5
From separated to joint variables: the hurdle-race problem

We consider a general *provisioning problem*, where an economic agent aims to determine an initial amount that would be needed in order to meet a series of future payment obligations with a sufficiently large probability. We assume a "hold-to-maturity" (actuarial) approach: the initial provision is such that upon investing it in a portfolio one is always able in the future to pay-off the cash flows. In contrast, the "available-for-sales" (financial) framework admits the possible fall in the arbitrage-free price of the series of cash flows over a given time period and use it as means of assessing risk and establishing buffers. Although the latter approach is at the core of the latest regulatory documents such as Basel II and Solvency II, it can induce crashes when they would not otherwise occur. Furthermore, we believe better risk measures are available, such as the so-called *coherent risk measures* [ADEH]. This does not mean that the financial approach is wrong but the actuarial framework is at least a complementing alternative. For a detailed criticism of those regulatory documents we refer the reader to [DEGK].

In [VDGK], the authors study the *provisioning problem* and impose minimum requirements of available capital at each period, called *hurdles*. In their framework, hurdles are modeled as separated chance constraints with given reliability levels, one for each period of time. They coined the term *hurdle-race problem* to describe the provisioning problem with hurdles. We start by summarizing the approach proposed in [VDGK] and stating their main result. Then we propose an alternative chance constrained model which requires that all the obligations should be met jointly with a given reliability level. To this end we make use of a joint chance constraint and refer to the problem as the *joint hurdle-race problem*. We propose to solve the corresponding problem using SAA and to compare the results with [VDGK]. In addition, we consider another generalization in which the hurdles are not determined by the model builder but are defined as discounted values of future obligations by stochastic risk-free rates.

5.1 The hurdle-race problem and comonotonicity

An insurer wants to determine the initial provision R_0 required to meet n future obligations, of costs $\alpha_1, \dots, \alpha_n$ at fixed, prescribed times $t_1, \dots, t_n$, for $t \in [0, t_n]$. Among obligations, the insurer may invest his capital, with random returns. More precisely, the stochastic return process $(Y_1, \dots, Y_n)$ such that 1 unit at time 0 grows to $\exp(Y_1 + \cdots + Y_j)$ at time t_j determines the evolution of capital R_j in time,

$$R_j = R_{j-1} \exp(Y_j) - \alpha_j, \qquad j = 1, \dots, n. \tag{5-1}$$

In [VDGK], the authors impose probabilistic constraints (the *hurdles*) that have to be met every time t_j, that is, provision R_j has to be larger than a deterministic value V_j with high probability $1 - \varepsilon_j$. They formulate the *hurdle-race problem* as follows:

$$\begin{aligned} \overline{R}_0 &= \operatorname*{Min}_{R_0 \geq V_0} R_0 \qquad\qquad (5\text{-}2) \\ \mathsf{Prob}\{R_j \geq V_j \,|\, R_0\} &\geq 1 - \varepsilon_j, \quad j = 1, \dots, n \end{aligned}$$

for given hurdles $V_0, V_1, \dots, V_n$ and given tolerances $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n \in [0, 1]$.

To determine the optimal provision $\overline{R}_0$ in (5-2) set

$$S_{[0,j]} = \sum_{i=1}^{j-1} \alpha_i \exp(-Y_1 - \cdots - Y_i) + (V_j + \alpha_j) \exp(-Y_1 - \cdots - Y_j), \tag{5-3}$$

the stochastically discounted value of the future obligations in the restricted time period $[0, j]$. Theorem 1 in [VDGK], below, gives the optimal solution of problem (5-2) in terms of the quantiles of the distributions of $S_{[0,j]}$.

Theorem 7 *The optimal initial provision $\overline{R}_0$ defined in (5-2) is given by*

$$\overline{R}_0 = \mathrm{Max}\{V_0, F^{-1}_{S_{[0,1]}}(1 - \varepsilon_1), F^{-1}_{S_{[0,2]}}(1 - \varepsilon_2), \dots, F^{-1}_{S_{[0,n]}}(1 - \varepsilon_n)\},$$

where $F_{S_{[0,j]}}$ is the cumulative distribution function of $S_{[0,j]}$, $j = 1, \dots, n$.

A basic ingredient in the proof is the simple fact that

$$\mathsf{Prob}\left\{R_j \geq V_j \mid R_0\right\} = \mathsf{Prob}\left\{S_{[0,j]} \leq R_0\right\}, \quad j = 1, \dots, n.$$

Thus, in order to determine the optimal $\overline{R}_0$ we are led to compute the quantiles of the random variables $S_{[0,j]}$, which is very hard in most relevant cases. For instance, if the random return process $(Y_1, \dots, Y_n)$ follows a multivariate normal distribution, we have that $S_{[0,j]}$ is a sum of lognormal distributions, a random variable with no known distribution. The approach in [VDGK] replaces the random variables $S_{[0,j]}$ by simpler random variables using *comonotonic* approximations, whose quantiles can be calculated explicitly. Such approximations assume the random vector is strongly correlated, with all the components depending on the same univariate random variable. A detailed description of the theory of comonotonicity can be found in [DDGa]. For examples and applications we refer the reader to [DDGb].

In [VDGK], numerical experiments are performed for the case in which the random variables (5-3) are sums of lognormals, and hence the return process $(Y_1, \dots, Y_n)$ follows a multivariate normal distribution. They compare their results with the values obtained from the empirical distribution of $S_{[0,j]}$. In the next section, in the more general *joint hurdle-race problem*, we apply SAA to obtain good candidate solutions. We compare the results for the joint hurdle-race to the ones obtained in [VDGK]. From now on we refer to this model (5-2) as the *separated hurdle-race problem*.

5.2 The joint hurdle-race problem

The definition of $\overline{R}_0$ in (5-2) does not reflect the proper safety requirements: for each fixed time t_j the probability of not satisfying a hurdle is small, but the probability of having missed one of the hurdles may remain high. Indeed, there is no guarantee that the optimal provision keeps the joint probability of missing at least one hurdle low. In [HEN], the author exemplifies the contrast between both models in a cash matching problem somewhat similar to the separated hurdle race problem (5-2).

We then consider the *joint hurdle-race* problem,

$$\overline{R}_0 = \operatorname*{Min}_{R_0 \geq V_0} R_0 \tag{5-4}$$

$$\mathsf{Prob}\{R_j \geq V_j,\ j = 1, \dots, n \,|\, R_0\} \geq 1 - \varepsilon \tag{5-5}$$

for $\varepsilon \in [0, 1]$. As opposed to problem (5-2), the optimal provision $\overline{R}_0$ in problem (5-4) is the smallest value such that with high probability *no* hurdle is violated.

Although we have a single constraint in (5-4) opposed to n constraints in (5-2), problem (5-4) is harder to solve: the joint probability calculation in (5-4) involves the computation of a quantile of the cumulative distribution of the random vector $(S_{[0,1]}, S_{[0,2]}, \dots, S_{[0,n]})$, an extremely difficult task. Even checking feasibility for a given candidate R_0 is usually hard.

We use SAA to obtain good candidate solutions of (5-4) and lower bounds for the optimal value. Indeed, (5-4) is a joint chance constrained problem so the techniques of the previous Chapters apply.

The joint hurdle-race problem is not explicitly written in format (1-1), but it can be easily converted by employing the max-function

$$\begin{aligned}\mathsf{Prob}\{R_1 \geq V_1, \ldots, R_n \geq V_n \mid R_0\} &= \mathsf{Prob}\left\{S_{[0,1]} \leq R_0, \ldots, S_{[0,n]} \leq R_0\right\} \\ = \mathsf{Prob}\left\{\max_{j=1,\ldots,n}\{S_{[0,j]}\} \leq R_0\right\} &= \mathsf{Prob}\left\{\max_{j=1,\ldots,n}\{S_{[0,j]}\} - R_0 \leq 0\right\}. \end{aligned} \tag{5-6}$$

Using (5-6) we have that problem (5-4) is a particular case of (1-1):

$$\begin{array}{ll} \underset{R_0 \geq V_0}{\text{Min}} & R_0 \\ \text{s.t.} & \mathsf{Prob}\left\{\max_{j=1,\ldots,n}\{S_{[0,j]}\} - R_0 \leq 0\right\} \geq 1-\varepsilon. \end{array} \tag{5-7}$$

But how do we solve (5-7) with SAA? Given a sample size N and a reliability level γ, the SAA formulation becomes a MILP as follows.

$$\begin{array}{lll} \underset{R_0 \geq V_0}{\text{Min}} & R_0 & \\ \text{s.t.} & (V_1+\alpha_1)\exp(-Y_1^s) - R_0 - Kz_s \leq 0, & s=1,\ldots,N, \\ & \vdots & \\ & \sum_{i=1}^j \alpha_{i(j)} \exp(-Y_1^s - \cdots - Y_i^s) - R_0 - Kz_s \leq 0, & s=1,\ldots,N, \\ & \vdots & \\ & \sum_{i=1}^n \alpha_{i(n)} \exp(-Y_1^s - \cdots - Y_n^s) - R_0 - Kz_s \leq 0, & s=1,\ldots,N, \\ & \sum_{s=1}^N z_s \leq N\gamma, & \\ & z^s \in \{0,1\}^N, & \end{array} \tag{5-8}$$

where K is a sufficiently large positive constant, Y_i^s are samples from the return process $(Y_1, \ldots, Y_n)$ and

$$\alpha_{i(j)} = \begin{cases} \alpha_i, & i \neq j, \\ V_j + \alpha_j, & i = j. \end{cases}$$

The feasibility check cannot be performed exactly and uses Monte-Carlo methods.

5.2.1
Numerical experiments

We compare separated and the joint hurdle-race problems. Following [VDGK], we performed experiments with $n = 40$ periods of investment and with normal iid returns Y_i with mean $\mu = \log 1.10$ and standard deviation $\sigma = 0.10$. The hurdles and the liabilities are equal to 10 and 0.8 respectively for all time periods. For the separated hurdles we choose $\varepsilon_j = 0.05$ for all periods and for the joint case we take $\varepsilon = 0.05$. Our numerical experiments showed that the stochastic lower bound approximation in [VDGK] is extremely accurate for the separated hurdle-race problem, and we obtained $\overline{R}_0 = 13.56411337$. An (stochastic) upper bound, obtained by the comonotonic approximation, is 14.48125099.

For the joint hurdle-race problem, we first choose values of γ and N. Following the empirical findings for the portfolio problem and the blending problem of the previous Chapter, we set $\gamma = \varepsilon/2 = 0.025$. We then compute the value of N for which the optimal solution of the SAA problem is feasible for the original problem with probability greater than 99% (see [CG]), which in this case is $N = 90$. This estimate is usually too conservative and should be regarded as an upper bound for the actual value of N to be used in the experiments. The best result was obtained for $N = 50$ and the smallest initial provision was $\overline{R}_0 = 15.81238194$. A Monte-Carlo experiment with 100 000 samples estimated the true probability of this candidate as 0.9527 and it is thus feasible for the original problem.

We compute statistical lower bounds for the optimal value of the joint hurdle-race problem following section 4.1.3. Fixing $M = 1\,000$ and L according to (3-12), the 99%-confidence lower bound for the optimal value is 15.302594. Obviously, the candidate solution obtained by SAA can be regarded as an upper bound for the true optimal value (if feasible).

Similar experiments were performed with $\varepsilon_j = 0.01$. The techniques in [VDGK] give 16.34858684 for the solution of the separated problem, with comonotonic upper bound[1] equal to 18.37208466. It is not clear how these methods should be extended to the joint hurdle problem. In this case, applying SAA with $N = 150$ and $\gamma = \varepsilon/2 = 0.005$, the new method obtained 18.21640345. The true probability was estimated by Monte-Carlo and was equal to exactly 1, with 100 000 samples. The 99%-confident lower bound obtained was 17.515987.

It is interesting to compare the solution for the joint case (5-4) with the one for the separate case (5-2). The best provisions obtained for the latter were less than those for the former in both experiments. One might be tempted to adopt the separated version as the best model since smaller provisions are usually preferred. However, the solution of the joint problem is obviously more robust. We proceed to substantiate this claim with some simulations.

The separated case was treated as in [VDGK], whereas we used SAA for the joint problem. For illustration, we first present small size simulation

[1]The comonotonic upper bound is constructed using conditional expectations of comonotonic random variables.

experiments that show the differences between the two approaches. Figures 5.1 and 5.2 show the estimated probability of violating each hurdle for the separated problem. The values in the y-axis are close to 0.05 and 0.01 respectively, in accordance with the chosen reliability levels. Figures 5.3 and 5.4 show 100 sample paths for each choice of ε in the separated case. In Figure 5.3, 12 paths violated one of the hurdles at least once, giving a violation probability of 0.12, significantly higher than the original 0.05 confidence level. In Figure 5.4, the probability of violation was 0.06, also higher than the original significance level 0.01. In the joint case, Figures 5.5 and 5.6 indicate that at least one constraint was violated only 3% and 1% of the time, in accordance with the joint reliability levels $\varepsilon = 0.05$ and $\varepsilon = 0.01$, a much more robust situation.

In order improve numerical accuracy, we ran the same experiments for 10 000 paths. For $\varepsilon_j = \varepsilon = 0.05$, the estimated probability of violation of at least one hurdle for the separated hurdle-race problem was 0.1173, much higher than the corresponding value 0.0328 for the joint version with the same reliability level. For $\varepsilon_j = \varepsilon = 0.01$, the estimated values were 0.0247 for the separated formulation and 0.0094 for the joint counterpart. In both cases the separated hurdle-race problem misses the joint reliability level by roughly twice the pre-determined reliability level, substantially underestimating the more robust provision given by SAA for the joint case.

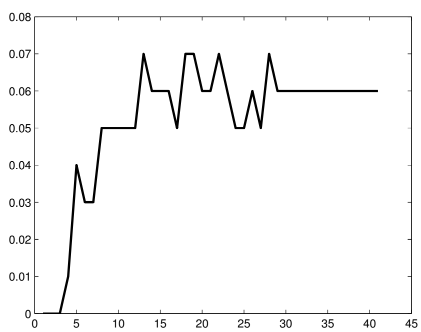

Figure 5.1: $\varepsilon = 0.05$.

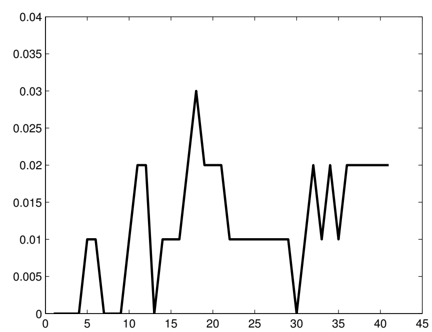

Figure 5.2: $\varepsilon = 0.01$.

5.3 Stochastic hurdles

We now consider a joint hurdle-race model with the possibility of modeling stochastic hurdles. Now we define the hurdles as the market consistent value of future liabilities when evaluating the portfolio at period j, i.e., the

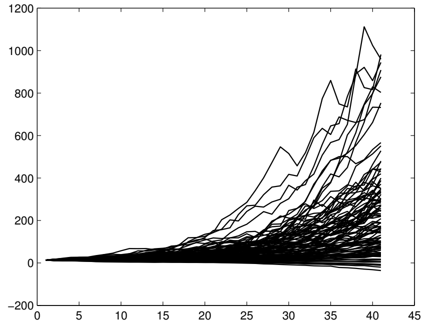

Figure 5.3: Sample paths for $\varepsilon = 0.05$, SHR.

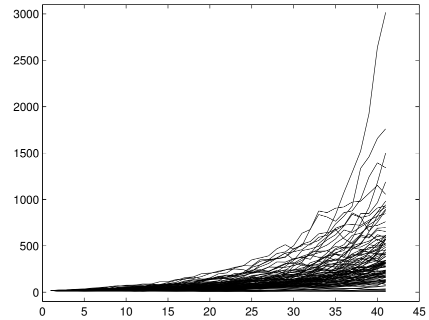

Figure 5.4: Sample paths for $\varepsilon = 0.01$, SHR.

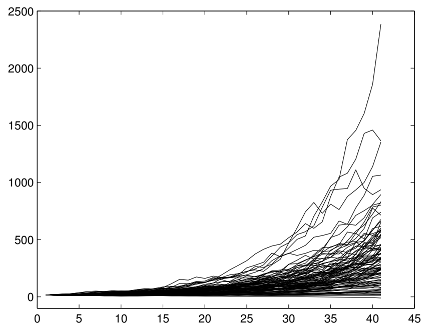

Figure 5.5: Sample paths for $\varepsilon = 0.05$, JHR.

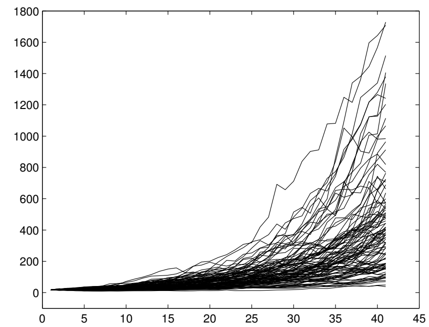

Figure 5.6: Sample paths for $\varepsilon = 0.01$, JHR.

hurdles will be the discounted cash flows at the stochastic risk-free rate. In particular, the hurdles V_j are not known at period j.

In addition to the stochastic return process $(Y_1, \ldots, Y_n)$, the model will have a risk-free rate process $(r_1, \ldots, r_n)$ that will determine the (stochastic) hurdles as follows.

$$
\begin{aligned}
V_0 &= \alpha_1 \exp(-r_1) + \alpha_2 \exp(-r_1 - r_2) + \cdots + \alpha_n \exp(-r_1 - \cdots - r_n), \\
V_1 &= \alpha_1 + \alpha_2 \exp(-r_2) + \cdots + \alpha_n \exp(-r_2 - \cdots - r_n), \\
&\vdots \\
V_n &= \alpha_n.
\end{aligned}
\tag{5-9}
$$

SAA obtains candidate solutions *with no additional computational effort*. In the numerics, the risk-free rate process was composed of iid normal random variables with mean $\mu_r = 0.04$ and standard deviation $\sigma_r = 0.05$. The stochastic return process is the same as described in section (5.2.1). For

$\varepsilon = 0.05$, the best solution was 21.68791898, with $N = 60$ and $\gamma = \varepsilon/2 = 0.025$. The estimated probability was 0.95275, using 20 000 samples. For $\varepsilon = 0.01$, the best solution was 25.17916015, with $N = 120$ and $\gamma = \varepsilon/2 = 0.005$. The estimated probability was exactly 1, with 20 000 samples.