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A
Regularization of real Lipschitz functions

In this section $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$a \leq \frac{f(x) - f(y)}{x - y} \leq b \ , \quad x \neq y.$$

For $\delta > 0$ define $\psi_\delta : \mathbb{R} \to \mathbb{R}$ by

$$\psi_\delta(x) = \frac{1}{\delta\sqrt{2\pi}} \exp\big(- \frac{1}{2}\Big(\frac{x}{\delta}\Big)^2\big).$$

Define

$$f_\delta(x) := \int_{\mathbb{R}} f(s)\psi_\delta(x - s)ds = \int_{\mathbb{R}} f(x - s)\psi_\delta(s)ds. \tag{A.1}$$

Proposition A.1 *There exists some function $c : \{x > 0\} \to \{x > 0\}$ such that $\delta \to 0$ implies that $c(\delta) \to 0$ and for all $x \in \mathbb{R}$ we have $|f(x) - f_\delta(x)| < c(\delta)$. Moreover, if $x \neq y$, then $a \leq \dfrac{f_\delta(x) - f_\delta(y)}{x - y} \leq b$.*

Proof: It is easy to see that

$$a \leq \frac{f_\delta(x) - f_\delta(y)}{x - y} \leq b \ , \quad x \neq y.$$

Now, we obtain the desired $c : \{x > 0\} \to \{x > 0\}$.

$$\begin{aligned} |f_\delta(x) - f(x)| &= \Big| \int_{\mathbb{R}} \big(f(x - s) - f(x)\big)\, \psi_\delta(s)ds \,\Big| \\ &\leq \int_{\mathbb{R}} |f(x - s) - f(x)|\, \psi_\delta(s)ds \\ &\leq (|a| + |b|) \int_{\mathbb{R}} |s|\, \psi_\delta(0 - s) = c(\delta). \end{aligned}$$

Observe that $\delta \to 0$ implies that $c(\delta) \to |0| = 0$.

■

Lemma A.2 *Suppose that*

$$\lim_{s\to-\infty}\frac{f(s)}{s}=a\ ,\qquad \lim_{s\to+\infty}\frac{f(s)}{s}=b.$$

Then,

$$\lim_{s\to-\infty}\frac{f_\delta(s)}{s}=a\ ,\qquad \lim_{s\to+\infty}\frac{f_\delta(s)}{s}=b$$

Proof: By proposition A.1, for all $x\in\mathbb{R}$,

$$|f_\delta(x)-f(x)|\leq c(\delta).$$

Dividing by x and making $x\to-\infty$ (resp. $+\infty$) we have that $f_\delta(x)/x\to a$ (resp. b).

∎

Lemma A.3 *Let $f:\mathbb{R}\to\mathbb{R}$ be a Lipschitz (thus, almost everywhere differentiable) convex function such that*

$$\liminf_{s\to-\infty}f'(s)=a<\limsup_{s\to+\infty}f'(s)=b.$$

Then, for all $\delta>0$, f_δ is such that $f''_\delta>0$.

Proof: Take $\psi_\delta:\mathbb{R}\to\mathbb{R}$ and $f_\delta:\mathbb{R}\to\mathbb{R}$ as defined above. As we have already seen, f_δ converges uniformly to f as $\delta\to0$.

We prove that f_δ is strictly convex. As f is Lipschitz, the fundamental theorem of calculus is valid for it. Integrating by parts we obtain

$$f_\delta{}'(x)=\int_{\mathbb{R}}f'(s)\psi_\delta(x-s).$$

As a consequence, noting that $\psi_\delta{}'$ is odd, $(\psi_\delta(-s)=-\psi_\delta(s))$

$$\begin{aligned}
f_\delta{}''(x)&=\int_{\mathbb{R}}f'(x-s)\psi_\delta{}'(s)ds\\
&=\int_{-\infty}^{0}f'(x-s)\psi_\delta{}'(s)ds+\int_{0}^{+\infty}f'(x-s)\psi_\delta{}'(s)ds\\
&=\int_{-\infty}^{0}f'(x-s)\psi_\delta{}'(s)ds+\int_{-\infty}^{0}f'(x+s)\psi_\delta{}'(-s)ds\\
&=\int_{-\infty}^{0}f'(x-s)\psi_\delta{}'(s)ds-\int_{-\infty}^{0}f'(x+s)\psi_\delta{}'(s)ds\\
&=\int_{-\infty}^{0}\big(f'(x-s)-f'(x+s)\big)\psi_\delta{}'(s)ds.
\end{aligned}$$

For $s < 0$, $f'(x-s) \geq f'(x+s)$ as f is convex. Also, $\psi_\delta(s) > 0$. This suffices to prove that f_δ is convex. To obtain that it is strictly convex, for each $x \in \mathbb{R}$, take $s_0(x) < 0$ so that, for $s < s_0(x)$ both $|f'(x-s) - b| \leq (b-a)/4$ and $|f'(x+s) - a| \leq (b-a)/4$ almost everywhere. Then, we have

$$f'(x-s) \geq \frac{3b+a}{4} , \quad -f'(x+s) \geq -\frac{3a+b}{4}.$$

Finally,

$$f_\delta''(x) \geq \int_{-\infty}^{s_0(x)} \Big(\frac{3b+a}{4} - \frac{3a+b}{4}\Big)\psi_\delta'(s)ds = \int_{-\infty}^{s_0(x)} \frac{b-a}{2}\psi_\delta'(s)ds > 0.$$

■

B
Spectral Theory

Definition B.1 *1. The spectrum of an operator defined in a complex Banach space, $T : Y \to Y$ is designated by $\sigma(T)$ which is a subset of $\mathbb{C}$.*

2. The resolvent of an operator defined in a complex Banach space $T : Y \to Y$ is designated by $\rho(T)$.

3. A spectral component of an operator defined in a complex Banach space $T : Y \to Y$ is a subset of $\sigma(T)$ which is both open and closed in $\sigma(T)$.

4. If a closed simple curve $\gamma : [0,1] \to \mathbb{C}$ is positively oriented, then its interior is the bounded set delimited by it (in the Jordan sense).

In the next results, $\gamma, \gamma_1, \gamma_2$ are allways simple closed rectifiable and positively oriented curves.

Theorem B.2 ([26] theorem 4-3) *Let Y be a complex Banach space and $T : Y \to Y$ be an operator. Let γ be a curve which lies in $\rho(T)$. Define*

$$P_\gamma(T) := \frac{1}{2\pi i}\int_\gamma (zI - T)^{-1} dz. \tag{B.1}$$

Then P is a projection ($PP = P$) and it commutes with every transformation which commutes with T. In particular, $Y = Ran(P) \oplus Ker(P)$.

Theorem B.3 ([26] theorem 4-6) *For the projection P defined in* (B.1)*, $P = 0$ if and only if the interior of the curve γ belongs to $\rho(T)$. Similarly, $P = I$ if, and only if, $\sigma(T)$ lies entirely in the interior of γ.*

Corollary B.4 *Consider $T \in \mathcal{L}(Y)$. Take two disjoint curves γ_1 and γ_2 with disjoint interior V_1 and V_2, respectively. The projection $P_{\gamma_1\cup\gamma_2}(T) := P_{\gamma_1}(T) + P_{\gamma_2}(T)$ equals 0 if and only if, $V_1, V_2 \subset \rho(T)$. Similarly, $P_{\gamma_1\cup\gamma_2}(T) = I$ if and only if $\sigma(T) \subset V_1 \cup V_2$.*

The proof of corollary B.4 follows usual techniques involving calculus in complex Banach spaces as well as theorem B.3.

Lemma B.5 *Take T_0, V_i, σ_i and γ_i as above. Suppose* $\dim \operatorname{Ran} P_{\gamma_1}(T_0) = k$. *Then in some ball $B_r(T_0)$, the map $T \mapsto P_{\gamma_1}(T)$ is well defined and Lipschitz,* $\dim \operatorname{Ran} P_{\gamma_1}(T) = k$ *and the operators $T - \lambda I$ for $\lambda \notin V_2$ are Fredholm of index zero.*

Note that $\dim \operatorname{Ran} P_{\gamma_1}(T_0)$ could be infinite, but we are interested in the case when it equals one. The continuity of $T \mapsto P_{\gamma_1}(T)$ does not depend on $P_{\gamma_1}(T_0)$ being finite dimensional and corresponds to a an appropriate notion of continuity of the invariant subspaces related to a spectral component of T.

Proof: As γ_1 is compact, it is not hard to see that there exists a ball $B(T_0) \subset \mathcal{L}(Y)$ for which, if $T \in B(T_0)$ and $z \in \gamma_1$, then $(T - zI)^{-1} \in \mathcal{L}(Y)$.

It follows that the projections $P_{\gamma_i}(T)$ are well defined in $B_r(T_0)$ and given by an integral along the same curve γ_1. For $T, \tilde{T} \in B_r(T_0)$, the Lispchitz estimate is easy:

$$\|P_{\gamma_1}(T) - P_{\gamma_1}(\tilde{T})\| \leq \frac{1}{2\pi} \int_{\gamma_1} \|(zI - T)^{-1} - (zI - \tilde{T})^{-1}\| dz$$

$$\leq \frac{c}{2\pi} \int_{\gamma_1} \|(zI - T)^{-1}(\tilde{T} - zI - T + zI)(zI - \tilde{T})^{-1}\| dz \leq \frac{\tilde{c}}{2\pi} \|T - \tilde{T}\|.$$

Here, we used the fact that the inverses $(zI - T)^{-1}$ are uniformly bounded for all $T \in B_r(T_0)$ and $z \in \gamma_1$. Now take r so that $\|P_{\gamma_1}(T) - P_{\gamma_1}(T_0)\| < 1$. This implies that $\dim \operatorname{Ran} P_{\gamma_1}(T) = k$ for $T \in B_r(T_0)$.

Now, split $Y = \operatorname{Ran} P_{\gamma_1}(T) \oplus \operatorname{Ran} P_{\gamma_2}(T)$, both terms being closed invariant subspaces under T. In particular, from theorem B.2, for $\lambda \notin V_2$, the restriction of $T - \lambda I$ to $\operatorname{Ran} P_{\gamma_2}(T)$ is invertible and the restriction to $\operatorname{Ran} P_{\gamma_1}(T)$ is a linear operator from a vector space of dimension k to itself — adding up, $T - \lambda I$ is a Fredholm operator of index zero.

■

Proposition B.6 *Let $T_0 \in \mathcal{L}(Y)$. Take $\sigma_1, \sigma_2 \subset \sigma(T_0)$ disjoint spectral components of T_0 such that $\sigma_1 \cup \sigma_2 = \sigma(T)$. Take disjoint curves $\gamma_1, \gamma_2 \subset \rho(T_0)$ with disjoint interior V_1, V_2 such that σ_1, σ_2 are contained,respectively, in V_1, V_2. Then, there exists a ball $B(T_0) \subset \mathcal{L}(Y)$ such that, for all $T \in B(T_0)$, we have $\sigma(T) = V_1 \cup V_2$.*

Proof: Observe that, for a curve γ containing $\sigma(T)$ we have

$$P_{\gamma_1}(T_0) + P_{\gamma_2}(T_0) = \int_{\gamma_1 \cup \gamma_2} (zI - T_0)^{-1} = \int_{\gamma} (zI - T_0)^{-1} = I. \tag{B.2}$$

We prove that, for T close enough to T_0 we have $P_{\gamma_1}(T) + P_{\gamma_2}(T) = I$, that is, by corollary B.4, $\sigma(T) \subset V_1 \cup V_2$.

Observing that $\{T_0 - zI : z \in \gamma_1 \cup \gamma_2\} \subset \mathcal{L}(Y)$ is compact, and that $(T_0 - zI)^{-1}$ well defined for all $z \in \gamma_1 \cup \gamma_2$, it is not hard to find a ball $B(T_0) \subset \mathcal{L}(Y)$ for which, for all $T \in B(T_0)$ and $z \in \gamma_1 \cup \gamma_2$, the operators $T - zI : Y \to Y$ are all invertible.

Suppose, to the contrary, that there exists a sequence $\{T_k\}_k \subset \mathcal{L}(Y)$ such that $T_k \to t_0$ and for all $k \in \mathbb{N}$ there exists some $\lambda_k \in \sigma(T_k) - (V_1 \cup V_2)$. Take $k \geq N$ great enough so that $T_k \in B(T_0)$, that is $(T_k - z)^{-1} \in \mathcal{L}(Y)$ for all $z \in \gamma_1 \cup \gamma_2$. It follows that, for $k \geq N$, both projections below are well defined

$$P_{\gamma_1}(T_k) = \int_{\gamma_1} (zI - T_k)^{-1} dz \ , \quad P_{\gamma_2}(T_k) = \int_{\gamma_2} (zI - T_k)^{-1} dz$$

From equation (B.2), $I = P_{\gamma_1 \cup \gamma_2}(T_0)$, so that for $k \geq N$, we have

$$\|I - P_{\gamma_1 \cup \gamma_2}(T_k)\| = \|P_{\gamma_1 \cup \gamma_2}(T_0) - P_{\gamma_1 \cup \gamma_2}(T_k)\|$$

$$\leq \frac{1}{2\pi} \int_{\gamma_1 \cup \gamma_2} \|(zI - T_0)^{-1} - (zI - T_k)^{-1}\|_{\mathcal{L}(Y)} dz$$

$$\leq \frac{1}{2\pi} \int_{\gamma_1 \cup \gamma_2} \|(zI - T_0)^{-1}(T_0 - T_k)(zI - T_k)^{-1}\|_{\mathcal{L}(Y)} dz \leq \frac{c}{2\pi} \|T_0 - T_k\|_{\mathcal{L}(Y)}.$$

Let $B(I) \subset \mathcal{L}(Y)$ for which every linear transformation in it is invertible. For $k \geq N_1$, we have that $P_{\gamma_1 \cup \gamma_2}(T)$ is invertible. It follows that $P_{\gamma_1 \cup \gamma_2}(T_k) = I$, that is, from corollary B.4, the spectrum of T_k lies in $V_1 \cup V_2$.

■

Proposition B.7 *Let X, Y be real reflexive Banach spaces, $X \subset Y$, X dense in Y and consider $T_0 : X \to Y$ an operator which is Fredholm of index 0. Suppose that there exist eigenvectors $\phi_0 \in X$, $\phi_0^* \in Y^*$ of T_0 and T_0^* associated to the eigenvalue $\lambda_0 \in \mathbb{R}$ such that $Ker(T_0 - \lambda_0 I)$ is one dimensional and λ_0 is an isolated eigenvalue. Then, for some ball $B(T_0) \subset \mathcal{L}(X, Y)$, there exists a C^∞ function $(\lambda, \phi, \phi^*) : B(T_0) \to \mathbb{R} \times X \times Y^*$ such that*

1. *$T \in B(T_0)$ is Fredholm of index 0,*
2. *$T\phi(T) = \lambda(T)\phi(T)$, with $\phi(T) \neq 0$ and $Ker(T - \lambda(T)) = \langle \phi(T) \rangle$,*
3. *$T^*\phi^*(T) = \lambda(T)\phi^*(T)$, with $\phi^*(T) \neq 0$ and $Ker(T^* - \lambda(T)) = \langle \phi^*(T) \rangle$,*
4. *$\lambda(T_0) = \lambda_0$, $\phi(T_0) = \phi_0$, $\phi^*(T_0) = \phi_0^*$.*

Proof: To prove item 1 we observe that the operators in $\mathcal{L}(X,Y)$ with Fredholm index equal to 0 form an open subset of $\mathcal{L}(X,Y)$. As T_0 is Fredholm of index 0, take $B(T_0)$ contained in that subset.

Take $l \in Y^*$ such that $l(\phi_0) = 1$ and $Ker(l) = Ran(T_0 - \lambda_0 I)$, which is possible because $\phi_0 \notin Ran(T - \lambda_0 I)$ as T_0 is Fredholm of index 0.

Consider the function

$$G : \mathcal{L}(X,Y) \times \mathbb{R} \times \{\phi_0 + Ker(l)\} \to Y \ , \quad (T, \lambda, \phi) \mapsto T\phi - \lambda\phi.$$

Note that G is a C^∞ function and that $G(T_0, \lambda_0, \phi_0) = 0$.

We want to show that, at (T_0, λ_0, ϕ_0), the derivative of G on λ and ϕ is invertible, so that, by the implicit function theorem, the level 0 of G near (T_0, λ_0, ϕ_0) can be written as $G(T, \lambda(T), \phi(T)) = 0$ where $\phi(T)$ and $\lambda(T)$ are C^∞ functions.

It is clear that

$$\Big(\frac{\partial G}{\partial\lambda\partial\phi}(T_0, \lambda_0, \phi_0)\Big)(\lambda, \phi) = T_0\phi - \lambda_0\phi - \lambda\phi_0.$$

We prove that it is an isomorphism from $\mathbb{R} \times Ker(l)$ to Y.

Suppose that $T_0\phi - \lambda_0\phi - \lambda\phi_0 = 0$ for some pair (λ, ϕ). Apply the functional l to both sides of the equation above. Note that $l(T_0\phi - \lambda_0\phi) = 0$ and $l(\lambda\phi_0) = \lambda$. It follows that $\lambda = 0$. Now, we have $T_0\phi - \lambda_0\phi = 0$ with $\phi \in Ker(l)$ with $\phi_0 \notin Ker(l)$. It follows that $\phi = 0$ and injectivity is proved.

Now we prove surjectivity. Take $g \in Y$. Write $g = w + t\phi_0 \in Ran(T_0 - \lambda_0) \oplus \langle\phi_0\rangle$. As $T_0 - \lambda_0 I : Ker(l) \cap X \to Ran(T_0 - \lambda_0)$ is an isomorphism, take $u \in Ker(l) \cap X$ such that $T_0 u - \lambda_0 u = w$ and $\lambda = t \in \mathbb{R}$.

Finally, by the implicit function theorem, there exists a neighbourghood $V(T_0)$ where we can define a C^∞ function

$$(\lambda, \phi) : V(T_0) \to V(\lambda_0) \times V(\phi_0)$$

such that, for $T \in V(T_0)$, $G(T, \lambda(T), \phi(T)) = 0$ if and only if $(\lambda(T), \phi(T)) = (\lambda, \phi)$ for all $(T, \lambda, \phi) \in V(T_0) \times V(\lambda_0) \times V(\phi_0)$.

Now we want to prove that there exist balls $B(T_0) \in V(T_0)$ and $B(\lambda_0) \subset V(\lambda_0)$ such that $\lambda(T)$ is simple and that for all $T \in B(T_0)$, we have $\sigma(T) \cap B(\lambda_0) = \lambda(T)$

Consider the complexification

$$T_{\mathbb{C}} : X_{\mathbb{C}} \to Y_{\mathbb{C}} \ , \quad u + iv \mapsto Tu + iTv.$$

Fix $z \in \rho(T_{0,\mathbb{C}}) \cap \mathbb{R}$. Consider the operators $(T - zI)^{-1}$ which are defined in a ball $B_1(T_{0,\mathbb{C}}) \subset \mathcal{L}(Y_{\mathbb{C}})$. By hypothesis, the eigenvalue λ_0 is also a simple isolated eigenvalue of $T_{0,\mathbb{C}}$, that is, $(\lambda_0 - z)^{-1}$ is a simple isolated eigenvalue of $(T_{0,\mathbb{C}} - zI)^{-1}$. Take a curve $\gamma \in \rho(T_{0,\mathbb{C}})$ such that the interior of γ intersects $\sigma((T_{0,\mathbb{C}} - zI)^{-1})$ only at $(\lambda_0 - z)^{-1}$. Then, there exists a ball around $T_{0,\mathbb{C}}$ such that $P_\gamma((T_{\mathbb{C}} - zI)^{-1})$ has unidimensional kernel, by lemma B.5. It follows that $(T_{\mathbb{C}} - z)^{-1}$ has a single eigenvalue in the interior of the curve γ. That is, there exists some ball $B(T_0) \subset \mathcal{L}(X, Y)$ such that for all $T \in B(T_0)$ there exists a single eigenvalue (maybe complex) in the interior of γ.

Take a possibly smaller ball $B_1(T_0) \subset B(T_0) \cap V(T_0)$, where $V(T_0)$ was given by the implicit function theorem. From the arguments above, it follows that, for all $T \in B_1(T_0)$, the only eigenvalue belonging to $V(\lambda_0) \cap \lambda(B(T_0))$ is $\lambda(T)$, which is real.

Now, apply the same result to the operator $T_0^* : Y^* \to X^*$ with $T_0^*\phi_0^* = \lambda_0\phi_0^*$ to obtain $B(T_0^*) \subset \mathcal{L}(X, Y)$ in which we can define a C^∞ function

$$(\lambda^*, \phi^*) : B(T_0^*) \to V(\lambda_0) \times V(\phi_0^*)$$

satisfying, for all $S \in B(T_0^*)$, $S\phi^*(S) = \lambda^*(S)\phi^*(S)$ and $\lambda(T_0^*) = \lambda_0$ and $\phi^*(T_0^*) = \phi_0^*$ and that, if there exists some eigenvalue $\lambda \in V(\lambda_0)$ for an operator $S \in B(T_0^*)$, then $\lambda^*(S) = \lambda$.

Take a possibly smaller ball $B_2(T_0) \subset B_1(T_0)$ such that for all $T \in B_2(T_0)$ we have $T^* \in B(T_0^*)$. Note that $\lambda^*(T^*) = \lambda(T)$. It follows that the function

$$(\lambda, \phi, \phi^*) : B_1(T_0) \to V(\lambda_0) \times V(\phi_0) \times V(\phi_0^*)$$

is well defined with the components λ and ϕ being C^∞. As $\phi^* : B(T_0^*) \to V(\phi_0^*)$ is C^∞ and $T \mapsto T^*$ is linear, it follows that $T \mapsto \phi^*(T^*)$ is C^∞.

■

Proposition B.8 *Let X, Y be complex reflexive Banach spaces, $X \subset Y$, X dense in Y and consider $T_0 : X \to Y$ an operator which is Fredholm of index 0. Suppose that there exists an eigenvector $\phi_0 \in X$ of T_0 associated to the eigenvalue $\lambda_0 \in \mathbb{C}$ such that $Ker(T_0 - \lambda_0 I)$ is one dimensional and λ_0 is an isolated eigenvalue. Then, for some ball $B(T_0) \subset \mathcal{L}(X, Y)$, there exists a C^∞ function $(\lambda, \phi) : B(T_0) \to \mathbb{C} \times X$ such that*

1. *$T \in B(T_0)$ is Fredholm of index* 0,

2. $T\phi(T) = \lambda(T)\phi(T)$, *with* $\phi(T) \neq 0$ *and* $Ker(T - \lambda(T)) = \langle \phi(T) \rangle$,

3. $\lambda(T_0) = \lambda_0$, $\phi(T_0) = \phi_0$.

The proof is analogous to the one of proposition B.7, but with the simplification that we are not interested in showing that the eigenfunction associated to $\lambda(T)$ of the adjoint operator T^* is differentiable on T.

C
Folds

C.1
Differentiable fold

Proposition C.1 (differentiable fold) *Let $\mathcal{B}$ be a Banach space. Let*

$$G : \mathcal{B} \times \mathbb{R} \to \mathcal{B} \times \mathbb{R} \ , \quad (z,t) \mapsto (z, h(z,t))$$

be a C^2 function. Suppose that, for every fixed $z \in \mathcal{B}$, the map $(t) \mapsto h_z(t) := h(z,t)$ satisfies $\lim_{|t|\to\infty} h(z,t) = -\infty$. Then, the following propositions are equivalent

$$\textit{if } h_z'(c(z)) := \frac{\partial h}{\partial t}(z, c(z)) = 0, \textit{ then } h''(z, c(z)) := \frac{\partial^2 h}{\partial t^2}(z, c(z)) < 0$$

there exist C^1 diffeomorphisms $\Psi_1, \Psi_2 : B \times \mathbb{R} \to B \times \mathbb{R}$ such that
$$(\Psi_2 \circ G \circ \Psi_1)(z,t) = (z, -t^2).$$

Proof: The proof follows a few simple (but technical) steps. We give a one dimensional counterpart of the proof so that the reader gets familiarized with the ideas. By one dimensional counterpart we mean: let $h : \mathbb{R} \to \mathbb{R}$ be C^2 such that $\lim_{|t|\to\infty} h(t) = -\infty$. Then, the assertions below are equivalent.

$$\textit{if } h'(t) = 0, \textit{ then } h''(t) < 0$$

there exist C^1 diffeomorphisms $\psi_1, \psi_2 : \mathbb{R} \to \mathbb{R}$ such that $\psi_2(h(\psi_1(s))) = -s^2$.

($\Longleftarrow$) Suppose that there exist diffeomorphisms $\psi_1, \psi_2 : \mathbb{R} \to \mathbb{R}$ such that $\psi_1(h(\psi_1(s))) = -s^2$.

First we prove that h has a unique critical point. By the behaviour of h at infinity, it has some critical point. Let $c_0 \in \mathbb{R}$ be a critical point of h. Let $s_0 \in \mathbb{R}$ be such that $\psi_1(s_0) = c_0$. Then, $h'(c_0) = h'(\psi(s_0)) = 0$.

Clearly,

$$frac\partial\partial t\psi_2 \circ h \circ \psi_1(s) = -2s$$

implies that 0 is the only critical point of $\psi_2 \circ h \circ \psi_1$. Since ψ_1, ψ_2 are diffeomorphisms, it follows that $\psi_1(0) = c_0$ is the only critical point of h. That is, $h'(t) = 0$ if and only if $t = c_0$.

Since h has a unique critical point, c_0 and $\lim_{|t|\to\infty} h(t) = -\infty$, then its critical point must be a local maximum, in other words, $h''(c_0) \leq 0$.

Now we prove that $h''(c_0) < 0$. For $s \in \mathbb{R}$, we have that

$$\psi_2'(h(\psi_1(s)))h'(\psi_1(s))\psi_1'(s) = -2s.$$

Use Taylor's formula with a remainder in the C^1 function $h' \circ \psi_1$ to obtain, for s near 0,

$$h'(\psi_1(s)) = h'(\psi_1(0)) + h''(\psi_1(0))s + o(s) = h''(c_0)s + o(s).$$

It follows that

$$\psi_2'(h(\psi_1(s)))\frac{h''(c_0)s + o(s)}{s}\psi_1'(s) = \frac{-2s}{s} = -2.$$

Take the limit $s \to 0$ to obtain

$$\psi_2'(h(c_0))h''(c_0)\psi_1'(0) = -2$$

so that $h''(c_0) \neq 0$. Since $h''(c_0) \leq 0$ we have proved that $h''(c_0) < 0$.

($\Longrightarrow$) Suppose that, if c_0 is a critical point of h, then $h''(c_0) < 0$. It is clear that h has a single critical point. Let c_0 be the unique critical point of h. Consider the function $t \mapsto h_1(t) := h(t + c_0)$, that is, a translation in the argument. Now, make a translation in the counterdomain $t \mapsto h_2(t) := h(t + c_0) - h(c_0)$. It follows that $h_2(0) = 0$ and $h_2'(0) = h'(c_0) = 0$. Using a technique due to Hadamard we obtain that for all $t \in \mathbb{R}$,

$$h_2(t) = h_2(t) - h_2(0) = t\int_0^1 h_2'(rt)dr \leq 0.$$

Also, $h_2'(rt) = h_2'(rt) - h_2'(0) = tr\int_0^1 h_2''(rst)ds$, so that, for all $t \in \mathbb{R}$,

$$h_2(t) = t^2\int_0^1 r\int_0^1 h_2''(rst)ds\, dr =: t^2\gamma(t) \leq 0.$$

We claim that $t \mapsto t\sqrt{-\gamma(t)}$ is a diffeomorphism. First we check that it is well defined, that is, for all $t \in \mathbb{R}$ we have $\gamma(t) \leq 0$. This is easy: since $h_2(t)$ has a unique critical point $c_0 = 0$, $h_2(c_0) = 0$ and $\lim_{|t|\to\infty} h_2(t) = -\infty$. Now we show that $\gamma(t) < 0$. For $t \neq 0$, it is clear that $\gamma(t) < 0$. For $t = 0$, we use continuity of γ and the fact that $h_2''(0) < 0$. Again, use Taylor's remainder formula, for $t \neq 0$ near 0,

$$h_2(t) = h_2(0) + h_2'(0)t + h_2''(0)\frac{t^2}{2} + o(t^2) = h_2''(0)\frac{t^2}{2} + o(t^2).$$

Dividing by t^2 and making $t \to 0$, we obtain

$$\gamma(t) = \frac{h_2(t)}{t^2} = \frac{h''(0)}{2} + \frac{o(t^2)}{t^2} \to \gamma(0) = \frac{h''(0)}{2} < 0,$$

so we have proved that $\gamma(t) < 0$. It is also easy to see that $t \mapsto t\sqrt{-\gamma(t)}$ is surjective using the fact that $\lim_{|t|\to\infty} h_2(t) = -\infty$.

Finally, we check that $t \mapsto t\sqrt{-\gamma(t)}$ is stricly increasing. To that end we prove that its derivative is everywhere positive. A simple calculation shows that

$$t\sqrt{-\gamma(t)} := \begin{cases} -\sqrt{-h_2(t)}\ , & \text{if } t < 0 \\ \sqrt{-h_2(t)}\ , & \text{if } t \geq 0 \end{cases} .$$

If $t > 0$, then we differentiate $t\sqrt{-\gamma(t)}$ and obtain

$$-\frac{h_2'(t)}{2\sqrt{-h_2(t)}} > 0,$$

since $h_2'(t) < 0$. Analogously, for $t < 0$,

$$\frac{h_2'(t)}{2\sqrt{-h_2(t)}} > 0$$

since $h_2'(t) > 0$. The difficulty now is to prove that $t\sqrt{-\gamma(t)}$ is differentiable at 0 and that its derivative is positive, thus proving that γ is a diffeomorphism. We use the mean value theorem and

$$\lim_{t\uparrow 0} \frac{\partial - \sqrt{-h_2(t)}}{\partial t} = \sqrt{-h''(c_0)/2} = \lim_{t\downarrow 0} \frac{\partial\sqrt{-h_2(t)}}{\partial t}$$

to obtain our result. Note that $h_2''(0) = h''(c_0) < 0$.

First, consider, for $t < 0$ near 0,

$$
\begin{aligned}
&-\frac{h_2'(t)}{2\sqrt{-h_2(t)}} \\
&= -\frac{h_2'(0) + h_2''(0)t + o(t)}{2\sqrt{-h_2(0) - h_2'(0)t - h''(0)t^2/2 - o(t^2)}} \\
&= -\frac{h_2''(0)t + o(t)}{2t\sqrt{-h_2''(0)/2 - o(t^2)/t^2}} \\
&= -\frac{h_2''(0) + o(t)/t}{2\sqrt{-h_2''(0)/2 - o(t^2)/t^2}} \to -\frac{h_2''(0)}{2\sqrt{-h_2''(0)/2}} = \sqrt{-h_2''(0)/2} > 0
\end{aligned}
$$

Now, for $t > 0$ near 0,

$$
\begin{aligned}
\frac{h_2'(t)}{2\sqrt{-h_2(t)}} &= \frac{h_2'(0) + h_2''(0)t + o(t)}{2\sqrt{-h_2(0) - h_2'(0)t - h_2''(0)t^2/2 - o(t^2)}} \\
&= \frac{h_2''(0)t + o(t)}{-2t\sqrt{-h_2''(0)/2 - o(t^2)/t^2}} \\
&= -\frac{h_2''(0) + o(t)/t}{2\sqrt{-h_2''(0)/2 - o(t^2)/t^2}} \to -\frac{h_2''(0)}{2\sqrt{-h_2''(0)/2}} \\
&= \sqrt{-h_2''(0)/2} > 0.
\end{aligned}
$$

Now, by the mean value theorem, for $\xi(t) = t\sqrt{\gamma(t)}$ and $0 < c(t) < t$,

$$
\Big|\, \frac{\xi(t) - \xi(0)}{t} - \sqrt{-h''(c_0)/2}\,\Big| = \Big|\, \xi'(c(t)) - \sqrt{-h''(c_0)/2}\,\Big|
$$

with the right hand side going to 0 as $t \to 0$, by the limits we obtained before.

Consider $\xi^{-1} : \mathbb{R} \to \mathbb{R}$. Observe that, by the definition of ξ^{-1},

$$
\begin{aligned}
h_2(\xi^{-1}(s)) &= -\xi^{-1}(s)^2\gamma(\xi^{-1}(s)) \\
&= -\big(\xi^{-1}(s)\sqrt{\gamma(\xi^{-1}(s))}\big)^2 = -\xi(\xi^{-1}(s))^2 = -s^2.
\end{aligned}
$$

It follows that $\psi_1(s) = \xi^{-1}(s) + c_0$ and $\psi_2(t) = t - h(c_0)$. Clearly, ψ_1 and ψ_2 are diffeomorphisms.

Now we consider the higher dimensional case. Note that the maps ψ_1 and ψ_2 have C^∞ dependence on the parameter c_0. The thing to worry about is how c_0 depend on the component $z \in \mathcal{B}$. So, we can consider the changes of variables $\Psi_1(c(z), t)$ and $\Psi_2(c(z), t)$ where $c(z)$ is the unique critical point associated to the height h_z. If these functions have C^1 dependence on z, then we will have proved the implication ($\Longrightarrow$).

The upshot here is that the parameter $c(z)$ does not depend on t, so that our changes of variables will be invertible, and hence diffeomorphisms. We refrain from giving more details on that — we just prove that $c(z)$ has C^1 dependence on z. That said, we prove that the critical set is the graph of a C^1 function $c : \mathcal{B} \to \mathbb{R}$. Let $c(z) \in \mathbb{R}$ be the unique point such that $h'(z, c(z)) = 0$.

Clearly, the derivative of the function G at a point (z, t) is not invertible if, and only if, $h'(z, t) = 0$. Consider the function C^1 function $h' : \mathcal{B} \times \mathbb{R} \to \mathbb{R}$. Consider the level $h'(z, c(z)) = 0$. As $h''(z, t_z) < 0$, the implicit function theorem provides that there exists a C^1 function $z \in \mathcal{B} \mapsto \tilde{c}(z) \in \mathbb{R}$ such that $h'(z, c(z)) = 0$. Again, by hypothesis, $\tilde{c}(z) = c(z)$, that is, $z \in \mathcal{B} \mapsto c(z) \in \mathbb{R}$ is C^1.

Finally we prove ($\Longleftarrow$) for the multidimensional case, which is trickier than the one dimensional case since it involves 2×2 matrices of operators.

Suppose that there exist diffeomorphisms $\Psi_1, \Psi_2 : \mathcal{B} \times \mathbb{R} \to \mathcal{B} \times \mathbb{R}$ such that $\Psi_2(G(\Psi_1(z, s))) = (z, -s^2)$. The Jacobian of the composite function is give by

$$D\Psi_2(G(\Psi_1(z, s)))DG(\Psi_1(z, s))D\Psi_1(z, s) = \begin{bmatrix} I & 0 \\ 0 & -2s \end{bmatrix}.$$

Clearly, fixed $z \in \mathcal{B}$, the only critical point of $\Psi_2 \circ G \circ \Psi_1$ is the point $(z, 0) \in \mathcal{B} \times \mathbb{R}$.

With that information in hand, we prove that, at a critical point c_0 of h_z, we have that $h_z''(c_0) \leq 0$. By contradiction, suppose that there exists a critical point c_0 of h_z such that $h_z''(c_0) > 0$. Since $\lim_{|t| \to \infty} h_z(t) = -\infty$, there exists some other critical point c_1 of h_z.

Observe that, since Ψ_2 and Ψ_1 are diffeomorphisms and

$$DG(z, t) := \begin{bmatrix} I & 0 \\ H(t) & h_z'(t) \end{bmatrix},$$

that $D(\Psi_2 \circ G \circ \Psi_1)(z, t)$ is not invertible if, and only if, $h_z'(t) = 0$.

Take $s_0 \neq s_1$ such that $\Psi_1(z, s_0) = (z, c_0)$ and $\Psi_1(z, s_1) = (z, c_1)$ which is possible because Ψ_1 is a diffeomorphism. Then we have that both $(z, s_0) \neq (z, s_1)$ are critical points of $\Psi_2 \circ G \circ \Psi_1$ which is a contradiction.

Now, we prove that $h_z''(c_0) < 0$ if c_0 is a critical point of h_z. Set, for fixed $z \in \mathcal{B}$, with the capital letters representing operators from $\mathcal{B}$ to $\mathcal{B}$ and the small ones representing real numbers,

$$(D\Psi_2)(G(\Psi_1(z,t))) := \begin{bmatrix} A_2(t) & b_2(t) \\ C_2(t) & d_2(t) \end{bmatrix}, \quad D\Psi_1(z,t) := \begin{bmatrix} A_1(t) & b_1(t) \\ C_1(t) & d_1(t) \end{bmatrix},$$

$$(DG)(\Psi_1(z,t)) := \begin{bmatrix} I & 0 \\ H(t) & h_z'(t) \end{bmatrix}.$$

Then, we conclude that

$$\begin{bmatrix} (A_2 + b_2 H)A_1 + b_2 h_z' C_1 & b_1(A_2 + b_2 H) + b_2 d_1 h_z' \\ (C_2 + d_2 H)A_1 + d_2 h_z' C_1 & b_1(C_2 + d_2 H) + d_2 d_1 h_z' \end{bmatrix}(t) = \begin{bmatrix} I & 0 \\ 0 & -2t \end{bmatrix}$$

All the terms in the left side matrix depend on t, and we use the notation (t) so that the equation fits in one line.

Suppose, by contradiction, that $h_z''(c_0) = 0$. Take the term m_{11} of matrix $D(\Psi_2 \circ G \circ \Psi_1)(t)$

$$(A_2(t) + b_2(t)H(t))A_1(t) + b_2(t)h_z'(t)C_1(t) \tag{C.1}$$

and observe that, as $t \to c_0$, the unique critical point of h_z, we have that

$$(A_2(t) + b_2(t)H(t))A_1(t) + b_2(t)h_z'(t)C_1(t) \to (A_2(c_0) + b_2(c_0)H(c_0))A_1(c_0) = I$$

so that $A_2(c_0) + b_2(c_0)H(c_0) \neq 0$.

On the other hand, divide both terms in the equation (C.1) by t and note that, as $t \to c_0$,

$$\frac{b_1(t)}{t}(A_2(t) + b_2(t)H(t)) + b_2(t)d_1(t)\frac{h_z'(t)}{t} \to 0$$

so that $b_1(t)/t \to 0$.

Finally, as $t \to c_0$ we have, for the term m_{44} of the matrix $D(\Psi_2 \circ G \circ \Psi_1)(t)$,

$$\frac{b_1(t)}{t}(C_2(t) + d_2(t)H(t)) + d_2(t)d_1(t)\frac{h_z'(t)}{t} \to -2$$

which is a contradiction since both $b_1(t)/t$ and $h_z'(t)/t$ converge to 0. It follows that $h_z''(c_0) < 0$.

■

C.2
Topological fold

Proposition C.2 *Let $\mathcal{B}$ be a Banach space. Let $G : \mathcal{B} \times \mathbb{R} \to \mathcal{B} \times \mathbb{R}$ be a Lipschitz function such that $G(z,t) = (z, h(z,t))$. Suppose that, for all $z \in H_Y$ we have $\lim_{|t|\to\infty} h(z,t) = -\infty$ and that every local extreme point of h is a strict maximum point. Then, there exist homeomorphisms Ψ_1, Ψ_2 such that*

$$\Psi_2(G(\Psi_1(z,s))) = (z, -s^2)$$

Proof: We proceed in a similar way as we did to prove proposition C.1. We consider a real function $h : \mathbb{R} \to \mathbb{R}$ which is continuous, $\lim_{|t|\to\infty} h(t) = -\infty$ and that every local extreme of h is a strict maximum point. We want to obtain homeomorphisms $\psi_1, \psi_2 : \mathbb{R} \to \mathbb{R}$ such that $\psi_2(h(\psi_1(s))) = -s^2$.

First we observe that h reaches its maximum at a single point c_0. Then, consider the translation in the domain given by $s \mapsto h_1(s) := h(s + c_0)$. It follows that h_1 reaches its maximum at 0.

Now consider the translation of h_1 in the counterdomain given by $t \mapsto h_2(t) = h(t + c_0) - h(c_0)$. Note that $\lim_{|t|\to\infty} h_2(t) = -\infty$ and that it reaches its maximum at 0 with $h_2(0) = 0$, that is, $h_2 \le 0$. Morever, h_2 is strictly increasing for $t \le c_0$ and strictly decreasing for $t \ge 0$. Consider the homeomorphism

$$\xi(s) := \begin{cases} -\sqrt{-h_2(s)}, & \text{if } s \le 0 \\ \sqrt{-h_2(s)}, & \text{if } s > 0 \end{cases} .$$

Now, observe that, for $s > 0$

$$\sqrt{-h_2(\xi^{-1}(s))} = \xi(\xi^{-1}(s)) = s,$$

so that, $h_2(\xi^{-1}(s)) = -s^2$. Similary, for $s \le 0$, $h_2(\xi^{-1}(s)) = -s^2$. Define $\psi_1(s) = \xi^{-1}(s) + c_0$ and $\psi_2(t) = t - h(c_0)$.

Now we proceed to the multidimensional case. If c_0 varies continuously on the parameter z we obtain the homeomorphisms $\Psi_1(z,s) = (z, \xi^{-1}(s) + c(z))$ and $\Psi_2(z,t) = (z, t - h(z, c(z)))$ where $c(z)$ is the unique height at which $h(z, c(z)) = \max_{t\in\mathbb{R}}\{h(z,t)\}$.

This is a consequence of G being Lipschitz and the behaviour of $t \mapsto h(z,t)$ at infinity. Take $z_k \to z_0$ with $z_k \in \mathcal{B}$. Consider the sequence $c(z_k)$. We want to prove that $c(z_k) \to c(z_0)$. First, observe that $h(z_k, c(z_k)) \ge$

$h(z_k, c(z_0)) \to h(z_0, c(z_0))$, that is, $h(z_k, c(z_k))$ is bounded. Put the following piece together

$$|h(z_k, c(z_k)) - h(z_0, c(z_k))| \leq C\|z_k - z_0\|_{\mathcal{B}} \to 0.$$

Now, if $|c(z_k)| \to \infty$, then $h(z_k, c(z_k)) \to -\infty$, contradicting $h(z_0, c(z_k))$ being bounded — hence $\{c(z_k)\}_k$ is bounded.

Take any subsequence of $\{c(z_k)\}_k$. Relabel it and call it $\{c(z_i)\}_i$. It has a convergent subsequence $\{c(z_{i_j})\}_j$ with limit c_0. We prove that $c(z_{i_j}) \to c(z_0)$. Note that $h(z_{i_j}, c(z_{i_j})) \geq h(z_{i_j}, c(z_0))$. By continuity of h, it follows that $h(z_0, c_0) \geq h(z_0, c(z_0))$. Since $c(z_0)$ is the maximum point of $t \mapsto h(z_0, t)$, it follows that $c_0 = c(z_0)$. As the argument above is valid for any subsequence of $\{c(z_k)\}_k$, we have that $c(z_k) \to c(z_0)$.

■