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Annex A
Resulting space discretization

The weak formulation of the differential equations gives the following equalities (repeated from equations presented in Chapter 3 of the main document):

$$\int_{\Omega} \delta\varepsilon.\sigma'\, d\Omega - \int_{\Omega} \delta\varepsilon.m.p\, d\Omega + \int_{\Gamma_d} [\![\delta u]\!]\,(t_F - p_F.n_{\Gamma d})d\Gamma - \int_{\Gamma_t} \delta u.\bar{t}\, d\Gamma = 0 \tag{A.1}$$

$$\int_{\Omega} \nabla\delta p k_f \nabla p\, d\Omega + \int_{\Gamma_d} \delta p [\![\dot{w}]\!] n_{\Gamma d}\, d\Gamma + \int_{\Omega} \delta p.\nabla\dot{u}\, d\Omega + \int_{\Gamma_w} \delta p.\bar{q}\, d\Gamma = 0 \tag{A.2}$$

$$\int_{\Gamma_d} \frac{\partial\delta p_F}{\partial x'} k_{f_F}.2\mathrm{h}.\frac{\partial p_F}{\partial x'} d\Gamma - \int_{\Gamma_d} \delta p_F \mathrm{q}_F n_{\Gamma d}\, d\Gamma + \int_{\Gamma_d} \delta p_F.2\mathrm{h}.\langle\frac{\partial \dot{u}_{x'}}{\partial x'}\rangle\, d\Gamma + \int_{\Gamma_d} \delta p_F.[\![\dot{u}_{y'}]\!]\, d\Gamma = 0 \tag{A.3}$$

It may be admitted that the test functions δu, δp and δp_F follow the same discretization rules as the variables u, p and p_F. It is also considered that the vector of the nodal variables for each element node is given by $\bar{u}$. Although generalized for any number of enriched degrees of freedom, for the sake of clearness the discretization is developed for one enriched displacement variable a and one enriched pressure variable p_a. Eq. (A.4) to Eq. (A.16) present the discretization of the variables and their derivatives.

$$u = N_u^{std}\bar{u} + N_u^{enr}\bar{a} \tag{A.4}$$

$$\delta u = N_u^{std}\delta\bar{u} + N_u^{enr}\delta\bar{a} \tag{A.5}$$

$$[\![u]\!] = [\![N_u^{std}]\!]\bar{u} + [\![N_u^{enr}]\!]\bar{a} = [\![N_u^{enr}]\!]\bar{a} \tag{A.6}$$

$$\varepsilon = B_u^{std}\bar{u} + B_u^{enr}\bar{a} \tag{A.7}$$

$$\delta\varepsilon = B_u^{std}\delta\bar{u} + B_u^{enr}\delta\bar{a} \tag{A.8}$$

$$\nabla\dot{u} = B_u^{std}\dot{\bar{u}} + B_u^{enr}\dot{\bar{a}} \tag{A.9}$$

$$p = N_p^{std}\bar{p} + N_p^{enr}\overline{p_a} \tag{A.10}$$

$$\delta p = N_p^{std}\delta\bar{p} + N_p^{enr}\delta\overline{p_a} \tag{A.11}$$

$$\nabla p = B_p^{std}\bar{p} + B_p^{enr}\overline{p_a} \tag{A.12}$$

$$\delta\nabla p = B_p^{std}\delta\bar{p} + B_p^{enr}\delta\overline{p_a} \tag{A.13}$$

$$p_F = N_{p_F}^{std}\overline{p_F} \tag{A.14}$$

$$\delta p_F = N_{p_F}^{std}\delta\overline{p_F} \tag{A.15}$$

$$\nabla p_F = B_{p_F}^{std}\overline{p_F} \tag{A.16}$$

Replacing the variables in Eq. (3.14), the following equation is obtained

$$\begin{aligned}
&\int_\Omega \delta\varepsilon.D.\varepsilon\, d\Omega - \int_\Omega \delta\varepsilon.m.p\, d\Omega + \int_{\Gamma_d} [\![\delta u]\!]\,(t_F - p_F.n_{\Gamma d})d\Gamma \\
&\quad - \int_{\Gamma_t} \delta u.\bar{t}\, d\Gamma \\
&\quad = \int_\Omega (B_u^{std})^T\delta\bar{u}DB_u^{std}\bar{u}\, d\Omega \\
&\quad + \int_\Omega (B_u^{std})^T\delta\bar{u}DB_u^{enr}\bar{a}\, d\Omega \\
&\quad + \int_\Omega (B_u^{enr})^T\delta\bar{a}DB_u^{std}\bar{u}\, d\Omega \\
&\quad + \int_\Omega (B_u^{enr})^T\delta\bar{a}DB_u^{enr}\bar{a}\, d\Omega \\
&\quad - \int_\Omega m(B_u^{std})^T\delta\bar{u}N_p^{std}\bar{p}\, d\Omega \\
&\quad - \int_\Omega m(B_u^{std})^T\delta\bar{u}N_p^{enr}\overline{p_a}\, d\Omega \\
&\quad - \int_\Omega m(B_u^{enr})^T\delta\bar{a}N_p^{std}\bar{p}\, d\Omega \\
&\quad - \int_\Omega m(B_u^{enr})^T\delta\bar{a}N_p^{enr}\overline{p_a}\, d\Omega \\
&\quad + \int_{\Gamma_d} [\![N_u^{enr}]\!]^T\,\delta\bar{a}\, D_F\, [\![N_u^{enr}]\!]\bar{a}\, d\Gamma \\
&\quad + \int_{\Gamma_d} [\![N_u^{enr}]\!]^T\,\delta\bar{a}\,(-p_F n_{\Gamma d})d\Gamma \\
&\quad - \int_{\Gamma_t} (N_u^{std})^T\,\delta\bar{u}\,\bar{t}\, d\Gamma - \int_{\Gamma_t} (N_u^{enr})^T\,\delta\bar{a}\,\bar{t}\, d\Gamma \\
&\quad = 0
\end{aligned} \tag{A.17}$$

Assembling the test functions, it gives

$$\begin{aligned}\delta\bar{u}\Bigg\{&\int_{\Omega} (B_u^{std})^T D B_u^{std}\bar{u}\, d\Omega + \int_{\Omega} (B_u^{std})^T D B_u^{enr}\bar{a}\, d\Omega \\ &- \int_{\Omega} m(B_u^{std})^T N_p^{std}\bar{p}\, d\Omega \\ &- \int_{\Omega} m(B_u^{std})^T N_p^{enr}\bar{c}\, d\Omega - \int_{\Gamma_t} (N_u^{std})^T\ \bar{t}\, d\Gamma\Bigg\} \\ &+ \delta\bar{a}\Bigg\{\int_{\Omega} (B_u^{enr})^T D B_u^{std}\bar{u}\, d\Omega \\ &+ \int_{\Omega} (B_u^{enr})^T D B_u^{enr}\bar{a}\, d\Omega \\ &- \int_{\Omega} m(B_u^{enr})^T N_p^{std}\bar{p}\, d\Omega \\ &- \int_{\Omega} m(B_u^{enr})^T N_p^{enr}\overline{p_a}\, d\Omega \\ &+ \int_{\Gamma_d} [\![N_u^{enr}]\!]^T D_F\, [\![N_u^{enr}]\!]\, \bar{a}\, d\Gamma \\ &- \int_{\Gamma_d} [\![N_u^{enr}]\!]^T (p_F n_{\Gamma d}) d\Gamma - \int_{\Gamma_t} (N_u^{enr})^T\, \bar{t}\, d\Gamma\Bigg\} \\ &= 0\end{aligned} \tag{A.18}$$

Considering that this condition is valid for any test function, the term within the brackets must equal zero. Arranging the terms into a matrix form, the following relation is obtained

$$\begin{bmatrix} K_{uu} & K_{ua} \\ K_{au} & K_{aa}\end{bmatrix}\begin{Bmatrix}\bar{u} \\ \bar{a}\end{Bmatrix} - \begin{bmatrix} Q_{up} & Q_{uc} \\ Q_{ap} & Q_{ac}\end{bmatrix}\begin{Bmatrix}\bar{p} \\ \overline{p_a}\end{Bmatrix} = \begin{Bmatrix} f_u^{ext} \\ f_a^{ext}\end{Bmatrix} - \begin{Bmatrix} f_u^{int} \\ f_a^{int}\end{Bmatrix} \tag{A.19}$$

where

$$K_{uu} = \int_{\Omega} (B_u^{std})^T D B_u^{std}\, d\Omega \tag{A.20}$$

$$K_{ua} = \int_{\Omega} (B_u^{std})^T D B_u^{enr}\, d\Omega \tag{A.21}$$

$$K_{au} = \int_{\Omega} (B_u^{enr})^T D B_u^{std}\, d\Omega \tag{A.22}$$

$$K_{aa} = \int_{\Omega} (B_u^{enr})^T D B_u^{enr}\, d\Omega \tag{A.23}$$

$$Q_{up} = \int_{\Omega} (B_u^{std})^T\, m\, N_p^{std}\, d\Omega \tag{A.24}$$

$$Q_{uc} = \int_{\Omega} (B_u^{std})^T m N_p^{enr} \, d\Omega \tag{A.25}$$

$$Q_{ap} = \int_{\Omega} (B_u^{enr})^T m N_p^{std} \, d\Omega \tag{A.26}$$

$$Q_{ac} = \int_{\Omega} (B_u^{enr})^T m N_p^{enr} \, d\Omega \tag{A.27}$$

$$f_u^{ext} = -\int_{\Gamma_t} (N_u^{std})^T \; \bar{t} \, d\Gamma \tag{A.28}$$

$$f_a^{ext} = \int_{\Gamma_t} (N_u^{enr})^T \, \bar{t} \, d\Gamma \tag{A.29}$$

$$f_u^{int} = 0 \tag{A.30}$$

$$f_a^{int} = \int_{\Gamma_d} [\![N_u^{enr}]\!]^T D_F \, \bar{a} \, d\Gamma - \int_{\Gamma_d} [\![N_u^{enr}]\!]^T (p_F n_{\Gamma d}) d\Gamma \tag{A.31}$$

$$m = \{1 \quad 1 \quad 0\}^T \tag{A.32}$$

Generalizing the equations and terms, it gives

$$[K]\{\bar{\mathbb{U}}\} - [Q]\{\bar{\mathbb{P}}\} + f_{\mathbb{U}}^{int} - f_{\mathbb{U}}^{ext} = 0 \tag{A.33}$$

$$K_{\beta\gamma} = \int_{\Omega} \left(B_u^{\beta}\right)^T D B_u^{\gamma} \, d\Omega \tag{A.34}$$

$$Q_{\beta\zeta} = \int_{\Omega} \left(B_u^{\beta}\right)^T m \, N_p^{\zeta} \, d\Omega \tag{A.35}$$

$$f_{\beta}^{int} = \int_{\Gamma_d} \left[\!\left[N_u^{\beta}\right]\!\right]^T D_F \, \bar{\beta} \, d\Gamma - \int_{\Gamma_d} \left[\!\left[N_u^{\beta}\right]\!\right]^T (p_F n_{\Gamma d}) d\Gamma \tag{A.36}$$

$$f_{\beta}^{ext} = \int_{\Gamma_t} \left(N_u^{\beta}\right)^T \; \bar{t} \, d\Gamma \tag{A.37}$$

For the continuity in the porous region, the replacement of Eqs. (A.4) to (A.16) in Eq. (A.2) gives

$$
\begin{aligned}
\int_{\Omega} \nabla\delta p k_f \nabla p \, d\Omega + \int_{\Gamma_d} \delta p.\mathrm{c}(p - p_F) n_{\Gamma d} \, d\Gamma + \int_{\Omega} \delta p. \nabla\dot{u} \, d\Omega \\
+ \int_{\Gamma_w} \delta p.\bar{q} \, d\Gamma \\
= \int_{\Omega} \left(B_p^{std}\right)^T \delta\bar{p} \, k_f \, B_p^{std} \bar{p} \, d\Omega \\
+ \int_{\Omega} \left(B_p^{std}\right)^T \delta\bar{p} \, k_f \, B_p^{enr} \overline{p_a} \, d\Omega \\
+ \int_{\Omega} \left(B_p^{enr}\right)^T \delta\overline{p_a} \, k_f B_p^{std} \bar{p} \, d\Omega \\
+ \int_{\Omega} \left(B_p^{enr}\right)^T \delta\overline{p_a} \, k_f \, B_p^{enr} \overline{p_a} \, d\Omega \\
+ \int_{\Gamma_d} \left(N_p^{std}\right)^T \delta\bar{p} \, c \, N_p^{std} \, \bar{p} \, d\Gamma \\
+ \int_{\Gamma_d} \left(N_p^{std}\right)^T \delta\bar{p} \, c \, N_p^{enr} \, \overline{p_a} \, d\Gamma \\
+ \int_{\Gamma_d} \left(N_p^{enr}\right)^T \delta\overline{p_a} \, c \, N_p^{std} \, \bar{p} \, d\Gamma \\
+ \int_{\Gamma_d} \left(N_p^{enr}\right)^T \delta\overline{p_a} \, c \, N_p^{enr} \, \overline{p_a} \, d\Gamma \\
- \int_{\Gamma_d} \left(N_p^{std}\right)^T \delta\bar{p} \, c \, N_{p_F}^{std} \, \overline{p_F} \, d\Gamma \\
- \int_{\Gamma_d} \left(N_p^{enr}\right)^T \delta\overline{p_a} \, c \, N_{p_F}^{std} \, \overline{p_F} \, d\Gamma \\
+ \int_{\Omega} \left(N_p^{std}\right)^T \delta\bar{p} \, m \, B_u^{std} \dot{\bar{u}} \, d\Omega \\
+ \int_{\Omega} \left(N_p^{std}\right)^T \delta\bar{p} \, m \, B_u^{enr} \dot{\bar{a}} \, d\Omega \\
+ \int_{\Omega} \left(N_p^{enr}\right)^T \delta\overline{p_a} \, m \, B_u^{std} \dot{\bar{u}} \, d\Omega \\
+ \int_{\Omega} \left(N_p^{enr}\right)^T \delta\overline{p_a} \, m \, B_u^{enr} \dot{\bar{a}} \, d\Omega \\
+ \int_{\Gamma_w} \left(N_p^{std}\right)^T \delta\bar{p} \, \bar{q} \, d\Gamma + \int_{\Gamma_w} \left(N_p^{enr}\right)^T \delta\overline{p_a} \, \bar{q} \, d\Gamma \\
= 0
\end{aligned}
\tag{A.38}
$$

Assembling the test functions, it gives

$$
\begin{aligned}
\delta\bar{p}\Bigg\{\int_{\Omega} & \left(B_p^{std}\right)^T k_F B_p^{std}\bar{p}\, d\Omega + \int_{\Omega} \left(B_p^{std}\right)^T k_F B_p^{enr}\overline{p_a}\, d\Omega \\
& + \int_{\Gamma_w} \left(N_p^{std}\right)^T \bar{q}\, d\Gamma + \int_{\Omega} \left(N_p^{std}\right)^T m B_u^{std}\dot{\bar{u}}\, d\Omega \\
& + \int_{\Omega} \left(N_p^{std}\right)^T m B_u^{enr}\dot{\bar{a}}\, d\Omega \\
& + \int_{\Gamma_d} \left(N_p^{std}\right)^T c\, N_p^{std}\, \bar{p}\, d\Gamma \\
& + \int_{\Gamma_d} \left(N_p^{std}\right)^T c\, N_p^{enr}\, \overline{p_a}\, d\Gamma \\
& - \int_{\Gamma_d} \left(N_p^{std}\right)^T c\, N_{p_F}^{std}\, \overline{p_F}\, d\Gamma \Bigg\} \\
& + \delta\overline{p_a}\Bigg\{\int_{\Omega} \left(B_p^{enr}\right)^T k_F B_p^{std}\bar{p}\, d\Omega \\
& + \int_{\Omega} \left(B_p^{enr}\right)^T k_F B_p^{enr}\overline{p_a}\, d\Omega + \int_{\Gamma_w} \left(N_p^{enr}\right)^T \bar{q}\, d\Gamma \\
& + \int_{\Omega} \left(N_p^{enr}\right)^T m B_u^{std}\dot{\bar{u}}\, d\Omega \\
& + \int_{\Omega} \left(N_p^{enr}\right)^T m B_u^{enr}\dot{\bar{a}}\, d\Omega \\
& + \int_{\Gamma_d} \left(N_p^{enr}\right)^T c\, N_p^{std}\, \bar{p}\, d\Gamma \\
& + \int_{\Gamma_d} \left(N_p^{enr}\right)^T c\, N_p^{enr}\, \overline{p_a}\, d\Gamma \\
& - \int_{\Gamma_d} \left(N_p^{enr}\right)^T c\, N_{p_F}^{std}\, \overline{p_F}\, d\Gamma \Bigg\} = 0
\end{aligned}
\tag{A.39}
$$

Arranging the terms into a matrix form, the following relation is obtained

$$
\begin{aligned}
\begin{bmatrix} Q_{pu} & Q_{pa} \\ Q_{cu} & Q_{ca} \end{bmatrix}\begin{Bmatrix} \dot{\bar{u}} \\ \dot{\bar{a}} \end{Bmatrix} + \begin{bmatrix} H_{pp} + L_{pp} & H_{pc} + L_{pc} \\ H_{cp} + L_{cp} & H_{cc} + L_{cc} \end{bmatrix}\begin{Bmatrix} \bar{p} \\ \overline{p_a} \end{Bmatrix} + \begin{bmatrix} L_{pp_F} \\ L_{cp_F} \end{bmatrix}\{\overline{p_F}\} \\
= \begin{Bmatrix} q_p^{ext} \\ q_c^{ext} \end{Bmatrix}
\end{aligned}
\tag{A.40}
$$

where

$$Q_{pu} = \int_{\Omega} \left(N_p^{std}\right)^T m\, B_u^{std}\, d\Omega \tag{A.41}$$

$$Q_{pa} = \int_{\Omega} \left(N_p^{std}\right)^T m\, B_u^{enr}\, d\Omega \tag{A.42}$$

$$Q_{cu} = \int_{\Omega} \left(N_p^{enr}\right)^T m\, B_u^{std}\, d\Omega \tag{A.43}$$

$$Q_{ca} = \int_{\Omega} \left(N_p^{enr}\right)^T m\, B_u^{enr}\, d\Omega \tag{A.44}$$

$$H_{pp} = \int_{\Omega} \left(B_p^{std}\right)^T k_F\, B_p^{std}\, d\Omega \tag{A.45}$$

$$H_{pc} = \int_{\Omega} \left(B_p^{std}\right)^T k_F\, B_p^{enr}\, d\Omega \tag{A.46}$$

$$H_{cp} = \int_{\Omega} \left(B_p^{enr}\right)^T k_F B_p^{std}\, d\Omega \tag{A.47}$$

$$H_{cc} = \int_{\Omega} \left(B_p^{enr}\right)^T k_F\, B_p^{enr}\, d\Omega \tag{A.48}$$

$$L_{pp_F} = \int_{\Gamma_d} \left(N_p^{std}\right)^T c\, N_{p_F}^{std}\, d\Gamma \tag{A.49}$$

$$L_{cp_F} = \int_{\Gamma_d} \left(N_p^{enr}\right)^T c\, N_{p_F}^{std}\, d\Gamma \tag{A.50}$$

$$L_{pp} = \int_{\Gamma_d} \left(N_p^{std}\right)^T c\, N_p^{std}\, d\Gamma \tag{A.51}$$

$$L_{pc} = \int_{\Gamma_d} \left(N_p^{std}\right)^T c\, N_p^{enr}\, d\Gamma \tag{A.52}$$

$$L_{cp} = \int_{\Gamma_d} \left(N_p^{enr}\right)^T c\, N_p^{std}\, d\Gamma \tag{A.53}$$

$$L_{cc} = \int_{\Gamma_d} \left(N_p^{enr}\right)^T c\, N_p^{enr}\, d\Gamma \tag{A.54}$$

$$q_p^{ext} = \int_{\Gamma_w} \left(N_p^{std}\right)^T \bar{q}\, d\Gamma \tag{A.55}$$

$$q_c^{ext} = \int_{\Gamma_w} \left(N_p^{enr}\right)^T \bar{q}\, d\Gamma \tag{A.56}$$

Generalizing the equations and terms, it gives

$$[Q^T]\left\{\dot{\bar{\mathbb{U}}}\right\} + [H + L1]\{\bar{\mathbb{P}}\} - [L2]\{\bar{\mathbb{P}}_F\} - q_{\mathbb{P}}^{ext} = 0 \tag{A.57}$$

$$H_{\delta\zeta} = \int_{\Omega} \left(B_p^{\delta}\right)^T k_F\, B_p^{\zeta}\, d\Omega \tag{A.58}$$

$$L1_{\delta\zeta} = \int_{\Gamma_d} \left(N_p^{\delta}\right)^T c\, N_P^{\zeta}\, d\Gamma \tag{A.59}$$

$$L2_{\delta p_F} = \int_{\Gamma_d} \left(N_p^{\delta}\right)^T c\, N_{p_F}^{std}\; d\Gamma \tag{A.60}$$

$$q_{\delta}^{ext} = \int_{\Gamma_w} \left(N_p^{\delta}\right)^T \bar{q}_w\, d\Gamma \tag{A.61}$$

For the continuity in the fracture region, the replacement of Eqs. (A.4) to (A.16) in Eq. (A.3) gives

$$\begin{aligned}
&\int_{\Gamma_d} \frac{\partial \delta p_F}{\partial x'} k_{f_F}.\,2\mathrm{h}.\frac{\partial p_F}{\partial x'}\, d\Gamma + \int_{\Gamma_d} \delta p_F.\,\mathrm{c}(p_F - p) n_{\Gamma d}\, d\Gamma \\
&\quad + \int_{\Gamma_d} \delta p_F.\,2\mathrm{h}.\langle \frac{\partial \dot{u}_{x'}}{\partial x'} \rangle\, d\Gamma + \int_{\Gamma_d} \delta p_F.\,[\![\dot{u}_{y'}]\!]\, d\Gamma \\
&\quad = \int_{\Gamma_d} \left(B_{p_F}^{std}\right)^T t_{\Gamma_d}\, \delta\overline{p_F}\, (2\mathrm{h}) k_{fF}\, \nabla\overline{p_F}\, t_{\Gamma_d}\; d\Gamma \\
&\quad - \int_{\Gamma_d} \left(N_{p_F}^{std}\right)^T \delta\overline{p_F}\, c\, N_p^{std}\, \bar{p}\, d\Gamma \\
&\quad - \int_{\Gamma_d} \left(N_{p_F}^{std}\right)^T \delta\overline{p_F}\, c\, N_p^{enr}\, \overline{p_a}\, d\Gamma \\
&\quad + \int_{\Gamma_d} \left(N_{p_F}^{std}\right)^T \delta\overline{p_F}\, c\, N_{p_F}^{std}\, \overline{p_F}\, d\Gamma \\
&\quad + \int_{\Gamma_d} \left(N_{p_F}^{std}\right)^T t_{\Gamma_d} \delta\overline{p_F}\, (2\mathrm{h}) \langle \nabla\dot{u} \rangle\, t_{\Gamma_d}\; d\Gamma \\
&\quad + \int_{\Gamma_d} \left(N_{p_F}^{std}\right)^T \delta\overline{p_F}\, [\![\dot{u}]\!] n_{\Gamma_d}\, d\Gamma = 0
\end{aligned} \tag{A.62}$$

Assembling the test functions, it gives

$$\begin{aligned}
\delta\overline{p_F} \Bigg\{ &\int_{\Gamma_d} \left(B_{p_F}^{std}\right)^T t_{\Gamma_d}\, (2\mathrm{h}) k_{fF}\, \nabla\overline{p_F}\, t_{\Gamma_d}\; d\Gamma \\
&+ \int_{\Gamma_d} \left(N_{p_F}^{std}\right)^T t_{\Gamma_d}\, (2\mathrm{h}) \langle \nabla\dot{u} \rangle\, t_{\Gamma_d}\; d\Gamma \\
&+ \int_{\Gamma_d} \left(N_{p_F}^{std}\right)^T [\![\dot{u}]\!] n_{\Gamma_d}\, d\Gamma \\
&- \int_{\Gamma_d} \left(N_{p_F}^{std}\right)^T c\, N_p^{std}\, \bar{p}\, d\Gamma \\
&- \int_{\Gamma_d} \left(N_{p_F}^{std}\right)^T c\, N_p^{enr}\, \overline{p_a}\, d\Gamma \\
&+ \int_{\Gamma_d} \left(N_{p_F}^{std}\right)^T c\, N_{p_F}^{std}\, \overline{p_F}\, d\Gamma \Bigg\}
\end{aligned} \tag{A.63}$$

Arranging the terms into a matrix form, the following relation is obtained

$$[L_{p_Fp} \quad L_{p_Fc}]\begin{Bmatrix}\bar{p}\\ \overline{p_a}\end{Bmatrix} + [H_{p_Fp_F} + L_{p_Fp_F}]\{\overline{p_F}\} = q_{p_F}^{int} \tag{A.64}$$

where

$$H_{p_Fp_F} = \int_{\Gamma_d} \left(B_{p_F}^{std}\right)^T t_{\Gamma_d}\,(2\mathrm{h})k_{Fd}\,\nabla\overline{p_F}\,t_{\Gamma_d}\;d\Gamma \tag{A.65}$$

$$L_{p_Fp_F} = \int_{\Gamma_d} \left(N_{p_F}^{std}\right)^T c\,N_{p_F}^{std}\;d\Gamma \tag{A.66}$$

$$L_{p_Fp} = \int_{\Gamma_d} \left(N_{p_F}^{std}\right)^T c\,N_{p}^{std}\;d\Gamma \tag{A.67}$$

$$L_{p_Fc} = \int_{\Gamma_d} \left(N_{p_F}^{std}\right)^T c\,N_{p}^{enr}\;d\Gamma \tag{A.68}$$

$$q_{p_F}^{int} = \int_{\Gamma_d} \left(N_{p_F}^{std}\right)^T t_{\Gamma_d}\,(2\mathrm{h})\langle\nabla\dot{\bar{u}}\rangle\,t_{\Gamma_d}\;d\Gamma + \int_{\Gamma_d} \left(N_{p_F}^{std}\right)^T [\![\dot{\bar{u}}]\!]n_{\Gamma_d}\,d\Gamma \tag{A.69}$$

Generalizing the equations and terms, it gives

$$-[L2^T]\{\overline{\mathbb{P}}\} + [H_F + L3]\{\overline{\mathbb{P}}_F\} - q_{\overline{\mathbb{P}}_F}^{int} = 0 \tag{A.70}$$

where

$$L2_{\delta P_F} = \int_{\Gamma_d} \left(N_{p}^{\delta}\right)^T c\,N_{p_F}^{std}\;d\Gamma \tag{A.71}$$

$$L3 = \int_{\Gamma_d} \left(N_{p_F}^{std}\right)^T c\,N_{p_F}^{std}\;d\Gamma \tag{A.72}$$

$$H_F = \int_{\Gamma_d} \left(B_{p_F}^{std}\right)^T t_{\Gamma_d}\,(2\mathrm{h})k_{Fd}\,B_{p_F}^{std}\,t_{\Gamma_d}\;d\Gamma \tag{A.73}$$

$$q_{p_F}^{int} = \int_{\Gamma_d} \left(N_{p_F}^{std}\right)^T t_{\Gamma_d}\,(2\mathrm{h})\langle\nabla\dot{\bar{u}}\rangle\,t_{\Gamma_d}\;d\Gamma + \int_{\Gamma_d} \left(N_{p_F}^{std}\right)^T [\![\dot{\bar{u}}]\!]n_{\Gamma_d}\,d\Gamma \tag{A.74}$$

The values related with velocity are defined as

$$
\begin{aligned}
\langle \nabla \dot{\bar{u}} \rangle = \frac{\nabla \dot{\bar{u}}^{+} + \nabla \dot{\bar{u}}^{-}}{2} \\
&= \frac{1}{2}\left[(B_u^{std}\dot{\bar{u}} + B_u^{enr}\dot{\bar{a}})^{+} + (B_u^{std}\dot{\bar{u}} + B_u^{enr}\dot{\bar{a}})^{-}\right] \\
&= \frac{1}{2}\left[2.B_u^{std}\dot{\bar{u}} + B_u^{std}(H^{+} + H^{-})\dot{\bar{a}}\right] \\
&= B_u^{std}\dot{\bar{u}} + B_u^{std}\langle H \rangle \dot{\bar{a}} \\
&= B_u^{std}\frac{(\bar{u}_n - \bar{u}_{n-1})}{\Delta t} + B_u^{std}\langle H \rangle \frac{(\bar{a}_n - \bar{a}_{n-1})}{\Delta t}
\end{aligned}
\tag{A.75}
$$

$$
\begin{aligned}
[\![\dot{\bar{u}}]\!] = \dot{\bar{u}}^{+} - \dot{\bar{u}}^{-} \\
&= \left(N_u^{std^{+}}\dot{\bar{u}} + N_u^{std^{+}}H^{+}\dot{\bar{a}}\right) \\
&\quad - (N_u^{std^{-}}\dot{\bar{u}} + N_u^{std^{-}}H^{-}\dot{\bar{a}}) = N_u^{std}[\![H]\!]\dot{\bar{a}} \\
&= N_u^{std}[\![H]\!]\frac{(\bar{a}_n - \bar{a}_{n-1})}{\Delta t}
\end{aligned}
\tag{A.76}
$$

Given that

$$N_u^{std^{+}} = N_u^{std^{-}} = N_u^{std} \tag{A.77}$$

$$N_u^{enr+} = N_u^{std^{+}}H^{+} = N_u^{std}H^{+} \tag{A.78}$$

$$N_u^{enr-} = N_u^{std^{-}}H^{-} = N_u^{std}H^{-} \tag{A.79}$$

$$B_u^{std^{+}} = B_u^{std^{-}} = B_u^{std} \tag{A.80}$$

$$B_u^{enr+} = B_u^{std^{+}}H^{+} = B_u^{std}H^{+} \tag{A.81}$$

$$B_u^{enr-} = B_u^{std^{-}}H^{-} = B_u^{std}H^{-} \tag{A.82}$$

Where H^{+} and H^{-} represent the values of the enrichment function *H* in the fracture top and bottom faces, respectively. In $\bar{u}_n - \bar{u}_{n-1}$, *n* represents the current increment and *n-1* the previous one.

Generalizing a number of degrees of freedom equal to *ndof*, it gives

$$\langle \nabla \dot{\bar{u}} \rangle = \sum_{k}^{ndof} \langle B_u^k \rangle \frac{(\bar{k}_n - \bar{k}_{n-1})}{\Delta t} \tag{A.83}$$

$$[\![\dot{\bar{u}}]\!] = \sum_{k}^{ndof} [\![N_u^k]\!] \frac{(\bar{k}_n - \bar{k}_{n-1})}{\Delta t} \tag{A.84}$$

with $\langle B_u^u \rangle = B_u^{std}$ and $\langle B_u^a \rangle = B_u^{std}\langle H_a \rangle$

Annex B
Newton-Raphson Algorithm

The Jacobian of derivatives is given by

$$J = \begin{bmatrix} \frac{\partial \Psi_{\mathbb{U}}}{\partial \overline{\mathbb{U}}} & \frac{\partial \Psi_{\mathbb{U}}}{\partial \overline{\mathbb{P}}} & \frac{\partial \Psi_{\mathbb{U}}}{\partial \overline{\mathbb{P}}_F} \\ \frac{\partial \Psi_{\mathbb{P}}}{\partial \overline{\mathbb{U}}} & \frac{\partial \Psi_{\mathbb{P}}}{\partial \overline{\mathbb{P}}} & \frac{\partial \Psi_{\mathbb{P}}}{\partial \overline{\mathbb{P}}_F} \\ \frac{\partial \Psi_{\overline{\mathbb{P}}_F}}{\partial \overline{\mathbb{U}}} & \frac{\partial \Psi_{\overline{\mathbb{P}}_F}}{\partial \overline{\mathbb{P}}} & \frac{\partial \Psi_{\overline{\mathbb{P}}_F}}{\partial \overline{\mathbb{P}}_F} \end{bmatrix} = \begin{bmatrix} K + \frac{\partial f_{\mathbb{U}}^{int}}{\partial \overline{\mathbb{U}}} & -Q + \frac{\partial f_{\mathbb{U}}^{int}}{\partial \overline{\mathbb{P}}} & \frac{\partial f_{\mathbb{U}}^{int}}{\partial \overline{\mathbb{P}}_F} \\ \frac{1}{\Delta t} Q^T & (H + L) & L \\ -\frac{\partial q_{\overline{\mathbb{P}}_F}^{int}}{\partial \overline{\mathbb{U}}} & L^T - \frac{\partial q_{\overline{\mathbb{P}}_F}^{int}}{\partial \overline{\mathbb{P}}} & (H_F + L_F) - \frac{\partial q_{\overline{\mathbb{P}}_F}^{int}}{\partial \overline{\mathbb{P}}_F} \end{bmatrix} \tag{B.1}$$

Multiplying the second and third lines for Δt, it gives

$$J = \begin{bmatrix} \frac{\partial \Psi_{\mathbb{U}}}{\partial \overline{\mathbb{U}}} & \frac{\partial \Psi_{\mathbb{U}}}{\partial \overline{\mathbb{P}}} & \frac{\partial \Psi_{\mathbb{U}}}{\partial \overline{\mathbb{P}}_F} \\ \frac{\partial \Psi_{\mathbb{P}}}{\partial \overline{\mathbb{U}}} & \frac{\partial \Psi_{\mathbb{P}}}{\partial \overline{\mathbb{P}}} & \frac{\partial \Psi_{\mathbb{P}}}{\partial \overline{\mathbb{P}}_F} \\ \frac{\partial \Psi_{\overline{\mathbb{P}}_F}}{\partial \overline{\mathbb{U}}} & \frac{\partial \Psi_{\overline{\mathbb{P}}_F}}{\partial \overline{\mathbb{P}}} & \frac{\partial \Psi_{\overline{\mathbb{P}}_F}}{\partial \overline{\mathbb{P}}_F} \end{bmatrix} = \begin{bmatrix} K + \frac{\partial f_{\mathbb{U}}^{int}}{\partial \overline{\mathbb{U}}} & -Q + \frac{\partial f_{\mathbb{U}}^{int}}{\partial \overline{\mathbb{P}}} & \frac{\partial f_{\mathbb{U}}^{int}}{\partial \overline{\mathbb{P}}_F} \\ Q^T & \Delta t(H + L) & \Delta t. L \\ -\Delta t \frac{\partial q_{\overline{\mathbb{P}}_F}^{int}}{\partial \overline{\mathbb{U}}} & \Delta t \left(L^T - \frac{\partial q_{\overline{\mathbb{P}}_F}^{int}}{\partial \overline{\mathbb{P}}} \right) & \Delta t(H_F + L_F) - \Delta t \frac{\partial q_{\overline{\mathbb{P}}_F}^{int}}{\partial \overline{\mathbb{P}}_F} \end{bmatrix} \tag{B.2}$$

Re-scaling the problem formulation for one standard u and one enriched degree of freedom a, the derivatives for the mechanical equation are

$$\frac{\partial f_{\mathbb{U}}^{int}}{\partial \overline{\mathbb{U}}} = \begin{Bmatrix} \frac{\partial}{\partial \bar{u}} \\ \frac{\partial}{\partial \bar{a}} \end{Bmatrix} \begin{Bmatrix} 0 \\ \int_{\Gamma_d} [\![N_u^{enr}]\!]^T \left(D_F\, \bar{a} - p_F n_{\Gamma d}\right) d\Gamma \end{Bmatrix}^T = \begin{bmatrix} 0 & 0 \\ 0 & \int_{\Gamma_d} [\![N_u^{enr}]\!]^T D_F\, [\![N_u^{enr}]\!] d\Gamma \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & T_a \end{bmatrix} \tag{B.3}$$

$$\frac{\partial f_{\mathbb{U}}^{int}}{\partial \overline{\mathbb{P}}} = \begin{Bmatrix} \frac{\partial}{\partial \bar{p}} \\ \frac{\partial}{\partial \overline{p_a}} \end{Bmatrix} \begin{Bmatrix} 0 \\ \int_{\Gamma_d} [\![N_u^{enr}]\!]^T \left(D_F\, \bar{a} - p_F n_{\Gamma d}\right) d\Gamma \end{Bmatrix}^T = 0 \tag{B.4}$$

$$\frac{\partial f_{\mathbb{U}}^{int}}{\partial \overline{\mathbb{P}}_F} = \frac{\partial}{\partial \overline{p_F}} \begin{Bmatrix} 0 \\ \int_{\Gamma_d} [\![N_u^{enr}]\!]^T \left(D_F\, \bar{a} - p_F n_{\Gamma d}\right) d\Gamma \end{Bmatrix} = \begin{Bmatrix} 0 \\ -\int_{\Gamma_d} [\![N_u^{enr}]\!]^T n_{\Gamma d}\, N_{p_F}^{std} d\Gamma \end{Bmatrix} = \begin{Bmatrix} 0 \\ -Q_{ap_F} \end{Bmatrix} \tag{B.5}$$

The derivatives for the continuity equation in the fracture are

$$\frac{\partial q_{\overline{\mathbb{P}}_F}^{int}}{\partial \overline{\mathbb{U}}} = \begin{Bmatrix} \frac{\partial}{\partial \bar{u}} \\ \frac{\partial}{\partial \bar{a}} \end{Bmatrix} \left\{ \int_{\Gamma_d} \left(N_{p_F}^{std}\right)^T t_{\Gamma_d}\, (2\mathrm{h}) \langle \nabla \dot{u} \rangle\, t_{\Gamma_d}\ d\Gamma + \int_{\Gamma_d} \left(N_{p_F}^{std}\right)^T [\![\dot{u}]\!] n_{\Gamma_d}\, d\Gamma \right\}$$
$$= \begin{Bmatrix} \int_{\Gamma_d} \left(N_{p_F}^{std}\right)^T t_{\Gamma_d}\, (2\mathrm{h}) \left(B_u^{std} \frac{1}{\Delta t}\right) t_{\Gamma_d}\ d\Gamma \\ \int_{\Gamma_d} \left(N_{p_F}^{std}\right)^T t_{\Gamma_d}\, (2\mathrm{h}) \left(B_u^{std} \langle H \rangle \frac{1}{\Delta t}\right) t_{\Gamma_d}\ d\Gamma + \int_{\Gamma_d} \left(N_{p_F}^{std}\right)^T N_u^{std} [\![H]\!] \frac{1}{\Delta t} n_{\Gamma_d}\, d\Gamma \end{Bmatrix}$$
$$= \begin{Bmatrix} \frac{1}{\Delta t} S_{P_f u} \\ \frac{1}{\Delta t} S_{P_f a} + \frac{1}{\Delta t} V_{P_f a} \end{Bmatrix} \tag{B.6}$$

$$\frac{\partial q_{\overline{\mathbb{P}}_F}^{int}}{\partial \overline{\mathbb{P}}} = \begin{Bmatrix} \frac{\partial}{\partial \bar{p}} \\ \frac{\partial}{\partial \overline{p_a}} \end{Bmatrix} \left\{ \int_{\Gamma_d} \left(N_{p_F}^{std}\right)^T t_{\Gamma_d}\, (2\mathrm{h}) \langle \nabla \dot{u} \rangle\, t_{\Gamma_d}\ d\Gamma + \int_{\Gamma_d} \left(N_{p_F}^{std}\right)^T [\![\dot{u}]\!] n_{\Gamma_d}\, d\Gamma \right\} = 0 \tag{B.7}$$

$$\frac{\partial q_{\overline{\mathbb{P}}_F}^{int}}{\partial \overline{\mathbb{P}}_F} = \frac{\partial}{\partial \overline{p_F}} \left(\int_{\Gamma_d} \left(N_{p_F}^{std}\right)^T t_{\Gamma_d}\, (2\mathrm{h}) \langle \nabla \dot{u} \rangle\, t_{\Gamma_d}\ d\Gamma + \int_{\Gamma_d} \left(N_{p_F}^{std}\right)^T [\![\dot{u}]\!] n_{\Gamma_d}\, d\Gamma \right) = 0 \tag{B.8}$$

Substituting the derivatives in the Jacobian, it gives

$$J = \begin{bmatrix} K_{uu} & K_{ua} & -Q_{up} & -Q_{uc} & 0 \\ K_{au} & K_{aa}+T_a & -Q_{ap} & -Q_{ac} & -Q_{ap_F} \\ Q_{pu} & Q_{pa} & \Delta t(H_{pp}+L_{pp}) & \Delta t(H_{pc}+L_{pc}) & \Delta t.L_{pp_F} \\ Q_{cu} & Q_{ca} & \Delta t(H_{cp}+L_{cp}) & \Delta t(H_{cc}+L_{cc}) & \Delta t.L_{cp_F} \\ -S_{P_fu} & -\left(S_{P_fa}+V_{P_fa}\right) & \Delta t.L_{p_Fp} & \Delta t.L_{p_Fc} & \Delta t(H_{p_Fp_F}+L_{p_Fp_F}) \end{bmatrix} \tag{B.9}$$

If both porous and fracture material constitutive behaviour are such that their matrices K and T are symmetric, the Jacobian may be symmetric if the following simplifications are considered:

- The lines relative to pore and fracture pressures (third, fourth and fifth lines) are multiplied by -1
- $S_{P_fu} = 0$
- $\left(S_{P_fa} + V_{P_fa}\right) = Q_{p_Fa} = {Q_{ap_F}}^T$

The resulting Jacobian matrix is then given by

$$J = \begin{bmatrix} K_{uu} & K_{ua} & -Q_{up} & -Q_{uc} & 0 \\ K_{au} & K_{aa}+T_a & -Q_{ap} & -Q_{ac} & -Q_{ap_F} \\ -Q_{pu} & -Q_{pa} & -\Delta t(H_{pp}+L_{pp}) & -\Delta t(H_{pc}+L_{pc}) & -\Delta t.L_{pp_F} \\ -Q_{cu} & -Q_{ca} & -\Delta t(H_{cp}+L_{cp}) & -\Delta t(H_{cc}+L_{cc}) & -\Delta t.L_{cp_F} \\ 0 & -Q_{p_Fa} & -\Delta t.L_{p_Fp} & -\Delta t.L_{p_Fc} & -\Delta t(H_{p_Fp_F}+L_{p_Fp_F}) \end{bmatrix} \tag{B.10}$$

Generalizing the terms, it gives

$$J = \begin{bmatrix} K+T & -Q & -Q_F \\ -Q^T & -\Delta t(H+L1) & \Delta t.L2 \\ -{Q_F}^T & \Delta t.L2^T & -\Delta t.(H_F+L3) \end{bmatrix} \tag{B.11}$$

Where the matrices are given by Eqs. (A.34), (A.35), (A.58) to (A.60), (A.71) to (A.73), and

$$T_{\beta\gamma} = \int_{\Gamma_d} \left[\!\left[N_u^\beta \right]\!\right]^T D_F \left[\!\left[N_u^\gamma \right]\!\right] d\Gamma \tag{B.12}$$

$$Q_{F_{\beta p_F}} = \int_{\Omega} \left[\!\left[N_u^\beta \right]\!\right]^T n_{\Gamma d}\, N_{p_F}^{std}\, d\Omega \tag{B.13}$$