$, by defining $u_\lambda := \min\{u_{\lambda,1}, u_{\lambda,2}\}$, as the minimum of strong supersolutions greater or equal than α, so $u_\lambda \geq \alpha$ in Ω. Therefore, theorem 6.3 would give us a solution u of (P_λ) such that $\alpha \leq u \leq u_\lambda \lneqq u_{\lambda,1}$, which contradicts the minimality of $u_{\lambda,1}$ and implies (6.21).

To finish the proof, define $v := u_{\lambda,2} - u_{\lambda,1} \gneqq 0$ in Ω by (6.21). Then, since $u_{\lambda,1}$ and $u_{\lambda,2}$ are strong, v satisfies, almost everywhere in Ω,

$$
\begin{aligned}
-\mathcal{L}^-[v] \geq & - F[u_{\lambda,2}] + F[u_{\lambda,1}] \\
= & \lambda c(x) v + \langle M(x) Dv, Dv \rangle + \langle M(x) Dv, Du_{\lambda,1} \rangle + \langle M(x) Du_{\lambda,1}, Dv \rangle \\
\geq & - 2\mu_2 |Du_{\lambda,1}| \, |Dv|.
\end{aligned}
$$

Hence, v is a nonnegative strong solution of $\mathcal{M}^-(D^2 v) - \widetilde{b}(x)|Dv| \leq 0$ in Ω, for $\widetilde{b} = b + 2\mu_2 |Du_{\lambda,1}| \in L^p_+(\Omega)$. Then SMP gives us that $v > 0$ in Ω, since $v \not\equiv 0$. Now, Hopf lemma and $v = 0$ on $\partial\Omega$ imply $\partial_\nu v|_{\partial\Omega} > 0$, so $v \gg 0$ in Ω. ∎

As far as uniqueness is concerned, from theorem 1(iii) in [12], if the coercive problem for $\lambda \leq 0$ has a strong solution u_λ, it is the unique L^p-viscosity solution of (P_λ). So, under (H_2), u_λ is strong, then unique, in the L^p-viscosity sense, for all $\lambda < 0$. Observe that, in this case, we must have $\{(\lambda, u_\lambda), \ \lambda \leq 0\} \subset \mathcal{C}$. In other words, the projection of $\mathcal{C}$ on the λ-axis contains $(-\infty, 0]$, as in theorems 1.1 and 1.2 in [13] for the Laplacian.

We finish the proof of *4.* with the following claim.

Claim 6.20 *If* $\bar{\lambda} < +\infty$ *and F is convex in* (r, p, X), *the solution* $u_{\bar{\lambda}}$ *of* $(P_{\bar{\lambda}})$, *obtained in claim 6.18, is unique.*

Proof. Suppose, in order to obtain a contradiction, that there exist two different solutions u_1 and u_2 of $(P_{\bar{\lambda}})$, both strong by (H_2). Consider $\beta = \beta_{\bar{\lambda}}$ defined as $\frac{1}{2}(u_1 + u_2)$. Then, a.e. in Ω,

$$
\begin{aligned}
-F[\beta] &\geq -\{F[u_1] + F[u_2]\}/2 \\
&= \bar{\lambda}c(x)\beta + h(x) + \{\langle M(x)Du_1, Du_1\rangle + \langle M(x)Du_2, Du_2\rangle\}/2 \\
&\gneqq \bar{\lambda}c(x)\beta + h(x) + \langle M(x)D\beta, D\beta\rangle
\end{aligned}
$$

using also the convexity of $p \mapsto \langle M(x)p, p\rangle$. Hence β is a strong supersolution of $(P_{\bar{\lambda}})$ which is not a solution. Let us see that it is strict. Set $U := \beta - u$, where $u \in E$ is an L^p-viscosity subsolution of $(P_{\bar{\lambda}})$ with $u \leq \beta$ in Ω. Thus, U is an L^p-viscosity solution of

$$
\begin{aligned}
-\mathcal{L}^-[U] &\gneqq \bar{\lambda}c(x)U - \langle M(x)DU, DU\rangle + \langle M(x)DU, D\beta\rangle + \langle M(x)D\beta, DU\rangle \\
&\geq -\mu_2|DU|^2 - 2\mu_2|D\beta|\,|DU|,
\end{aligned}
$$

and so, by lemma 2.20, the function $w := \frac{1}{m}(1 - e^{-mU})$, where $m = \mu_2/\lambda_P$, is a nonnegative L^p-viscosity solution of $\mathcal{M}^-(D^2w) - \widetilde{b}(x)|Dw| \lneqq 0$ in Ω, with $\widetilde{b} = b + 2\mu_2|D\beta| \in L^p_+(\Omega)$. Then SMP gives us $w > 0$ in Ω. Now, Hopf and $w \geq 0$ on $\partial\Omega$ imply that $\partial_\nu v|_{\partial\Omega} > 0$ in the boundary points where $w = 0$, and so $w \gg 0$ on Ω. Consequently, $U \gg 0$ in Ω and therefore β is a strict strong supersolution of $(P_{\bar{\lambda}})$.

Consider also $\alpha = \alpha_{\bar{\lambda}}$ the strict strong subsolution of $(P_{\bar{\lambda}})$ given by proposition 6.9 and look at the set $\bar{\mathcal{O}} = \{\alpha \ll u \ll \beta\} = \{\alpha_{\bar{\lambda}} \ll u \ll \beta_{\bar{\lambda}}\}$. Again, by the $C^{1,\alpha}$ estimates in chapter 3,

$$
\|u\|_{C^{1,\alpha}(\bar{\Omega})} \leq C \quad \text{for all } u \in [\alpha, \beta] \ L^p\text{-visc. sol. of } (P_\lambda), \ \lambda \in [\bar{\lambda}, \bar{\lambda} + 1] \quad (6.22)
$$

for some $C > 0$ that depends on the L^∞-norm of α. Then, by theorem 6.3, we obtain $R > C$ such that $\deg(I - \mathcal{T}_{\bar{\lambda}}, \bar{\mathcal{S}}, 0) = 1$, where $\bar{\mathcal{S}} = \bar{\mathcal{O}} \cap \mathcal{B}_R$.

We claim that there exists $\varepsilon > 0$ such that

$$
\deg(I - \mathcal{T}_\lambda, \bar{\mathcal{S}}, 0) = 1, \quad \text{for all } \lambda \in [\bar{\lambda}, \bar{\lambda} + \varepsilon]. \quad (6.23)
$$

As in the proof of claim 6.14, we will verify that there exists some $\varepsilon \in (0, 1)$ such that there is no fixed points of $\mathcal{T}_\lambda$ on the boundary of $\bar{\mathcal{S}}$, for all λ in the preceding interval. Indeed, if this were not the case, there would exist a sequence $\lambda_k \to \bar{\lambda}$ with the respective solutions u_k of (P_{λ_k}) belonging to $\partial\bar{\mathcal{S}}$. Say $\lambda_k \in [\bar{\lambda}, \bar{\lambda} + 1]$ for $k \geq k_0$. Then, since $\alpha \leq u_k \leq \beta$ in Ω, by (6.22) we must

have $u_k \in \partial\bar{\mathcal{O}}$ for $k \geq k_0$, which means that for each such k,

$$\max_{\overline{\Omega}} (\alpha - u_k) = 0 \quad \text{or} \quad \max_{\overline{\Omega}} (u_k - \beta) = 0. \tag{6.24}$$

By (6.22) and the compact inclusion $C^{1,\alpha}(\overline{\Omega}) \subset E$, $u_k \to u$ in E for some $u \in E$, up to a subsequence. This u is an L^p-viscosity solution of $(P_{\bar{\lambda}})$ by stability (proposition 2.12); and $\alpha \leq u \leq \beta$ in Ω by taking the limit as $k \to +\infty$ in the corresponding inequalities for u_k. Thus $\alpha \ll u \ll \beta$ in Ω, since α and β are strict. Passing to limits in (6.24), we obtain that u touches α or β, which contradicts the definition of $\alpha \ll u \ll \beta$.

Hence, obtaining (6.23) it is just a question of applying homotopy invariance in λ in the interval $[\bar{\lambda}, \bar{\lambda} + \varepsilon]$. Next, with (6.23) at hand, we repeat exactly the same argument done in claim 6.16 to obtain the existence of a second solution $u_{\lambda,2}$ of (P_λ), for all $\lambda \in [\bar{\lambda}, \bar{\lambda} + \varepsilon]$. But this, finally, contradicts the definition of $\bar{\lambda}$. ∎

6.4
Proof of Theorem 5.4

Suppose $u_0 \leq 0$ with $cu_0 \lneqq 0$ in Ω and (H_2).

Claim 6.21 u_0 *is a strict strong supersolution of* (P_λ), *for all* $\lambda > 0$.

Proof. Since $\lambda c(x)u_0 \lneqq 0$ in Ω, u_0 is a strong supersolution of (P_λ) which is not a solution. To see that it is strict, we take $u \in E$ an L^p-viscosity subsolution of (P_λ) such that $u \leq u_0$ in Ω, and set $U := u_0 - u$. Then, since u_0 is strong, U is an L^p-viscosity supersolution of

$$-\mathcal{L}^-[U] \geq \lambda c(x)U - \langle M(x)DU, DU \rangle + \langle M(x)Du_0, DU \rangle + \langle M(x)Du_0, DU \rangle$$
$$\geq -\mu_2 |DU|^2 - 2\mu_2 |Du_0| |DU|.$$

Moreover, $\mathcal{M}^-(D^2w) - \tilde{b}(x)|Dw| \leq 0$ in Ω in the L^p-viscosity sense, where $\tilde{b} = b + 2\mu_2 |Du_0| \in L^p_+(\Omega)$ and $w = \frac{1}{m}\{1 - e^{-mU}\}$, $m = \mu_2/\lambda_P$, by lemma 2.20. Using SMP and the fact that u_0 is not a solution of (P_λ), we have $w > 0$ in Ω. Since $w \geq 0$ on $\partial\Omega$, at points belonging to $\partial\Omega$ such that $w > 0$ we are done. If in turn $x_0 \in \partial\Omega$ is such that $w(x_0) = 0$, then $\partial_\nu U(x_0) > 0$ by Hopf. Thus $w \gg 0$ and so $U \gg 0$ in Ω. ∎

We now prove that for all $\lambda > 0$, (P_λ) has at least two solutions, $u_{\lambda,1}$ and $u_{\lambda,2}$, with $u_{\lambda,1} \ll u_0$ and $u_{\lambda,1} \ll u_{\lambda,2}$.

Fix a $\lambda > 0$. From lemma 6.9 and claim 6.21, we get a pair of strong strict sub and supersolutions, $\alpha = \alpha_\lambda$ and u_0, which implies, by theorem 6.3, the existence of a first solution $u_{\lambda,1} \in \mathcal{S}$, where

$$\mathcal{S} = \mathcal{S}_\lambda = \{u \in C_0^1(\overline{\Omega}); \ \alpha \ll u \ll u_0 \ \text{ in } \Omega\} \cap \mathcal{B}_R \ \text{ for some } R > 0.$$

Remark 6.22 *We already know, from theorem 5.3, that at least two solutions exist. Here we redefine, as in [15], the set $\mathcal{S}$ in order to obtain more precise informations about Σ. Notice that, with this new definition, we automatically have $u_{\lambda,1} \ll u_0 \leq 0$.*

Fix a $\lambda > 0$ and set $\Lambda_2 := 2\lambda$. As in the proof of claim 6.16, we observe that, by lemma 6.8, $(P_{\lambda,k})$ has no solutions for $k \geq 1$. Moreover, for h replaced by $h + k\widetilde{c}$, theorem 5.6 gives us an L^∞ a priori bound for solutions of $(P_{\lambda,k})$ for every $k \in [0,1]$, which depends on λ. This provides, by the $C^{1,\alpha}$ global estimates (theorem 3.1), an a priori bound for solutions in E, i.e. $\|u\|_E < R_0$ for every u solution of $(P_{\lambda,k})$, for all $k \in [0,1]$, where $R_0 > R$ also depends on λ. By the homotopy invariance of the degree,

$$\deg(I - \mathcal{T}_\lambda, \mathcal{B}_{R_0}, 0) = \deg(I - \mathcal{T}_{\lambda,0}, \mathcal{B}_{R_0}, 0) = \deg(I - \mathcal{T}_{\lambda,1}, \mathcal{B}_{R_0}, 0) = 0.$$

Therefore, by the excision property of the degree

$$\deg(I - \mathcal{T}_\lambda, \mathcal{B}_{R_0} \setminus \mathcal{S}, 0) = \deg(I - \mathcal{T}_\lambda, \mathcal{B}_{R_0}, 0) - \deg(I - \mathcal{T}_\lambda, \mathcal{S}, 0) = -1$$

and the existence of a second solution $u_{\lambda,2} \in \mathcal{B}_{R_0} \setminus \mathcal{S}$ is derived.

Since the argument above can be done for any $\lambda > 0$, we obtain the existence of at least two solutions for every positive λ. Exactly the same reasoning from claims 6.19 and 6.17 applies to check that $u_{\lambda,1} \ll u_{\lambda,2}$ in Ω and to get their behavior when $\lambda \to 0^+$, respectively, since $\bar{\lambda}$ is the same from theorem 5.3. Of course, here $\bar{\lambda} = +\infty$.

Claim 6.23 *For $\lambda_1 < \lambda_2$, we have $u_{\lambda_2,1} \ll u_{\lambda_1,1}$ in Ω.*

Proof. For fixed $\lambda_1 < \lambda_2$ note that $\lambda_1 c(x) u_{\lambda_1,1} \gneqq \lambda_2 c(x) u_{\lambda_1,1}$ since $u_{\lambda,1} < 0$. Then, $u_{\lambda_1,1}$ is a strong supersolution of (P_{λ_2}) which is not a solution and, in particular, $u_{\lambda_1,1} \neq u_{\lambda_2,1}$.

We first infer that $u_{\lambda_2,1} \lneqq u_{\lambda_1,1}$ in Ω. In fact, similarly to the argument in the proof of claim 6.19, recall that $\alpha = \alpha_{\lambda_2}$, given by lemma 6.9, is such that $\alpha \leq u$ for every strong supersolution of (P_{λ_2}), and in particular $\alpha \leq u_{\lambda_1,1}$. Remember also that $u_{\lambda_2,1}$ is the minimal strong solution such that $u_{\lambda_2,1} \geq \alpha$ in Ω. Now, if there was a point $x_0 \in \Omega$ such that $u_{\lambda_2,1}(x_0) > u_{\lambda_1,1}(x_0)$, by defining $\beta := \min\{u_{\lambda_1,1}, u_{\lambda_2,1}\}$, as the minimum of strong supersolutions of (P_{λ_2}) not less than α, we have $\alpha \leq \beta$ in Ω. Thus, theorem 6.3 provides a solution u of (P_{λ_2}) such that $\alpha \leq u \leq \beta \lneqq u_{\lambda_2,1}$ in Ω, contradicting the minimality of $u_{\lambda_2,1}$.

Proceeding as usual, the function $v := u_{\lambda_1,1} - u_{\lambda_2,1}$ is a nonnegative strong supersolution of $\mathcal{M}^-(D^2v) - \widetilde{b}(x)|Dv| \leq 0$ in Ω, then SMP gives us that $v > 0$ in Ω, since $v \not\equiv 0$. Next, Hopf and $v = 0$ on $\partial\Omega$ imply that $\partial_\nu v|_{\partial\Omega} > 0$, hence $v \gg 0$ in Ω. $\blacksquare$

Remark 6.24 *Notice that $u \ll u_0$ in Ω, for every nonpositive L^p-viscosity subsolution u of (P_λ) in E. Indeed, since $\lambda c(x)u \leq 0$ in Ω, u is also an L^p-viscosity subsolution of (P_0). By remark 5.9 we have $u \leq u_0$, since u_0 is strong. Now, by claim 6.21 and definition of strict supersolution, we get $u \ll u_0$ in Ω. In particular, $u \equiv 0$ is never a solution of (P_λ), for any $\lambda > 0$ (recall that $cu_0 \not\equiv 0$).*

Claim 6.25 *In addition to the hypotheses of theorem 5.4, suppose that F is convex in (r, p, X). Then, (P_λ) has at most one nonpositive solution. In particular, $\max_{\overline{\Omega}} u_{\lambda,2} > 0$.*

Proof. Suppose, in order to obtain a contradiction, that there exist two different nonpositive solutions u_1 and u_2, strong by (H_2). By remark 6.24 we know that $u_1 \ll u_0$ and $u_2 \ll u_0$ in Ω. We can assume that they are ordered, in the sense that $u_1 \lneqq u_2$. Indeed, observe that $\max\{u_1, u_2\} \leq u_0$, then theorem 6.3 yields a solution u_3 of (P_λ) with $u_2 \leq \max\{u_1, u_2\} \leq u_3 \leq u_0 \leq 0$ in Ω. Thus, if the solutions u_1 and u_2 do not satisfy $u_1 \lneqq u_2$, then there is a point $x_0 \in \Omega$ with $u_1(x_0) > u_2(x_0)$, which implies that $u_2 \not\equiv \max\{u_1, u_2\}$ and so $u_2 \lneqq u_3$; in this case we just replace u_1, u_2 by u_2, u_3 respectively.

Since $u_2 \ll 0$ (from $u_2 \ll u_0$ and $\partial_\nu u_0 \leq 0$), the quantity

$$\tau := \inf\{t > 0;\ (1+t)u_2 \leq u_1 \text{ in } \Omega\}$$

is well defined and finite. Further, $\tau > 0$, since $u_2 - u_1 \gneqq 0$, so this infimum is attained. Then, by setting $w := \frac{1}{\tau}\{(1+\tau)u_2 - u_1\}$, we have that $w \leq 0$ satisfies $u_2 = \frac{\tau}{1+\tau}w + \frac{1}{1+\tau}u_1$ and it is a strong subsolution of

$$
\begin{aligned}
F[w] &\geq \frac{1+\tau}{\tau}F[u_2] - \frac{1}{\tau}F[u_1] = \frac{1+\tau}{\tau}\{\lambda c(x)u_2 + \langle M(x)Du_2, Du_2\rangle + h(x)\} \\
&\quad - \frac{1}{\tau}\{\lambda c(x)u_1 + \langle M(x)Du_1, Du_1\rangle + h(x)\} \\
&\geq \lambda c(x)w + \langle M(x)Dw, Dw\rangle + h(x) \quad \text{in } \Omega,
\end{aligned}
$$

since F is convex in (r, p, X) and $p \mapsto \langle M(x)p, p\rangle$ is convex in p. Now, by remark 6.24 we have $w \ll u_0$ in Ω, i.e. $w = w_\tau < 0$ in Ω. Then, there exists a little bit smaller $t \in (0, \tau)$ such that $w_t < 0$ in Ω (see the argument in proposition 4.4, by taking a compact set with small measure containing the boundary). Therefore, this last contradicts the definition of τ as a minimum. $\blacksquare$

6.5
Proof of Theorem 5.5

Suppose, for the time being, just $u_0 \geq 0$ with $cu_0 \gneq 0$ in Ω and $(SC)_0$.

Claim 6.26 *We have $u \gg u_0$, for every nonnegative L^p-viscosity supersolution $u \in E$ of (P_λ), for all $\lambda > 0$.*

Proof. Notice that $\lambda c(x)u \geq 0$ in Ω implies that u is an L^p-viscosity supersolution of (P_0). Since u_0 is strong, by remark 5.9, $u \geq u_0$ in Ω. But u_0 is not a solution of (P_λ) for $\lambda > 0$ since $cu_0 \not\equiv 0$, which means that $u_0 \not\equiv u$.

Set $v := u - u_0$ in Ω. Then, using $M \geq 0$, we see that v is a nonnegative L^p-viscosity supersolution of

$$\mathcal{M}^-(D^2v) - \widetilde{b}(x)|Dv| \leq 0 \quad \text{in } \Omega$$

with $\widetilde{b} := b + 2\mu_2\,|Du_0|$, as usual. By SMP, $v > 0$ in Ω. If $v > 0$ on $\partial\Omega$ it is done; if on the other side there exists $x_0 \in \partial\Omega$ with $v(x_0) = 0$, we apply Hopf lemma to obtain $\partial_\nu v(x_0) > 0$. Therefore, $v \gg 0$ in Ω. ∎

Claim 6.27 *(P_λ) has no nonnegative L^p-viscosity solutions for λ large.*

Proof. Let $\lambda \geq \widetilde{\lambda}_1$, where $\widetilde{\lambda}_1 = \widetilde{\lambda}_1^+\,(\widetilde{\mathcal{L}}^-(c),\Omega) > 0$ is the principal weighted eigenvalue of

$$\widetilde{\mathcal{L}}^-[v] := \mathcal{M}^-(D^2v) + \widetilde{b}(x)|Dv|\,, \quad \widetilde{b}(x) := b(x) + 2\mu_2\,|Du_0| \in L_+^p(\Omega),$$

associated to $\widetilde{\varphi}_1 = \widetilde{\varphi}_1^+\,(\widetilde{\mathcal{L}}^-(c),\Omega) \in W^{2,p}(\Omega)$, from proposition 4.3, i.e.

$$(6.25)\qquad \begin{cases} (\widetilde{\mathcal{L}}^- + \widetilde{\lambda}_1 c)[\widetilde{\varphi}_1] &=& 0 &\text{in} & \Omega \\ \widetilde{\varphi}_1 &>& 0 &\text{in} & \Omega \\ \widetilde{\varphi}_1 &=& 0 &\text{on} & \partial\Omega \end{cases}$$

Suppose, in order to obtain a contradiction, that there exists a nonnegative L^p-viscosity solution u of (P_λ) and set $v := u - u_0$ in Ω. By claim 6.26, it follows that $v \gg 0$ in Ω.

Since u_0 is strong, we can use it as a test function into the definition of L^p-viscosity supersolution of u, together with $(SC)_0$ and $\mu_2 I \geq M \geq 0$, to obtain that

$$-\mathcal{L}^-[v] \geq \lambda c(x)v + \lambda c(x)u_0 + \langle M(x)Dv, Dv \rangle + \langle M(x)Dv, Du_0 \rangle$$
$$+ \langle M(x)Du_0, Dv \rangle \gneq \widetilde{\lambda}_1 c(x)v - 2\mu_2|Du_0|\,|Dv|$$

since $c(x)u_0 \gneqq 0$, and so v satisfies

$$\begin{cases} (\widetilde{\mathcal{L}}^- + \widetilde{\lambda}_1 c)\,[v] & \lneqq & 0 & \text{in} & \Omega \\ v & > & 0 & \text{in} & \Omega \end{cases} \tag{6.26}$$

in the L^p-viscosity sense. As the proof of lemma 6.8, applying proposition 4.4 (and remark 4.6) to (6.26) and (6.25), we get $v = t\widetilde{\varphi}_1$ for $t > 0$. But this contradicts the first line in (6.26), since $(\widetilde{\mathcal{L}}^- + \widetilde{\lambda}_1 c)\,[t\widetilde{\varphi}_1] = 0$ in Ω. ■

Define

$$\bar{\lambda} := \sup\{\lambda\,;\ (P_\lambda) \text{ has an } L^p\text{-viscosity solution } u_\lambda \geq 0 \text{ in } \Omega\}$$

which is finite, by claim 6.27. Of course it is well defined and nonnegative, since $u_0 \geq 0$. Also, by the definition of $\bar{\lambda}$, (P_λ) has no nonnegative solutions for $\lambda > \bar{\lambda}$.

It is a subtle but important detail that $\bar{\lambda}$ is a positive number. In fact, by the existence of the continuum (theorem 5.3) we know that, for λ small, there exists a solution u_λ of (P_λ) such that $u_\lambda \not\equiv 0$, since $\|u_0\|_E > 0$. But why can we infer that $u_\lambda \geq 0$ for small λ positive? This is the subject of the next claim.

Consider (H_2) from now. Then, L^p-viscosity solutions of (P_λ) are strong.

Claim 6.28 $\bar{\lambda} > 0$.

Proof. Let $\lambda \in (0, \Lambda_0)$, where Λ_0 is such that there exists a nontrivial solution of (P_λ) in this interval, as indicated above. Suppose $\Lambda_0 \leq \min\{1, 1/C_0\}$, where C_0 is a lower bound for the solutions of (P_λ) such that $u \geq -C_0$, for all $\lambda \in [0, 1]$. We are supposing here $C_0 > 0$, otherwise every solution of (P_λ) would be nonnegative for $\lambda \leq 1$.

Suppose firstly that $h \geq c$. In this case every nontrivial solution u of (P_λ) satisfies

$$\begin{cases} -\mathcal{L}^-[u] & \geq & -F[u] \geq c(x)(1 - \Lambda_0\,C_0) + \langle M(x)Du, Du \rangle \geq 0 & \text{in} & \Omega \\ u & = & 0 & & \text{on} & \partial\Omega. \end{cases}$$

Then $u \geq 0$ in Ω by ABP, for all $\lambda \in (0, \Lambda_0)$.

In the general case, let u_c a nontrivial nonnegative solution of the problem

$$\begin{cases} -F[u_c] & = & \widetilde{h}(x) + \lambda c(x)u_c + \langle M(x)Du_c, Du_c \rangle \geq 0 & \text{in} & \Omega \\ u_c & = & 0 & & \text{on} & \partial\Omega \end{cases}$$

for $\widetilde{h} := \max\{h, c\} \geq c$. Notice that u_c is strong under (H_2), since $|\widetilde{h}| \leq |h| + c$. Moreover, since $\widetilde{h} \geq h$, then u_c is a supersolution of (P_λ). Therefore $u_c \geq u_0$

by claim 6.26. Further, since $\lambda c(x)u_0 \geq 0$, u_0 is a strong subsolution of (P_λ). Thus, applying theorem 6.3, we obtain an L^p-viscosity solution u of (P_λ) with $u_0 \leq u \leq u_c$ in Ω. In particular this solution is nonnegative and nontrivial. ∎

Remark 6.29 *Another way to prove claim 6.5 is through arguments on first eigenvalues, which provide an estimate on the smallness of λ.*

Set $v := u_0 - u$, with u an L^p-viscosity solution of (P_λ) for positive λ such that $\lambda < (C_A \|c\|_{L^p(\Omega)})^{-1}$, where C_A is the constant from ABP for $\mu = 0$. Of course, negativity of v yields $u \geq u_0 \geq 0$ in Ω. So, in order to obtain a contradiction, suppose that $\sup_\Omega v > 0$.

Notice that v is an L^p-viscosity solution of $\widetilde{\mathcal{L}}^+[v] \geq -\lambda c(x)v^+$ in Ω, with $v = 0$ on $\partial\Omega$. Thus, as in the proof of proposition 3.4 in [12], we use ABP to obtain that $\sup_\Omega v \leq \lambda\, C_A\|c\|_{L^p(\Omega)} \sup_\Omega v^+$ which, by the choice of λ, yields a contradiction.

Claim 6.30 *For each $\lambda \in (0, \bar\lambda)$, (P_λ) has a well ordered strict pair of strong sub and supersolutions, namely $u_0 \ll \beta_\lambda$ in Ω.*

Proof. Let $\lambda \in (0, \bar\lambda)$. As the strict subsolution we just consider u_0 again, which is strong. Note that u_0 is strict, since for any supersolution $u \in E$ such that $u \geq u_0$ in Ω, we have $u \gg u_0$, by repeating the final paragraph in the proof of claim 6.26.

Note that, from the definition of $\bar\lambda$, there exists $\mu \in (\lambda, \bar\lambda)$ and a nonnegative solution u_μ of (P_μ). By claim 6.26, $u_\mu \gg u_0$ in Ω. On the other hand, since

$$c(x)(\mu - \lambda)\, u_\mu \geq c(x)(\mu - \lambda)\, u_0 \gneqq 0,$$

we have $\mu c(x)u_\mu \gneqq \lambda c(x)u_\mu$ and so u_μ is a supersolution of (P_λ) which is not a solution. In addition, u_μ is strict because if $u \in E$ is an L^p-viscosity subsolution of (P_λ) with $u \leq u_\mu$ in Ω, by defining $v = u_\mu - u$ and arguing as usual when we have a strong supersolution, v becomes an L^p-viscosity supersolution of

$$\mathcal{M}^-(D^2v) - \widetilde{b}(x)|Dv| - \mu_2\,|Dv|^2 \lneqq 0 \quad \text{in } \Omega$$

with $\widetilde{b} := b + 2\mu_2\,|Du_\mu| \in L^p_+(\Omega)$. Thus, $w = \frac{1}{m}\{1 - e^{-mv}\}$, for $m = \mu_2/\lambda_P$, is an L^p-viscosity supersolution of $\mathcal{M}^-(D^2w) - \widetilde{b}(x)|Dw| \lneqq 0$ in Ω by lemma 2.20. Then, SMP gives us $w > 0$ in Ω and so $w \gg 0$ in Ω, by applying Hopf at the boundary bounds where $w = 0$. Consequently, $v \gg 0$ in Ω.

Therefore, we can define $\beta_\lambda := u_\mu$ for $\mu = \mu(\lambda)$, for all $\lambda \in (0, \bar\lambda)$. ∎

Hence, by claim 6.30 and theorem 6.3, there exists a solution $u_{\lambda,1}$ of (P_λ) with $u_0 \leq u_{\lambda,1} \leq \beta_\lambda$ in Ω and $\deg(I - \mathcal{T}_\lambda, \mathcal{S}, 0) = 1$, for all $\lambda > 0$.

Next, we work a little bit more to construct the second solution $u_{\lambda,2}$ that also satisfies $u_{\lambda,2} \gg u_0$ but is not in $\mathcal{S}$ (as in [15]). For this, fix a $\lambda \in (0, \bar{\lambda})$ and consider the open subset of E defined by

$$\mathcal{D} = \{u \in C_0^1(\overline{\Omega}); \ u_0 \ll u \text{ in } \Omega\},$$

which contains the set $\mathcal{S}$ from theorem 6.3, since

$$\mathcal{S} = \{u \in \mathcal{B}_R \, ; \ u \ll \beta_\lambda \text{ in } \Omega\} \cap \mathcal{D}.$$

Analogously to the proof of claim 6.16, we obtain an a priori L^∞ bound for the solutions of $(P_{\lambda,k})$ which depends on λ but not on $k \in [0,1]$, related to $\Lambda_2 := \bar{\lambda} - \delta$ for some small $\delta > 0$. This provides a $R_0 = R_0(\lambda, \delta) > R$ which bounds the E-norm of the solutions, by the $C^{1,\alpha}$ estimates. Then, by the homotopy invariance of the degree in k and the fact that there is no solution for $k = 1$, we have $\deg(I - \mathcal{T}_\lambda, \mathcal{B}_{R_0} \cap \mathcal{D}, 0) = 0$. Therefore, by excision, $\deg(I - \mathcal{T}_\lambda, (\mathcal{B}_{R_0} \cap \mathcal{D}) \backslash \mathcal{S}, 0) = -1$, which provides a second solution $u_{\lambda,2} \in \mathcal{D} \backslash \mathcal{S}$, i.e. a solution that satisfies, by construction, $u_{\lambda,2} \gg u_0$ in Ω, for $\lambda \in (0, \bar{\lambda} - \delta]$, for every $\delta > 0$. In particular, this second solution is also nonnegative and nontrivial for all $\lambda < \bar{\lambda}$.

Under (H_2), theorem 6.3 (ii) allows us to choose $u_{\lambda,1}$ as the minimal strong solution between u_0 and β_λ. As the proof of claim 6.19, this implies (6.21). Indeed, $u_{\lambda,1} \neq u_{\lambda,2}$ and, if would exist $x_0 \in \Omega$ with $u_{\lambda,1}(x_0) > u_{\lambda,2}(x_0)$, by defining $\tilde{\beta} = \tilde{\beta}_\lambda := \min\{u_{\lambda,1}, u_{\lambda,2}, \beta_\lambda\}$, as the minimum of strong supersolutions greater or equal than u_0, we have $\tilde{\beta} \geq u_0$ in Ω. Also, $\tilde{\beta} \leq \beta_\lambda$ in Ω. By theorem 6.3 there exists a solution u of (P_λ) such that $u_0 \leq u \leq \tilde{\beta} \lneqq u_{\lambda,1}$, which contradicts the minimality of $u_{\lambda,1}$, since u is a solution which belongs to the order interval $[u_0, \beta_\lambda]$.

Therefore, defining $v = u_{\lambda,2} - u_{\lambda,1} \gneqq 0$ in Ω, we see that v is a nonnegative strong solution of $\mathcal{M}^-(D^2 v) - \tilde{b}(x)|Dv| \leq 0$ in Ω. Thus, since $v \not\equiv 0$, SMP yields $v > 0$ in Ω and Hopf concludes that $v \gg 0$ in Ω, i.e.

$$u_{\lambda,1} \ll u_{\lambda,2} \text{ in } \Omega, \text{ for all } \lambda \in (0, \bar{\lambda}).$$

Claim 6.31 *For $\lambda_1 < \lambda_2$, we have $u_{\lambda_1,1} \ll u_{\lambda_2,1}$ in Ω.*

Proof. The proof is similar to the proof of claim 6.30, but a little bit simpler since both $u_{\lambda_1,1}$ and $u_{\lambda_2,2}$ are strong. We repeat it here to avoid confusions about notation. For fixed $\lambda_1 < \lambda_2$, we have $\lambda_2 \, c(x) \, u_{\lambda_2,1} \gneqq \lambda_1 \, c(x) \, u_{\lambda_2,1}$ since

$$(\lambda_2 - \lambda_1)c(x)u_{\lambda_2,1} \geq (\lambda_2 - \lambda_1)c(x)u_0 \gneqq 0.$$

Then, $u_{\lambda_2,1}$ is a supersolution of (P_{λ_1}) that is not a solution. In particular it follows that $u_{\lambda_1,1} \neq u_{\lambda_2,1}$.

Next $u_{\lambda_1,1} \lneqq u_{\lambda_2,1}$ in Ω. In fact, if there was a point $x_0 \in \Omega$ such that $u_{\lambda_1,1}(x_0) > u_{\lambda_2,1}(x_0)$, by defining $\tilde{\beta} = \min\{u_{\lambda_1,1}, u_{\lambda_2,1}, \beta_{\lambda_1}\}$, the minimum of strong supersolutions of (P_{λ_1}) larger than u_0, we would have that $u_0 \leq \tilde{\beta}$ in Ω. By theorem 6.3, there exists a solution u of (P_{λ_2}) such that $u_0 \leq u \leq \tilde{\beta} \lneqq u_{\lambda_1,1}$, which contradicts the minimality of $u_{\lambda_1,1}$, since u is a solution that belongs to the order interval $[u_0, \beta_{\lambda_1}]$.

Hence the function $v := u_{\lambda_2,1} - u_{\lambda_1,1} \gneqq 0$ is a strong supersolution of $\mathcal{M}^-(D^2 v) - \tilde{b}(x)|Dv| \leq 0$ in Ω, for $\tilde{b} = b + 2\mu_2|Du_{\lambda_1,1}|$. Then SMP yields $v > 0$ in Ω, since $v \not\equiv 0$. Next, Hopf and $v = 0$ on $\partial\Omega$ give us $\partial_\nu v|_{\partial\Omega} > 0$, from where $v \gg 0$ in Ω. ∎

The existence proof of at least one solution for $(P_{\bar{\lambda}})$ follows exactly the same lines as the proof of claim 6.18 since, in there, we only used the fact that there exists one sequence of solutions corresponding to a maximizing sequence of λ's converging to the supremum $\bar{\lambda}$. Furthermore, uniqueness is true if F is convex in (r, p, X), by following the proof of claim 6.20. Finally, the behavior of the solutions is the same as in claim 6.17 and this finishes the proof of theorem 5.5.

Remark 6.32 *Particular cases of theorems 5.4 and 5.5 are $h \lneqq 0$ and $h \gneqq 0$, respectively, when the hypothesis (H_0) holds. Indeed, if $h \gneqq 0$, then u_0 is a strong supersolution of*

$$-\mathcal{L}^-[u_0] \geq -F[u_0] = h(x) + \langle M(x)Du_0, Du_0 \rangle \gneqq 0 \quad \text{in } \Omega$$

with $u_0 = 0$ on $\partial\Omega$. Then SPM gives us $u_0 > 0$ in Ω and so $cu_0 \not\equiv 0$. Furthermore, by Hopf, $u_0 \gg 0$ in Ω.

On the other hand, if $h \lneqq 0$, then u_0 is a strong subsolution of

$$-\mathcal{L}^+[u_0] \leq -F[u_0] \lneqq \langle M(x)Du_0, Du_0 \rangle \quad \text{in } \Omega$$

and so $v_0 := \frac{1}{m}(e^{mu_0} - 1)$, for $m = \frac{\mu_2}{\lambda_P}$, is a strong subsolution of $\mathcal{L}^+[v_0] \gneqq 0$ in Ω by lemma 2.20, with $v_0 = 0$ on $\partial\Omega$. Again by SMP we get $v_0 < 0$ in Ω (then $v_0 \ll 0$ in Ω by Hopf) and so does u_0 (with $u_0 \ll 0$), from where $cu_0 \not\equiv 0$.

Notice that in the case $h \equiv 0$, we have that $u \equiv 0$ is a strong solution of (P_λ), for all $\lambda \in \mathbb{R}$. By theorem 1(iii) in [12], this is the unique L^p-viscosity solution for all $\lambda \leq 0$. By theorem 5.3 we obtain the existence of $\bar{\lambda}$ such that (P_λ) has at least one more nontrivial solution, for all $\lambda \in (0, \bar{\lambda})$.

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A
Notations

Here we list some notations used throughout the text.

- $\mathbb{S}^n$ is the set of $n \times n$ symmetric matrices with real entries, $\mathbb{S}^n \subset \mathbb{M}_{n \times n}(\mathbb{R})$;

- $B_r(x_0) = \{x \in \mathbb{R}^n; |x - x_0| < r\}$ is a ball with radius r in $\mathbb{R}^n$ centered at x_0, for $r > 0$, $B_r = B_r(0)$;

- $B_r^+ = B_r \cap \{x_n > 0\}$, $\mathbb{T}_r = B_r \cap \{x_n = 0\}$, for $r > 0$;

- $B_r^\nu(x_0) = B_r(x_0) \cap \{x_n > -\nu\}$, $\mathbb{T}_r^\nu(x_0) = B_r(x_0) \cap \{x_n = -\nu\}$, $r, \nu > 0$;

- ω is a Lipschitz modulus if $\omega(r) \leq \omega(1)r$ for all $r \geq 0$;

- Ω is a bounded domain (open and connected) in $\mathbb{R}^n$; $u : \Omega \to \mathbb{R}$;

- ∂_ν is the derivative in the direction of the interior unit normal;

- We say $f \in L_+^p(\Omega)$ to mean a real function $f : \Omega \to \mathbb{R}$ such that $f \in L^p(\Omega)$ and $f \geq 0$;

- $Du = (u_{x_1}, \ldots, u_{x_n})$ is the gradient of u;

- $D^2 u = (\partial_{ij} u)_{i,j=1}^n$ is the Hessian matrix of u, where $\partial_{ij} u = \frac{\partial^2 u}{\partial x_i \partial x_j}$;

- $\Delta u = \sum_{i=1}^n \partial_{ii} u = \mathrm{tr}(D^2 u)$ is the Laplacian of u;

- $u \in USC(\Omega)$ means that $\overline{\lim}_{y \to x} u(y) = u(x)$ for all $x \in \Omega$, and it is equivalent to have $\{x \in \Omega; u(x) < \alpha\}$ open for all $\alpha \in \mathbb{R}$;

- $\mathcal{L}^\pm[u] = \mathcal{M}^\pm(D^2 u) \pm b(x)|Du|$ denote both $\mathcal{L}^+[u] = \mathcal{M}^+(D^2 u) + b(x)|Du|$ and $\mathcal{L}^-[u] = \mathcal{M}^-(D^2 u) - b(x)|Du|$;

- In chapters 5 and 6, $E = C^1(\overline{\Omega})$ and $\mathcal{B}_r = \mathcal{B}_r^E(0) = \{u \in E; \|u\|_E < r\}$ is the ball in E, for any $r > 0$.

B
Quadratic ABP

In this chapter we give a simple proof of quadratic ABP in the case $p > n$, with μ a positive constant. The proof is a version of theorem 3.1 in [63] (for constant b). For more general conditions see also theorems 2.6 in [25] and 3.4 in [37] (see also theorem 3.3 of [24]), even for unbounded μ.

Proof of Proposition 2.6. We prove the result for subsolutions, since in the supersolution case we just need to apply the former result to $-u$. We can suppose $0 \in \Omega$ and consider $B = B_{2d}(0)$ for $d = \operatorname{diam}(\Omega)$. We define $g : B \to \mathbb{R}$,

$$g := f^- + d^{-\frac{n}{p}}\|f^-\|_{L^p(\Omega)} \text{ in } \Omega, \quad g := 0 \text{ in } B \setminus \Omega$$

then $\|g\|_{L^p(B)} \leq \|f^-\|_{L^p(\Omega)} + d^{-\frac{n}{p}}\|f^-\|_{L^p(\Omega)}|\Omega|^{\frac{1}{p}} \leq (1 + |B_1|^{\frac{1}{p}})\|f^-\|_{L^p(\Omega)}$. Also, by Holder's inequality, $\|g\|_{L^n(B)} \leq |B|^{\frac{1}{n}-\frac{1}{p}}\|g\|_{L^p(B)} \leq d^{1-\frac{n}{p}}C_{n,p}\|f^-\|_{L^p(\Omega)}$.

By theorem 3.5[1], there exists a unique strong solution $w \in W^{2,p}(B)$ of

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2 w) + b(x)|Dw| = -g(x) \text{ in } B$$

with $w = 0$ on ∂B, for $b \in L^p_+(B)$ defined as zero in $B \setminus \Omega$, with

$$\|w\|_{L^\infty(B)} \leq C_A\|g\|_{L^n(B)} \leq d^{1-\frac{n}{p}}C_{n,p}C_A\|f^-\|_{L^p(\Omega)},$$

where $C_A = dC_1$ is the constant from ABP in the L^n case, and also

$$\|w\|_{W^{2,p}(B')} \leq C\{\|w\|_{L^\infty(B)} + \|g\|_{L^p(B)}\} \leq C_2\|f^-\|_{L^p(\Omega)}$$

for all $B' \subset\subset B$, where $C_2 = C_2(n, p, \lambda, \Lambda, \|b\|_{L^p(\Omega)}, d, \operatorname{dist}(B', \partial B))$. By the Sobolev embedding $W^{2,p} \subset W^{1,\infty}$, the last inequality and the L^p estimate for g, we obtain, for $B' := B_d(0) \subset\subset B = B_{2d}(0)$,

$$\|Dw\|_{L^\infty(\Omega)} \leq \|Dw\|_{L^\infty(B')} \leq C_n\|w\|_{W^{2,p}(B')} \leq C_0\|f^-\|_{L^p(\Omega)}$$

where $C_0 = C_n C_2 = C_0(n, p, \lambda, \Lambda, \|b\|_{L^p(\Omega)}, d)$.

---

[1]Of course, we use that $C^{1,\alpha}$ regularity and theorem 3.5, for unbounded coefficients in the case $\mu = 0$, are already available. After, we come back here and show quadratic ABP for $\mu \neq 0$, which is needed to prove $C^{1,\alpha}$ regularity for $\mu \neq 0$.

We first claim that if $\mu \|f^-\|_{L^p(\Omega)} d^{\frac{n}{p}} \leq \delta =: 1/C_0^2$, then the function $v := u - w - \max_{\partial\Omega} u$ is an L^p-viscosity subsolution of

$$\mathcal{M}_{\lambda,\Lambda}^+(D^2 v) + \widetilde{b}(x)|Dv| + \mu|Dv|^2 \geq 0 \text{ in } \Omega, \quad v \leq 0 \text{ on } \partial\Omega,$$

where $\widetilde{b}(x) = b(x) + 2\mu\|Dw\|_{L^\infty(\Omega)} \in L^p(\Omega)$. To prove this, let $\phi \in W_{\text{loc}}^{2,p}(\Omega)$. Since w is a strong solution, we can write that

$$\begin{aligned}
\mathcal{M}_{\lambda,\Lambda}^+&(D^2(w+\phi)) + b(x)|D(w+\phi)| + \mu|D(w+\phi)|^2 \\
&\leq \{\mathcal{M}_{\lambda,\Lambda}^+(D^2 w) + b(x)|Dw|\} + \mathcal{M}_{\lambda,\Lambda}^+(D^2\phi) \\
&\quad + b(x)|D\phi| + 2\mu|Dw||D\phi| + \mu|D\phi|^2 + \mu|Dw|^2 \\
&\leq -g(x) + \{\mathcal{M}_{\lambda,\Lambda}^+(D^2\phi) + \widetilde{b}(x)|D\phi| + \mu|D\phi|^2\} + \mu\|\|Dw\|_{L^\infty(\Omega)}^2 \\
&\leq -f^-(x) - d^{-\frac{n}{p}}\|f^-\|_{L^p(\Omega)} + \mu\, C_0^2\|f^-\|_{L^p(\Omega)}^2 \\
&\quad + \{\mathcal{M}_{\lambda,\Lambda}^+(D^2\phi) + \widetilde{b}(x)|D\phi| + \mu|D\phi|^2\} \\
&\leq f(x) + \{\mathcal{M}_{\lambda,\Lambda}^+(D^2\phi) + \widetilde{b}(x)|D\phi| + \mu|D\phi|^2\} \text{ for a.e. } x \in \Omega
\end{aligned}$$

by the choice $\mu\, C_0^2\,\|f^-\|_{L^p(\Omega)}\, d^{\frac{n}{p}} \leq 1$. So, if $v - \phi$ attains a maximum at $x_0 \in \Omega$, then $u - (w + \phi)$ attains a maximum at $x_0 \in \Omega$, where $w + \varphi \in W_{\text{loc}}^{2,p}(\Omega)$. Hence, by the definition of u being an L^p-viscosity subsolution,

$$\begin{aligned}
0 &\leq \text{ess} \varlimsup_{x \to x_0} \left\{ \mathcal{M}_{\lambda,\Lambda}^+(D^2(w+\phi)) + b(x)|D(w+\phi)| + \mu|D(w+\phi)|^2 - f(x) \right\} \\
&\leq \text{ess} \varlimsup_{x \to x_0} \left\{ \mathcal{M}_{\lambda,\Lambda}^+(D^2\phi) + \widetilde{b}(x)|D\phi| + \mu|D\phi|^2 \right\}.
\end{aligned}$$

Of course, $v \leq 0$ on $\partial\Omega$ by construction and the claim is established.

Next, $v \leq 0$ in Ω. Indeed, $V := \frac{1}{m}(e^{mv} - 1)$ is an L^p-viscosity solution of $\mathcal{M}_{\lambda,\Lambda}^+(D^2 V) + \widetilde{b}(x)|DV| \geq 0$ in Ω by lemma 2.20, where $m = \mu/\lambda$. Further, $V \leq 0$ on $\partial\Omega$. Then by ABP we have $V \leq 0$ in Ω, as well as for v. Therefore,

$$u - \max_{\partial\Omega} u \leq w \leq \|w\|_{L^\infty(\Omega)} \leq C_B\, \|f^-\|_{L^p(\Omega)} \text{ in } \Omega,$$

where $C_B := d^{2-\frac{n}{p}}\, C_{n,p}\, C_1$ depends on n, p, λ, Λ, $\text{diam}(\Omega)$, $\|b\|_{L^p(\Omega)}$ and it remains bounded if these quantities are bounded. ∎

See the foregoing works [25], [37], [24] for a quadratic version of ABP for $p_0 < p \leq n$ (which is known as GMP); their proofs are based on the iterative method.

C
A property of functions

To prove $C^{1,\alpha}$ estimates, the usual way is to show that some function u, solution of a given equation in the L^p-viscosity sense, satisfies

$$|u(x) - u(y) - Du(y) \cdot (x - y)| \leq C|x - y|^{1+\alpha}, \quad |Du(y)| \leq C \qquad \text{(C.1)}$$

for all x, y in a suitable domain. How this implies that Du belongs to C^α and

$$|Du(x) - Du(y)| \leq C|x - y|^\alpha, \qquad \text{(C.2)}$$

is a well known universal fact which does not depend on the equation, it is only a property of functions. A direct proof for it can be found in [27]. We reproduce it below for completeness, adding some improvements done by Prof. Boyan Sirakov. We also refer to the appendix of [62] for an elementary proof.

In the one dimensional case $n = 1$, (C.2) is evident from (C.1), since for each $x, y \in B_{1/2} = (-1/2, 1/2)$, $x > y$, we have

$$\begin{cases} \max\{u(x) - u(y) - u'(y)(x - y), -u(x) + u(y) + u'(y)(x - y)\} \leq C|x - y|^{\alpha+1} \\ \max\{u(y) - u(x) - u'(x)(y - x), -u(y) + u(x) + u'(x)(y - x)\} \leq C|x - y|^{\alpha+1} \end{cases}$$

and thus, by summing, $|u'(x) - u'(y)|(x - y) \leq C|x - y|^{\alpha+1}$.

In the general case we argue by induction on n. Suppose (C.2) valid for each function of $n-1$ variables which satisfies (C.1) and let u be a differentiable function of n variables satisfying (C.1) for each $x, y \in B_{1/2}$. Fix $\bar{x}, \bar{y} \in B_{1/2}$. We want to show that

$$|Du(\bar{x}) - Du(\bar{y})| \leq C|\bar{x} - \bar{y}|^\alpha, \qquad \text{(C.3)}$$

where C depends on the right quantities. Without loss we can assume that $\bar{x}, \bar{y}$ belong to the plane $\{x_n = 0\}$ (otherwise just rotate the space). Set

$$v = u|_{x_n=0} : (\mathbb{R}^{n-1} \approx \mathbb{R}^n \cap \{x_n = 0\}) \cap B_1 \to \mathbb{R}$$

so the induction hypothesis holds for v and gives us

$$|D_{x'}u(x) - D_{x'}u(y)| \leq C|x-y|^\alpha \ , \ \text{for all } x, y \in \{x_n = 0\} \cap B_{1/2}.$$

where $x = (x', x_n)$. Thus we only need to show that

$$|\partial_{x_n}u(\bar{x}) - \partial_{x_n}u(\bar{y})| \leq C|\bar{x} - \bar{y}|^\alpha.$$

We can assume $|\bar{x} - \bar{y}| < 1/4$ (otherwise we are done since the gradient of u is bounded). Set $r = 2|\bar{x} - \bar{y}|$ and fix some point $z \in B_r(\bar{x}) \cap B_r(\bar{y}) \cap B_{1/2}$ such that $z_n = r/4$. Then

$$
\begin{aligned}
|\partial_{x_n}u(\bar{x}) - \partial_{x_n}u(\bar{y})| &= \frac{4}{r}|\partial_{x_n}u(\bar{x}) \cdot z_n - \partial_{x_n}u(\bar{y}) \cdot z_n| \\
&= \frac{4}{r}|u(z) - u(\bar{y}) - D_{x'}u(\bar{y}) \cdot (z' - \bar{y}') - \partial_{x_n}u(\bar{y}) \cdot z_n \\
&\quad - \{u(z) - u(\bar{y}) - D_{x'}u(\bar{y}) \cdot (z' - \bar{y}')\} \\
&\quad - \{u(z) - u(\bar{x}) - D_{x'}u(\bar{x}) \cdot (z' - \bar{x}') - \partial_{x_n}u(\bar{x}) \cdot z_n\} \\
&\quad + u(z) - u(\bar{x}) - D_{x'}u(\bar{x}) \cdot (z' - \bar{x}')| \\
&= \frac{4}{r}|u(z) - u(\bar{y}) - D_x u(\bar{y}) \cdot (z - \bar{y}) - \{u(z) - u(\bar{x}) - D_x u(\bar{x}) \cdot (z - \bar{x})\} \\
&\quad + u(\bar{y}) - u(\bar{x}) + D_{x'}u(\bar{y}) \cdot (z' - \bar{y}') - D_{x'}u(\bar{x}) \cdot (z' - \bar{x}')| \\
&\leq \frac{4}{r}\{|u(z) - u(\bar{y}) - D_x u(\bar{y}) \cdot (z - \bar{y})| + |u(z) - u(\bar{x}) - D_x u(\bar{x}) \cdot (z - \bar{x})| \\
&\quad + |u(\bar{y}) - u(\bar{x}) - D_{x'}u(\bar{x}) \cdot (\bar{y}' - \bar{x}')| + |(D_{x'}u(\bar{y}) - D_{x'}u(\bar{x})) \cdot (z' - \bar{y}')|\}
\end{aligned}
$$

where we subtracted and added $D_{x'}u(\bar{x}) \cdot (z' - \bar{y}')$ to get the last expression. Now we estimate each one of the above terms separately. By using (C.1),

$$|u(z) - u(\bar{y}) - D_x u(\bar{y}) \cdot (z - \bar{y})| \leq C|z - \bar{y}|^{\alpha+1} \leq Cr^{\alpha+1} = C2^{\alpha+1}|\bar{x} - \bar{y}|^{\alpha+1};$$

$$|u(z) - u(\bar{x}) - D_x u(\bar{x}) \cdot (z - \bar{x})| \leq C|z - \bar{x}|^{\alpha+1} \leq Cr^{\alpha+1} = C2^{\alpha+1}|\bar{x} - \bar{y}|^{\alpha+1};$$

$$|u(\bar{y}) - u(\bar{x}) - D_{x'}u(\bar{x}) \cdot (\bar{y}' - \bar{x}')| \leq C|\bar{x} - \bar{y}|^{\alpha+1}$$

since $D_{x'}u(\bar{x}) \cdot (\bar{y}' - \bar{x}') = D_x u(\bar{x}) \cdot (\bar{y} - \bar{x})$. Finally, we conclude by using, in the last modulus, the induction hypothesis

$$|D_{x'}u(\bar{y}) - D_{x'}u(\bar{x})| \, |z' - \bar{y}'| \leq C|\bar{x} - \bar{y}|^\alpha r = 2C|\bar{x} - \bar{y}|^{\alpha+1}.$$

Consequently, from (C.2), we derive the estimate

$$\|u\|_{1,\alpha} = \|u\|_\infty + \|Du\|_\infty + [Du]_\alpha \leq (1 + 2C)W,$$

with C replaced by CW (which is the appropriate constant), W being equal to $\|u\|_\infty$ plus another terms, depending on each case, local or global one.

D
The Local Maximum Principle for C-viscosity solutions

We present a detailed proof of theorem 2.2 in the case b is bounded and ν is continuous in B_3, following the ideas of [89], [90], [41], [33] and [91]. Consider here $\mathcal{L}^{\pm}[u] := \mathcal{M}_{\lambda,\Lambda}^{\pm}(D^2 u) \pm \gamma |Du|, \ \ \gamma \geq 0$.

Prof of Theorem 2.2. Let $1 \leq r < p$, for the time being. The conclusion is trivial if $u \equiv 0$ in B_2, thus suppose $u \not\equiv 0$. So $v := u^+ \gneq 0$ in B_2.

Case 1. $f \in C(B_3)$: In this case consider the sup-convolution of v as

$$v^\varepsilon := \sup_{y \in B_{5/2}} \left\{ v(y) - \frac{|x-y|^2}{2\varepsilon} \right\}.$$

Then, as in lemma 6.1 in [89], we know that $v^\varepsilon \in C^{0,1}(B_{5/2})$, it is twice differentiable a.e. in B_2 and $v^\varepsilon \to v$ uniformly in $\overline{B}$, for every $B \subset\subset B_{5/2}$. Moreover, from (2.6), v^ε satisfies, in the C-viscosity sense (and also a.e.),

$$\mathcal{L}^+[v^\varepsilon] + \nu(x) v_\varepsilon \geq -f_\varepsilon(x) \quad \text{in } B_{5/2}^\varepsilon = \{x \in B_{5/2}; \ \text{dist}(x, \partial\Omega) > \delta_\varepsilon\}$$

(and in particular in B_2 for $\varepsilon \leq \varepsilon_0$), for $\delta_\varepsilon = 2 \left(\varepsilon \|v\|_{L^\infty(B_{5/2})} \right)^{1/2}$ and

$$v_\varepsilon(x) := \sup_{|x-y| < \delta_\varepsilon} v^\varepsilon(y), \quad f_\varepsilon(x) := \sup_{|x-y| < \delta_\varepsilon} f^+(y). \tag{D.1}$$

Set $w_\varepsilon := \eta \, v^\varepsilon$ in $B_{2-\delta_\varepsilon}$, where $\eta \in [0,1]$ is a cut-off function defined as

$$\eta(x) = \eta_\varepsilon(x) := \frac{\left((2-\delta_\varepsilon)^2 - |x|^2 \right)^\beta}{(2-\delta_\varepsilon)^{2\beta}}$$

for some $\beta \geq 2$ to be chosen. Notice that, for $\delta = \delta_\varepsilon$, $D\eta(x) = -\frac{2\beta \, x}{(2-\delta)^2} \eta^{1-\frac{1}{\beta}}$ and $D^2\eta(x) = \frac{2\beta}{(2-\delta)^4} \left\{ 2(\beta-1) x \otimes x - ((2-\delta)^2 - |x|^2) I \right\} \eta^{1-\frac{2}{\beta}}$.

Thus it holds a.e. that

$$
\begin{aligned}
-\mathcal{L}^+[w_\varepsilon] &= \mathcal{M}^-(-\eta D^2 v^\varepsilon - 2D\eta \otimes Dv^\varepsilon - v^\varepsilon D^2\eta) - \gamma |\eta Dv^\varepsilon + v^\varepsilon D\eta| \\
&\leq \mathcal{M}^-(-\eta D^2 v^\varepsilon) + 2\mathcal{M}^+((-D\eta) \otimes Dv^\varepsilon) + \mathcal{M}^+(-v^\varepsilon D^2\eta) \\
&\quad - \eta\, \gamma |Dv^\varepsilon| + \gamma |v^\varepsilon D\eta| \\
&\leq -\eta\, \mathcal{L}^+[v^\varepsilon] + 2\Lambda |D\eta| \, |Dv^\varepsilon| + \gamma |v^\varepsilon| \, |D\eta| \\
&\quad + \frac{2\beta}{(2-\delta)^4} \left\{ 2(\beta-1) \mathcal{M}^+(-v^\varepsilon \, x \otimes x) + ((2-\delta)^2 - |x|^2) \mathcal{M}^+(v^\varepsilon I) \right\} \eta^{1-\frac{2}{\beta}}
\end{aligned}
$$

$$\leq \eta \left\{ \nu(x)v_\varepsilon + f_\varepsilon(x) \right\} + \frac{4\beta\Lambda}{(2-\delta)^2} \, |x| \, |Dv^\varepsilon| \, \eta^{1-\frac{1}{\beta}} + \frac{2\beta}{(2-\delta)^2} \, \gamma \, |x| \, |v^\varepsilon| \, \eta^{1-\frac{1}{\beta}}$$

$$+ \frac{2\beta\Lambda}{(2-\delta)^4} \, |v^\varepsilon| \left\{ 2(\beta-1)|x|^2 + n\left((2-\delta)^2 - |x|^2\right) \right\} \eta^{1-\frac{2}{\beta}} \qquad (\text{D.2})$$

since the only nonzero eigenvalue of $x \otimes x$ is $|x|^2$. Since v^ε is Lipschitz in $B_{5/2}$, it belongs to $W^{1,\infty}(B)$ [29, p. 294]. So u is differentiable a.e., its derivative coincides with the weak derivative a.e. [29, p. 295] and $Dv^\varepsilon \in L^\infty(B_{5/2})$. Thus, naming by H the right hand side of the above, we have that $H \in L^p(B_{2-\delta})$, since

$$\| \, \eta \, v_\varepsilon \nu \|_{L^p(B_{2-\delta})} \leq \| \nu \|_{L^{p_1}(B_2)} \| \, \eta \, v_\varepsilon \|_{L^{q_1}(B_{2-\delta})}, \quad \text{where } \frac{1}{p} = \frac{1}{p_1} + \frac{1}{q_1} \qquad (\text{D.3})$$

i.e. for $q_1 = \frac{p\,p_1}{p_1 - p} > p \geq n$, Holder's inequality and Young yield

$$\| \eta \, v_\varepsilon \|_{L^{q_1}(B_{2-\delta})} \leq \theta \| \, \eta \, v_\varepsilon \|_{L^{q_2}(B_{2-\delta})} + \theta^{-\mu} \| \, \eta \, v_\varepsilon \|_{L^r(B_{2-\delta})}, \mu = \frac{1/r - 1/q_1}{1/q_1 - 1/q_2} \qquad (\text{D.4})$$

where $1 \leq r \leq q_1 \leq q_2$ and $\theta > 0$.

Remark D.1 (*Contact sets in ABP in the case $\mu = 0$*) *If $b \in L^\infty_+(\Omega)$ or if $u \in W^{2,p}_{\text{loc}}(\Omega)$, the L^p norm of the function f can be taken over $\Gamma^+_u \cap \Omega^+$ (respectively $\Gamma^-_u \cap \Omega^-$), the upper (lower) contact set of the function u, where*

$$\Gamma^+_u \, (\Gamma^-_u) := \{ x \in \Omega; \, \exists p \in \mathbb{R}^n; \, u(y) \leq (\geq) \, u(x) + p \cdot (y - x), \, \forall \, y \in \Omega \}.$$

See the proofs of proposition 6.2 in [33] and theorem 9.1 in [41].

Now we use lemma 2.8 in [4] (see also lemma 7.10 in [33]) to obtain that w_ε is also an L^p-viscosity solution of (D.2). Then, by ABP and remark D.1,

$$\sup_{B_{2-\delta}} w_\varepsilon \leq C_A \, (2-\delta)^{2-\frac{n}{p}} \, \|H\|_{L^p(\Gamma^+_\varepsilon \cap \{w_\varepsilon > 0\})}. \qquad (\text{D.5})$$

Claim D.2 $w_\varepsilon(x) \geq |Dw_\varepsilon(x)| \, (2 - \delta - |x|), \quad \text{a.e. } x \in \Gamma^+_\varepsilon.$

Proof. Fix a point $x \in \Gamma^+_\varepsilon$ of differentiability of $w = w_\varepsilon$ with $|Dw(x)| > 0$ (otherwise it is obvious). Notice that $|x| = 2 - \delta - m_x$, $m_x \in (0, 2-\delta]$, so take $k_0 = k_0(x) \in \mathbb{N}$ such that $m_x > \frac{1}{k_0} \geq \frac{1}{k}$, for all $k \geq k_0$. Next choose

$$y_k = y_k(x) := x - t_k \frac{Dw(x)}{|Dw(x)|}, \quad \text{for } t_k = 2 - \delta - |x| - \tfrac{1}{k}.$$

Then $y_k \in B_{2-\delta}(0)$, since $|y_k| - |x| \leq |y_k - x| = 2 - \delta - |x| - \frac{1}{k} \Rightarrow |y_k| < 2 - \delta$, and so $0 \leq w(y_k) \leq w(x) + Dw(x) \cdot (y_k - x) = w(x) - t_k \, |Dw(x)|$ for all $k \geq k_0$, which proves claim D.2 by passing to limits as $k \to +\infty$. ∎

Then, using claim D.2 and definition of η, we have for a.e. $x \in \Gamma_\varepsilon^+$,

$$|Dv^\varepsilon| \leq \frac{1}{\eta}\left\{ |Dw_\varepsilon| + v^\varepsilon|D\eta| \right\} \leq \frac{1}{\eta}\left\{ \frac{w_\varepsilon}{2-\delta+|x|} + v^\varepsilon \frac{2\beta|x|}{(2-\delta)^2}\,\eta^{1-\frac{1}{\beta}} \right\}$$

$$\leq v^\varepsilon \eta^{-\frac{1}{\beta}}\left\{ \frac{2(2-\delta)}{(2-\delta)^{2/\beta}} + \frac{2\beta(2-\delta)}{(2-\delta)^2} \right\} \leq 4(1+\beta)\,v^\varepsilon \eta^{-\frac{1}{\beta}}$$

with $\varepsilon \leq \varepsilon_0 = \varepsilon(\|v\|_\infty)$ chosen such that $\delta = \delta_\varepsilon \leq 1$. Thus

$$|H| \leq \eta\,\nu(x)v_\varepsilon + \eta\,f_\varepsilon(x) + \frac{16\beta\Lambda}{2-\delta}(1+\beta)\,v^\varepsilon\,\eta^{1-\frac{2}{\beta}} + \frac{2\beta\gamma}{2-\delta}\,v^\varepsilon\,\eta^{1-\frac{1}{\beta}}$$

$$+ \frac{4\beta\Lambda}{(2-\delta)^2}(\beta-1+n)\,v^\varepsilon\eta^{1-\frac{2}{\beta}}$$

$$\leq \eta\,\nu(x)v_\varepsilon + \eta\,f_\varepsilon(x) + \left\{ 16\beta\Lambda(1+\beta) + 2\beta\gamma + 4\beta\Lambda(\beta-1+n) \right\} w_\varepsilon\,\eta^{-\frac{2}{\beta}}$$

$$\leq \eta\,\nu(x)v_\varepsilon + f_\varepsilon(x) + C\,w_\varepsilon^{1-\frac{2}{\beta}}\,(w_\varepsilon/\eta)^{\frac{2}{\beta}}$$

where $C = C(n,\beta,\gamma,\Lambda)$. Hence, for $L^p = L^p(B_{2-\delta})$,

$$\|H\|_{L^p(\Gamma_\varepsilon^+)} \leq \|\eta\,\nu(x)v_\varepsilon\|_{L^p} + \|f_\varepsilon\|_{L^p} + C\,(\sup_{\Gamma_\varepsilon^+} w_\varepsilon)^{1-\frac{2}{\beta}}\,\|(v^\varepsilon)^{\frac{2}{\beta}}\|_{L^p}.$$

Therefore, by the latter, (D.3) and (D.4) applied to (D.5), we obtain

$$\sup_{B_{2-\delta}} w_\varepsilon \leq C_0\,\theta\|\nu\|_{L^{p_1}(B_2)}\|\,\eta\,v_\varepsilon\|_{L^{q_2}(B_{2-\delta})} + C_0\,\theta^{-\mu}\|\nu\|_{L^{p_1}(B_2)}\|\,\eta\,v_\varepsilon\|_{L^r(B_2)}$$

$$+ C_0\,\|f_\varepsilon\|_{L^p(B_3)} + c_0\,(\sup_{\Gamma_\varepsilon^+} w_\varepsilon)^{1-\frac{2}{\beta}}\left(\int_{B_2}(v^\varepsilon)^r \right)^{1/p} \tag{D.6}$$

for $C_0 = 2^{1-\frac{n}{p}}\,C_A$, by choosing $\beta = 2p/r \geq 2$, since $1 \leq r \leq p$.

We are going to use Young's inequality in the form

$$ab \leq \sigma a^{p_0} + \sigma^{-\frac{q_0}{p_0}}b^{q_0}, \text{ for } p_0, q_0 \in (1,+\infty) \text{ such that } \frac{1}{p_0} + \frac{1}{q_0} = 1. \tag{D.7}$$

By taking $a = (\sup_{\Gamma_\varepsilon^+} w_\varepsilon)^{1-\frac{2}{\beta}}$, $b = 1$, $p_0 = 1/\left(1-\frac{2}{\beta}\right) = \frac{p}{p-r} > 1$, so $q_0 = \frac{p}{r} > 1$, and choosing $\sigma = \frac{1}{2c_0}\left(\int_{B_2}(v^\varepsilon)^r \right)^{-1/p}$, we have

$$c_0\,(\sup_{\Gamma_\varepsilon^+} w_\varepsilon)^{1-\frac{2}{\beta}}\left(\int_{B_2}(v^\varepsilon)^r \right)^{1/p} \leq \frac{1}{2}\sup_{B_{2-\delta}} w_\varepsilon + c\left(\int_{B_2}(v^\varepsilon)^r \right)^{1/r}. \tag{D.8}$$

Note that if $\|\nu\|_{L^{p_1}(B_2)} = 0$, the first two addends on the right hand side of (D.6) disappear. Therefore, in (D.6) we assume $\|\nu\|_{L^{p_1}(B_2)} > 0$ and set $\theta = (4C_0|B_2|^{1/q_2}\|\nu\|_{L^{p_1}(B_2)})^{-1}$; thus

$$\sup_{B_{2-\delta}} w_\varepsilon \leq \frac{1}{2}\sup_{B_{2-\delta}}(\eta\,v_\varepsilon) + c\,\|\nu\|_{L^{p_1}}^{\mu+1}\|\,\eta\,v_\varepsilon\|_{L^r} + 2C_0\,\|f_\varepsilon\|_{L^p} + 2c\left(\int_{B_2}(v^\varepsilon)^r \right)^{1/r}$$

and it follows that

$$\sup\nolimits_{B_{2-\delta}} w_\varepsilon - \tfrac{1}{2}\sup\nolimits_{B_{2-\delta}} (\eta\, v_\varepsilon) \le C\left\{\|f_\varepsilon\|_{L^p(B_3)} + \left(\textstyle\int_{B_2}(v^\varepsilon)^r\right)^{1/r}\right\}.$$

This produces the desired bound by letting $\varepsilon \to 0$, since $v^\varepsilon, v_\varepsilon \to v$ uniformly in B_2, $B_1 \subset B_{2-\delta_\varepsilon}$ for all $\varepsilon \le \varepsilon_0$ and $f_\varepsilon \to f^+$ in this case.

If $0 < r < 1$, we use the result just proved for $r = 2$ to write

$$\left(\textstyle\int_{B_2} v^2\right)^{1/2} = \left(\textstyle\int_{B_2} v^{2-r}v^r\right)^{1/2} \le (\sup\nolimits_{B_2} v)^{1-\frac{r}{2}}\left(\textstyle\int_{B_2} v^r\right)^{1/2}.$$

Then, by using (D.7) again with $p_0 = (1 - \frac{r}{2})^{-1} = \frac{2}{2-r}$, $q_0 = \frac{2}{r} > 1$ and $\sigma = \frac{1}{2C}(\int_{B_2} v^r)^{-1/2}$, we obtain

$$\sup\nolimits_{B_2} v \le 2C\left\{\|f^+\|_{L^p(B_3)} + \left(\textstyle\int_{B_2} v^r\right)^{1/r}\right\}.$$

On the other hand, if $r \ge p$, we just apply Holder's inequality on the right hand to derive the respective estimate with r.

Case 2. $f \in L^p(B_3)$: Let $f_k \in L^p(\Omega) \cap C(B_3)$ such that $f_k \to f^+$ in $L^p(B_3)$ and let $\psi_k \in W^{2,p}_{\mathrm{loc}}(B_3) \cap C(\overline{B_3})$ be the L^p-viscosity solution of

$$-\mathcal{L}^-[\psi_k] = f_k - f^+ \text{ in } B_3, \quad \psi_k = 0 \text{ on } \partial B_3$$

given, for instance, by theorem 4.6 in [38] (or by proposition 3.5). By ABP, $\|\psi_k\|_{L^\infty(B_2)} \to 0$. Now set $v_k := v + \psi_k$. So, since ψ_k is strong,

$$-f^+ - \nu(x)v \le \mathcal{L}^+[v] \le \mathcal{L}^+[v_k] - \mathcal{L}^-[\psi_k] = \mathcal{L}^+[v_k] + f_k - f^+$$

i.e. $\mathcal{L}^+[v_k] \ge \nu(x)v_k + f_k(x) = \nu(x)v_k + \widetilde{f}_k(x)$ in B_3 in the L^p-viscosity sense, where $\widetilde{f}_k := f_k + \nu(v - v_k) \in C(B_3)$. Then case 1 yields the result, by

$$\sup\nolimits_{B_2} v_k \le 2C\left\{\|\widetilde{f}_k^+\|_{L^p(B_3)} + \left(\textstyle\int_{B_2} v_k^r\right)^{1/r}\right\}$$

and letting $k \to +\infty$, since $\widetilde{f}_k = f_k - \nu(x)\psi_k \to f^+$ in $L^p(B_3)$. ∎

Remark D.3 *Notice that this version of LMP is sufficient for our purposes in section 5.2. We assumed $\nu(x)$ continuous in the proof above, but it is just a technicality when the functions c and h^+ in the problem (P_λ) are bounded, with $\nu(x) = \Lambda_2\, c_0\, (1 + m_2 v_2)\, |\ln(1 + m_2 v_2)|\, /m_2 v_2$ as in the proof of theorem 5.6. Indeed, say $c(x) \le c_0$, $h^+(x) \le h_0$, then $\nu\, v_2 \le |\nu|\, |v_2| \le \widetilde{\nu}\, |v_2|$, where $\widetilde{\nu}$, defined as $\widetilde{\nu}(x) = \nu(x)$ if $v_2 \ne 0$ and $\widetilde{\nu}(x) = \Lambda_2\, c_0$ if $v_2 = 0$, is a continuous function in B_2^+. Now observe that the proof above remains true if, instead of $\nu(x)u$, we have $\nu(x)|u|$, since $v = u^+$ is still an L^p-viscosity subsolution of (2.6), and the rest of the proof carries on in the same way.*