A
Matrizes auxiliares da Equação de Estado

As matrizes $\mathbf{N}_1$ e $\mathbf{N}_2$ definidas no Capítulo 3 tem a seguinte forma:

$$\mathbf{N_1} = \begin{bmatrix} 0 & 0 & -\mathcal{F}^{\beta,\alpha}(c_{13}) - \mathcal{B}^{\beta,\alpha}(c_{23}) \\ 0 & 0 & in\mathcal{B}^{\beta,\alpha}(c_{23}) \\ -\mathcal{F}^{\beta,\alpha}(c_{55}) & in\mathcal{B}^{\beta,\alpha}(c_{44}) & 0 \end{bmatrix}$$

$$\mathbf{N_2} = \begin{bmatrix} \mathbf{N}_{11} & \mathbf{N}_{12} & 0 \\ \mathbf{N}_{21} & \mathbf{N}_{22} & 0 \\ 0 & 0 & \mathcal{E}^{\beta,\alpha}(c_{55}) + n^2\mathcal{D}^{\beta,\alpha}(c_{44}) - \omega^2\mathcal{A}^{\beta,\alpha}(\rho) \end{bmatrix}$$

Onde:

$$\mathbf{N}_{11} = \mathcal{E}^{\beta,\alpha}(c_{11}) + \mathcal{C}^{\beta,\alpha}(c_{12}) + \mathcal{C}^{\alpha,\beta}(c_{12}) + D^{\beta,\alpha}(c_{22}) + n^2\mathcal{D}^{\beta,\alpha}(c_{66}) - \omega^2\mathcal{A}^{\beta,\alpha}(\rho)$$
$$\mathbf{N}_{12} = in\mathcal{C}^{\beta,\alpha}(c_{12}) + in\mathcal{D}^{\beta,\alpha}(c_{22}) - in(\mathcal{C}^{\alpha,\beta}(c_{66}) - \mathcal{D}^{\beta,\alpha}(c_{66}))$$
$$\mathbf{N}_{21} = in\mathcal{C}^{\beta,\alpha}(c_{66}) - in\mathcal{D}^{\beta,\alpha}(c_{66}) - in(\mathcal{C}^{\alpha,\beta}(c_{12}) + \mathcal{D}^{\beta,\alpha}(c_{22}))$$
$$\mathbf{N}_{22} = \mathcal{E}^{\beta,\alpha}(c_{66}) - \mathcal{C}^{\beta,\alpha}(c_{66}) + \mathcal{D}^{\beta,\alpha}(c_{66}) - \mathcal{C}^{\alpha,\beta}(c_{66}) + n^2\mathcal{D}^{\beta,\alpha}(c_{22}) - \omega^2\mathcal{A}^{\beta,\alpha}(\rho)$$

Inicialmente, as matrices $\mathcal{A}(c), \mathcal{B}(c), .., \mathcal{F}(c)$ são dependentes dos termos β e α como mostrado no Capítulo 3 e nas matrizes anteriores, onde β indica a posição da coluna e α a posição da fila. Após de armar estas matrizes considerando N camadas isotrópicas, elas são de dimensão $N+1$, sendo só dependentes da constante elástica. Estas matrizes tem a seguinte forma:

$$\mathcal{A}(c) = \begin{bmatrix} A_{11} & A_{12} & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ & A_{22} & A_{23} & 0 & 0 & 0 & 0 & \dots & 0 \\ & & A_{33} & A_{34} & 0 & 0 & 0 & \dots & \vdots \\ & & & \ddots & \ddots & 0 & 0 & \dots & \vdots \\ & & & & A_{jj} & A_{jj+1} & 0 & \dots & \vdots \\ & & sim & & & \ddots & \ddots & 0 & \vdots \\ & & & & & & \ddots & \ddots & 0 \\ & & & & & & & A_{NN} & A_{NN+1} \\ & & & & & & & & A_{N+1N+1} \end{bmatrix}$$

Onde

$$\begin{aligned} A_{11} &= \frac{1}{12}c^1 h_1(4r_1 + h_1) \\ A_{12} &= \frac{1}{12}c^1 h_1(2r_1 + h_1) \\ A_{22} &= \frac{1}{12}c^1 h_1(4r_1 + 3h_1) + \frac{1}{12}c^2 h_2(4r_2 + h_2) \\ A_{23} &= \frac{1}{12}c^2 h_2(2r_2 + h_2) \\ A_{33} &= \frac{1}{12}c^2 h_2(4r_2 + 3h_2) + \frac{1}{12}c^3 h_3(4r_3 + h_3) \\ A_{34} &= \frac{1}{12}c^3 h_3(2r_3 + h_3) \\ A_{jj} &= \frac{1}{12}c^{j-1} h_{j-1}(4r_{j-1} + 3h_{j-1}) + \frac{1}{12}c^j h_j(4r_j + h_j) \\ A_{jj+1} &= \frac{1}{12}c^j h_j(2r_j + h_j) \\ A_{N+1N+1} &= \frac{1}{12}c^N h_N(4r_N + 3h_N) \end{aligned}$$

$$\mathcal{B}(c) = \begin{bmatrix} \frac{1}{3}c^1h_1 & \frac{1}{6}c^1h_1 & 0 & 0 & \dots & 0 \\ & \frac{1}{3}c^1h_1 + \frac{1}{3}c^2h_2 & \frac{1}{6}c^2h_2 & 0 & \dots & 0 \\ & & \frac{1}{3}c^2h_2 + \frac{1}{3}c^3h_3 & \frac{1}{6}c^3h_3 & 0 & \vdots \\ & sim & & \ddots & \ddots & 0 \\ & & & & \frac{1}{3}c^{N-1}h_{N-1} + \frac{1}{3}c^Nh_N & \frac{1}{6}c^Nh_N \\ & & & & & \frac{1}{3}c^Nh_N \end{bmatrix}$$

$$\mathcal{C}(c) = \begin{bmatrix} -\frac{1}{2}c^1 & -\frac{1}{2}c^1 & 0 & 0 & \dots & 0 \\ \frac{1}{2}c^1 & \frac{1}{2}c^1 - \frac{1}{2}c^2 & -\frac{1}{2}c^2 & 0 & \dots & 0 \\ 0 & \frac{1}{2}c^2 & \frac{1}{2}c^2 - \frac{1}{2}c^3 & -\frac{1}{2}c^3 & 0 & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & & \frac{1}{2}c^{N-1} & \frac{1}{2}c^{N-1} - \frac{1}{2}c^N & -\frac{1}{2}c^N \\ 0 & 0 & \dots & 0 & \frac{1}{2}c^N & \frac{1}{2}c^N \end{bmatrix}$$

$$\mathcal{D}(c) = \begin{bmatrix} D_{11} & D_{12} & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ & D_{22} & D_{23} & 0 & 0 & 0 & 0 & \dots & 0 \\ & & D_{33} & D_{34} & 0 & 0 & 0 & \dots & \vdots \\ & & & \ddots & \ddots & 0 & 0 & \dots & \vdots \\ & & & & D_{jj} & D_{jj+1} & 0 & \dots & \vdots \\ & & sim & & & \ddots & \ddots & 0 & \vdots \\ & & & & & & \ddots & \ddots & 0 \\ & & & & & & & D_{NN} & D_{NN+1} \\ & & & & & & & & D_{N+1N+1} \end{bmatrix}$$

Onde

$$D_{11} = \frac{1}{2}\frac{c^1}{h_1^2}[2(r_1+h_1)^2 ln(\frac{r_1+h_1}{r_1}) - 2r_1h_1 - 3h_1^2]$$

$$D_{12} = \frac{1}{2}\frac{c^1}{h_1^2}[-2(r_1^2+r_1h_1) ln(\frac{r_1+h_1}{r_1}) + h_1^2 + 2r_1h_1]$$

$$D_{22} = \frac{1}{2}\frac{c^1}{h_1^2}[2r_1^2 ln(\frac{r_1+h_1}{r_1}) - 2r_1h_1 + h_1^2] + \frac{1}{2}\frac{c^2}{h_2^2}[2(r_2+h_2)^2 ln(\frac{r_2+h_2}{r_2}) - 2r_2h_2 - 3h_2^2]$$

$$D_{23} = \frac{1}{2}\frac{c^2}{h_2^2}[-2(r_2^2+r_2h_2) ln(\frac{r_2+h_2}{r_2}) + h_2^2 + 2r_2h_2]$$

$$D_{33} = \frac{1}{2}\frac{c^2}{h_2^2}[2r_2^2 ln(\frac{r_2+h_2}{r_2}) - 2r_2h_2 + h_2^2] + \frac{1}{2}\frac{c^3}{h_3^2}[2(r_3+h_3)^2 ln(\frac{r_3+h_3}{r_3}) - 2r_3h_3 - 3h_3^2]$$

$$D_{34} = \frac{1}{2}\frac{c^3}{h_3^2}[-2(r_3^2+r_3h_3) ln(\frac{r_3+h_3}{r_3}) + h_3^2 + 2r_3h_3]$$

$$D_{jj} = \frac{1}{2}\frac{c^{j-1}}{h_{j-1}^2}[2r_{j-1}^2 ln(\frac{r_{j-1}+h_{j-1}}{r_{j-1}}) - 2r_{j-1}h_{j-1} + h_{j-1}^2] + \frac{1}{2}\frac{c^j}{h_j^2}[2(r_j+h_j)^2 ln(\frac{r_j+h_j}{r_j}) - 2r_jh_j - 3h_j^2]$$

$$D_{jj+1} = \frac{1}{2}\frac{c^j}{h_j^2}[-2(r_j^2 + r_jh_j) ln(\frac{r_j+h_j}{r_j}) + h_j^2 + 2r_jh_j]$$

$$D_{N+1,N+1} = \frac{1}{2}\frac{c^N}{h_N^2}[2r_N^2 ln(\frac{r_N+h_N}{r_N}) - 2r_Nh_N]$$

$$\mathcal{E}(c) = \begin{bmatrix} E_{11} & E_{12} & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ & E_{22} & E_{23} & 0 & 0 & 0 & 0 & \dots & 0 \\ & & E_{33} & E_{34} & 0 & 0 & 0 & \dots & \vdots \\ & & & \ddots & \ddots & 0 & 0 & \dots & \vdots \\ & & & & E_{jj} & E_{jj+1} & 0 & \dots & \vdots \\ & & sim & & & \ddots & \ddots & 0 & \vdots \\ & & & & & & \ddots & \ddots & 0 \\ & & & & & & & E_{NN} & E_{NN+1} \\ & & & & & & & & E_{N+1N+1} \end{bmatrix}$$

Onde

$$E_{11} = \frac{1}{2}\frac{c^1}{h_1}(2r_1 + h_1)$$
$$E_{12} = -\frac{1}{2}\frac{c^1}{h_1}(2r_1 + h_1)$$
$$E_{22} = \frac{1}{2}\frac{c^1}{h_1}(2r_1 + h_1) + \frac{1}{2}\frac{c^2}{h_2}(2r_2 + h_2)$$
$$E_{23} = -\frac{1}{2}\frac{c^2}{h_2}(2r_2 + h_2)$$
$$E_{33} = \frac{1}{2}\frac{c^2}{h_2}(2r_2 + h_2) + \frac{1}{2}\frac{c^3}{h_3}(2r_3 + h_3)$$
$$E_{34} = -\frac{1}{2}\frac{c^3}{h_3}(2r_3 + h_3)$$
$$E_{jj} = \frac{1}{2}\frac{c^{j-1}}{h_{j-1}}(2r_{j-1} + h_{j-1}) + \frac{1}{2}\frac{c^j}{h_j}(2r_j + h_j)$$
$$E_{jj+1} = -\frac{1}{2}\frac{c^j}{h_j}(2r_j + h_j)$$
$$E_{N+1N+1} = \frac{1}{2}\frac{c^N}{h_N}(2r_N + h_N)$$

$$\mathcal{F}(c) = \begin{bmatrix} F_{11} & F_{12} & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ F_{21} & F_{22} & F_{23} & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & F_{32} & F_{33} & F_{34} & 0 & 0 & 0 & \dots & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & 0 & 0 & \dots & \vdots \\ 0 & 0 & 0 & F_{jj-1} & F_{jj} & F_{jj+1} & 0 & \dots & \vdots \\ 0 & 0 & 0 & 0 & \ddots & \ddots & \ddots & 0 & \vdots \\ \vdots & \vdots & \vdots & \vdots & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & 0 & F_{NN-1} & F_{NN} & F_{NN+1} \\ 0 & 0 & \dots & \dots & \dots & 0 & 0 & F_{N+1N} & F_{N+1N+1} \end{bmatrix}$$

onde:

$$
\begin{aligned}
F_{11} &= -\frac{1}{6}c^1(3r_1 + h_1) \\
F_{12} &= -\frac{1}{6}c^1(3r_1 + 2h_1) \\
F_{21} &= \frac{1}{6}c^1(3r_1 + h_1) \\
F_{22} &= \frac{1}{6}c^1(3r_1 + 2h_1) - \frac{1}{6}c^2(3r_2 + h_2) \\
F_{23} &= -\frac{1}{6}c^2(3r_2 + 2h_2) \\
F_{32} &= \frac{1}{6}c^2(3r_2 + h_2) \\
F_{33} &= \frac{1}{6}c^2(3r_2 + 2h_2) - \frac{1}{6}c^3(3r_3 + h_3) \\
F_{34} &= -\frac{1}{6}c^3(3r_3 + 2h_3) \\
F_{jj} &= \frac{1}{6}c^{j-1}(3r_{j-1} + 2h_{j-1}) - \frac{1}{6}c^j(3r_j + h_j) \\
F_{jj+1} &= -\frac{1}{6}c^j(3r_j + 2h_j) \\
F_{jj-1} &= \frac{1}{6}c^{j-1}(3r_{j-1} + h_{j-1}) \\
F_{N+1N+1} &= \frac{1}{6}c^N(3r_N + 2h_N)
\end{aligned}
$$

Nestas matrizes c^j e h_j representam a constante elástica e a espessura da camada j, e r_j representa o raio inferior da camada j.

B
Matrizes auxiliares $\underline{\boldsymbol{A}}_{\alpha}(r)$ e $\underline{\mathbf{L}}_{\alpha}(r)$

$$\underline{\boldsymbol{A}}_{\alpha}(r) =$$

$$\begin{bmatrix} k_L h_{n-1}^{(\alpha)}(x) - k_L \frac{n}{x} h_n^{(\alpha)}(x) & ik_z k_T h_{n-1}^{(\alpha)}(y) - ik_z k_T \frac{n}{y} h_n^{(\alpha)}(y) & ik_T \frac{n}{y} h_n^{(\alpha)}(y) \\ ik_L \frac{n}{x} h_n^{(\alpha)}(x) & -k_z k_T \frac{n}{y} h_n^{(\alpha)}(y) & k_T \frac{n}{y} h_n^{(\alpha)}(y) - k_T h_{n-1}^{(\alpha)}(y) \\ ik_z h_n^{(\alpha)}(x) & k_T^2 h_n^{(\alpha)}(y) & 0 \end{bmatrix}$$

$$\underline{\mathbf{L}}_{\alpha}(r) =$$

$$\begin{bmatrix} -2\mu k_L^2 f_n^{(\alpha)}(x) - \lambda \frac{\omega^2}{c_L^2} h_n^{(\alpha)}(x) & -i2\mu k_z k_T^2 f_n^{(\alpha)}(y) & i2\mu k_T^2 \frac{n}{y} g_n^{(\alpha)}(y) \\ i2\mu k_L^2 \frac{n}{x} g_n^{(\alpha)}(x) & -2\mu k_z k_T^2 \frac{n}{y} g_n^{(\alpha)}(y) & 2\mu k_T^2 (f_n^{(\alpha)}(y) - \frac{h_n^{(\alpha)}(y)}{2}) \\ i2\mu k_z k_L (h_{n-1}^{(\alpha)}(x) - \frac{n}{x} h_n^{(\alpha)}(x)) & \mu k_T (k_T^2 - k_z^2)(h_{n-1}^{(\alpha)}(y) - \frac{n}{y} h_n^{(\alpha)}(y)) & -\mu k_z k_T \frac{n}{y} h_n^{(\alpha)}(y) \end{bmatrix}$$

Nestas matrizes:

$x = k_L\, r$ e $y = k_T\, r$.

As funções $f_n^{(\alpha)}(x)$ e $g_n^{(\alpha)}(x)$ tem a seguinte forma:

$$f_n^{(\alpha)}(x) = \frac{1}{x} h_{n-1}^{(\alpha)}(x) + (1 - \frac{n(n+1)}{x^2}) h_n^{(\alpha)}(x)$$
$$g_n^{(\alpha)}(x) = h_{n-1}^{(\alpha)}(x) - \frac{1+n}{x} h_n^{(\alpha)}(x)$$

Onde:

$$h_n^{(1)}(x) = \begin{cases} e^{-ix} H_n^1(x), & \text{se } x^2 \geq 0 \\ -\frac{2}{\pi}(i)^{-n+1} e^{-ix} K_n(-ix), & \text{se } x^2 < 0 \end{cases}$$

$$h_n^{(2)}(x) = \begin{cases} e^{ix} H_n^2(x), & \text{se } x^2 \geq 0 \\ 2(i)^n [e^{ix} I_n(-ix)] + \frac{2}{\pi}(i)^{-n+1} e^{2ix} [e^{-ix} K_n(-ix)], & \text{se } x^2 < 0 \end{cases}$$

Onde $H_n^\alpha(x)$ é a função de Hankel de classe α. $I_n(x)$ e $K_n(x)$ são as funções de bessel de primeira e segunda classe, respectivamente.

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