



André Zaccur Uchôa Cavalcanti

**Results of Ambrosetti-Prodi type for
non-selfadjoint elliptic operators**

Tese de Doutorado

Thesis presented to the Postgraduate Program in Matemática of the Departamento de Matemática do Centro Técnico Científico da PUC–Rio, as partial fulfillment of the requirements for the degree of Doutor em Matemática.

Advisor : Prof. Boyan Slavchev Sirakov
Co–Advisor: Prof. Carlos Tomei

Rio de Janeiro
March 2015



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Abstract

Zaccur, André; Sirakov, Boyan (Advisor); Tomei, Carlos (Co-Advisor). **Results of Ambrosetti-Prodi type for non-selfadjoint elliptic operators**. Rio de Janeiro, 2015. 91p. D.Sc. Thesis — Departamento de Matemática, Pontifícia Universidade Católica do Rio de Janeiro.

The celebrated Ambrosetti-Prodi theorem studies perturbations of the Dirichlet Laplacian by a nonlinear function jumping over the principal eigenvalue of the operator. Various extensions of this landmark result were obtained for self-adjoint operators, in particular by Berger-Podolak in 1975, who gave a geometrical description of the solution set. In this thesis we show that similar theorems are valid for non self-adjoint operators. We employ techniques based on the maximum principle, which even let us obtain new results in the self-adjoint setting. In particular, we show that the semilinear operator is a fold. As a consequence, we obtain exact count of solutions for these operators even when the perturbation is non-smooth.

Keywords

Elliptic operators; Ambrosetti-Prodi; non-linear equations; global folds;

Resumo

Zaccur, André; Sirakov, Boyan; Tomei, Carlos. **Resultados do tipo Ambrosetti-Prodi para operadores elíticos não auto-adjuntos**. Rio de Janeiro, 2015. 91p. Tese de Doutorado — Departamento de Matemática, Pontifícia Universidade Católica do Rio de Janeiro.

O célebre teorema de Ambrosetti-Prodi estuda perturbações do Laplaciano sob condições de Dirichlet por funções não lineares que saltam sobre o autovalor principal do operador. Diversas extensões desse resultado foram obtidos para operadores auto-adjuntos, em particular por Berger-Podolak em 1975, que deram uma descrição geométrica do conjunto solução. Nós empregamos técnicas baseadas no princípio do máximo que nos permite obter novos resultados inclusive para o cenário auto-adjunto. Em particular, nós mostramos que o operador semi-linear é uma dobra global. Obtemos também uma contagem exata de soluções para esses operadores ainda quando a perturbação não é suave.

Palavras-chave

Operadores elíticos; Ambrosetti-Prodi; equações não lineares; dobras globais;

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