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Annex A
Resulting space discretization

The weak formulation of the differential equations gives the following equalities (repeated from equations presented in Chapter 3 of the main document):

\[
\int_{\Omega} \delta \epsilon \cdot \sigma' \, d\Omega - \int_{\Omega} \delta \epsilon \cdot m \cdot p \, d\Omega + \int_{\Gamma_d} \delta \mathbf{u} \cdot (t_F - p_F \cdot n_{\Gamma_d}) \, d\Gamma \\
- \int_{\Gamma_t} \delta \mathbf{u} \cdot \mathbf{t} \, d\Gamma = 0
\]  
\[(A.1)\]

\[
\int_{\Omega} \nabla \delta p k_F \nabla p \, d\Omega + \int_{\Gamma_d} \delta p [\bar{w}] n_{\Gamma_d} \, d\Gamma + \int_{\Omega} \delta p \cdot \nabla \bar{u} \, d\Omega \\
+ \int_{\Gamma_w} \delta p \cdot \bar{q} \, d\Gamma = 0
\]
\[(A.2)\]

\[
\int_{\Gamma_d} \frac{\partial \delta p_F}{\partial x'} k_F \cdot 2h \cdot \frac{\partial p_F}{\partial x} \, d\Gamma - \int_{\Gamma_d} \delta p_F q_F n_{\Gamma_d} \, d\Gamma + \\
\int_{\Gamma_d} \delta p_F \cdot 2h \cdot \left( \frac{\partial \bar{u}_x'}{\partial x} \right) \, d\Gamma + \int_{\Gamma_d} \delta p_F \cdot [\bar{u}_y'] \, d\Gamma = 0
\]
\[(A.3)\]

It may be admitted that the test functions \( \delta u, \delta p \) and \( \delta p_F \) follow the same discretization rules as the variables \( u, p \) and \( p_F \). It is also considered that the vector of the nodal variables for each element node is given by \( \bar{u} \). Although generalized for any number of enriched degrees of freedom, for the sake of clearness the discretization is developed for one enriched displacement variable \( a \) and one enriched pressure variable \( p_a \). Eq. (A.4) to Eq. (A.16) present the discretization of the variables and their derivatives.

\[
u = N_{\text{std}}^{u} \bar{u} + N_{\text{enr}}^{u} \bar{\alpha}
\]
\[(A.4)\]

\[
\delta u = N_{\text{std}}^{\delta u} \delta \bar{u} + N_{\text{enr}}^{\delta u} \delta \bar{\alpha}
\]
\[(A.5)\]

\[
[\mathbf{u}] = [N_{\text{std}}^{u}][\bar{u}] + [N_{\text{enr}}^{u}][\bar{\alpha}] = [N_{\text{enr}}^{u}] [\bar{\alpha}]
\]
\[(A.6)\]

\[
\varepsilon = B_{\text{std}}^{u} \bar{u} + B_{\text{enr}}^{u} \bar{\alpha}
\]
\[(A.7)\]
\[ \delta \varepsilon = B_{u}^{\text{std}} \delta \bar{u} + B_{u}^{\text{enr}} \delta \bar{a} \]  
(A.8)

\[ \nabla \delta \bar{u} = B_{u}^{\text{std}} \delta \bar{u} + B_{u}^{\text{enr}} \delta \bar{a} \]  
(A.9)

\[ p = N_{p}^{\text{std}} \bar{p} + N_{p}^{\text{enr}} \bar{p}_{a} \]  
(A.10)

\[ \delta p = N_{p}^{\text{std}} \delta \bar{p} + N_{p}^{\text{enr}} \delta \bar{p}_{a} \]  
(A.11)

\[ \nabla \delta p = B_{p}^{\text{std}} \delta \bar{p} + B_{p}^{\text{enr}} \delta \bar{p}_{a} \]  
(A.12)

\[ \delta \nabla p = B_{p}^{\text{std}} \delta \bar{p} + B_{p}^{\text{enr}} \delta \bar{p}_{a} \]  
(A.13)

\[ p_{F} = N_{pF}^{\text{std}} \bar{p}_{F} \]  
(A.14)

\[ \delta p_{F} = N_{pF}^{\text{std}} \delta \bar{p}_{F} \]  
(A.15)

\[ \nabla p_{F} = B_{pF}^{\text{std}} \bar{p}_{F} \]  
(A.16)

Replacing the variables in Eq. (3.14), the following equation is obtained:

\[
\int_{\Omega} \delta \varepsilon. D. \varepsilon. d\Omega - \int_{\Omega} \delta \varepsilon. m. p. d\Omega + \int_{\Gamma_{d}} \left[ \delta \bar{u} \right] \left( t_{F} - p_{F}. n_{\Gamma_{d}} \right) d\Gamma \\
- \int_{\Gamma_{t}} \delta \bar{u}. \bar{t} \; d\Gamma \\
= \int_{\Omega} (B_{u}^{\text{std}})^{T} \delta \bar{u} \bar{D}_{u} B_{u}^{\text{std}} \; d\Omega \\
+ \int_{\Omega} (B_{u}^{\text{std}})^{T} \delta \bar{u} \bar{D}_{u}^{\text{enr}} \; d\Omega \\
+ \int_{\Omega} (B_{u}^{\text{enr}})^{T} \delta \bar{a} \bar{D}_{u} B_{u}^{\text{std}} \; d\Omega \\
+ \int_{\Omega} (B_{u}^{\text{enr}})^{T} \delta \bar{a} \bar{D}_{u}^{\text{enr}} \; d\Omega \\
- \int_{\Omega} m(B_{u}^{\text{std}})^{T} \delta \bar{u} N_{p}^{\text{std}} \bar{p} \; d\Omega \\
- \int_{\Omega} m(B_{u}^{\text{std}})^{T} \delta \bar{u} N_{p}^{\text{enr}} \bar{p}_{a} \; d\Omega \\
- \int_{\Omega} m(B_{u}^{\text{enr}})^{T} \delta \bar{a} N_{p}^{\text{std}} \bar{p} \; d\Omega \\
- \int_{\Omega} m(B_{u}^{\text{enr}})^{T} \delta \bar{a} N_{p}^{\text{enr}} \bar{p}_{a} \; d\Omega \\
+ \int_{\Gamma_{d}} \left[ N_{u}^{\text{enr}} \right]^{T} \delta \bar{a} \left[ D_{F} \left[ N_{u}^{\text{enr}} \right] \right] \; d\Gamma \\
+ \int_{\Gamma_{d}} \left[ N_{u}^{\text{enr}} \right]^{T} \delta \bar{a} \left( -p_{F}. n_{\Gamma_{d}} \right) \; d\Gamma \\
- \int_{\Gamma_{t}} (N_{u}^{\text{std}})^{T} \delta \bar{u} \; d\Gamma - \int_{\Gamma_{t}} (N_{u}^{\text{enr}})^{T} \delta \bar{a} \; d\Gamma \\
= 0
\]  
(A.17)
Assembling the test functions, it gives

\[
\delta \vec{u} \left\{ \int_\Omega (B_u^{\text{std}})^T D B_u^{\text{std}} \, d\Omega + \int_\Omega (B_u^{\text{std}})^T D B_u^{\text{en}} \, \vec{a} \, d\Omega \right\} \\
- \int_\Omega m(B_u^{\text{std}})^T N_p^{\text{std}} \vec{p} \, d\Omega \\
- \int_\Omega m(B_u^{\text{en}})^T N_p^{\text{en}} \vec{c} \, d\Omega - \int_{\Gamma_t} (N_u^{\text{std}})^T \vec{\xi} \, d\Gamma \\
+ \delta \vec{a} \left\{ \int_\Omega (B_u^{\text{en}})^T D B_u^{\text{std}} \, \vec{u} \, d\Omega \right\} \\
+ \int_\Omega (B_u^{\text{en}})^T D B_u^{\text{en}} \, \vec{a} \, d\Omega \\
- \int_\Omega m(B_u^{\text{en}})^T N_p^{\text{std}} \vec{p} \, d\Omega \\
- \int_\Omega m(B_u^{\text{en}})^T N_p^{\text{en}} \vec{p} \, d\Omega \\
+ \int_{\Gamma_d} \left[ N_u^{\text{en}} \right]^T D_F \left[ N_u^{\text{en}} \right] \, \vec{\alpha} \, d\Gamma \\
- \int_{\Gamma_d} \left[ N_u^{\text{en}} \right]^T \left( p_F n_{\Gamma_d} \right) d\Gamma - \int_{\Gamma_t} \left( N_u^{\text{en}} \right)^T \vec{\xi} \, d\Gamma \right\} \\
= 0
\]  

(A.18)

Considering that this condition is valid for any test function, the term within the brackets must equal zero. Arranging the terms into a matrix form, the following relation is obtained

\[
\begin{bmatrix}
K_{uu} & K_{ua} \\
K_{au} & K_{aa}
\end{bmatrix}
\begin{Bmatrix}
\{\vec{u}\} \\
\{\vec{a}\}
\end{Bmatrix}
- \begin{bmatrix}
Q_{up} \\
Q_{ap}
\end{bmatrix}
\begin{Bmatrix}
\vec{p} \\
\vec{\alpha}
\end{Bmatrix}
= \begin{Bmatrix}
f_u^{\text{ext}} \\
f_{alpha}^{\text{ext}}
\end{Bmatrix}
- \begin{Bmatrix}
f_u^{\text{int}} \\
f_{alpha}^{\text{int}}
\end{Bmatrix}
\]  

(A.19)

where

\[
K_{uu} = \int_\Omega (B_u^{\text{std}})^T D B_u^{\text{std}} \, d\Omega
\]  

(A.20)

\[
K_{ua} = \int_\Omega (B_u^{\text{std}})^T D B_u^{\text{en}} \, d\Omega
\]  

(A.21)

\[
K_{au} = \int_\Omega (B_u^{\text{en}})^T D B_u^{\text{std}} \, d\Omega
\]  

(A.22)

\[
K_{aa} = \int_\Omega (B_u^{\text{en}})^T D B_u^{\text{en}} \, d\Omega
\]  

(A.23)

\[
Q_{up} = \int_\Omega (B_u^{\text{std}})^T m N_p^{\text{std}} \, d\Omega
\]  

(A.24)
\[ Q_{uc} = \int_{\Omega} (B_u^{std})^T mN_p^{enr} \, d\Omega \]  \hspace{1cm} (A.25)

\[ Q_{ap} = \int_{\Omega} (B_u^{enr})^T mN_p^{std} \, d\Omega \]  \hspace{1cm} (A.26)

\[ Q_{ac} = \int_{\Omega} (B_u^{enr})^T mN_p^{enr} \, d\Omega \]  \hspace{1cm} (A.27)

\[ f_u^{ext} = -\int_{\Gamma_t} (N_u^{std})^T \, \bar{t} \, d\Gamma \]  \hspace{1cm} (A.28)

\[ f_a^{ext} = \int_{\Gamma_t} (N_u^{enr})^T \, \bar{t} \, d\Gamma \]  \hspace{1cm} (A.29)

\[ f_u^{int} = 0 \]  \hspace{1cm} (A.30)

\[ f_a^{int} = \int_{\Gamma_d} [N_u^{enr}]^T D_F \, \bar{a} \, d\Gamma - \int_{\Gamma_d} [N_u^{enr}]^T (p_F n_{r_d}) \, d\Gamma \]  \hspace{1cm} (A.31)

\[ m = \{1 \quad 1 \quad 0\}^T \]  \hspace{1cm} (A.32)

Generalizing the equations and terms, it gives

\[ [K]\{\bar{U}\} - [Q]\{\bar{P}\} + f_u^{int} - f_u^{ext} = 0 \]  \hspace{1cm} (A.33)

\[ K_{\beta\gamma} = \int_{\Omega} (B_u^\beta)^T D B_u^\gamma \, d\Omega \]  \hspace{1cm} (A.34)

\[ Q_{\beta\zeta} = \int_{\Omega} (B_u^\beta)^T m N_p^\zeta \, d\Omega \]  \hspace{1cm} (A.35)

\[ f_\beta^{int} = \int_{\Gamma_d} [N_u^\beta]^T D_F \, \bar{\beta} \, d\Gamma - \int_{\Gamma_d} [N_u^\beta]^T (p_F n_{r_d}) \, d\Gamma \]  \hspace{1cm} (A.36)

\[ f_\beta^{ext} = \int_{\Gamma_t} (N_u^\beta)^T \, \bar{t} \, d\Gamma \]  \hspace{1cm} (A.37)

For the continuity in the porous region, the replacement of Eqs. (A.4) to (A.16) in Eq. (A.2) gives
\[
\int_{\Omega} \nabla \delta p k_f \nabla d\Omega + \int_{\Gamma_d} \delta p_c (p - p_F) n_{\Gamma_d} d\Gamma + \int_{\Omega} \delta p \nabla \hat{u} d\Omega
\]
\[
+ \int_{\Gamma_w} \delta p \bar{q} d\Gamma
\]
\[
= \int_{\Omega} (B_p^{std})^T \delta \bar{p} k_f B_p^{std} \bar{p} d\Omega
\]
\[
+ \int_{\Omega} (B_p^{std})^T \delta \bar{p} k_f B_p^{enr} \bar{p}_a d\Omega
\]
\[
+ \int_{\Omega} (B_p^{enr})^T \delta \bar{p}_a k_f B_p^{std} \bar{p}_a d\Omega
\]
\[
+ \int_{\Omega} (B_p^{enr})^T \delta \bar{p}_a k_f B_p^{enr} \bar{p}_a d\Omega
\]
\[
+ \int_{\Gamma_d} (N_p^{std})^T \delta \bar{p} c N_p^{std} \bar{p}_a d\Gamma
\]
\[
+ \int_{\Gamma_d} (N_p^{enr})^T \delta \bar{p}_a c N_p^{enr} \bar{p}_a d\Gamma
\]
\[
- \int_{\Gamma_d} (N_p^{std})^T \delta \bar{p} c N_p^{std} \bar{p}_F d\Gamma
\]
\[
- \int_{\Gamma_d} (N_p^{enr})^T \delta \bar{p}_a c N_p^{enr} \bar{p}_F d\Gamma
\]
\[
+ \int_{\Omega} (N_p^{std})^T \delta \bar{p} m B_u^{std} \hat{u} d\Omega
\]
\[
+ \int_{\Omega} (N_p^{enr})^T \delta \bar{p}_a m B_u^{enr} \hat{u} d\Omega
\]
\[
+ \int_{\Omega} (N_p^{enr})^T \delta \bar{p}_a m B_u^{std} \hat{u} d\Omega
\]
\[
+ \int_{\Omega} (N_p^{std})^T \delta \bar{p}_a \bar{q} d\Gamma + \int_{\Gamma_w} (N_p^{enr})^T \delta \bar{p}_a \bar{q} d\Gamma
\]
\[
= 0 \quad (A.38)
\]
Assembling the test functions, it gives

\[
\delta p \left\{ \int_{\Omega} (B_{p}^{\text{std}})^T k_F B_{p}^{\text{std}} \hat{p} \, d\Omega + \int_{\Omega} (B_{p}^{\text{std}})^T k_F B_{p}^{\text{emr}} \bar{p}_a \, d\Omega \\
+ \int_{r_w} (N_{p}^{\text{std}})^T \bar{q} \, d\Gamma + \int_{r_w} (N_{p}^{\text{std}})^T m B_{u}^{\text{std}} \hat{u} \, d\Omega \\
+ \int_{r_d} (N_{p}^{\text{std}})^T m B_{u}^{\text{emr}} \hat{a} \, d\Omega \\
+ \int_{r_d} (N_{p}^{\text{emr}})^T c N_{p}^{\text{std}} \hat{p} \, d\Gamma \\
+ \int_{r_d} (N_{p}^{\text{emr}})^T c N_{p}^{\text{emr}} \bar{p}_a \, d\Gamma \\
- \int_{r_d} (N_{p}^{\text{emr}})^T c N_{p}^{\text{std}} \bar{p}_a \, d\Gamma \right\} \\
+ \delta p \left\{ \int_{\Omega} (B_{p}^{\text{emr}})^T k_F B_{p}^{\text{std}} \hat{p} \, d\Omega \\
+ \int_{\Omega} (N_{p}^{\text{emr}})^T k_F B_{p}^{\text{emr}} \bar{p}_a \, d\Omega + \int_{r_w} (N_{p}^{\text{emr}})^T \bar{q} \, d\Gamma \\
+ \int_{\Omega} (N_{p}^{\text{emr}})^T m B_{u}^{\text{emr}} \hat{u} \, d\Omega \\
+ \int_{\Omega} (N_{p}^{\text{emr}})^T m B_{u}^{\text{emr}} \hat{a} \, d\Omega \\
+ \int_{r_d} (N_{p}^{\text{emr}})^T c N_{p}^{\text{emr}} \hat{p} \, d\Gamma \\
+ \int_{r_d} (N_{p}^{\text{emr}})^T c N_{p}^{\text{emr}} \bar{p}_a \, d\Gamma \\
- \int_{r_d} (N_{p}^{\text{emr}})^T c N_{p}^{\text{std}} \bar{p}_a \, d\Gamma \right\} = 0 
\]

(A.39)

Arranging the terms into a matrix form, the following relation is obtained

\[
\begin{bmatrix} Q_{pu} & Q_{pa} \\ Q_{cu} & Q_{ca} \end{bmatrix} \begin{bmatrix} \hat{u} \\ \hat{a} \end{bmatrix} = \begin{bmatrix} H_{pp} + L_{pp} & H_{pc} + L_{pc} \\ H_{cp} + L_{cp} & H_{cc} + L_{cc} \end{bmatrix} \begin{bmatrix} \bar{p} \\ \bar{p}_a \end{bmatrix} + \begin{bmatrix} L_{ppr} \\ L_{cpp} \end{bmatrix} \begin{bmatrix} \bar{p}_r \\ \bar{p}_a \end{bmatrix} 
\]

(A.40)

where

\[
Q_{pu} = \int_{\Omega} (N_{p}^{\text{std}})^T m B_{u}^{\text{std}} \, d\Omega \\
Q_{pa} = \int_{\Omega} (N_{p}^{\text{std}})^T m B_{u}^{\text{emr}} \, d\Omega 
\]

(A.41)  

(A.42)
Generalizing the equations and terms, it gives
\[
[Q^r]\{\ddot{U}\} + [H + L1]\{\ddot{\mathbf{P}}\} - [L2]\{\dddot{\mathbf{P}}_f\} - q_{ext}^r = 0 \tag{A.57}
\]
\[
H_{\delta\zeta} = \int_{\Omega} (B_p^\delta)^T k_F B_p^\zeta \mathrm{d}\Omega \tag{A.58}
\]
\[
L1_{\delta\zeta} = \int_{\Gamma_d} (N_p^\delta)^T c N_p^\zeta \mathrm{d}\Gamma \tag{A.59}
\]
\[ L2 \delta p_F = \int_{\Gamma_d} \left( N^d_p \right)^T c N_{\text{std}} d\Gamma \] \tag{A.60}

\[ q^\text{ext}_\delta = \int_{\Gamma_w} \left( N^d_p \right)^T \bar{q}_w d\Gamma \] \tag{A.61}

For the continuity in the fracture region, the replacement of Eqs. (A.4) to (A.16) in Eq. (A.3) gives

\[
\int_{\Gamma_d} \frac{\partial \delta p_F}{\partial x'} k_{\text{ff}} \cdot 2h \frac{\partial p_F}{\partial x'} d\Gamma + \int_{\Gamma_d} \delta p_F \cdot c(p_F - p)n_{\Gamma_d} d\Gamma \\
+ \int_{\Gamma_d} \delta p_F \cdot 2h \nabla \bar{u}' \cdot \nabla p_F t_{\Gamma_d} d\Gamma \\
= \int_{\Gamma_d} \left( B^\text{std}_{pF} \right)^T t_{\Gamma_d} \delta \overline{p_F} \left( 2h \right) k_{\text{ff}} \nabla \overline{p_F} t_{\Gamma_d} d\Gamma \\
- \int_{\Gamma_d} \left( N^\text{std}_{pF} \right)^T \delta \overline{p_F} c N^\text{std}_{pF} \left( \overline{p}_p \right) d\Gamma \\
- \int_{\Gamma_d} \left( N^\text{std}_{pF} \right)^T \delta \overline{p_F} c N^\text{enr}_{pF} \left( \overline{p}_a \right) d\Gamma \\
+ \int_{\Gamma_d} \left( N^\text{std}_{pF} \right)^T t_{\Gamma_d} \delta \overline{p_F} \left( 2h \right) \nabla \bar{u} t_{\Gamma_d} d\Gamma \\
+ \int_{\Gamma_d} \left( N^\text{std}_{pF} \right)^T \left[ \overline{\nabla u} \right] n_{\Gamma_d} d\Gamma = 0 \] \tag{A.62}

Assembling the test functions, it gives

\[
\overline{p_F} \left\{ \int_{\Gamma_d} \left( B^\text{std}_{pF} \right)^T t_{\Gamma_d} \left( 2h \right) k_{\text{ff}} \nabla \overline{p_F} t_{\Gamma_d} d\Gamma \\
+ \int_{\Gamma_d} \left( N^\text{std}_{pF} \right)^T t_{\Gamma_d} \left( 2h \right) \nabla \bar{u} t_{\Gamma_d} d\Gamma \\
+ \int_{\Gamma_d} \left( N^\text{std}_{pF} \right)^T \left[ \overline{\nabla u} \right] n_{\Gamma_d} d\Gamma \right\} \tag{A.63}
\]

Arranging the terms into a matrix form, the following relation is obtained
\[
[L_{p_{FP}} \ L_{p_{FC}}] \{\bar{P} \bar{P}_d\} + [H_{p_{FPF}} + L_{p_{FPF}}]\{\bar{P}_F\} = q_{p_{F}}^{\text{int}} \tag{A.64}
\]

where

\[
H_{p_{FPF}} = \int_{\Gamma_d} (B_{p_{FP}}^{\text{std}})^T t_{\Gamma_d} (2h) k_{\Gamma_d} \nabla \bar{P}_F t_{\Gamma_d} d\Gamma \tag{A.65}
\]

\[
L_{p_{FPF}} = \int_{\Gamma_d} (N_{p_{FP}}^{\text{std}})^T c N_{p_{FP}}^{\text{std}} d\Gamma \tag{A.66}
\]

\[
L_{p_{FPF}} = \int_{\Gamma_d} (N_{p_{FP}}^{\text{std}})^T c N_{p}^{\text{std}} d\Gamma \tag{A.67}
\]

\[
L_{p_{FC}} = \int_{\Gamma_d} (N_{p_{FP}}^{\text{std}})^T c N_{p}^{\text{env}} d\Gamma \tag{A.68}
\]

\[
q_{p_{F}}^{\text{int}} = \int_{\Gamma_d} (N_{p_{FP}}^{\text{std}})^T t_{\Gamma_d} (2h) (\nabla \hat{u}) t_{\Gamma_d} d\Gamma
\]
\[+ \int_{\Gamma_d} (N_{p_{FP}}^{\text{std}})^T [\hat{u}] n_{\Gamma_d} d\Gamma \tag{A.69}\]

Generalizing the equations and terms, it gives

\[-[L_{2}^T]\{\bar{P}\} + [H_{F} + L_{3}]\{\bar{P}_F\} - q_{p_{F}}^{\text{int}} = 0 \tag{A.70}\]

where

\[
L_{2}^{P_{F}} = \int_{\Gamma_d} (N_{p}^{\text{std}})^T c N_{p_{FP}}^{\text{std}} d\Gamma \tag{A.71}
\]

\[
L_{3} = \int_{\Gamma_d} (N_{p_{FP}}^{\text{std}})^T c N_{p_{FP}}^{\text{std}} d\Gamma \tag{A.72}
\]

\[
H_{F} = \int_{\Gamma_d} (B_{p_{FP}}^{\text{std}})^T t_{\Gamma_d} (2h) k_{\Gamma_d} B_{p_{FP}}^{\text{std}} t_{\Gamma_d} d\Gamma \tag{A.73}
\]

\[
q_{p_{F}}^{\text{int}} = \int_{\Gamma_d} (N_{p_{FP}}^{\text{std}})^T t_{\Gamma_d} (2h) (\nabla \hat{u}) t_{\Gamma_d} d\Gamma
\]
\[+ \int_{\Gamma_d} (N_{p_{FP}}^{\text{std}})^T [\hat{u}] n_{\Gamma_d} d\Gamma \tag{A.74}\]

The values related with velocity are defined as
\[ \langle \nabla \ddot{u} \rangle = \nabla \ddot{u}^+ + \nabla \ddot{u}^- \]

\[
= \frac{1}{2} \left[ (B_u^{\text{std}} \ddot{u} + B_u^{\text{env}} \dot{\alpha})^+ + (B_u^{\text{std}} \ddot{u} + B_u^{\text{env}} \dot{\alpha})^- \right]
\]

\[
= \frac{1}{2} \left[ 2B_u^{\text{std}} \ddot{u} + B_u^{\text{std}}(H^+ + H^-) \dot{\alpha} \right]
\]

\[
= B_u^{\text{std}} (\ddot{u} + \ddot{u}^+ + \ddot{u}^-) + B_u^{\text{std}} \langle H \rangle \dot{\alpha}
\]

\[
= B_u^{\text{std}} \frac{(\bar{u}_n - \bar{u}_{n-1})}{\Delta t} + B_u^{\text{std}} \langle H \rangle \frac{(\bar{\alpha}_n - \bar{\alpha}_{n-1})}{\Delta t}
\]

\[ \llbracket \ddot{u} \rrbracket = \hat{u}^+ - \hat{u}^-
\]

\[
= \left( N_u^{\text{std}^+} \ddot{u} + N_u^{\text{std}^+} H^+ \dot{\alpha} \right)
- \left( N_u^{\text{std}^-} \ddot{u} + N_u^{\text{std}^-} H^- \dot{\alpha} \right)
= N_u^{\text{std}^+} \llbracket H \rrbracket \dot{\alpha}
\]

\[ \text{(A.76)} \]

Given that

\[ N_u^{\text{std}^+} = N_u^{\text{std}^-} = N_u^{\text{std}} \]

\[ N_u^{\text{env}^+} = N_u^{\text{std}^+} H^+ = N_u^{\text{std}} H^+ \]

\[ N_u^{\text{env}^-} = N_u^{\text{std}^-} H^- = N_u^{\text{std}} H^- \]

\[ B_u^{\text{std}^+} = B_u^{\text{std}^-} = B_u^{\text{std}} \]

\[ B_u^{\text{env}^+} = B_u^{\text{std}^+} H^+ = B_u^{\text{std}} H^+ \]

\[ B_u^{\text{env}^-} = B_u^{\text{std}^-} H^- = B_u^{\text{std}} H^- \]

\[ \text{(A.77)} \]

\[ \text{(A.78)} \]

\[ \text{(A.79)} \]

\[ \text{(A.80)} \]

\[ \text{(A.81)} \]

\[ \text{(A.82)} \]

Where \( H^+ \) and \( H^- \) represent the values of the enrichment function \( H \) in the fracture top and bottom faces, respectively. In \( \bar{u}_n - \bar{u}_{n-1} \), \( n \) represents the current increment and \( n-1 \) the previous one.

Generalizing a number of degrees of freedom equal to \( \text{ndof} \), it gives

\[ \langle \nabla \ddot{u} \rangle = \sum_k^{\text{ndof}} \langle B_u^k \rangle \frac{(\bar{k}_n - \bar{k}_{n-1})}{\Delta t} \]

\[ \text{(A.83)} \]

\[ \llbracket \ddot{u} \rrbracket = \sum_k^{\text{ndof}} \llbracket N_u^k \rrbracket \frac{(\bar{k}_n - \bar{k}_{n-1})}{\Delta t} \]

\[ \text{(A.84)} \]

with \( \langle B_u^k \rangle = B_u^{\text{std}} \) and \( \langle B_u^a \rangle = B_u^{\text{std}} \langle H_a \rangle \)
Annex B
Newton-Raphson Algorithm

The Jacobian of derivatives is given by

\[
J = \begin{bmatrix}
\frac{\partial \psi_U}{\partial \mathbf{U}} & \frac{\partial \psi_U}{\partial \mathbf{P}} & \frac{\partial \psi_U}{\partial \mathbf{F}} \\
\frac{\partial \psi_F}{\partial \mathbf{U}} & \frac{\partial \psi_F}{\partial \mathbf{P}} & \frac{\partial \psi_F}{\partial \mathbf{F}} \\
\frac{\partial \psi_{\mathbf{F}_F}}{\partial \mathbf{U}} & \frac{\partial \psi_{\mathbf{F}_F}}{\partial \mathbf{P}} & \frac{\partial \psi_{\mathbf{F}_F}}{\partial \mathbf{F}}
\end{bmatrix}
\]

(B.1)

Multiplying the second and third lines for \( \Delta \) it gives

\[
J = \begin{bmatrix}
\frac{\partial f_U^{\text{int}}}{\partial \mathbf{U}} & \frac{\partial f_U^{\text{int}}}{\partial \mathbf{P}} & \frac{\partial f_U^{\text{int}}}{\partial \mathbf{F}} \\
\frac{\partial q_{\mathbf{F}_F}^{\text{int}}}{\partial \mathbf{U}} & \frac{\partial q_{\mathbf{F}_F}^{\text{int}}}{\partial \mathbf{P}} & \frac{\partial q_{\mathbf{F}_F}^{\text{int}}}{\partial \mathbf{F}}
\end{bmatrix}
\]

(B.2)

Re-scaling the problem formulation for one standard \( u \) and one enriched degree of freedom \( a \), the derivatives for the mechanical equation are
\[
\frac{\partial f_{U}^{\text{int}}}{\partial \bar{U}} = \left\{ \frac{\partial}{\partial \bar{u}} \right\} \left\{ \int_{\Gamma_d} \left[ N_{\text{enr}}^u \right]^T (D_F \bar{a} - p_F n_{r_d}) d\Gamma \right\}^T = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \int_{\Gamma_d} \left[ N_{\text{enr}}^u \right]^T D_F \left[ N_{\text{enr}}^u \right] d\Gamma = \begin{bmatrix} 0 \\ T_a \end{bmatrix}
\] (B.3)

\[
\frac{\partial f_{U}^{\text{int}}}{\partial \bar{p}} = \left\{ \frac{\partial}{\partial \bar{p}} \right\} \left\{ \int_{\Gamma_d} \left[ N_{\text{enr}}^u \right]^T (D_F \bar{a} - p_F n_{r_d}) d\Gamma \right\}^T = 0
\] (B.4)

\[
\frac{\partial f_{U}^{\text{int}}}{\partial \bar{p}_F} = \frac{\partial}{\partial \bar{p}_F} \left\{ \int_{\Gamma_d} \left[ N_{\text{enr}}^u \right]^T (D_F \bar{a} - p_F n_{r_d}) d\Gamma \right\} = \left\{ -\int_{\Gamma_d} \left[ N_{\text{enr}}^u \right]^T n_{r_d} N_{p_F}^{\text{std}} d\Gamma \right\} = \{-Q_{ap_F}\}
\] (B.5)

The derivatives for the continuity equation in the fracture are

\[
\frac{\partial q_{p_F}^{\text{int}}}{\partial \bar{U}} = \left\{ \frac{\partial}{\partial \bar{u}} \right\} \left\{ \int_{\Gamma_d} \left( N_{\text{enr}}^{\text{std}} \right)^T \left( t_{r_d} (2h)(\nabla \bar{u}) \right) t_{r_d} d\Gamma + \int_{\Gamma_d} \left( N_{\text{enr}}^{\text{std}} \right)^T [\bar{u}] n_{r_d} d\Gamma \right\} = \left\{ \begin{bmatrix} \int_{\Gamma_d} \left( N_{\text{enr}}^{\text{std}} \right)^T \left( B_{u}^{\text{enr}}(H) \frac{1}{\Delta t} \right) t_{r_d} d\Gamma \\ \int_{\Gamma_d} \left( N_{\text{enr}}^{\text{std}} \right)^T \left( 2h \right) t_{r_d} d\Gamma + \int_{\Gamma_d} \left( N_{\text{enr}}^{\text{std}} \right)^T [\bar{u}] n_{r_d} d\Gamma \end{bmatrix} \right\} = \left\{ \begin{bmatrix} \frac{1}{\Delta t} S_{p_F u} \\ \frac{1}{\Delta t} S_{p_F a} + \frac{1}{\Delta t} V_{p_F a} \end{bmatrix} \right\}
\] (B.6)

\[
\frac{\partial d_{p_F}^{\text{int}}}{\partial \bar{p}} = \left\{ \frac{\partial}{\partial \bar{p}} \right\} \left\{ \int_{\Gamma_d} \left( N_{\text{enr}}^{\text{std}} \right)^T t_{r_d} (2h)(\nabla \bar{u}) \right\} = 0
\] (B.7)

\[
\frac{\partial q_{p_F}^{\text{int}}}{\partial \bar{p}_F} = \frac{\partial}{\partial \bar{p}_F} \left\{ \int_{\Gamma_d} \left( N_{\text{enr}}^{\text{std}} \right)^T \left( t_{r_d} (2h)(\nabla \bar{u}) \right) t_{r_d} d\Gamma + \int_{\Gamma_d} \left( N_{\text{enr}}^{\text{std}} \right)^T [\bar{u}] n_{r_d} d\Gamma \right\} = 0
\] (B.8)

Substituting the derivatives in the Jacobian, it gives
If both porous and fracture material constitutive behaviour are such that their matrices $K$ and $T$ are symmetric, the Jacobian may be symmetric if the following simplifications are considered:

- The lines relative to pore and fracture pressures (third, fourth and fifth lines) are multiplied by -1
- $S_{Pa} = 0$
- $(S_{Pa} + V_{Pa}) = Q_{Pa} = Q_{app}^T$

The resulting Jacobian matrix is then given by

$$
J = \begin{bmatrix}
K_{uu} & K_{ua} & -Q_{up} & -Q_{uc} & 0 \\
K_{au} & K_{aa} + T_a & -Q_{ap} & -Q_{ac} & -Q_{app} \\
Q_{pu} & Q_{pa} & \Delta t(H_{pp} + L_{pp}) & \Delta t(H_{pc} + L_{pc}) & \Delta t.L_{ppf} \\
Q_{cu} & Q_{ca} & \Delta t(H_{cp} + L_{cp}) & \Delta t(H_{cc} + L_{cc}) & \Delta t.L_{cpp} \\
-S_{Pf, u} - (S_{Pf, a} + V_{Pf, a}) & \Delta t.L_{ppf} & \Delta t.L_{ppc} & \Delta t(H_{ppf} + L_{ppf}) &
\end{bmatrix}
$$

(B.9)

Generalizing the terms, it gives

$$
J = \begin{bmatrix}
K + T & -Q & -Q_F \\
-Q^T & -\Delta t(H + L_1) & \Delta t.L_2 \\
-Q_F^T & \Delta t.L_2^T & -\Delta t(H_F + L_3)
\end{bmatrix}
$$

(B.10)

Where the matrices are given by Eqs. (A.34), (A.35), (A.58) to (A.60), (A.71) to (A.73), and

$$
T_{\beta Y} = \int_{\Gamma_d} \left[ N_u^\beta \right]^T D_F \left[ N_u^\gamma \right] d\Gamma
$$

(B.12)

$$
Q_{Fpp,F} = \int_\Omega \left[ N_u^\beta \right]^T n_{\Gamma_d} N_{pp}^{std} d\Omega
$$

(B.13)