# Study of an embarked vibro-impact system: mathematical modeling and nonlinear analysis

In this chapter a general overview of the state of the art of impact mechanics is shown, as well as some methodologies for impact phenomenon modeling and the numerical integration of discontinuous differential equations. The hammer modeling is also addressed in this chapter, for both test rig configurations. The parameter identification is discussed and the comparison between numerical analysis and the experimental results is performed.

#### 3.1 Brief impact theory

Literature on contact dynamics and impact analysis is vast and spans many diverse disciplines. It is not the concern of this thesis to fully describe impact phenomena, but to provide necessary information to understand the results of this research. To provide a broader context for this subject, references are listed throughout the text, including previous master's dissertations [2] [91] [32] and doctoral theses [29] [17] [41] at the Department of Mechanical Engineering of PUC-Rio.

Impact is a complex phenomena that occurs when two or more bodies undergo a collision. This phenomena is important in many different areas: machine design, robotics and multi-body analysis are a few examples [34] [63]. Characteristics of impact are: very brief duration, high levels of force, rapid dissipation of energy and high values of acceleration and deceleration. During impact, the system presents discontinuities in geometry and some material properties may be modified by the impact itself. *Contact* is a more ambiguous term, although it is frequently used interchangeably with *impact* [62].

A typical energy flow associated with an impact is shown in Figure 3.1.



Figure 3.1: Energy flow associated with impact [15].

In general, two different approaches are used to describe impact. The first approach, referred to as *impulse-momentum* or *discrete* method, assumes that the interaction between the objects occurs in a short time period and that the configuration of impacting bodies does not change significantly. The dynamic analysis is divided into two intervals: before and after impact. To model the energy transfer and dissipation various coefficients are employed, mainly the coefficient of restitution and the impulse ratio. Application of these methods has been confined primarily to impact between rigid bodies and are referred to as impulse-momentum or discrete methods [62] [34] [15]. The extension to flexible systems and the extension case involving multiple contacts and intermittent contact is quite complicated.

The second approach (referred to as *force-based* method or *continuous* analysis) is based on the concept that interaction forces will act in a continuous manner during the impact. The analysis may be performed in the usual way by simply adding the contact forces to the equations of motion during their action period. This allows for a better description of the real behavior of a system. More importantly, this approach is naturally suitable for contact modeling and complex contact scenarios involving multiple contacts and bodies. This approach is referred to as continuous analysis or force-based methods [34] [15] and implies a multi-scale analysis within the time domain.

#### 3.1.1 Force-based method (or continuous analysis)

Application of the impulse-momentum methods to model the impact dynamics of rigid bodies leads to several problems. First, in the presence of Coulomb friction, cases arise in which no solution or multiple solutions exist. These ambiguities have been attributed to the approximate nature of Coulomb's model and the inadequacy of the rigid body model, but no clear explanation has been found [34]. The second problem is that energy conservation principles may be violated during frictional impacts, as shown by Stronge [76], as a consequence of the definition of the coefficient of restitution. Finally, the discrete approach is not easily extendible to generic multi-body systems. The use of compliance or continuous contact models where the impact force is a function of local indentation can overcome the problems encountered in the discrete formulation [34].

Different models have been postulated to represent the interaction force at the surfaces of two contacting bodies [53] [48] [17] [41] [61] [15] [76] [10] [34] [40] [63] [69] [81]. The first model was developed by Hertz [34], in which an elastostatic theory was used to calculate local indentation without the use of damping. The corresponding relationship between the impact force and the indentation is allowed to be nonlinear. In the first and simplest model of damping, referred to as the spring-dashpot model [15] [69] [81], the contact force is represented by a linear spring-damper element. Hunt and Crossley [40] showed that a linear damping model does not truthfully represent the physical nature of the energy transfer process. Thus, they proposed a model based on Hertz's theory of contact with a non-linear damping force defined in terms of local penetration and its corresponding rate. Other damping models have also been proposed to describe totally or partially plastic impacts [10] [34].

Contact stiffness and damping forces are dependent on at least two parameters: the coefficient of stiffness and the coefficient of damping. For simple contact between two bodies, the former is determined by the geometry and the material of the contacting bodies, while the coefficient of damping can be related to the coefficient of restitution. An important advantage of continuous contact dynamics analysis is the possibility of using one of many models available in literature.

#### 3.1.2 Continuous contact dynamics models

The continuous model, also referred to as the *compliant contact model*, overcomes the problems associated with the discrete models. The basis of the continuous formulation for contact dynamics is accounting for the deformation of the bodies during impact or contact. In a large class of continuous models, this is done by defining the normal contact force,  $F_i$ , as an explicit function of local indentation,  $\delta$  and its derivative,

$$F_i = F_i(\delta, \dot{\delta}) = F_{\delta}(\delta) + F_{\dot{\delta}}(\dot{\delta}) \tag{3-1}$$

Four existing contact force models are summarized here.

#### Hertz's model.

This is a nonlinear model, limited to impacts with elastic deformation, and in its original form does not include damping. With this model, the contact process can be pictured as two rigid bodies interacting via a nonlinear spring along the line of impact. The hypotheses state that deformation is concentrated in the vicinity of the contact area, elastic wave motion is neglected, and the total mass of each body moves with the velocity of its mass center. Therefore, this impact model does not consider rigid body rotation, only pure translation. The impact force is defined as [76] [73]

$$F_i = k_i \delta^n av{3-2}$$

where  $k_i$  is the impact stiffness and n is a constant. The term  $\delta^n$  is the nonlinear term. Both parameters  $k_i$  and n depend on material and geometric properties and are computed by using elastostatic theory. For instance, in the case of two spheres in central impact, n = 1.5. Stiffness  $k_i$  is defined in terms of Poisson's ratio, Young's moduli and the radii of the two spheres [73]. Since the Hertzian model does not account for energy dissipation, its equivalent coefficient of restitution is 1. Therefore, this model can be used only for low impact speeds and hard materials [34].

#### Spring-dashpot model.

In this model, the impact is schematically represented with a linear damper (dashpot) for energy dissipation in parallel with a linear spring for elastic behavior [15] [34]. The contact force is defined as

$$F_i = k_i \delta + c_i \dot{\delta} , \qquad (3-3)$$

where  $k_i$  is the impact stiffness and  $c_i$  is the impact damping. The impact force profile for this model is represented schematically in Figure 3.2. This model has two weaknesses:

- 1. The contact force at the beginning of impact (point A, Figure 3.2) is discontinuous, because of the damping term. In a more realistic model, both elastic and damping forces should be initially at the magnitude of zero and increase over time;
- 2. As the objects are separating (point B, Figure 3.2), i.e., the indentation approaches zero, their relative velocity tends to be negative. As a result, a negative force is present that holds the objects.



Figure 3.2: Spring-dashpot model [34].

Although the spring-dashpot model is not physically realistic, its simplicity has made it a popular choice [34] [30] [68] [69] [71] [81] [84]. It provides a reasonable method for capturing the energy dissipation associated with the normal forces without explicitly considering plastic deformation issues.

#### Hunt and Crossley model.

To overcome the problems of the spring-dashpot model and to retain the advantages of the Hertz's model, an alternative model for energy dissipation was introduced by Hunt and Crossley [40]. It includes a nonlinear (hysteresis) damping term, and the impact/contact force is modeled as

$$F_i(\delta, \dot{\delta}) = k_i \delta^n + c_i \delta^n \dot{\delta} = k_i \delta^n (1 + \lambda_i \dot{\delta}) , \quad \lambda_i = \frac{c_i}{k_i} , \qquad (3-4)$$

where  $k_i$  is the impact stiffness,  $c_i$  is the impact damping, n is a constant term and  $\lambda_i$  is the damping/ stiffness ratio.

As with the spring-dashpot model, the damping parameter  $c_i$  can be related to the coefficient of restitution, since both are related to the energy dissipated by the impact process. An important aspect of this model is that damping depends on the indentation. This is physically sound since contact area increases with deformation and a plastic region is more likely to develop for larger indentations. Another advantage is that the contact force has no discontinuities at initial contact and separation, therefore it begins and ends with the correct value of zero. This model has been studied and used by several authors [40] [29] [2]. The phase plane of indentation and the impact force profile of a rigid mass impacting a surface where the impact is modeled using the Hunt & Crossley model are shown in Figure 3.3.



Figure 3.3: Impact of a rigid mass against a rigid wall, for different velocities before impact [2]: a) phase plane; b) contact force profile. Parameters:  $m = 2Kg, k_i = 2.1 \cdot 10^8 N/m, n = 1.6, \lambda_i = 0.6s.$ 

#### Lankarani model.

Lankarani and Nikravesh [45] developed a contact force model with hysteresis damping for impact in multi-body systems. This model uses the general trend of the Hertz contact law, in which a hysteresis damping function was incorporated in the model that represents the energy dissipated during the impact. This model can be expressed as

$$F_{i} = k_{i}\delta^{n} \left[ 1 + \frac{3(1-e^{2})}{4} \frac{\dot{\delta}}{\dot{\delta}^{(-)}} \right] , \qquad (3-5)$$

where  $k_i$  is the impact stiffness, n is a constant term, e is Newton's coefficient of restitution and  $\dot{\delta}^{(-)}$  is the instant velocity before impact.

In this model, as in the Hunt and Crossley impact model, the hysteresis damping function assumes that the loss of energy in impact is all due to the material damping of the colliding bodies. An advantage of the Lankarani model is that it shows a direct relationship between the coefficient of restitution and the contact force.

#### 3.2 Numerical integration of discontinuous ordinary differential equations

Physical phenomenon, such as impact and dry friction in mechanical systems, are often studied by means of mathematical models with some kind of discontinuity or non-smoothness. Systems which can be described by a set of first-order ordinary equations with a discontinuous right-hand side form a sub-class of discontinuous dynamical systems and are addressed as *Filippov systems* or *non-smooth systems of Filippov-type* [51]. The focus of this section is to describe one methodology of solving such systems.

Non-smooth nonlinearity is abundant in nature, being usually related to either the friction phenomenon or the contacts between system components. Therefore, physical systems with dry friction and impact operate in different modes, and the transition from one mode to another can often be idealized as an instantaneous or discrete transition. Since the time scale of this transition is much smaller than the scale of the individual mode dynamics, their modeling results in non-smooth systems. Non-smooth systems appear in many kinds of engineering systems and also in everyday life. Examples include the stick-slip oscillations of a violin string or mechanical brakes [48].

The mathematical modeling and numerical simulations of non-smooth systems present many difficulties, which make their description unusually complex. Moreover, the dynamical behavior of these systems shows a variety of responses. Therefore, non-smooth systems present scientific and technological interests that motivated different researches [48] [50] [51] [52] [69] [84] [88].

Prevailing scientific literature presents many reports dealing with non-smooth systems, concerning the mathematical modeling, the proposition of numerical algorithms, and experimental approaches employed to verify the results. The idea that non-smooth systems can be considered as continuous in a finite number of continuous subspaces, and that system parameters do not change in an abrupt manner, inspired some authors to try to describe non-smooth systems by a smoothed form. Wiercigroch et al [82] [81] [83] [84] [85] [86] [87] [88] [89], Leine et al [48] [49] [50] [51], Franca et al [29] [30], Savi et al [68] [69] [71] [72] and Glocker [38] [35] [36] propose interesting approaches to deal with mathematical discontinuity.

The mathematical model applied throughout this work uses a smoothed switch model, proposed by Leine [48] on the study of stick-slip vibrations. Basically, the switch model treats non-smooth systems by defining sets of ordinary differential equations. Each set is related to a subspace of the physical system. Leine's innovative idea is the definition of transition regions that governs the dynamical response. Therefore, each subspace has its own ordinary differential equations. Besides, each transition region also has its governing equations, defined in order to smooth the system dynamics. The use of this approach smoothes the discontinuities and allows the use of classical numerical procedures. Numerical investigations of single-degree-of-freedom systems with a discontinuous support show efficiency and allow analysis of many aspects related to the non-smooth system dynamics.

#### 3.2.1 Filippov solution concept

The Filippov solution considers a dynamical system described by a set of ordinary differential equations

$$\dot{\mathbf{x}}(t) = f(t, \mathbf{x}(t)), \qquad \mathbf{x}(t) \in \mathbb{R}^n, \tag{3-6}$$

where  $\mathbf{x}$  is the state vector and  $f(t, \mathbf{x}(t))$  is the right-hand side vector describing state vector's time derivative. The time derivative vector  $f(t, \mathbf{x})$ may be discontinuous in  $\mathbf{x}$ . The solution  $\mathbf{x}(t)$  presented by Filippov leads to a differential equation with a discontinuous right-hand side, continuous in time [48] [49]. Systems with a discontinuous solution (occurring for systems with impulse-momentum or discrete methods, as mentioned in previous section) are not described by the theory of Filippov. This section is restricted to the Filippov concept where the right-hand side of the differential equation is discontinuous at a number of hyper-surfaces. For the case of a single hyper-surface, the state space  $\mathbb{R}^n$  is split into two subspaces  $\nu_+$  and  $\nu_-$  by a hyper-surface  $\Sigma$  in a way that  $\mathbb{R}^n = \nu_+ \cup \nu_- \cup \Sigma$ . The hyper-surface is called the *switching boundary* and is defined by a scalar switching function  $h(\mathbf{x})$ . The state  $\mathbf{x}(t)$  is in  $\Sigma$  when

$$h(\mathbf{x}) = 0 \Longleftrightarrow \mathbf{x} \in \Sigma . \tag{3-7}$$

The subspaces  $\nu_+$  and  $\nu_-$  and the switching boundary  $\Sigma$  can be formulated as

$$\begin{cases} \nu_{-} = \{ \mathbf{x} \in \mathbb{R}^{n} \mid h(\mathbf{x}) < 0 \}, \\ \Sigma = \{ \mathbf{x} \in \mathbb{R}^{n} \mid h(\mathbf{x}) = 0 \}, \\ \nu_{+} = \{ \mathbf{x} \in \mathbb{R}^{n} \mid h(\mathbf{x}) > 0 \}. \end{cases}$$
(3-8)

The switching boundary is assumed to be continuous but it can be allowed to be non-smooth.

The right-hand side of the dynamics  $\dot{\mathbf{x}}(t) = f(t, \mathbf{x}(t))$  is assumed to be locally continuous for all  $\mathbf{x} \notin \Sigma$ . In this way it is able to consider the following *n*-dimensional nonlinear system with a discontinuous right-hand side

$$\dot{\mathbf{x}}(t) = f(t, \mathbf{x}(t)) = \begin{cases} f_{-}(t, \mathbf{x}(t)), & \mathbf{x} \in \nu_{-}, \\ f_{+}(t, \mathbf{x}(t)), & \mathbf{x} \in \nu_{+}, \end{cases}$$
(3-9)

with initial condition  $\mathbf{x}(0) = \mathbf{x}_0$ . As mentioned, the right-hand side  $f(t, \mathbf{x}(t))$ is assumed to be discontinuous but is continuous piecewise, and smooth on  $\nu_-$  and  $\nu_+$ , and discontinuous on  $\Sigma$ . The system described in Equation (3-9) does not define  $f(t, \mathbf{x}(t))$  if  $\mathbf{x}(t)$  is on  $\Sigma$ . To overcome such problem the set-valued extension  $F(t, \mathbf{x})$  is defined:

$$\dot{\mathbf{x}}(t) = F(t, \mathbf{x}(t)) = \begin{cases} f_{-}(t, \mathbf{x}(t)), & \mathbf{x} \in \nu_{-}, \\ \overline{co}\{f_{-}(\mathbf{x}), f_{+}(\mathbf{x})\}, & \mathbf{x} \in \Sigma, \\ f_{+}(t, \mathbf{x}(t)), & \mathbf{x} \in \nu_{+}, \end{cases}$$
(3-10)

where the convex set with two right-hand sides  $f_{-}$  and  $f_{+}$  can be written as

$$\overline{co}\{f_{-}(\mathbf{x}), f_{+}(\mathbf{x})\} = \{(1-q)f_{-}(\mathbf{x}) + qf_{+}(\mathbf{x}), \forall q \in [0,1]\}$$
(3-11)

The extension (or convexification) of a discontinuous system into a convex differential inclusion is know as *Filippov's convex method* [48].

#### 3.2.2 Numerical integration method: the *Switch Model*

This section presents one technique for the integration of differential equations with a discontinuous right-hand side. The technique presented is the switch model, however this is not the only technique available. Other examples are the event-driven integration method, time-stepping methods and augmented Lagrangian approach, widely discussed in [49] [46] [47]. First, the smoothing method is briefly discussed, and the disadvantages of the smoothing method motivates the use of the Switch Model.

#### Smoothing Method.

Consider the following differential inclusion

$$\dot{\mathbf{x}}(t) = F(t, \mathbf{x}(t)) = \begin{cases} f_{-}(t, \mathbf{x}(t)), & \mathbf{x} \in \nu_{-} \\ \overline{co}\{f'_{-}(\mathbf{x}), f'_{+}(\mathbf{x})\}, & \mathbf{x} \in \Sigma \\ f_{+}(t, \mathbf{x}(t)), & \mathbf{x} \in \nu_{+} \end{cases}$$
(3-12)

with

$$\nu_{-} = \mathbf{x} \in \mathbb{R}^{n} \mid h(\mathbf{x}) < 0$$
  

$$\Sigma = \mathbf{x} \in \mathbb{R}^{n} \mid h(\mathbf{x}) = 0$$
  

$$\nu_{+} = \mathbf{x} \in \mathbb{R}^{n} \mid h(\mathbf{x}) > 0$$
(3-13)

A difficulty is faced when trying to numerically integrate a discontinuous system of Filippov-type with an integration algorithm for ordinary differential equations (ODE). An ODE integrator will chatter around the hyper-surface, computing points alternating between  $\nu_{-}$  and  $\nu_+$ . If the integration algorithm is equipped with a variable step size, then it will find a reasonable approximation to the exact solution but it will take a considerable computational time. The discontinuous system can therefore not efficiently be numerically solved using the integration method of ODE due to the presence of discontinuity. The discontinuous vector field is often approximated by a smoothed vector field, as described in the works of Leine [48] and Wiercigroch [81]. The discontinuous right-hand side  $f(t, \mathbf{x})$  may typically contain a sign-function. A common smoothing approximation of the sign-function, frequently used to model Coulomb friction, uses an arc-tangent function. However, if the system has chaotic behavior, the results using the smoothing method will probably be incorrect, as shown in the work of Wiercigroch [81]. Figure 3.4 shows the difference in the Poincaré topology when a discontinuous system is modeled using a smoothing function under different parameter values (Figure 3.4(a), (b) and (c)), compared to the system modeled with the discontinuity (Figure 3.4(d)) [81].

The advantage of the smoothing method is its easiness of use, as standard integration techniques can be directly applied and no additional



Figure 3.4: Change of Poincaré map topology of the smoothed system under different parameter values (a), (b) and (c); and for discontinuous system (d). From the work of Wiercigroch [81].

programming has to be done. The main disadvantage of this method is that it yields stiff differential equations, which are numerically time consuming, not to mention that in the case of nonlinear systems the smoothing method may lead to incorrect results, especially if the system has chaotic regions [81].

#### Switch Model.

This numerical technique is used for integrating differential inclusions without the problems of solving stiff differential equations. The methodology introduces a vector field in the transition surface, such that the state of the system is pushed to the middle of the surface, thereby avoiding numerical instabilities.

Consider a vector field which contains a switching boundary, as shown on Equation (3-10). The switching boundary contains a part with a transversal intersection from  $\nu_{-}$  to  $\nu_{+}$ . The switch model constructs a 'band' or 'boundary layer' with thickness  $2\eta$  around  $\Sigma$ , that allows for an efficient numerical approximation.

$$\overline{co}\{f_{-}(\mathbf{x}), f_{+}(\mathbf{x})\}, \quad x \in \Sigma$$
(3-14)

This assures that the system passes through the discontinuity when the transition from  $\nu_{-}$  to  $\nu_{+}$  takes place, and vice-versa. Thickness parameter  $\eta$  should be chosen sufficiently small, which means that  $\eta$  is small so it will not influence the solution. The switch model maintains the continuity of the state vector and yields a set of non-stiff ordinary differential equations. This is the method used to deal with the discontinuity caused by the impacts in the mathematical modeling, therefore this method will be revisited in future sections, applied specifically to the stated problem.

Figure 3.5 shows the simulation results of a single DOF mass-spring system with a base excitation and discontinuous support, modeled using the switch model. It is observed the narrow band  $\eta$  between sub-spaces  $\nu_{-}$ and  $\nu_{+}$ . Such information is shown in Figure 3.6, where the transition from sub-space  $\nu_{-}$  to  $\nu_{+}$ , passes though hyper-surface  $\Sigma$ .



Figure 3.5: One DOF system with discontinuous support and base excitation, application of the Switch model.

#### 3.3

### Hammer supported by wires: mathematical modeling and comparison between numerical simulation and experimental results

The mathematical modeling and the comparison between numerical simulation and experimental results of the first test rig are presented in this section. The purpose of the first test rig (hammer supported by wires) was to eliminate some phenomenon caused by the beam springs bending



Figure 3.6: One DOF system with discontinuous support and base excitation, transition between sub-spaces during one impact.

vibration. At first it was thought that the two peaks in the impact force profile were caused by the beam bending vibration. Subsequently the actual cause of such phenomenon is the axial vibration of the hammer. Therefore, to satisfactorily identify the impact parameters, a second hammer was built, so the axial vibration was eliminated. In this way, only the geometry of the impact bodies and the materials involved will affect the impact dynamic behavior [15]. Once the impact parameters are identified, these may be carried in other modeling without loss of information. Following the impact identification, the mathematical modeling is proposed. The test rig parameter identification is performed and the numerical simulations are compared to the experimental results, including some nonlinear analysis.

#### 3.3.1 Impact model

To adopt an impact model capable of describing the real impact observed during the experiments in this work, some hypotheses are adopted:

- 1. impact is central and co-linear, i.e., the center of mass of both bodies lies on the impact line;
- 2. impact velocity is along the impact line;
- 3. tangential contact force is always zero, because impact is co-linear and there is no impact velocity component in the tangential axis.

In this way, all impact models presented in earlier sections are in line with the hypotheses adopted. All models listed will be used to describe the test rig impact behavior, and from comparing the data one model will be chosen.

#### 3.3.2 Impact force: parameters identification

To better identify the impact parameters, a second impact body (hammer) was built (see Figure 2.3 for original hammer configuration). The idea is to remove the influence of the axial vibration behavior while characterizing and identifying the impact force. The new hammer is made completely of steel. See Figure 3.7.





Figure 3.7: Impact parameters identification, new hammer.

Using the same screw with knurled nut and impact force sensor, impact conditions will be reproducible, because impact depends only on the materials and geometries involved during contact. To identify the impact force parameters, a simple experiment was performed, where a well-known initial condition was imposed on the system. The experimental data was compared to the numerical simulation. All impact models listed were compared to the experimental data. An optimization procedure was performed, using the *fminsearch* Matlab function, attempting to find the parameters that minimize the error between numerical simulation and experimental data. Error function selected was the difference between peaks in the impact force profile. Charts in Figure 3.8 show a typical response of the system during impact.



Figure 3.8: New hammer released from a initial condition of 3 mm (approximately), impact force and acceleration of first impact.

Charts above show that the new hammer is rigid, because there is no difference in the profiles of impact force and acceleration. However, it is important to notice that although the new hammer is rigid, a similar impact profile is observed. So, it is possible to affirm that existing flexibility is not caused by the hammer. Therefore, the cart impact surface, where the impact force sensor is mounted, is responsible for the flexibility. The frequency of this oscillation is within the 3 kHz range, observed during the hammer modal analysis presented in Chapter 2 (see Figure 2.61). When comparing the numerical simulation and the experimental data for the impact parameter identification, a time difference between impacts will be noted, caused by the impact surface flexibility. This flexibility will not be taken into account during the mathematical modeling.

The numerical simulation and experimental data for the impact force parameter identification are show in Figures 3.9 and 3.10. As mentioned, all four impact force models were tested. The Hertz model, see Equation (3-2), was obviously discarded, because it does not take into account any loss of energy. The Hunt & Crossley model and the Lankarani model, Equations (3-4) and (3-5) respectively, better represent the impact force profile close to the physical reality. These models present a strong nonlinear behavior, which makes parameter identification difficult.

Figures 3.9 and 3.10 show the comparison between experimental data and numerical simulation for the case where the hammer is released from two different initial conditions. The impact is modeled using the spring-dashpot model.

Parameter	Value	Unit
Impact stiffness, $k_i$	$5.5\cdot 10^6$	N/m
Impact damping, $c_i$	$1.2\cdot 10^3$	Ns/m



Figure 3.9: Impact force parameters identification. Comparison between experimental data and numerical simulation. Impact modeled using the spring-dashpot model.



Figure 3.10: Impact force parameters identification. Comparison between experimental data and numerical simulation. Impact modeled using the spring-dashpot model.

Although the spring-dashpot model, Equation (3-3), is not capable of reproducing the real impact force profile over time due to the jump caused by the damping force, this model generated the best results. Impact parameters for this model are listed in Table 3.1.

#### 3.3.3 Mathematical modeling and parameter identification

The mathematical modeling of the first test rig, i.e., hammer supported by wires, is presented below. Using the Lagrange equation [42], with  $\theta$  as the generalized coordinate, see Figure 3.11, the hammer can be modeled as a single pendulum embarked in a cart with prescribed movement  $(y(t) = A_0 \sin \Omega t)$ , where the impact surface is also moving within the system. See Figure 3.11.



Figure 3.11: Model of hammer supported by wires, physical representation.

For the situation of no impact, i.e.,  $l\sin\theta - gap > 0$ , equation of motion is

$$ml^{2}\ddot{\theta} - mlA_{0}\Omega^{2}\cos\theta\sin\Omega t + mgl\sin\theta = 0.$$
(3-15)

Because impact is modeled using continuous analysis, when the hammer is impacting the cart  $(l \sin \theta - gap \leq 0)$ , the equation of motion will slightly change to

$$ml^{2}\ddot{\theta} - mlA_{0}\Omega^{2}\cos\theta\sin\Omega t + mgl\sin\theta = -F_{i}l; \ F_{i} = k_{i}\delta + c_{i}\dot{\delta}, \quad (3-16)$$

where the penetration  $\delta$  and the velocity of penetration,  $\dot{\delta}$  are described as

$$\delta = l \sin \theta - gap,$$
  

$$\dot{\delta} = l \dot{\theta} \cos \theta.$$
(3-17)

It is important to emphasize that the generalized coordinate  $\theta$  (and therefore  $\dot{\theta}$ ) is embarked on the cart. To compare the numerical results with

Parameter	Value	Unit
Natural frequency, $\omega$	1.82	Hz
Hammer mass, $m$	0.298	kg
Cart mass, $M$	5.38	kg
Wire length, $l$	75	mm
Excitation amplitude, $A_0$	0.89	mm

Table 3.2: Hammer supported by wires - parameters identification.

the experimental data, where the linear displacement is measured outside the cart, the following transformations must take place:

$$x = l\sin\theta + A_0\sin(\Omega t)$$
  

$$\dot{x} = l\dot{\theta}\cos\theta + A_0\Omega\cos(\Omega t)$$
(3-18)

The system without impact presents some degree of damping. However, the mathematical modeling does not take this into consideration. For the system without impact, the test rig parameters are identified and the hammer natural frequency is obtained. These results are shown in Table 3.2.

#### 3.3.4 Numerical results and comparison between numerical simulation and experimental results

According to the Filippov theory [48] [51] [49], the mathematical modeling presented is described by a differential equation with a discontinuous right-hand side. Therefore, the state space  $\dot{\mathbf{x}} = f(\mathbf{x}), \mathbf{x} \in \mathbf{R}^n$ may be split into two subspaces  $\Gamma_-$  and  $\Gamma_+$ , separated by a hyper-surface  $\Sigma$ . Hyper-surface is defined by a scalar function  $h(\mathbf{x})$ . Consequently, the state space  $\mathbf{x}$  is in  $\Sigma$  when  $h(\mathbf{x}) = 0$ . Hence, it is possible to define the subspaces  $\Gamma_-$  and  $\Gamma_+$ , as well as the hyper-surface  $\Sigma$ , using the sets:

$$\Gamma_{-} = \{ \mathbf{x} \in \mathbf{R}^{n} \mid h(\mathbf{x}) < 0 \}$$
  

$$\Sigma = \{ \mathbf{x} \in \mathbf{R}^{n} \mid h(\mathbf{x}) = 0 \}$$
  

$$\Gamma_{+} = \{ \mathbf{x} \in \mathbf{R}^{n} \mid h(\mathbf{x}) > 0 \}$$
  
(3-19)

Some physical systems need different interfaces in order to perform a correct description of the transitions. The impact force model used in the mathematical modeling is an example. Due to the nature of the impact model, the contact between the mass and the support occurs whenever the linear displacement becomes equal to the contact gap. However, the mass loses contact with the support when the contact force vanishes. Two indicator functions are used to define the system subspaces [68]:

$$h_{\alpha}(\theta,\theta) = l\sin\theta - gap$$

$$h_{\beta}(\theta,\dot{\theta}) = k_{i}\delta + c_{i}\dot{\delta},$$
(3-20)

where the penetration  $\delta$  and velocity of penetration  $\dot{\delta}$  are already defined in Equation (3-17).

The mass is not in contact with the support if the state vector  $\mathbf{x} = (\theta, \dot{\theta}) \in \Gamma_{-}$ , in other words:

$$\Gamma_{-} = \{ \mathbf{x} \in \mathbf{R}^2 \mid h_{\alpha}(\theta, \dot{\theta}) < 0 \text{ or } h_{\beta}(\theta, \dot{\theta}) < 0 \}.$$
(3-21)

For the case when there is contact between the mass and the support:

$$\Gamma_{+} = \{ \mathbf{x} \in \mathbf{R}^{2} \mid h_{\alpha}(\theta, \dot{\theta}) > 0 \text{ and } h_{\beta}(\theta, \dot{\theta}) > 0 \}.$$
(3-22)

The hyper-surface  $\Sigma$  consists of the conjunction of two surfaces,  $\Sigma_{\alpha}$ and  $\Sigma_{\beta}$ . The hyper-surface  $\Sigma_{\alpha}$  defines the transition from  $\Gamma_{-}$  to  $\Gamma_{+}$ , i.e., when the mass initiates the contact with the support  $(l\sin\theta - gap = 0)$ ,

$$\Sigma_{\alpha} = \{ \mathbf{x} \in \mathbf{R}^2 \mid h_{\alpha}(\theta, \dot{\theta}) = 0 \text{ and } h_{\beta}(\theta, \dot{\theta}) \ge 0 \}.$$
 (3-23)

Surface  $\Sigma_{\beta}$  defines the transition from  $\Gamma_{+}$  to  $\Gamma_{-}$  as the contact is lost when the impact force vanishes:

$$\Sigma_{\beta} = \{ \mathbf{x} \in \mathbf{R}^2 \mid h_{\alpha}(\theta, \dot{\theta}) \ge 0 \text{ and } h_{\beta}(\theta, \dot{\theta}) = 0 \}.$$
 (3-24)

Consequently, the state equation of this discontinuous system is written as follows:

$$\dot{\mathbf{x}} = f(\mathbf{x}, t) = \begin{cases} f_{-}(\mathbf{x}, t), \mathbf{x} \in \Gamma_{-} \\ \overline{co} \{ f_{-}(\mathbf{x}, t), f_{+}(\mathbf{x}, t) \}, \mathbf{x} \in \Sigma \\ f_{+}(\mathbf{x}, t), \mathbf{x} \in \Gamma_{+} \end{cases}$$
(3-25)

where

$$f_{-}(\mathbf{x},t) = \begin{bmatrix} \dot{\theta} \\ \frac{A_0 \Omega^2}{l} \cos \theta \sin \Omega t - \frac{g}{l} \sin \theta \end{bmatrix}; \mathbf{x} \in \Gamma_{-}$$
(3-26)

$$f_{+}(\mathbf{x},t) = \begin{bmatrix} \dot{\theta} \\ \frac{A_{0}\Omega^{2}}{l}\cos\theta\sin\Omega t - \frac{g}{l}\sin\theta - \frac{1}{ml}(k_{i}\delta + c_{i}\dot{\delta}) \end{bmatrix} ; \mathbf{x} \in \Gamma_{+} \quad (3-27)$$

$$\overline{co}\{f_{-}(\mathbf{x},t),f_{+}(\mathbf{x},t)\} = \begin{bmatrix} \dot{\theta} \\ \frac{A_{0}\Omega^{2}}{l}\cos\theta\sin\Omega t - \frac{g}{l}\sin\theta - \frac{1}{ml}(c_{i}\dot{\delta}) \end{bmatrix}; in \Sigma_{\alpha}$$
(3-28)

$$\overline{co}\{f_{-}(\mathbf{x},t),f_{+}(\mathbf{x},t)\} = \begin{bmatrix} \dot{\theta} \\ \frac{A_{0}\Omega^{2}}{l}\cos\theta\sin\Omega t - \frac{g}{l}\sin\theta \end{bmatrix}; in \Sigma_{\beta} \qquad (3-29)$$

This approach allows one to deal with non-smooth systems by employing a smoothed system. Leine [49] also defines a finite region that considers transition between subspaces. Therefore, a region of small relative displacement is defined as  $|l \sin \theta - gap| < \eta$  as well as  $|k_i \delta + c_i \dot{\delta}| < \eta$ , where  $\eta \ll 1$ . The finite region is useful for numerical simulations because an exact instant of impact will not be computed. The thickness parameter of the narrow band  $\eta$  needs to be chosen according to the physical problem [68].

The comparison between numerical simulation and experimental data starts with the chart of the non-dimensional force  $(F_i/mg)$  in the non-dimensional frequency domain  $(\Omega/\omega)$ , for each gap imposed on the test rig. These results are shown in Figures 3.12, 3.13 and 3.14. The methodology applied to identify is the same as that used during the experimental analysis, where for each excitation frequency the maximum impact force is detected, regardless of the impact force behavior.



Figure 3.12: Hammer supported by wires, comparison between numerical simulation and experimental results. Non dimensional force *versus* non dimensional frequency. Gap 0.0 mm.



Figure 3.13: Hammer supported by wires, comparison between numerical simulation and experimental results. Non dimensional force *versus* non dimensional frequency. Gap 1.0 mm.



Figure 3.14: Hammer supported by wires, comparison between numerical simulation and experimental results. Non dimensional force *versus* non dimensional frequency. Gap 2.4 mm.

From the numerical analysis and the comparison between numerical simulation and experimental data shown in Figure 3.12, it is possible to identify, numerically, the frequency bands observed in the test rig, as the range of excitation frequencies is covered. For the first frequency band, characterized by one impact per excitation cycle, the time range used to capture the experimental displacements (cart + hammer) was too short, showing eventually one or two cycles of excitation, which is not enough to perform a satisfactory comparison of displacement over time, phase plane and Poincaré map.

#### 3.3.5

### Hammer displacement, phase plane and Poincaré map; comparison between numerical simulation and experimental results

#### Poincaré map.

In mathematics, particularly in dynamical systems, a first recurrence map or Poincaré map, named after Henri Poincaré (29 April 1854 - 17 July 1912), is the intersection of a periodic orbit in the state space of a continuous dynamical system with a certain lower dimensional subspace, called the Poincaré section, transversal to the flow of the system. More precisely, one considers a periodic orbit with initial conditions on the Poincaré section and observes the point at which these orbits first return to the section, thus the name first recurrence map [90]. See Figure 3.15.



Figure 3.15: Poincaré map [90].

A Poincaré map can be interpreted as a discrete dynamical system with a state space that is one dimension smaller than the original continuous dynamical system. It preserves many properties of periodic and quasiperiodic orbits of the original system and has a lower dimensional state space that is often used for analyzing the original system. In practice this is not always possible as there is no general method to construct a Poincaré map [90].

With the exception of the first frequency band (z = 1/1) for the 0.0 mm gap, where the number of excitation cycles was insufficient, the experimental data is compared to numerical simulations. Figures 3.16 to 3.23 show the phase plane and Poincaré map. Charts show good agreement, especially in every frequency band, even showing satisfactory agreement after bifurcation. Figures 3.18 and 3.21 show the comparison for the first frequency band (z = 1/1), while Figures 3.17 and 3.19 show the comparison for the second frequency band (z = 1/2), all in different gap conditions. Figures 3.16 and 3.20 show the comparison after the bifurcation.



Figure 3.16: Hammer supported by wires, gap 0.0 mm, non dimensional frequency 2.7. Numerical (blue)/ experiment (red) comparison: a) Hammer displacement; b) Phase plane (solid line) and Poincaré map (dots).



Figure 3.17: Hammer supported by wires, gap 0.0 mm, non dimensional frequency 3.8. Numerical simulation (blue)/ experimental data (red) comparison: a) Hammer displacement; b) Phase plane (solid line) and Poincaré map (dots).



Figure 3.18: Hammer supported by wires, gap 1.0 mm, non dimensional frequency 1.5. Numerical simulation (blue)/ experimental data (red) comparison: a) Hammer displacement; b) Phase plane (solid line) and Poincaré map (dots).



Figure 3.19: Hammer supported by wires, gap 1.0 mm, non dimensional frequency 3.3. Numerical simulation (blue)/ experimental data (red) comparison: a) Hammer displacement; b) Phase plane (solid line) and Poincaré map (dots).



Figure 3.20: Hammer supported by wires, gap 1.0 mm, non dimensional frequency 4.7. Numerical simulation (blue)/ experimental data (red) comparison: a) Hammer displacement; b) Phase plane (solid line) and Poincaré map (dots).



Figure 3.21: Hammer supported by wires, gap 2.4 mm, non dimensional frequency 1.2. Numerical simulation (blue)/ experimental data (red) comparison: a) Hammer displacement; b) Phase plane (solid line) and Poincaré map (dots).



Figure 3.22: Hammer supported by wires, gap 2.4 mm, non dimensional frequency 1.4. Numerical simulation (blue)/ experimental data (red) comparison: a) Hammer displacement; b) Phase plane (solid line) and Poincaré map (dots).



Figure 3.23: Hammer supported by wires, gap 2.4 mm, non dimensional frequency 2.7. Numerical simulation (blue)/ experimental data (red) comparison: a) Hammer displacement; b) Phase plane (solid line) and Poincaré map (dots).

Overall comparison shows a satisfactory agreement, especially concerning the impact resonance and the maximum impact force for the z = 1/1 behavior, for each gap imposed.

#### 3.3.6

## Further numerical analysis: bifurcation diagrams, Peterka map and basins of attraction

Although the impact force charts shown in Figure 3.12 give some important information regarding the impact force amplitude and the impact

resonance, such charts provide neither information about the characteristics of the impact force, nor details on the transition between frequency bands. To better visualize the behavior of this dynamical system, two nonlinear tools are used. One is the Peterka map [60] [57] [56] [55], shown in Figure 3.24, which provides information about the characteristic of the impact force as the gap is varied and the range of excitation frequencies is covered. From this chart one can see the areas where the two frequencies bands occur, as noted by the red (z = 1/1) and green (z = 1/2) areas. The second tool is a bifurcation diagram [75], where the bifurcation in the z = 1/1 behavior is observed in more detail, showing the conditions of pitch-fork bifurcation (for gap 0.0 mm) and chaotic behavior (gaps 1.0 mm and 2.4 mm).



Figure 3.24: Hammer supported by wires, numerical result, Peterka map.



Figure 3.25: Hammer supported by wires, numerical (bright colors) / experiment (dots) comparison, Peterka map.

It is important to emphasize that in the Peterka map each gap/ frequency combination correspond to one numerical simulation. In order

to generate a satisfactory image resolution, the Peterka map was built in steps of 0.01 mm in gap and 0.01 Hz in frequency, generating 209916 points (therefore 209916 simulations). The computational effort to generate such map was around 120 hours in a PC equipped with an Intel Core 2 DUO 1.80 GHz with 2 Gb of RAM.

Although the Peterka map provides important information about the condition of impact, no information regarding the impact magnitude is given. To overcome this problem, a slight variance of the Peterka map is suggested. The relevant impact condition is z = 1/1. Therefore, just this area in the Peterka map is addressed. For each gap/ frequency combination, the nondimensional impact force  $(F_i/mg)$  is obtained and plotted in colors, see Figure 3.26.



Figure 3.26: Hammer supported by wires, Peterka map of z = 1/1 with impact force magnitude addressed.

This chart provides several important facts about the system behavior and it confirms some aspects observed during the experimental analysis. First, it confirms that the impact force when the hammer is excited in its natural frequency generates impact forces that are 3 times smaller in magnitude than the maximum force. It also shows that the maximum impact force for each given gap does not occur at the z = 1/1 boundary, except for high values of gap. Finally, the chart confirms the recommendation to operate in the field using the 0.0 mm gap, because the magnitude of the impact force is in the same value as the impact force in higher gap values. In addition, non-zero gap values are known to present nonlinear jumps.



Figure 3.27: Hammer supported by wires, gap 0.0 mm, bifurcation diagrams: a) Hammer displacement, Numerical simulation (blue)/ experimental data (red) comparison; b) Impact force (numerical).



Figure 3.28: Hammer supported by wires, gap 0.0 mm, bifurcation diagrams, hammer displacement, details of Figure 3.27(a).



Figure 3.29: Hammer supported by wires, gap 0.0 mm, bifurcation diagrams, Impact force, details of Figure 3.27(b).



Figure 3.30: Hammer supported by wires, gap 1.0 mm, bifurcation diagrams: a) Hammer displacement, Numerical simulation (blue)/ experimental data (red) comparison; b) Impact force (numerical).



Figure 3.31: Hammer supported by wires, gap 1.0 mm, bifurcation diagrams, Hammer displacement, details of Figure 3.30(a).



Figure 3.32: Hammer supported by wires, gap 1.0 mm, bifurcation diagrams, Impact force, details of Figure 3.30(b).



Figure 3.33: Hammer supported by wires, gap 2.4 mm, bifurcation diagrams: a) Hammer displacement, Numerical simulation (blue)/ experimental data (red) comparison; b) Impact force (numerical).



Figure 3.34: Hammer supported by wires, gap 2.4 mm, bifurcation diagrams, hammer displacement, details of Figure 3.33(a).



Figure 3.35: Hammer supported by wires, gap 2.4 mm, bifurcation diagrams, Impact force, details of Figure 3.33(b).

Finally, the presence of impact and the gap between the hammer and the cart induces nonlinearities, and therefore nonlinear phenomena arise, specifically in the transition between frequency bands. One of these phenomena is the change of the basins of attraction [75] for some gap conditions. In the Peterka map, Figure 3.24, for a gap condition higher than 1.5, there is an area between the z = 1/1 (red) and z = 1/2 (green) regions that are characterized by various impact conditions, which are dependent on the system initial condition, as verified by the experiment, Figure 2.20. In some conditions, where the system is excited, when a small amount of energy is inserted, the system will enter an impact condition for some time and then return to its non-impact condition. However, for a similar condition, when a small amount of energy is inserted, the system will enter an impact condition and remain in this state, as shown in Figure 2.20.

This area in the Peterka map can be better visualized with the use of basins of attraction [75], defined as the set of initial conditions  $\mathbf{x}_0$  such that  $\mathbf{x}(t) \to \mathbf{x}^*$  as  $t \to \infty$ .



Figure 3.36: Hammer supported by wires, basins of attraction; condition of impact (blue) / no impact (red): a)  $\Omega/\omega = 2.00 \ gap/A_0 = 2.00$ ; b)  $\Omega/\omega = 2.00 \ gap/A_0 = 1.50$ .



Figure 3.37: Hammer supported by wires, basins of attraction; condition of impact (blue) / no impact (red): a)  $\Omega/\omega = 1.75 \ gap/A_0 = 2.50$ ; b)  $\Omega/\omega = 1.75 \ gap/A_0 = 3.50$ .



Figure 3.38: Hammer supported by wires, basins of attraction; condition of impact (blue) / no impact (red): a)  $\Omega/\omega = 2.15 \ gap/A_0 = 1.60$ ; b)  $\Omega/\omega = 2.25 \ gap/A_0 = 1.50$ .

The computational effort to generate each basins of attraction was around 20 hours in a PC equipped with an Intel Core 2 DUO 1.80 GHz with 2 Gb of RAM.

#### 3.4 Hammer supported by beam springs - mathematical modeling and comparison between numerical simulation and experimental results

#### 3.4.1 Mathematical modeling and parameter identification

The mathematical modeling of this test rig is presented below. A simple mass-spring-damper system with base excitation [42] is used. See Figure 3.39.



Figure 3.39: Model of hammer supported by beam springs, physical representation.

For the situation of no impact, i.e., x - (y + gap) > 0, equation of motion is

$$m\ddot{x} + c(\dot{x} - \dot{y}) + k(x - y) + k(x - y)^3 = 0, \qquad (3-30)$$

where

$$y = A_0 \sin(\Omega t),$$
  

$$\dot{y} = A_0 \Omega \cos(\Omega t).$$
(3-31)

Although the hammer support has changed, the impact set up remains the same as the last experiment, because the materials and geometries of the impact bodies remain the same. So the same impact model will be used. Therefore, when the hammer is impacting the cart  $(x - (y + gap) \leq 0)$ , the equation of motion will slightly change to

$$m\ddot{x} + c(\dot{x} - \dot{y}) + k(x - y) + k(x - y)^3 = -F_i,$$
  

$$F_i = k_i \delta + c_i \dot{\delta},$$
(3-32)

Parameter	Value	Unit
Hammer mass, $m$	0.298	kg
Cart mass, $M$	5.38	kg
Excitation amplitude, $A_0$	0.89	mm
Damping ratio, $\zeta$	0.004	
Couplings Distance	Value	$\mathbf{Unit}$
$170 \mathrm{mm}$		
Natural frequency, $\omega$	4.50	Hz
Damping coefficient, $\zeta$	0.06	Ns/m
Couplings Distance	Value	Unit
$150 \mathrm{\ mm}$		
Natural frequency, $\omega$	5.25	Hz
Damping coefficient, $\zeta$	0.08	Ns/m
Couplings Distance	Value	$\mathbf{Unit}$
$135 \mathrm{\ mm}$		
Natural frequency, $\omega$	6.50	Hz
Damping coefficient, $\zeta$	0.09	Ns/m

Table 3.3: Hammer supported by beam springs - parameters identification.

where the penetration  $\delta$  and the velocity of penetration  $\dot{\delta}$  are described as

$$\delta = x - (y + gap),$$
  

$$\dot{\delta} = \dot{x} - \dot{y}.$$
(3-33)

In this modeling, the hammer displacement x is taken from a fixed frame of reference. Consequently the numerical results will be directly compared to the experimental data.

The parameter identification of this test rig takes into account some results from the previous experiment (hammer mass, cart mass and excitation amplitude). The first natural frequency of the beam springs assembly [42]represents the hammer stiffness in the mathematical model, see Equation (3-30):

$$k = m\omega^2, \tag{3-34}$$

where m is the hammer mass and  $\omega$  is the first natural frequency of the bending vibration of the beam springs, determined experimentally. For the damping identification, only viscous damping is considered, and because the damping ratio is related to material damping, the same ratio is adopted for all three beam spring lengths. The value of the damping ratio was extracted from previous work [2]. As in the mathematical modeling of the hammer supported by wires, the Filippov theory must be used in order to numerically integrate the equations of motion. Because the same impact model is used in both mathematical models, similar indicator functions are used. The contact between the mass and the support occurs whenever the linear displacement becomes equal to contact gap. In the other hand, the mass loses contact with the support when the contact force vanishes. The tow indicator functions are described as [68]:

$$h_{\alpha}(x, \dot{x}) = A_0 \sin(\Omega t) + gap,$$
  

$$h_{\beta}(x, \dot{x}) = k_i \delta + c_i \dot{\delta},$$
(3-35)

where the penetration  $\delta$  and velocity of penetration  $\dot{\delta}$  are defined in Equation (3-33). The same sub-spaces and hyper surfaces are used, refer to Equations (3-21), (3-22), (3-23), (3-24). Consequently, the state equation of this discontinuous system is written according to Equation (3-25), where

$$f_{-}(\mathbf{x},t) = \begin{bmatrix} \dot{x} \\ \frac{1}{m} \left( -c\dot{\Delta} - k\Delta - k\Delta^{3} \right) \end{bmatrix}; \mathbf{x} \in \Gamma_{-}$$
(3-36)

$$f_{+}(\mathbf{x},t) = \begin{bmatrix} \dot{x} \\ \frac{1}{m} \left( -c\dot{\Delta} - k\Delta - k\Delta^{3} - F_{i} \right) \end{bmatrix} ; \mathbf{x} \in \Gamma_{+}$$
(3-37)

$$\overline{co}\{f_{-}(\mathbf{x},t),f_{+}(\mathbf{x},t)\} = \begin{bmatrix} \dot{x} \\ \frac{1}{m} \left(-c\dot{\Delta} - k\Delta - k\Delta^{3} - (c_{i}\dot{\delta})\right) \end{bmatrix}; in \Sigma_{\alpha} \quad (3-38)$$

$$\overline{co}\{f_{-}(\mathbf{x},t),f_{+}(\mathbf{x},t)\} = \begin{bmatrix} \dot{x} \\ \frac{1}{m} \left(-c\dot{\Delta} - k\Delta - k\Delta^{3}\right) \end{bmatrix}; in \Sigma_{\beta} \qquad (3-39)$$

where  $\Delta = x - y$  and  $\dot{\Delta} = \dot{x} - \dot{y}$ .

#### 3.4.2 Comparison between numerical simulation and experimental results

The comparison between numerical simulation and experimental results starts with the chart of the non-dimensional force  $(F_i/mg)$  in the frequency domain  $(\Omega/\omega)$ , for each stiffness and gap imposed on the test rig. These results are shown in Figures 3.40, 3.41 and 3.42. The methodology applied in order to identify the impact force is the same performed for the experimental data, where for each excitation frequency the maximum impact force is detected, regardless of the impact force behavior.



Figure 3.40: Hammer supported by beam springs, couplings distance 170 mm. Numerical/ experiment comparison. Non dimensional force *versus* non dimensional frequency. a) Gap 0.0 mm. b) Gap 1.0 mm. c) Gap 3 mm.



Figure 3.41: Hammer supported by beam springs, couplings distance 150 mm. Numerical/ experiment comparison. Non dimensional force *versus* non dimensional frequency. a) Gap 0.0 mm. b) Gap 1.0 mm. c) Gap 3 mm.



Figure 3.42: Hammer supported by beam springs, couplings distance 135 mm. Numerical/ experiment comparison. Non dimensional force *versus* non dimensional frequency. a) Gap 0.0 mm. b) Gap 1.0 mm. c) Gap 3 mm.

Simulation results show satisfactory agreement with the experimental data. For the beam springs length of 170 mm, Figure 3.40, the simulation

captures well the maximum impact force and also the presence of the nonlinear jump, for the 3.0 mm gap configuration (Figure 3.40c). For the gap 1.0 mm configuration the comparison between numerical simulation and experimental data is not as satisfactory as compared to the other gap configurations. However such disagreement is due to the unexpected experimental data, as mentioned in the analysis on Chapter 2, where the behavior impact force magnitude in the frequency domain for the 1.0 mm gap was not in between 0.0 mm and 3.0 mm gap, as expected.

For the beam spring length of 150 mm, the agreement is better for the 1.0 mm gap configuration. For the 0.0 mm gap configuration, the agreement is satisfactory until the excitation frequency is twice the value of the natural frequency of the hammer. For the 3.0 mm gap configuration, agreement is also satisfactory, although the nonlinear jump is detected with a 10 % error in frequency. For the beam spring length of 135 mm, the agreement is satisfactory up to a non-dimensional frequency of 1.7, for the cases of 0.0 mm and 1.0 mm gap.

### 3.4.3 Nonlinear analysis: bifurcation diagrams, Peterka map and basins of attraction

In this subsection some nonlinear tools are used to investigate the hammer behavior, starting with the bifurcation diagrams, shown in Figures 3.43, 3.44 and 3.45.



Figure 3.43: Hammer springs, couplings distance 170 mm, bifurcation diagrams, hammer displacement, numerical simulation (blue)/ experimental data (red) comparison: a) gap 0.0 mm; b) gap 1.0 mm; c) gap 3.0 mm.



Figure 3.44: Hammer springs, couplings distance 150 mm, bifurcation diagrams, hammer displacement, numerical simulation (blue)/ experimental data (red) comparison: a) gap 0.0 mm; b) gap 1.0 mm; c) gap 3.0 mm.



Figure 3.45: Hammer springs, couplings distance 135 mm, bifurcation diagrams, hammer displacement, numerical simulation (blue)/ experimental data (red) comparison: a) gap 0.0 mm; b) gap 1.0 mm; c) gap 3.0 mm.

Two interesting issues can be observed. First, the bifurcation diagrams of the experimental data in all stiffness / gap combinations present a group of dispersed points, even in regions where a steady behavior was observed (z = 1/1 for example). So this behavior is not related to any chaotic condition. The second observation is related to the non-agreement between numerical simulation and experiment data. The simulation results present higher amplitudes than the experimental data. The phenomena that justifies both issues is the energy distribution in the bending vibration modes of the beam springs after each impact. Such energy distribution is shown as a dispersion of the experimental points in the bifurcation map, even in a steady-state condition. Because the mathematical model considers only the first bending vibration mode, the amplitude obtained by numerical simulation is always higher than the experimental data, even when the impact force is presenting equivalent values. This difference appears to be higher as the beam springs stiffness increases. Even taken into account this amplitude difference, the transition between frequency bands can be qualitatively observed.

Figures 3.46, 3.48 and 3.50 show the Peterka map, determined numerically. For all hammer stiffness, the map shows similar characteristics to the Peterka map supported by wires (Figure 3.24), even the instability region after the z = 1/1 region in values of non-dimensional gap higher than 1.5 (Figures 3.46(b), 3.48(b) and 3.50(b)). Figures 3.47, 3.49 and 3.51 present the Peterka map for frequency band z = 1/1 showing the magnitude of the impact force. The same observations stated for the hammer supported by wires can be applied here. One more conclusion observed is the similarity in the shape pattern of the maps regardless of the stiffness imposed (Figures 3.46(a), 3.48(a) and 3.50(a)). The maps are identical in all cases, except for the chaotic region between regions z = 1/1 and z = 1/2. This is an indication that the impact force behavior is somehow not dependent on the hammer stiffness. Independence is not completely due to the non-dimensional gap, which takes into consideration the hammer's first natural bending vibration frequency, a function of the system stiffness.



Figure 3.46: Hammer springs, couplings distance 170 mm; a) Peterka map;b) Detail of map.



Figure 3.47: Hammer springs, couplings distance 170 mm, Peterka map of z = 1/1 with impact force magnitude addressed.



Figure 3.48: Hammer springs, couplings distance 150 mm; a) Peterka map; b) Detail of map.



Figure 3.49: Hammer springs, couplings distance 150 mm, Peterka map of z = 1/1 with impact force magnitude addressed.



Figure 3.50: Hammer springs, couplings distance 135 mm; a) Peterka map; b) Detail of map.



Figure 3.51: Hammer springs, couplings distance 150 mm, Peterka map of z = 1/1 with impact force magnitude addressed.

Figures 3.46(b), 3.48(b) and 3.50(b) show some basins of attraction of the instability region presented in the Peterka map.



Figure 3.52: Hammer springs, couplings distance 170 mm, basins of attraction; condition of impact (blue) / no impact (red): a)  $\Omega/\omega = 1.80$   $gap/A_0 = 1.60$ ; b)  $\Omega/\omega = 2.20 \ gap/A_0 = 1.50$ .



Figure 3.53: Hammer springs, couplings distance 150 mm, basins of attraction; condition of impact (blue) / no impact (red): a)  $\Omega/\omega = 1.80$  $gap/A_0 = 1.74$ ; b)  $\Omega/\omega = 2.10 \ gap/A_0 = 1.40$ .



Figure 3.54: Hammer springs, couplings distance 135 mm, basins of attraction; condition of impact (blue) / no impact (red): a)  $\Omega/\omega = 2.15$   $gap/A_0 = 1.60$ ; b)  $\Omega/\omega = 2.25 \ gap/A_0 = 1.50$ .

#### 3.5 Final remarks

This chapter presented a general view of the state of the art of impact mechanics, as well as some methodologies to model the impact, and the numerical integration of discontinuous ordinary differential equations. The hammer modeling for both test rigs was also addressed in this chapter. The parameter identification was discussed and the comparison between numerical analysis and the experimental results was performed.

In both test rigs, the comparison between numerical simulation and experimental data was satisfactory, showing the capability of the mathematical model to predict the impact resonance for each stiffness/ gap combination. Nonlinear tools were used to understand the impact force behavior, including bifurcation diagrams, basins of attraction, Poincaré maps and Peterka maps. In this chapter a new methodology was proposed to better visualize each impact force behavior in the Peterka map, plotting one impact force behavior at a time (in this case only the z = 1/1 region was plotted), adding colors to the third coordinate  $F_i$ , the impact force magnitude. This methodology provided important information regarding the hammer behavior and it confirmed some aspects observed during the experiment analysis. After considering the experimental data and the nonlinear tools, it is recommended to operate with 0.0 mm gap, because the magnitude of the developed impact forces is in the same range as the impact force in higher gap values. However, in higher gap values nonlinear jump occurs, which is not the case for the 0.0 mm gap, where approximate values of the maximum impact force occur for a wider frequency range.

For the second test rig, i.e., hammer supported by beam springs, the mathematical model is capable of qualitatively determining the frequency bands, and predicting the impact force magnitude in the frequency domain for each stiffness/ gap combination. However, the mathematical model did not predict the hammer displacement well, due to the energy distribution in the bending vibration modes of the beam springs following each impact. This energy distribution was observed as a dispersion of the experimental points in the bifurcation map, even in steady-state condition. The mathematical model considers only the first bending vibration mode, so the amplitude of the mathematical model is always higher than the experimental data, even when the impact force presents equivalent values. This difference appears to be greater as the beam springs stiffness increases.

One last aspect that was observed in the case of the hammer supported by beam springs was the similarity in the shape pattern of the Peterka maps regardless of the hammer stiffness imposed, except for the chaotic region between frequency bands z = 1/1 and z = 1/2. This is an indication that the impact force behavior is somehow not dependent on the hammer stiffness.