Pontifícia Universidade Católica
do Rio de Janeiro

Edison Fausto Cuba Huamani

Affine Minimal Surfaces with Singularities

Dissertação de Mestrado

Dissertation presented to the Programa de Pós–graduação em Matemática of PUC-Rio in partial fulfillment of the requirements for the degree of Mestre em Matemática.

Advisor: Prof. Marcos Craizer

Rio de Janeiro
September 2017

Pontifícia Universidade Católica
do Rio de Janeiro

Edison Fausto Cuba Huamani

Affine Minimal Surfaces with Singularities

Dissertation presented to the Programa de Pós–graduação em Matemática of PUC-Rio in partial fulfillment of the requirements for the degree of Mestre em Matemática. Approved by the undersigned Examination Committee.

Prof. Marcos Craizer
Advisor
Departamento de Matemática – PUC-Rio

Prof. Ricardo Sá Earp
Departamento de Matemática – PUC-Rio

Prof. Ronaldo Alves Garcia
Universidade Federal de Goiás – UFG

Prof. David Martinez
Departamento de Matemática – PUC-Rio

Prof. Márcio da Silveira Carvalho
Vice Dean of Graduate Studies
Centro Técnico Científico – PUC-Rio

Rio de Janeiro, September 14th, 2017

Edison Fausto Cuba Huamani

Has a degree in Mathematics from Universidad Nacional de San Agustín, Arequipa - Perú.

Bibliographic data

Cuba Huamani, Edison Fausto

Affine Minimal Surfaces with Singularities / Edison Fausto Cuba Huamani; advisor: Marcos Craizer. – Rio de janeiro: PUC-Rio, Departamento de Matemática , 2017.

v., 79 f: il. color. ; 30 cm

Dissertação (mestrado) - Pontifícia Universidade Católica do Rio de Janeiro, Departamento de Matemática .

Inclui bibliografia

1. Matemática – Dissertação. 2. Linhas de Curvatura Planas;. 3. Superfícies Afins Mínimas;. 4. Superfícies Afins Maximais;. 5. Aplicações Afins Imprópias;. 6. Swallowtails;. 7. Cuspidal Edges.. I. Craizer, Marcos. II. Pontifícia Universidade Católica do Rio de Janeiro. Departamento de Matemática . III. Título.

CDD: 620.11

To my parents, for their support
and encouragement.

Acknowledgments

I would like to thank my fathers and brothers who have sacrificed a lot. We have worked together to move forward, thank you very much to all five and the whole family.

I would like to thank my advisor Prof. Marcos Craizer for the continuous support, for his patience, motivation, and immense knowledge.

Besides my advisor, I would like to thank the rest of my thesis committee for their suggestions and contributions, and for accepting the invitation.

My sincere thanks also goes to Creuza and all administrative staff of the Department of Mathematics for helping me to fulfill all the bureaucracy requirements.

To my many friends who have supported me throughout the process. I will always appreciate all they have done, especially Rodrigo, Thiago and Simone.

I dedicate this work and give special thanks to my favorite person.

I would like to express my sincere gratitude to CAPES for the aid granted.

Abstract

Cuba Huamani, Edison Fausto; Craizer, Marcos (Advisor). **Affine Minimal Surfaces with Singularities**. Rio de Janeiro, 2017. 79p. Dissertação de mestrado – Departamento de Matemática , Pontifícia Universidade Católica do Rio de Janeiro.

In this work we study surfaces with zero affine mean curvature. They are called affine minimal surfaces and for convex surfaces, they are also called affine maximal surfaces. We prove that an euclidean minimal surface is also an affine minimal surface if and only if the curvature lines of the conjugate euclidean minimal surface are planar. For an affine maximal surface, we describe how to recover it from the conormal vector field along a given curve. For some choices of the conormal vector, the maximal surface is singular and we describe conditions under which the singularities are cuspidal edges or swallowtails.

Keywords

Planar Curvature Lines; Affine Minimal Surfaces; Affine Maximal Surfaces; Improper Affine Maps; Swallowtails; Cuspidal Edges.

Resumo

Cuba Huamani, Edison Fausto; Craizer, Marcos. **Superfícies Mínimas Afins com Singularidades**. Rio de Janeiro, 2017. 79p. Dissertação de Mestrado – Departamento de Matemática , Pontifícia Universidade Católica do Rio de Janeiro.

Neste trabalho, estudamos superfícies com curvatura média afim zero. Elas são chamadas de superfícies mínimas afins e para superfícies convexas, também são chamadas de superfícies máximas afins. Provamos que uma superfície mínima euclidiana também é uma superfície mínima afim se, e somente se, as linhas de curvatura da superfície mínima euclidiana conjugada são planas. Para uma superfície máxima afim, descrevemos como recuperá-la do campo de vetor conormal ao longo de uma determinada curva. Para algumas escolhas do vector conormal, a superfície máxima é singular e descrevemos as condições sob as quais as singularidades são arestas cuspidais ou swallowtails.

Palavras-chave

Linhas de Curvatura Planas; Superfícies Afins Mínimas; Superfícies Afins Maximais; Aplicações Afins Imprópias; Swallowtails; Cuspidal Edges.

Table of contents

List of figures

1
Introduction

The family of affine maximal surfaces in $\mathbb{R}^3$ is an important subject in geometric analysis, since they are extremals of a geometric functional and the associated Euler-Lagrange equation is a non linear fourth order partial differential equation, which generalizes the Hessian one equation, see [7].

We introduce the notion of affine maximal map with a conformal representation, which generalizes the Weierstrass formula for improper affine spheres.

We also prove that the curvature lines of the original minimal surface corrrespond to asymptotic lines of its conjugate surface, the above equivalence shows that the conjugate of the minimal surface with planar curvature lines is an affine minimal surface. See for example [8] and [10].

Furthermore, we take the solution of the affine Cauchy problem and give the conditions to the existence and uniqueness of affine maximal maps with the desired singularities. In particular, we characterize when an analytic curve of $\mathbb{R}^3$ is the singular curve of some affine maximal map with cuspidal edges or swallowtails. See [1] and [7].

1.1
Literature Review

After Calabi's work, the use of geometric methods in studying PDEs of affine differential geometry was continued by different authors and the affine Bernstein problem was solved affirmatively.

This lack of global regular examples has led to a recent study of affine maximal maps, that is, affine minimal surfaces with some singularities. This has revealed an interesting global theory, where the solution of the affine Cauchy problem shows the existence of an important amount of affine maximal surfaces with singular curves and isolated singularities.

The study of minimal surfaces with planar curvature lines is a classical subject, having been studied by Bonnet, Enneper, and Eisenhart in the late 19-th century as recorded in [3] and [4].

1.2 Dissertation Outline

This dissertation is organized in five chapters. Chapter 1 has the literature review and dissertation outline. In the Chapter 2 we study the basic concepts of Euclidean differential geometry.

Chapter 3 contains a detailed explanation of the affine differential geometry and affine maximal surfaces. We introduce the notion of Berwald-Blaschke metric, affine conormal map, first fundamental affine form. We also give introduce the notion of affine minimal map and improper affine spheres. It also covers the principal mathematical tools used to understand the affine differential geometry.

In Chapter 4 we study minimal surfaces with planar curvature lines, then we revisit the subject from a different point of view. After calculating their metric functions using an analytical method, we recover the Weierstrass data, and we give parametrizations for these surfaces.

Chapter 5 contains one of the main results of this dissertation. It is the study of surfaces with singularities, where we can take the solution of the affine Cauchy problem and give the conditions to the existence and uniqueness of affine maximal maps with the desired singularities. In particular, we also characterize when an analytic curve of $\mathbb{R}^3$ is the singular curve of some improper affine sphere with prescribed cuspidal edges and swallowtails.

2
Euclidean Differential Geometry

In this chapter we give a brief introduction to the differential geometry of surfaces in three-dimensional euclidean space. The main purpose of this introduction is to provide the reader with the basic notions of differential geometry and with the essential formulas that will be needed later on. Section 2.1 discusses the notion of surfaces. Moreover, the notions of tangent space, as well as tangent and normal vector fields are defined. See [5] for more details.

2.1
Regular surfaces

There are many approaches that one can take to introduce surfaces. Some people immediately build the formalism of differentiable manifolds, or some others, first encounter surfaces in $\mathbb{R}^3$ as the solution set to the equation $F(x, y, z) = 0$ with three variables, and other present surfaces as the images of vector functions of two variables. See [5].

Definition 2.1.1 *A subset $S \subset \mathbb{R}^3$ is a* ***regular surface*** *if, for each $p \in S$, there exists a neighborhood V in $\mathbb{R}^3$ and a map $\Psi : \Omega \to V \cap \mathbb{R}^3$ of an open set $\Omega \subset \mathbb{R}^2$ onto $V \cap S \subset \mathbb{R}^3$ such that*

1. *Ψ is differentiable.*

2. *Ψ is a homeomorphism.*

3. *For each $q \in \Omega$, the differential $d\Psi_q : \mathbb{R}^2 \to \mathbb{R}^3$ is one to one.*

Definition 2.1.2 *Let S be a regular surface, $p \in S$, and consider all the curves defined on S passing through p. We define the tangent plane at p, denoted by T_pS, as the vector space of dimension 2 which contains all vectors tangent to the family of curves at the point p.*

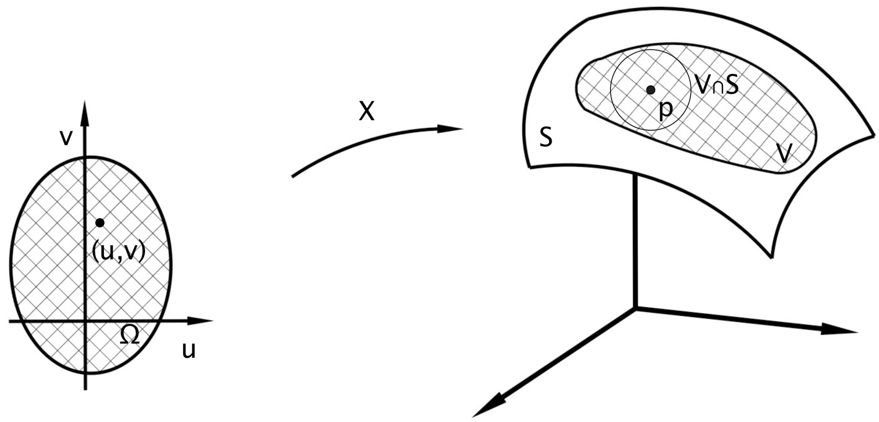

Figure 2.1: Parametrization of a Regular Surface.

Given $p \in S$ and let $\alpha : (-\varepsilon, \varepsilon) \to S$ be a differentiable parametrized curve, with $\alpha(0) = p$. The velocity vector $\alpha'(0)$ is called the vector tangent to S at p. The choice of any parametrization Ψ of S determines a base $\{\Psi_u, \Psi_v\}$ of T_pS, called the base associated with Ψ.

Definition 2.1.3 *Let $p \in S$, the quadratic form $I_p : T_pS \to \mathbb{R}$, defined by:*

$$I_p(w) = \langle w, w \rangle = \|w\|^2 \geq 0,$$

is called the ***first euclidean fundamental form*** *of the regular surface S at p.*

The first euclidean fundamental form can be expressed in the base $\{\Psi_u, \Psi_v\}$ associated with a parametrization $\Psi(u, v)$ at p, as follows: Let $w = \alpha'(0) = \Psi_u u' + \Psi_v v' \in T_pS$. Then,

$$\begin{aligned} I_p(w) &= \langle \Psi_u u' + \Psi_v v', \Psi_u u' + \Psi_v v' \rangle \\ &= E(u')^2 + 2Fu'v' + G(v')^2 \end{aligned}$$

where, $E = \langle \Psi_u, \Psi_u \rangle, F = \langle \Psi_u, \Psi_v \rangle$ and $G = \langle \Psi_v, \Psi_v \rangle$ are the coefficients of the first fundamental euclidean form in the base $\{\Psi_u, \Psi_v\}$ of T_pS.

Definition 2.1.4 *A regular surface S is* ***orientable*** *if it is possible to cover S with a family of coordinate neighborhoods so that if a point $p \in S$ is in two neighborhoods of this family, then the change of coordinates has positive Jacobian at p. The choice of family that satisfies this condition is called an orientation of S, and S is called oriented. If it is not possible to find such a family then S is called nonorientable.*

Fixed a parametrization, $\Psi : \Omega \subset \mathbb{R}^2 \to S$, we calculate the *normal euclidean vector* at each point $q \in \Psi(U)$, as:

$$\mathbf{N}(q) = \frac{\Psi_u \times \Psi_v}{\|\Psi_u \times \Psi_v\|}(q).$$

Definition 2.1.5 *Let $S \subset \mathbb{R}^3$ be a surface with an orientation. The* ***Gauss map*** *is defined to be $\boldsymbol{N} \colon S \to S^2 \subset \mathbb{R}^3$ is defined to be $p \to N(p)$.*

The Gauss map can be defined (globally) if and only if the surface is orientable. The Gauss map can always be defined locally (i.e. on a small piece of the surface).

The differential of the Gauss map $d\mathbf{N}_p : T_pS \to T_pS$, is a self-adjoint linear application, that is

$$\langle d\mathbf{N}_p(w_1), w_2\rangle = \langle w_1, d\mathbf{N}_p(w_2)\rangle, \quad w_1, w_2 \in T_pS.$$

Therefore, we can associate $d\mathbf{N}_p$ with a quadratic form Q in T_pS, given by

$$Q(w) = \langle d\mathbf{N}_p(w), w\rangle, \quad w \in T_pS.$$

Definition 2.1.6 *Let $p \in S$, the quadratic form $II_p : T_pS \to \mathbb{R}$, defined by*

$$II_p(w) = -\langle d\boldsymbol{N}_p(w), w\rangle,$$

is called the ***second euclidean fundamental form*** *of the regular surface S at p.*

The second euclidean fundamental form can be expressed in the base $\{\Psi_u, \Psi_v\}$ associated with a parametrization $\Psi(u, v)$. In fact, let $\mathbf{N}$ be the normal vector to S at $p \in S$ and $\alpha(s) = \Psi(u(s), v(s))$ a parametrized curve in S, with $\alpha(0) = p$. Therefore, the tangent vector to $\alpha(s)$ at p is $\alpha' = \Psi_u u' + \Psi_v v'$. Let's indicate by $\mathbf{N}$ the restriction of the normal vector to the curve $\alpha(s)$. In this way, we have

$$\langle \mathbf{N}(s), \alpha'(s)\rangle = 0 \Rightarrow \langle \mathbf{N}(s), \alpha''(s)\rangle = -\langle \mathbf{N}'(s), \alpha'(s)\rangle.$$

Let $w = \alpha'(0) = \Psi_u u' + \Psi_v v' \in T_pS$. Then,

$$\begin{aligned} II_p(w) &= -\langle d\mathbf{N}(\alpha'), \alpha'(0)\rangle \\ &= -\langle \mathbf{N}'(0), \alpha'(0)\rangle \\ &= -\langle \mathbf{N}(0), \alpha''(0)\rangle \\ &= -\langle \mathbf{N}(0), \Psi_{uu}(u')^2 + \Psi_u u'' + 2\Psi_{uv}u'v' + \Psi_{vv}(v')^2 - \Psi_v v''\rangle. \end{aligned}$$

Since $\langle \mathbf{N}, \Psi_u \rangle = \langle \mathbf{N}, \Psi_v \rangle = 0$. It follows that

$$II_p(w) = e(u')^2 + 2fu'v' + g(v')^2,$$

where, $e = \langle \mathbf{N}, \Psi_{uu} \rangle$, $f = \langle \mathbf{N}, \Psi_{uv} \rangle$ and $g = \langle \mathbf{N}, \Psi_{vv} \rangle$ are the coefficients of the second fundamental euclidean form in the base $\{\Psi_u, \Psi_v\}$ of T_pS.

Definition 2.1.7 *Let $p \in S$, and let $d\boldsymbol{N}_p : T_pS \to T_pS$ be the differential of the Gauss map. The determinant of $d\boldsymbol{N}_p$ is called the* ***euclidean Gaussian curvature*** *K of S at p. We observe that the euclidean Gaussian curvature can be obtained using the coefficients of the first and second fundamental form as follows:*

$$K = \frac{eg - f^2}{EG - F^2}.$$

Definition 2.1.8 *The* ***mean curvature*** *is the trace of $d\boldsymbol{N}_p$. In other words*

$$H = \frac{-1}{2}(a_{11} + a_{22}) = \frac{1}{2}\frac{eG - 2fF + gE}{EG - F^2}.$$

2.2 The Gauss Theorem and the compatibility equations

Here we are going to assign to each point of a surface a natural trihedron given by the vectors Ψ_u, Ψ_v and $\mathbf{N}$. Let S be a regular and oriented surface. Let $\Psi : \Omega \subset \mathbb{R}^2 \to S$ be a parametrization in the orientation of a surface S. See [5] for more details.

By expressing the derivatives of the vectors Ψ_u, Ψ_v and $\mathbf{N}$ in the basis $\{\Psi_u, \Psi_v, \mathbf{N}\}$, we obtain

$$\begin{aligned}
\Psi_{uu} &= \Gamma^1_{11}\Psi_u + \Gamma^2_{11}\Psi_v + e\mathbf{N}\\
\Psi_{uv} &= \Gamma^1_{12}\Psi_u + \Gamma^2_{12}\Psi_v + f\mathbf{N}\\
\Psi_{vv} &= \Gamma^1_{22}\Psi_u + \Gamma^2_{22}\Psi_v + g\mathbf{N}\\
\mathbf{N}_u &= a_{11}\Psi_u + a_{21}\Psi_u\\
\mathbf{N}_v &= a_{12}\Psi_u + a_{22}\Psi_u,
\end{aligned}$$

where the $a_{ij}, \quad i,j = 1,2$ are the coefficients of the Weingarten map and the coefficients $\Gamma^k_{i,j}, \quad i,j,k = 1,2$ are called the Christoffel symbols of S with parametrization $\Psi(u,v)$ and those symbols are symmetric relative to the lower indices and e, f and g are the coefficients of the second fundamental form of S.

Where the coefficients a_{ij} are given by

$$a_{11} = \frac{fF - eG}{EG - F^2}, \qquad a_{12} = \frac{gF - fG}{EG - F^2},$$
$$a_{21} = \frac{eF - fE}{EG - F^2}, \qquad a_{22} = \frac{fF - eG}{EG - F^2}.$$

To determine the Christoffel symbols we take the inner product with Ψ_u and Ψ_v obtaining the system

$$\begin{cases} \Gamma^1_{11}E + \Gamma^2_{11}F = \langle \Psi_{uu}, \Psi_u \rangle = \frac{1}{2}E_u, \\ \Gamma^1_{11}F + \Gamma^2_{11}G = \langle \Psi_{uu}, \Psi_v \rangle = F_u - \frac{1}{2}E_v, \\ \Gamma^1_{12}E + \Gamma^2_{12}F = \langle \Psi_{uv}, \Psi_u \rangle = \frac{1}{2}E_v, \\ \Gamma^1_{12}F + \Gamma^2_{12}G = \langle \Psi_{uv}, \Psi_v \rangle = \frac{1}{2}G_u, \\ \Gamma^1_{22}E + \Gamma^2_{22}F = \langle \Psi_{vv}, \Psi_u \rangle = F_v - \frac{1}{2}G_u, \\ \Gamma^1_{22}F + \Gamma^2_{22}G = \langle \Psi_{vv}, \Psi_v \rangle = \frac{1}{2}G_v. \end{cases} \tag{2.1}$$

It is possible to solve the above system to compute the Christoffel symbols in terms of the first fundamental form E, F and G and their derivatives.
We can obtain relations between the Christoffel symbols and their derivatives, since

$$(\Psi_{uu})_v = (\Psi_{uv})_u \qquad \text{and} \qquad (\Psi_{vv})_u = (\Psi_{uv})_v,$$

in each case, comparing the coefficients of Ψ_u, Ψ_v, and $\mathbf{N}$. With this, we obtain the famous **Gauss equations**

$$-EK = (\Gamma^2_{12})_u - (\Gamma^2_{11})_v + \Gamma^1_{12}\Gamma^2_{11} + (\Gamma^2_{12})^2 - \Gamma^2_{11}\Gamma^2_{22} - \Gamma^1_{11}\Gamma^2_{12} \tag{2.2}$$

and

$$GK = (\Gamma^1_{22})_u - (\Gamma^1_{12})_v + \Gamma^1_{22}\Gamma^1_{11} + \Gamma^2_{22}\Gamma^1_{12} - (\Gamma^1_{12})^2 - \Gamma^2_{12}\Gamma^1_{22}. \tag{2.3}$$

Similarly, the **Mainardi-Codazzi equation**

$$e_v - f_u = e\Gamma^1_{12} + f(\Gamma^2_{12} - \Gamma^1_{11}) - g\Gamma^2_{11}.$$

and

$$f_v - g_u = e\Gamma^1_{22} + f(\Gamma^2_{22} - \Gamma^1_{12}) - g\Gamma^2_{12}.$$

The Mainardi-Codazzi and Gauss equations are known under the name of compatibility equations.

Now, we state a fundamental theorem of existence of isothermal coordinates.

Theorem 2.2.1 *Always there exist isothermal coordinate system on any re-*

gular surface.

The proof of this theorem is delicate and will not be proved here. Its proof may be found in [14].

A natural question is whether there exist further relations of compatibility between the first and the second fundamental forms besides those already obtained. The following theorem stated shows that the answer is negative. In other words, we would obtain no further relations between the coefficients E, F, G, e, f, g and their derivatives. Actually, the Theorem is more explicit and asserts that the knowledge of the first and second fundamental forms determines a surface locally.

Theorem 2.2.2 (Bonnet) *Let E, F, G, e, f, g be differentiable functions, defined in an open set $V \subset \mathbb{R}^2$, with $E > 0$ and $G > 0$. Assume that the given functions satisfy formally the Gauss and Mainardi-Codazzi equations and that $EG - F^2 > 0$. Then, for every $q \in V$ there exists a neighborhood $U \subset V$ of q and a diffeomorphism $x : U \to x(U) \subset \mathbb{R}^3$ such that the regular surface $x(U) \subset \mathbb{R}^3$ has E, F, G and e, f, g as coefficients of the first and second fundamental forms, respectively. Furthermore, if U is connected and if*

$$\bar{x} : U \to \bar{x}(U) \subset \mathbb{R}^3$$

is another diffeomorphism satisfying the same conditions, then there exist a translation T and a proper linear orthogonal transformation ρ in $\mathbb{R}^3$ such that $\bar{x} = T \circ \rho \circ x$.

2.3 Euclidean minimal surfaces

The study of minimal surfaces is an active area of research in mathematics with many unsolved problems. Some standard examples of minimal surfaces in $\mathbb{R}^3$ are the plane, the Enneper surface, the catenoid, the helicoid, and Bonnet family. See [5]

Definition 2.3.1 *A **minimal surface** is a parametrized surface of class C^2 that satisfies the regularity condition and for which the mean curvature H is identically 0.*

Definition 2.3.2 *Let $\Psi(u, v)$ be a regular surface of class $C^2(\Omega, \mathbb{R}^3)$. It has **conformal parameters** u and v if*

$$|\Psi_u|^2 = |\Psi_v|^2 \quad \textit{and} \quad \langle \Psi_u, \Psi_v \rangle = 0.$$

2.3.1
The Hopf differential

In this section we associate to each minimal surface a holomorphic 2-form, the so-called Hopf differential, that informs us about the distribution of the umbilical points on the surface. Let $\Psi : \Omega \to \mathbb{R}^3$ be an immersion of an orientable surface. Consider conformal coordinates (u, v) defined on an open subset V of Ω. So, the mean curvature is given by

$$H = \frac{e+g}{2E}. \tag{2.4}$$

In terms of the Christoffel symbols, the second derivatives of Ψ are:

$$\begin{aligned} \Psi_{uu} &= \frac{E_u}{2E}\Psi_u - \frac{E_v}{2E}\Psi_v + e\mathbf{N} \\ \Psi_{uv} &= \frac{E_v}{2E}\Psi_u + \frac{E_u}{2E}\Psi_v + f\mathbf{N} \\ \Psi_{vv} &= -\frac{E_u}{2E}\Psi_u + \frac{E_v}{2E}\Psi_v + g\mathbf{N}. \end{aligned}$$

On the other hand, the derivatives of the normal vector are

$$\mathbf{N}_u = -\frac{e}{E}\Psi_u - \frac{f}{E}\Psi_v \qquad \text{and} \qquad \mathbf{N}_v = -\frac{f}{E}\Psi_u - \frac{g}{E}\Psi_v. \tag{2.5}$$

Furthermore, the Codazzi equations are

$$\begin{aligned} e_v - f_u = \mathbf{N}_v \cdot \Psi_{uu} - \mathbf{N}_u \cdot \Psi_{uv} &= \frac{E_v}{2E}(e+g) = E_v H \\ f_v - g_u = \mathbf{N}_v \cdot \Psi_{uv} - \mathbf{N}_u \cdot \Psi_{vv} &= -\frac{E_u}{2E}(e+g) = -E_u H. \end{aligned}$$

By differentiating $2EH = e + g$ with respect to u and v, we have

$$2E_uH + 2EH_u = e_u + g_u, \qquad 2E_vH + 2EH_v = e_v + g_v.$$

With both expressions, the Codazzi equations now read as

$$(e-g)_u + 2f_v = 2EH_u, \qquad (e-g)_v - 2f_u = -2EH_v, \tag{2.6}$$

respectively. Let us introduce the complex notation $z = u + iv$, $\bar{z} = u - iv$ and

$$\partial_z = \frac{1}{2}(\partial_u - i\partial_v), \qquad \partial_{\bar{z}} = \frac{1}{2}(\partial_u + i\partial_v).$$

Define

$$Q(z, \bar{z}) = e - g - 2if.$$

Equations (2.6) are then simplified as

$$Q_{\bar{z}} = EH_z. \tag{2.7}$$

We point out that the zeroes of Q are the umbilical points of the immersion since $Q(p) = 0$ if and only if $E = G$, $F = 0$, $e = g$ and $f = 0$ at p. We express Q as follows. Let us consider the derivatives X_z and N_z

$$\Psi_z = \frac{1}{2}(\Psi_u - i\Psi_v), \qquad N_z = \frac{1}{2}(N_u - iN_v).$$

Thus, we have

$$\Psi_z \cdot \mathbf{N}_z = -\frac{1}{4}(e - g - 2if) = -\frac{1}{4}Q(z, \bar{z}) \tag{2.8}$$

and then

$$Q(z, \bar{z}) = -4\Psi_z \cdot N_z.$$

We study Q under a change of conformal coordinates $w = h(z)$, where h is a holomorphic function. Then $\Psi_z = h'(z)\Psi_w$, $N_z = h'(z)N_w$ and

$$Q(z, \bar{z}) = -4\Psi_z \cdot N_z = -4h'(z)^2\Psi_w \cdot N_w = h'(z)^2 Q(w, \bar{w}).$$

Hence

$$Q(w, \bar{w})dw^2 = Q(w, \bar{w})h'(z)^2 dz^2 = Q(z, \bar{z})dz^2.$$

This equality means that Qdz^2 defines a global quadratic differential form on the surface.

Definition 2.3.3 *The differential form Qdz^2 is called the **Hopf differential**.*

Theorem 2.3.1 *A conformal immersion $\Psi : \Omega \to \mathbb{R}^3$ has constant mean curvature if and only if Qdz^2 is holomorphic. In such case, either the set of umbilical points is formed by isolated points, or the immersion is umbilical.*

Proof. Given a conformal parametrization $\Psi(u, v)$, From (2.7), we have $Q_{\bar{z}} = 0$, which is equivalent to saying that Q is holomorphic on V . Thus the umbilical points agree with the zeroes of a holomorphic function. This means either $Q = 0$ on V or the umbilical points are isolated. In the first case, an argument of connectedness proves that the set of umbilical points is an open and closed set of Ω and so Ω is an umbilical surface.

■

A direct consequence of the above theorem (2.3.1) is the Hopf theorem.

Theorem 2.3.2 (Hopf) *The only compact constant mean surface of genus 0 in $\mathbb{R}^3$ is the standard sphere.*

Proof. Since the genus of Ω is zero, the uniformization theorem says that its conformal structure is conformally equivalent to the usual structure of $\mathbb{C}$. This is defined by the parametrizations z in $\mathbb{C}$ and $w = 1/z$ in $\bar{\mathbb{C}} - \{0\}$. The intersection domain of both charts is $\mathbb{C} - \{0\}$ and in this open set, the Hopf differential Q is

$$Q(z) = w'(z)^2 Q(w) = \frac{1}{z^4} Q(w).$$

It follows that

$$\lim_{z\to\infty} Q(z) = \lim_{z\to\infty} \left(\frac{1}{z^4}\right) Q(w=0) = 0.$$

This shows that by writing $Q = Q(z)$ in terms of the parametrization z on $\mathbb{C}$, Q can be extended to ∞ by letting $Q(\infty) = 0$. Therefore Q is a bounded holomorphic function on $\bar{\mathbb{C}}$. Liouville's theorem asserts that the only bounded holomorphic functions on $\bar{\mathbb{C}}$ are constant. As $Q(\infty) = 0$, then $Q = 0$ on $\bar{\mathbb{C}}$ and all points are umbilical. Finally, the only umbilical closed surface in a space form is the sphere.

■

Let $\Omega \subset \mathbb{R}^2$ be a simply-connected domain with coordinates (u, v), and let $\Psi : \Omega \to \mathbb{R}^3$ be a conformally immersed surface. Since $\Psi(u, v)$ is conformal,

$$ds^2 = e^{2w}(du^2 + dv^2)$$

for some $w : \Omega \to \mathbb{R}$. Then the mean curvature H is,

$$H := \frac{1}{2e^{2w}}(\Psi_{uu} + \Psi_{vv}) \cdot \mathbf{N}.$$

The Hopf differential factor is given by

$$Q = \frac{1}{4}(\Psi_{uu} - 2i\Psi_{uv} - \Psi_{vv}) \cdot \mathbf{N}.$$

The Gauss-Weingarten equations are

$$\begin{cases} \Psi_{uu} = w_u\Psi_u - w_v\Psi_v + (Q + \bar{Q})\mathbf{N} \\ \Psi_{vv} = -w_u\Psi_u + w_v\Psi_v - (Q + \bar{Q})\mathbf{N} \\ \Psi_{uv} = w_v\Psi_u + w_u\Psi_v + i(Q - \bar{Q})\mathbf{N} \\ \mathbf{N}_u = -e^{-2w}(Q + \bar{Q})\Psi_u - ie^{-2w}(Q - \bar{Q})\Psi_v \\ \mathbf{N}_v = -ie^{-2w}(Q - \bar{Q})\Psi_u + e^{-2w}(Q + \bar{Q})\Psi_v. \end{cases} \tag{2.9}$$

Since, $\bar{Q} = \dfrac{1}{4}(\Psi_{uu} + 2i\Psi_{uv} - \Psi_{vv}) \cdot \mathbf{N}$ and $E = e^{2w}$, furthermore, $w = \dfrac{1}{2}\log E$

and its derivatives are

$$w_u = \frac{1}{2}\frac{E_u}{E}, \qquad \text{and} \qquad w_v = \frac{1}{2}\frac{E_v}{E}.$$

Note also that, $Q + \bar{Q} + He^{2w} = \frac{1}{4}(e - g - 2if) + \frac{1}{4}(e - g + 2if) + \frac{e+g}{2} = \frac{1}{2}(e-g)+\frac{1}{2}(e+g) = e$ and $g = -(Q+\bar{Q}+He^{2w})$ and $f = i(Q-\bar{Q}) = \Psi_{uv}\cdot\mathbf{N}$. The first and second fundamental form are

$$I = e^{2w}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$$II = \begin{pmatrix}Q + \bar{Q} + He^{2w} & i(Q - \bar{Q})\\ i(Q - \bar{Q}) & -(Q + \bar{Q}) + He^{2w}\end{pmatrix}$$

The principal curvatures k_1 and k_2 are the eigenvalues of the matrix $II.I^{-1}$. This gives the following expressions for the mean and the Gaussian curvatures

$$H = \frac{1}{2}(k_1 + k_2) = \frac{1}{2}tr(II.I^{-1}),$$

$$K = k_1k_2 = det(II.I^{-1}).$$

Now, we compute the Gaussian curvature in terms of the Hopf differential and the mean curvature.

$$\begin{aligned}
K = k_1k_2 &= det(II.I^{-1})\\
&= det\left(\frac{1}{e^{2w}}\begin{pmatrix}Q + \bar{Q} + He^{2w} & i(Q - \bar{Q})\\ i(Q - \bar{Q}) & -(Q + \bar{Q}) + He^{2w}\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right)\\
&= \frac{1}{e^{4w}}\left[(He^{2w})^2 - (Q + \bar{Q})^2 + Q^2 + \bar{Q}^2 - 2Q\bar{Q}\right]\\
&= \frac{1}{e^{4w}}(H^2e^{4w} - 4Q\bar{Q})\\
&= H^2 - 4Q\bar{Q}e^{-4w}.
\end{aligned}$$

Therefore, the Gauss-Codazzi equations are

$$\Delta w - 4Q\bar{Q}e^{-2w} = 0 \qquad \text{and} \qquad Q_{\bar{z}} = 0,$$

where $\Delta w = w_{uu} + w_{vv}$. Remember that $H = 0$. Thus, $K = -4Q\bar{Q}e^{-4w}$ or equivalently

$$Ke^{2w} = -4Q\bar{Q}e^{-2w}.$$

We need to prove that $\Delta w + Ke^{2w} = 0$. From equation (2.3)

$$\begin{aligned} GK &= (\Gamma^1_{22})_u - (\Gamma^1_{12})_v + \Gamma^1_{22}\Gamma^1_{11} + \Gamma^2_{22}\Gamma^1_{12} - (\Gamma^1_{12})^2 - \Gamma^2_{12}\Gamma^1_{22} \\ &= -w_{uu} - w_{vv} - w_u^2 + w_v^2 - w_v^2 + w_u^2, \\ &= -w_{uu} - w_{vv}, \end{aligned}$$

since, $E = G = e^{2w}$ and $F = 0$. Thus, the Christoffel symbols are

$$\begin{aligned} \Gamma^1_{11} &= w_u & \Gamma^2_{11} &= -w_v, \\ \Gamma^1_{12} &= w_v & \Gamma^2_{12} &= w_u, \\ \Gamma^1_{22} &= -w_u & \Gamma^2_{22} &= w_v. \end{aligned}$$

From equation (2.7), we have that $Q_{\bar{z}} = EH_z$ then $Q_{\bar{z}} = 0$. Thus, we obtained the desired result.

A point p of the surface Ψ is called umbilic if the principal curvatures at this point coincide $k_1(p) = k_2(p)$. The Hopf differential vanishes $Q(p) = 0$ exactly at umbilic points of the surface.

Coordinates in which both fundamental forms are diagonal are called curvature line coordinates and the corresponding parametrization (not necessarily conformal) is called a curvature line parametrization. A curvature line parametrization always exists in a neighborhood of a non-umbilic point. Near umbilic points, curvature lines form more complicated patterns.

Note that the Gauss-Codazzi equations implies that the Hopf differential factor Q is holomorphic. Moreover, the Gauss-Codazzi equation is invariant under the deformation $Q \mapsto \lambda^{-2}Q$ for $\lambda \in \mathbb{S}^1 \subset \mathbb{C}$. In fact, when $\Psi(u,v)$ is a minimal surface in $\mathbb{R}^3$, $\lambda \in \mathbb{S}^1$ allows us to create a single-parameter family of minimal surfaces $\Psi_\lambda(u,v)$ associated to $\Psi(u,v)$, called the associated family. All the surfaces $\Psi_\lambda(u,v)$ are isometric and have the same constant mean curvature. In particular if $\lambda^{-2} = i$, then the new surface is called the conjugate surface of Ψ.

From equation (2.9) we have that

$$\mathbf{N}_u = -e^{-2w}(Q + \bar{Q})\Psi_u \quad \text{and} \quad \mathbf{N}_v = e^{-2w}(Q + \bar{Q})\Psi_v. \tag{2.10}$$

Thus, $Q - \bar{Q} = 0$. This means that Q is real. Recall from (2.7) we have that Q is holomorphic. To prove that a real valued holomorphic function $Q = a + ib = a$

is constant, we used Cauchy Riemann equations

$$a_u = b_v = a_v = v_u = 0.$$

Note that both partial derivatives are zero. Thus $Q' \equiv 0$. We can conclude that Q is constant. Hence, we can normalize the Hopf differential such that $Q = -\frac{1}{2}$, and the Gauss-Weingarten equations become,

$$\begin{cases} \Psi_{uu} = w_u \Psi_u - w_v \Psi_v - \mathbf{N} \\ \Psi_{vv} = -w_u \Psi_u + w_v \Psi_v + \mathbf{N} \\ \Psi_{uv} = w_v \Psi_u + w_u \Psi_v \\ \mathbf{N}_u = e^{-2w} \Psi_u \\ \mathbf{N}_v = -e^{-2w} \Psi_v. \end{cases} \tag{2.11}$$

where $k_1 = -e^{-2w}$ and $k_2 = e^{-2w}$ are the principal curvatures of Ψ.
Furthermore, the Gauss equation becomes the following Liouville equation

$$\Delta w - e^{-2w} = 0.$$

Example 2.3.1 (Enneper surface with planar curvature lines) *The most common parametrization for Enneper surface is*

$$\Psi(u, v) = (u - \frac{1}{3}u^3 + uv^2, \frac{1}{3}v^3 - v - u^2 v, u^2 - v^2).$$

First, we show that this surface has isothermal coordinates. For this, we compute its derivatives

$$\Psi_u = (1 - u^2 + v^2, -2uv, 2u) \qquad \text{and} \qquad \Psi_v = (2uv, -1 - u^2 + v^2, -2v).$$

and then we obtain the coeffients of the first fundamental form

$$E = \langle \Psi_u, \Psi_u \rangle = (1 + u^2 + v^2)^2 = G = e^{2w}$$

and

$$F = \langle \Psi_u, \Psi_v \rangle = 2uv(1 - u^2 + v^2) - 2uv(-1 - u^2 + v^2) - 4uv = 0.$$

Since $E = G$ and $F = 0$, we have that $\Psi(u, v)$ is isothermal. The normal vector is

$$\begin{aligned}\mathbf{N} &= \frac{\Psi_u \wedge \Psi_v}{\|\Psi_u \wedge \Psi_v\|} = \frac{(-2u(1+u^2+v^2), 2v(1+u^2+v^2), -(u^2+v^2+1)(u^2+v^2-1))}{(1+u^2+v^2)^2} \\ &= \frac{(-2u, -2v, 1-u^2-v^2)}{e^w} = \frac{(-2u, -2v, 1-u^2-v^2)}{1+u^2+v^2}.\end{aligned}$$

The coefficients of the second fundamental form are $e = 2$, $g = -2$ and $f = 0$. Note also that both fundamental form are diagonal, then the curvature lines are the coordinate curves. The principal curvatures are

$$k_1 = -\frac{2}{(1+u^2+v^2)^2} = -e^{-2w}, \qquad k_2 = \frac{2}{(1+u^2+v^2)^2} = e^{-2w}.$$

The Hopf differential is $Q = \frac{1}{4}(e-g) = 1$. Using, equation (2.10) we can compute the derivatives of the normal vector

$$\mathbf{N}_u = -e^{-2w}(Q+\bar{Q})\Psi_u = -\frac{1}{(1+u^2+v^2)^2}(1-u^2+v^2, -2uv, 2u),$$

$$\mathbf{N}_v = e^{-2w}(Q+\bar{Q})\Psi_v = \frac{1}{(1+u^2+v^2)^2}(2uv, -1-u^2+v^2, -2v).$$

Recall that $E = e^{2w} = (1+u^2+v^2)^2$, then $w = \log(1+u^2+v^2)$, its derivatives are

$$w_u = \frac{2u}{1+u^2+v^2}, \qquad w_v = \frac{2v}{1+u^2+v^2},$$

$$w_{uu} = \frac{2v^2-2u^2+2}{(1+u^2+v^2)^2} \quad \text{and} \quad w_{vv} = \frac{2u^2-2v^2+2}{(1+u^2+v^2)^2}.$$

Note also that Enneper surface satisfies the Gauss-Codazzi equations. It is obvious that $Q_{\bar{z}} = 0$, since $H = 0$ and

$$\Delta w - 4Q\bar{Q}e^{-2w} = \frac{4}{(1+u^2+v^2)^2} - 4e^{-2w} = 4e^{-2w} - 4e^{-2w} = 0.$$

Now we compute the auto-intersections for the Enneper surface given by

$$\Psi(u,v) = (u - \frac{1}{3}u^3 + uv^2, v - \frac{1}{3}v^3 + u^2v, u^2 - v^2).$$

For this, we introduce polar coordinates ρ and ϕ. It is

$$u = \rho\cos\phi \quad \text{and} \quad v = \rho\sin\phi \quad \text{for} \quad (\rho,\phi) \in (0,\infty)\times(0,2\pi).$$

Thus, Enneper surface has a parametric representation

$$X(\rho,\phi) = \left(\rho\cos\phi - \frac{\rho^3}{3}\cos(3\phi), \rho\sin\phi + \frac{\rho^3}{3}\sin(3\phi), \rho^2\cos(2\phi)\right). \tag{2.12}$$

Note also that the components $x_i(\rho,\phi)$ of this surface satisfy the relation

$$x_1^2(\rho,\phi)+x_2^2(\rho,\phi)+\frac{4}{3}x_3^2(\rho,\phi)=\left(\rho+\frac{\rho^3}{3}\right)^2.$$

The points of self–intersection of Enneper surface given by the parametric representation (2.12) must satisfy

$$X(\rho_1,\phi_1)=X(\rho_2,\phi_2),\quad \text{that is}\quad x_i(\rho_1,\phi_1)=x_i(\rho_2,\phi_2)\quad \text{for}\quad k=1,2,3.$$

From (2.12) we have that

$$\rho_1+\frac{\rho_1^3}{3}=\rho_2+\frac{\rho_2^3}{3}.$$

The function $f(t)=t+\dfrac{t^3}{3}$ is obviously injective, this implies $\rho_1=\rho_2=\rho$. Thus it follows from $x_3(\rho,\phi_1)=x_3(\rho,\phi_2)$ that $\cos(2\phi_1)=\cos(2\phi_2)$, then $\phi_2=\pi-\phi_1$ or $\phi_2=2\pi-\phi_2$.
If $\phi_2=\pi-\phi_1$, then $x_1(\rho,\phi_1)=x_1(\rho,\pi-\phi_1)$ implies

$$\cos\phi_1-\frac{\rho^2}{3}\cos(3\phi_1)=\cos(\pi-\phi_1)-\frac{\rho^2}{3}\cos(3(\pi-\phi_1))=-\left(\cos\phi_1-\frac{\rho^2}{3}\cos(3\phi_1)\right)$$

that is $x_1(\rho,\phi_1)=-x_1(\rho,\phi_1)=f_1(\rho,\phi_1)=0$.
If $\phi_2=2\pi-\phi_1$, then it can similarly be shown that $x_2(\rho,\phi_1)=-x_2(\rho,\phi_1)=f_2(\rho,\phi_1)=0$.

Thus, the lines of self–intersection of Enneper surface are

$$f_1(\rho,\phi_1)=\cos\phi-\frac{\rho^2}{3}\cos(3\phi)=0\quad\text{and}\quad f_2(\rho,\phi_1)=\sin\phi+\frac{\rho^2}{3}\sin(3\phi)=0.$$

Consequently they are in the planes $u=0$ and $v=0$.
The asymptotic lines satisfies

$$2(du)^2-2(dv)^2=0$$

Its solution is $u=\pm v+c$, where $c\in\mathbb{R}$ is a constant. Since $F=0$ and $f=0$. The curvature lines are $u=c_1$ and $v=c_2$, where c_1 and $c_2\in\mathbb{R}$ are constants.

Example 2.3.2 (Catenoid) *The parametrization of the catenoid is*

$$\Psi(u,v)=(a\cosh v\cos u, a\cosh v\sin u, av),\qquad 0<u<2\pi,\quad -\infty<v<\infty.$$

First, we need to turn $\pi/2$. Thus, the new parametrization is

$$\begin{aligned}\Psi(u,v) &= (a\cosh v\cos(u-\pi/2), a\cosh v\sin(u-\pi/2), -av), \quad 0<u-\pi/2<2\pi, \quad -\infty<v<\infty. \\ &= (a\cosh v\sin u, -a\cosh v\cos u, -av).\end{aligned}$$

Its derivatives are

$$\Psi_u = (a\cosh v\cos u, a\cosh v\sin u, 0),$$

$$\Psi_v = (a\sinh v\sin u, -a\sinh v\cos u, -a).$$

It is easily checked that $E = G = a^2\cosh^2 v$, $F = 0$ and $\Psi_{uu} + \Psi_{vv} = 0$. Thus, the catenoid is a minimal surface. Next, we want to compute the normal vector

$$\begin{aligned}\mathbf{N} &= \frac{\Psi_u \wedge \Psi_v}{\|\Psi_u \wedge \Psi_v\|} = \frac{(a^2\cosh v\cos u, a^2\cosh v\sin u, -a^2\cosh v\sinh v)}{a^2\cosh^2 v} \\ &= \left(\frac{\cos u}{\cosh v}, \frac{\sin u}{\cosh v}, -\frac{\sinh v}{\cosh v}\right).\end{aligned}$$

and

$$\Psi_{uu} = (-a\cosh v\cos u, -a\cosh v\sin u, 0),$$

$$\Psi_{uv} = (-a\sinh v\sin u, -a\sinh v\cos u, 0),$$

$$\Psi_{uu} = (a\cosh v\cos u, a\cosh v\sin u, 0).$$

Then, the coefficients of the second fundamental form are $e = -a$, $g = a$ and $f = 0$. Note also that both fundamental form are diagonal, then the curvature lines are the coordinate curves. The principal curvatures are

$$k_1 = \frac{e}{E} = -\frac{1}{a\cosh^2 v}, \qquad k_2 = \frac{g}{G} = \frac{1}{a\cosh^2 v}.$$

The Hopf differential is $Q = \frac{1}{4}(e-g) = -\frac{a}{2}$. Recall that $E = e^{2w} = a^2\cosh^2 v$, then $w = \log(a\cosh v)$, its derivatives are

$$w_v = \frac{\sinh v}{\cosh v}, \qquad w_{vv} = \frac{1}{\cosh^2 v}, \quad w_u = w_{uu} = 0.$$

Using, equation (2.10) we can compute the derivatives of the normal vector

$$\mathbf{N}_u = -e^{-2w}(Q+\bar{Q})\Psi_u = \frac{1}{a\cosh^2 v}(a\cosh v\cos u, a\cosh v\sin u, 0),$$

$$\mathbf{N}_v = e^{-2w}(Q+\bar{Q})\Psi_v = \frac{-1}{a\cosh^2 v}(a\sinh v\sin u, -a\sinh v\cos u, -a).$$

We can also verify that catenoid satisfies the Gauss-Codazzi equations. It is obvious that $Q_{\bar{z}} = 0$, since $H = 0$ and

$$\Delta w - 4Q\bar{Q}e^{-2w} = w_{vv} - 4Q\bar{Q}e^{-2w} = \frac{1}{\cosh^2 v} - \frac{4a^2}{4}e^{-2w} = \frac{1}{\cosh^2 v} - \frac{1}{\cosh^2 v} = 0.$$

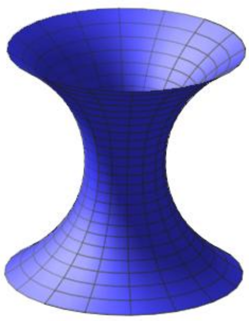

Figure 2.2: Catenoid.

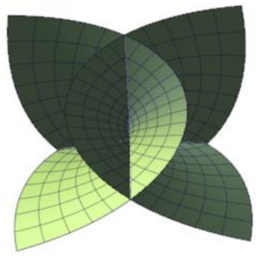

Figure 2.3: Enneper surface.

3
Affine Differential Geometry

In affine differential geometry we study properties of surfaces in 3-dimensional space that are invariant under affine transformations. See [6] and [7] for more details.

3.1
Berwald-Blaschke metric

The Berwald-Blaschke metric is invariant for Affine transformations and also independent of system of coordinates. This metric is a quadratic form. We shall see that this quadratic form might not be positive definite (non-convex case).
The Berwald-Blaschke metric is given by

$$h = \frac{Ldu^2 + 2Mdudv + Ndv^2}{|LN - M^2|^{1/4}},$$

where L, M and N are given by

$$L = [\Psi_u, \Psi_v, \Psi_{uu}], \quad M = [\Psi_u, \Psi_v, \Psi_{uv}], \quad N = [\Psi_u, \Psi_v, \Psi_{vv}].$$

From now on, we shall assume that the surface is non-degenerate, that is, $LN - M^2 \neq 0$.

3.1.1
Relation between the first fundamental affine form and the coefficients of the second euclidean fundamental form

There is a relation between the first fundamental affine form and the coefficients l_{ij} of the second euclidean fundamental form. In fact,

$$l_{ij} = \mathbf{N} \cdot \Psi_{ij} = \left(\frac{\Psi_u \wedge \Psi_v}{\|\Psi_u \wedge \Psi_v\|} \right) \cdot \Psi_{ij} = \frac{[\Psi_u, \Psi_v, \Psi_{ij}]}{\|\Psi_u \wedge \Psi_v\|},$$

where $\mathbf{N} = \dfrac{\Psi_u \wedge \Psi_v}{\|\Psi_u \wedge \Psi_v\|}$ is the euclidean normal vector.

The euclidean Gaussian curvature can be obtained as follows:

$$K = \frac{det(l_{ij})}{\|\Psi_u \wedge \Psi_v\|^4}. \tag{3.1}$$

1. $K < 0 \Longleftrightarrow LN - M^2 < 0,$
2. $K = 0 \Longleftrightarrow LN - M^2 = 0,$
3. $K > 0 \Longleftrightarrow LN - M^2 > 0.$

Points where $LN - M^2$ are negative, zero or positive are called, respectively, hyperbolic, parabolic or elliptical. Since, by hypothesis $LN - M^2 \neq 0$, we are working only in convex case (elliptical points) and non-convex case (hyperbolic points).

3.2 Affine normal and conormal maps

Definition 3.2.1 (Affine conormal field) *Let S be a regular surface with non-degenerate points and let $\Psi : \Omega \subset \mathbb{R}^2 \to S \subset \mathbb{R}^3$ be a parametrization, we define the affine conormal field, given by the expression*

$$\nu = \frac{\Psi_u \wedge \Psi_v}{|LN - M^2|^{1/4}}, \tag{3.2}$$

where L, M and N are the coefficients of the first fundamental affine form. Note also that it is uniquely determined up to sign.

By definition, we can see that $\nu \cdot d\Psi = 0$. Let $\rho^4 = \pm(LN - M^2)$, where the signal $\pm$ depends on the point being elliptical or hyperbolic. Using this notation we have

$$\nu = \frac{\Psi_u \wedge \Psi_v}{\rho}. \tag{3.3}$$

Definition 3.2.2 (Affine normal vector) *We define the affine normal vector by the following equations*

$$\begin{cases} \nu \cdot \xi = 1 \\ \nu_u \cdot \xi = 0 \\ \nu_v \cdot \xi = 0. \end{cases} \tag{3.4}$$

Observe that, the affine normal vector does not belong to the tangent plane to the surface S.

Differentiating the first equation in (3.4), we can easily verify that $\nu \cdot \xi_u = 0$ and $\nu \cdot \xi_v = 0$. Hence, there exists a function $\delta : \Omega \to \mathbb{R}$ such that

$$\xi = \delta(\nu_u \wedge \nu_v).$$

Now, we can compute the inner product of the affine normal and conormal to obtain the function δ as follows

$$1 = \nu \cdot \xi = \delta[\nu, \nu_u, \nu_v].$$

Hence, $\delta = \dfrac{1}{[\nu, \nu_u, \nu_v]}$, i.e., the affine normal vector is

$$\xi = \frac{1}{[\nu, \nu_u, \nu_v]} \nu_u \wedge \nu_v. \tag{3.5}$$

Note also that the affine normal vector is uniquely determined up to sign. For each point $x \in \Omega$ we take the line through x in the direction of the affine normal vector $\xi(x)$. This line, which is independent of the choice of sign for ξ, is called the affine normal through x.

Example 3.2.1 *The affine normal vector is constant in the elliptic and hyperbolic paraboloids.*

Considering the parametrization $X(u,v) = \left(u, v, \frac{1}{2}(u^2+v^2)\right)$ of the elliptical paraboloid. Notice that:

$$\begin{aligned} X_u &= (1,0,u), & X_{uu} &= (0,0,1), & X_{vv} &= (0,0,1), \\ X_v &= (0,1,v), & X_{uv} &= (0,0,0). \end{aligned}$$

Thus, we have to,

$$\rho = [X_u, X_v, X_{uu}][X_u, X_v, X_{vv}] - [X_u, X_v, X_{uv}]^2 = 1.$$

Hence,

$$\nu = \frac{X_u \wedge X_v}{\rho} = (-u, -v, 1).$$

Therefore, $\nu_u = (-1, 0, 0)$ and $\nu_v = (0, -1, 0)$, then

$$\xi = \frac{1}{[\nu, \nu_u, \nu_v]} (\nu_u \wedge \nu_v) = (0,0,1). \tag{3.6}$$

In analogous way, considering the parametrization of the hyperbolic paraboloid as $S(u,v) = \left(u, v, \frac{1}{2}(u^2 - v^2)\right)$, it is shown that the affine normal at any point of this surface is the vector $(0,0,1)$.

3.2.1 Affine curvatures

The curvatures describe the variation of the normal vector. We saw that $\nu \cdot \xi_u = \nu \cdot \xi_v = 0$. That is, the derivatives ξ_u and ξ_v are orthogonal to ν. In

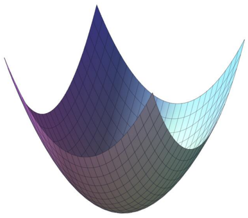

Figure 3.1: Elliptic paraboloid.

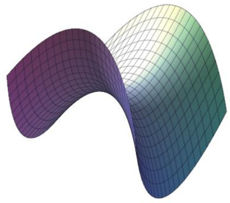

Figure 3.2: Hyperbolic paraboloid.

particular ξ_u and $\xi_v \in T_pS$. Therefore, we can define the Shape Operator S as follows $S : T_pS \to T_pS$ given by $S_p(v) = -D_v\xi$.
Since ξ_u and ξ_v are tangents to the surface we have that there are functions $b_{ij} : \Omega \to \mathbb{R}, \quad i, j = 1, 2$, such that

$$\xi_u = b_{11}\Psi_u + b_{21}\Psi_v,$$

$$\xi_v = b_{12}\Psi_u + b_{22}\Psi_v.$$

Therefore,

$$D\xi(\alpha') = (b_{11}u' + b_{12}v')\Psi_u + (b_{21}u' + b_{22}v')\Psi_v,$$

hence,

$$D\xi \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix}$$

This shows that in the basis $\{\Psi_u, \Psi_v\}$, the Shape Operator $S_p(v) = D_v\xi$ is given by the matrix $B = (b_{ij})$, $i, j = 1, 2$. Notice that this matrix is not necessarily symmetric.

Definition 3.2.3 *The coefficients b_{ij} form a matrix $B = (b_{ij})$, whose determinant and the half of the trace are respectively the Gaussian and Mean affine curvatures. Hence,*

$$\mathcal{K} = \det B = b_{11}b_{22} - b_{12}b_{21},$$

$$\mathcal{H} = \frac{1}{2}trB = \frac{1}{2}(b_{11} + b_{22}).$$

3.3 Isothermal coordinates

In affine differential geometry, isothermal coordinates means that the metric locally has the form

$$h = \rho(du^2 + dv^2).$$

where ρ is a smooth function. In this case, we are considering a definite metric (convex case).
Hence, we are studying coordinates with the following property.

$$[\Psi_u, \Psi_v, \Psi_{uu}] = [\Psi_u, \Psi_v, \Psi_{vv}] = \rho^2 \qquad \text{and} \quad [\Psi_u, \Psi_v, \Psi_{uv}] = 0. \tag{3.7}$$

Now, we state and prove an important result in affine differential geometry.

Theorem 3.3.1 (Lelieuvre Formula) *In locally convex surfaces with isothermal coordinates we have that*

$$\Psi_u = \nu \wedge \nu_v \qquad \textit{and} \qquad \Psi_v = -\nu \wedge \nu_u. \tag{3.8}$$

Proof. From (3.2), we have that

$$\nu = \frac{\Psi_u \wedge \Psi_v}{|LN - M^2|^{1/4}} = \frac{\Psi_u \wedge \Psi_v}{|L|^{1/2}} = \frac{\Psi_u \wedge \Psi_v}{\rho},$$

since $L = N$ and $M = 0$. The derivative of the conormal vector with respect to v is

$$\nu_v = \frac{\rho[(\Psi_{uv} \wedge \Psi_v) + (\Psi_u \wedge \Psi_{vv})] - (\Psi_u \wedge \Psi_v)\rho_v}{\rho^2}$$

Now, using $(A \wedge B) \wedge (A \wedge C) = [A, B, C]A$. It is a straight-forward computation that

$$\begin{aligned}
\nu \wedge \nu_v &= \left(\frac{1}{\rho}\Psi_u \wedge \Psi_v\right) \wedge \left(\frac{\rho[(\Psi_{uv} \wedge \Psi_v) + (\Psi_u \wedge \Psi_{vv})] - (\Psi_u \wedge \Psi_v)\rho_v}{\rho^2}\right) \\
&= \left(\frac{1}{\rho^2}\Psi_u \wedge \Psi_v\right) \wedge (\Psi_{uv} \wedge \Psi_v) + \left(\frac{1}{\rho^2}\Psi_u \wedge \Psi_v\right) \wedge (\Psi_u \wedge \Psi_{vv}) - \left(\frac{\rho_v}{\rho^3}\Psi_u \wedge \Psi_v\right) \wedge (\Psi_u \wedge \Psi_v) \\
&= \frac{1}{\rho^2}[\Psi_v, \Psi_u, \Psi_{uv}]\Psi_v + \frac{1}{\rho^2}[\Psi_u, \Psi_v, \Psi_{vv}]\Psi_u - \frac{\rho_v}{\rho^3}[\Psi_u, \Psi_v, \Psi_v]\Psi_u \\
&= \frac{1}{\rho^2}[\Psi_u, \Psi_v, \Psi_{vv}]\Psi_u = \frac{1}{\rho^2}\rho^2\Psi_u = \Psi_u.
\end{aligned}$$

In the same way, we can prove that $\Psi_v = -\nu \wedge \nu_u$.

∎

Now, we prove that

$$\rho = [\nu, \nu_u, \nu_v]. \tag{3.9}$$

From Lelieuvre Formula (Theorem 3.3.1) we can obtain the following formula to compute the conormal vector

$$\begin{aligned}
\Psi_u \wedge \Psi_v &= (\nu \wedge \nu_v) \wedge (-\nu \wedge \nu_u) \\
&= -[\nu, \nu_v, \nu_u]\nu \\
&= [\nu, \nu_u, \nu_v]\nu.
\end{aligned}$$

Thus,

$$\frac{\Psi_u \wedge \Psi_v}{[\nu, \nu_u, \nu_v]} = \nu.$$

From (3.3) we have $\nu = \dfrac{1}{\rho}\Psi_u \wedge \Psi_v$. Now we can conclude that

$$\rho = [\nu, \nu_u, \nu_v].$$

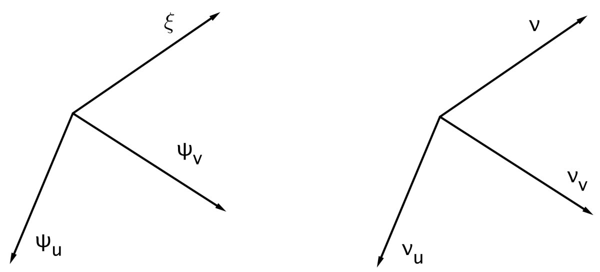

Figure 3.3: Affine Trihedron and its dual.

We also can compute the derivatives of the conormal vector ν_u and ν_v as follows

$$\begin{aligned}
\Psi_v \wedge \xi &= (-\nu \wedge \nu_u) \wedge (\frac{-1}{\rho}\nu_v \wedge \nu_u) \\
&= \frac{1}{\rho}(\nu_u \wedge \nu) \wedge (\nu_u \wedge \nu_v) \\
&= \frac{1}{\rho}[\nu_u, \nu, \nu_v]\nu_u \\
&= -\nu_u.
\end{aligned}$$

Similarly, we also can compute the derivative of the conormal vector ν with respect to v.

$$\begin{aligned}
\Psi_u \wedge \xi &= (\nu \wedge \nu_v) \wedge (\frac{-1}{\rho}\nu_v \wedge \nu_u) \\
&= \frac{1}{\rho}(\nu_v \wedge \nu) \wedge (\nu_v \wedge \nu_u) \\
&= \frac{1}{\rho}[\nu_v, \nu, \nu_u]\nu_v \\
&= \nu_v.
\end{aligned}$$

or equivalently

$$\nu_u = \xi \wedge \Psi_v, \qquad \text{and} \qquad \nu_v = \Psi_u \wedge \xi. \tag{3.10}$$

We can obtain another formula for the affine metric ρ using Lelieuvre Formula and the formula for the affine normal vector

$$
\begin{aligned}
[\Psi_u, \Psi_v, \xi] &= [\nu \wedge \nu_v, -\nu \wedge \nu_u, \frac{1}{\rho}\nu_u \wedge \nu_v] \\
&= -\frac{1}{\rho}((\nu \wedge \nu_v) \wedge (\nu \wedge \nu_u)) \cdot (\nu_u \wedge \nu_v) \\
&= -\frac{1}{\rho}[\nu, \nu_v, \nu_u]\nu \cdot (\nu_u \wedge \nu_v) \\
&= \frac{\rho}{\rho}\nu \cdot (\nu_u \wedge \nu_v) \\
&= [\nu, \nu_u, \nu_v].
\end{aligned}
$$

Thus,

$$\rho = [\nu, \nu_u, \nu_v] = [\Psi_u, \Psi_v, \xi].$$

Now, we find a relation between the conormal vector ν and the mean affine curvature $\mathcal{H}$.

Theorem 3.3.2 *Let $\Psi : \Omega \to \mathbb{R}^3$ be a smooth function with isothermal coordinates. We have that*

$$\nu_{uu} + \nu_{vv} = 2\rho\mathcal{H}\nu. \tag{3.11}$$

where ρ is the affine metric, ν is the conormal vector and $\mathcal{H}$ is the mean affine curvature. In other words $\triangle_h \nu = -\mathcal{H}\nu$, where $\triangle_h \nu := -\dfrac{\nu_{uu} + \nu_{vv}}{2\rho}$.

Proof. From Lelieuvre Formula we have that

$$\Psi_u = \nu \wedge \nu_v, \qquad \Psi_v = -\nu \wedge \nu_u.$$

Then, computing its derivatives with respect to v and u respectively we have

$$\Psi_{uv} = \nu \wedge \nu_{vv}, \qquad \Psi_{vu} = -\nu \wedge \nu_{uu}.$$

Ψ is a smooth function, then $\Psi_{uv} = \Psi_{vu}$. Hence,

$$\nu \wedge \nu_{vv} = -\nu \wedge \nu_{uu}$$

Thus,

$$\nu \wedge (\nu_{uu} + \nu_{vv}) = 0$$

Now, we can conclude that $\nu_{uu} + \nu_{vv} = \alpha\nu$.
From the definition of affine normal vector $\nu \cdot \xi = 1$. Thus,

$$(\nu_{uu} + \nu_{vv}) \cdot \xi = \alpha.$$

Equation (3.4) gives us $\nu_v \cdot \xi = 0$, differentiating with respect to v, we have

$$\nu_{vv} \cdot \xi + \nu_v \cdot \xi_v = 0,$$

also we have that $\nu_u \cdot \xi = 0$, differentiating with respect to u, we have

$$\nu_{uu} \cdot \xi + \nu_u \cdot \xi_u = 0.$$

Now, using Definition (3.2.3) we have that

$$\begin{aligned}
\nu_{uu} \cdot \xi &= -\nu_u \cdot \xi_u \\
&= -\nu_u \cdot (b_{11}\Psi_u + b_{21}\Psi_v) \\
&= -b_{11}\nu_u \cdot (\nu \wedge \nu_v) - b_{21}\nu_u \cdot (-\nu \wedge \nu_u) \\
&= -b_{11}[\nu_u, \nu, \nu_v] + b_{21}[\nu_u, \nu, \nu_u] \\
&= b_{11}\rho.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\nu_{vv} \cdot \xi &= -\nu_v \cdot \xi_v \\
&= -\nu_v \cdot (b_{12}\Psi_u + b_{22}\Psi_v) \\
&= -b_{12}\nu_v \cdot (\nu \wedge \nu_v) - b_{22}\nu_v \cdot (-\nu \wedge \nu_u) \\
&= -b_{12}[\nu_v, \nu, \nu_v] + b_{22}[\nu_v, \nu, \nu_u] \\
&= b_{22}\rho.
\end{aligned}$$

Then,

$$(\nu_{uu} + \nu_{vv}) \cdot \xi = \nu_{uu} \cdot \xi + \nu_{vv} \cdot \xi = (b_{11} + b_{22})\rho = 2\rho\mathcal{H}$$

Hence, we can conclude that

$$\nu_{uu} + \nu_{vv} = 2\rho\mathcal{H}\nu.$$

Furthermore,

$$\triangle_h \nu = -\mathcal{H}\nu,$$

since $\triangle_h \nu := -\dfrac{\nu_{uu} + \nu_{vv}}{2\rho}$.

Thus, we obtained a relationship between the mean affine curvature $\mathcal{H}$ and the Laplacian of the affine conormal vector.

∎

Furthermore, from Theorem (3.3.1), we know that $\Psi_u = \nu \wedge \nu_v$ differentiating with respect to u and $\Psi_v = -\nu \wedge \nu_u$ differentiating with respect to v, we obtain

$$\Psi_{uu} = (\nu_u \wedge \nu_v) + (\nu \wedge \nu_{vu}),$$

and

$$\Psi_{vv} = (-\nu_v \wedge \nu_u) - (\nu \wedge \nu_{uv}).$$

Hence,

$$\Psi_{uu} + \Psi_{vv} = 2\nu_u \wedge \nu_v = 2\rho\frac{1}{\rho}\nu_u \wedge \nu_v = 2\rho\xi.$$

Thus, we proved that

$$\Psi_{uu} + \Psi_{vv} = 2\rho\xi.$$

From Theorem (3.3.2), we have that $\nu_{uu} \cdot \xi = -\nu_u \cdot \xi_u = b_{11}\rho$ and $\nu_{vv} \cdot \xi = -\nu.\xi_v = b_{22}\rho$. Using the definition of affine normal vector (3.4), $\nu_v \cdot \xi = 0$, differentiating with respect to u, we have

$$\nu_{vu} \cdot \xi + \nu_v \cdot \xi_u = 0.$$

Hence,

$$\begin{aligned}
\nu_{vu} \cdot \xi &= -\nu_v \cdot \xi_u \\
&= -\nu_v \cdot (b_{11}\Psi_u + b_{21}\Psi_v) \\
&= -b_{11}\nu_v \cdot (\nu \wedge \nu_v) - b_{21}\nu_v \cdot (-\nu \wedge \nu_u) \\
&= -b_{11}[\nu_v, \nu, \nu_v] + b_{21}[\nu_v, \nu, \nu_u] \\
&= b_{21}\rho.
\end{aligned}$$

Note also from Definition (3.4), we have that $\nu_u \cdot \xi = 0$, differentiating with respect to v, we have

$$\nu_{uv} \cdot \xi + \nu_u \cdot \xi_v = 0.$$

Thus,

$$\begin{aligned}
\nu_{uv} \cdot \xi &= -\nu_u \cdot \xi_v \\
&= -\nu_u \cdot (b_{12}\Psi_u + b_{22}\Psi_v) \\
&= -b_{12}\nu_u \cdot (\nu \wedge \nu_v) - b_{22}\nu_u \cdot (-\nu \wedge \nu_u) \\
&= -b_{12}[\nu_u, \nu, \nu_v] + b_{22}[\nu_u, \nu, \nu_u] \\
&= b_{12}\rho.
\end{aligned}$$

Hence, in the convex case we have that $b_{12} = b_{21}$.

Now, we will find other formulas to compute the coefficients of the matrix B. Recall from (3.10) we have that, $\nu_u = \xi \wedge \Psi_v$ and $\nu_v = \Psi_u \wedge \xi$ and every entry (b_{ij}) of the matrix B is given by the following formula

$$\begin{aligned}
b_{11} &= -\frac{1}{\rho}\nu_u \cdot \xi_u = -\frac{1}{\rho}[\xi, \Psi_v, \xi_u] = \frac{1}{\rho}[\xi_u, \Psi_v, \xi] \\
b_{12} &= -\frac{1}{\rho}\nu_u \cdot \xi_v = -\frac{1}{\rho}[\xi, \Psi_v, \xi_v] = \frac{1}{\rho}[\xi_v, \Psi_v, \xi] \\
b_{21} &= -\frac{1}{\rho}\nu_v \cdot \xi_u = -\frac{1}{\rho}[\Psi_u, \xi, \xi_u] = \frac{1}{\rho}[\Psi_u, \xi_u, \xi] \\
b_{22} &= -\frac{1}{\rho}\nu_v \cdot \xi_v = -\frac{1}{\rho}[\Psi_u, \xi, \xi_v] = \frac{1}{\rho}[\Psi_u, \xi_v, \xi].
\end{aligned}$$

Hence, we have calculated the coefficients of the Shape Operator.

Recall that $\nu \cdot \xi = 1$ and $\nu \cdot \xi_u = \nu \cdot \xi_v = 0$. Hence ξ_u and $\xi_v \in T_pS$. To see that $B = D_v\xi$ is self-adjoint, it is sufficient to prove that

$$\xi_u \cdot \nu_v = \xi_v \cdot \nu_u.$$

We know that $\nu \cdot \xi_u = 0$ and $\nu \cdot \xi_v = 0$, then differentiating both equations with respect to v and u respectively, we have

$$\nu_v \cdot \xi_u + \nu \cdot \xi_{vu} = 0 \qquad \text{and} \qquad \nu_u \cdot \xi_v + \nu \cdot \xi_{uv} = 0.$$

Thus,

$$\nu_v \cdot \xi_u = -\nu \cdot \xi_{uv} = \nu_u \cdot \xi_v.$$

Hence, we can conclude that the matrix $B = D_v\xi$ is self-adjoint.

3.4 Asymptotic coordinates

For non-convex case, we can take asymptotic parameters (u, v), i.e.,

$$h = 2\rho du dv,$$

where ρ is a smooth function. Hence, we are studying coordinates with the following property.

$$[\Psi_u, \Psi_v, \Psi_{uu}] = [\Psi_u, \Psi_v, \Psi_{vv}] = 0, \qquad [\Psi_u, \Psi_v, \Psi_{uv}] = \rho^2. \tag{3.12}$$

Theorem 3.4.1 (Lelieuvre Formula) *In locally hyperbolic surfaces with asymptotic coordinates, we have that*

$$\Psi_u = \nu \wedge \nu_u \qquad and \qquad \Psi_v = -\nu \wedge \nu_v. \tag{3.13}$$

Proof. From (3.2), we have that

$$\nu = \frac{\Psi_u \wedge \Psi_v}{|LN - M^2|^{1/4}} = \frac{\Psi_u \wedge \Psi_v}{|M|^{1/2}} = \frac{\Psi_u \wedge \Psi_v}{\rho},$$

since $L = N = 0$ and $M = \rho^2$. The derivative of the conormal vector respect to v is

$$\nu_v = \frac{\rho[(\Psi_{uv} \wedge \Psi_v) + (\Psi_u \wedge \Psi_{vv})] - (\Psi_u \wedge \Psi_v)\rho_v}{\rho^2}.$$

Now, using $(A \wedge B) \wedge (A \wedge C) = [A, B, C]A$. It is a straight-forward computation that

$$\begin{aligned}
\nu \wedge \nu_v &= \left(\frac{1}{\rho}\Psi_u \wedge \Psi_v\right) \wedge \left(\frac{\rho[(\Psi_{uv} \wedge \Psi_v) + (\Psi_u \wedge \Psi_{vv})] - (\Psi_u \wedge \Psi_v)\rho_v}{\rho^2}\right) \\
&= \left[\left(\frac{1}{\rho^2}\Psi_u \wedge \Psi_v\right) \wedge (\Psi_{uv} \wedge \Psi_v)\right] + \left[\left(\frac{1}{\rho^2}\Psi_u \wedge \Psi_v\right) \wedge (\Psi_u \wedge \Psi_{vv})\right] - \left[\left(\frac{\rho_v}{\rho^3}\Psi_u \wedge \Psi_v\right) \wedge (\Psi_u \wedge \Psi_v)\right] \\
&= \frac{1}{\rho^2}[\Psi_v, \Psi_u, \Psi_{uv}]\Psi_v + \frac{1}{\rho^2}[\Psi_u, \Psi_v, \Psi_{vv}]\Psi_u - \frac{\rho_v}{\rho^3}[\Psi_u, \Psi_v, \Psi_v]\Psi_u \\
&= \frac{1}{\rho^2}[\Psi_v, \Psi_u, \Psi_{uv}]\Psi_v \\
&= -\frac{1}{\rho^2}\rho^2\Psi_v \\
&= -\Psi_v.
\end{aligned}$$

In the same way, we can prove that $\Psi_u = \nu \wedge \nu_u$.

■

Now, we shall prove that

$$\rho = [\nu, \nu_v, \nu_u]. \tag{3.14}$$

From Lelieuvre Formula (Theorem 3.4.1) it follows that

$$\begin{aligned}
\Psi_u \wedge \Psi_v &= (\nu \wedge \nu_u) \wedge (-\nu \wedge \nu_v) \\
&= -[\nu, \nu_u, \nu_v]\nu
\end{aligned}$$

Thus,

$$-\frac{\Psi_u \wedge \Psi_v}{[\nu, \nu_u, \nu_v]} = \nu,$$

Now using (3.3), we have $\nu = \dfrac{1}{\rho}\Psi_u \wedge \Psi_v$, so

$$\rho = [\nu, \nu_v, \nu_u].$$

Hence, we have been obtained the affine normal and conormal vectors.

$$\xi = \frac{1}{\rho}\nu_v \wedge \nu_u \qquad \text{and} \qquad \nu = \frac{1}{\rho}\Psi_u \wedge \Psi_v.$$

We also can compute the derivatives of the conormal vector ν as follows

$$\begin{aligned}\Psi_v \wedge \xi &= (-\nu \wedge \nu_v) \wedge (\frac{1}{\rho}\nu_v \wedge \nu_u)\\ &= \frac{1}{\rho}(\nu_v \wedge \nu) \wedge (\nu_v \wedge \nu_u)\\ &= \frac{1}{\rho}[\nu_v, \nu, \nu_u]\nu_v\\ &= -\nu_v.\end{aligned}$$

In analogous way, we can compute the derivative of the conormal vector ν with respect to v.

$$\begin{aligned}\Psi_u \wedge \xi &= (\nu \wedge \nu_u) \wedge (\frac{1}{\rho}\nu_v \wedge \nu_u)\\ &= \frac{1}{\rho}(\nu_u \wedge \nu) \wedge (\nu_u \wedge \nu_v)\\ &= \frac{1}{\rho}[\nu_u, \nu, \nu_v]\nu_u\\ &= \nu_u.\end{aligned}$$

Hence, we can conclude that,

$$\nu_v = \xi \wedge \Psi_v \qquad \text{and} \qquad \nu_u = \Psi_u \wedge \xi. \tag{3.15}$$

We can get another formula for the affine metric ρ using Lelieuvre Formula and the formula for the affine normal vector

$$\begin{aligned}[\Psi_u, \Psi_v, \xi] &= \left[\nu \wedge \nu_u, -\nu \wedge \nu_v, \frac{1}{\rho}\nu_v \wedge \nu_u\right]\\ &= -\frac{1}{\rho}((\nu \wedge \nu_u) \wedge (\nu \wedge \nu_v)) \cdot (\nu_v \wedge \nu_u)\\ &= -\frac{1}{\rho}[\nu, \nu_u, \nu_v]\nu \cdot (\nu_v \wedge \nu_u) = \frac{\rho}{\rho}\nu \cdot (\nu_v \wedge \nu_u) = [\nu, \nu_v, \nu_u].\end{aligned}$$

Thus,

$$\rho = [\nu, \nu_v, \nu_u] = [\Psi_u, \Psi_v, \xi].$$

Now we can find a relation between the conormal vector ν and the mean affine curvature in the indefinite case.

Theorem 3.4.2 *Let $\Psi : \Omega \to \mathbb{R}^3$ be a smooth function with asymptotic parameters. And let ν the affine conormal vector. Let $\mathcal{H}$ the mean affine curvature. We have that*

$$\nu_{uv} = \rho\mathcal{H}\nu. \tag{3.16}$$

where $\rho = [\nu, \nu_v, \nu_u] > 0$ is the affine metric. In other words, $\triangle_h \nu = -2\mathcal{H}\nu$, where $\triangle_h \nu := -\frac{\nu_{uv}}{2\rho}$.

Proof. From Lelieuvre Formula we have that

$$\Psi_u = \nu \wedge \nu_u, \qquad \Psi_v = -\nu \wedge \nu_v.$$

Then, computing its derivatives respect to v and u respectively we have

$$\Psi_{uv} = \nu_v \wedge \nu_u + \nu \wedge \nu_{uv}, \qquad \Psi_{vu} = -\nu_u \wedge \nu_v - \nu \wedge \nu_{vu}.$$

Ψ is a smooth function, then $\Psi_{uv} = \Psi_{vu}$. Hence,

$$\nu \wedge \nu_{uv} = -\nu \wedge \nu_{vu}.$$

Thus,

$$\nu \wedge \nu_{uv} = 0.$$

Now, we can conclude that $\nu_{uv} = \alpha\nu$.

Using the above equation and the definition of affine normal vector $\nu \cdot \xi = 1$. We have that,

$$\nu_{uv} \cdot \xi = \alpha.$$

From (3.4), we know that $\nu_v \cdot \xi = 0$. Differentiating with respect to u, we have

$$\nu_{vu} \cdot \xi + \nu_v \cdot \xi_u = 0,$$

also we have that $\nu_u \cdot \xi = 0$, differentiating with respect to v, we have

$$\nu_{uv} \cdot \xi + \nu_u \cdot \xi_v = 0.$$

Using Definition 3.2.3 we have

$$\begin{aligned}
\nu_{vu} \cdot \xi &= -\nu_v \cdot \xi_u \\
&= -\nu_v \cdot (b_{11}\Psi_u + b_{21}\Psi_v) \\
&= -b_{11}\nu_v \cdot (\nu \wedge \nu_u) - b_{21}\nu_v \cdot (-\nu \wedge \nu_v) \\
&= -b_{11}[\nu_v, \nu, \nu_u] + b_{21}[\nu_v, \nu, \nu_v] \\
&= b_{11}\rho.
\end{aligned}$$

Similarly,

$$\begin{aligned}\nu_{uv}\cdot\xi &= -\nu_u\cdot\xi_v\\ &= -\nu_u\cdot(b_{12}\Psi_u + b_{22}\Psi_v)\\ &= -b_{12}\nu_u\cdot(\nu\wedge\nu_u) - b_{22}\nu_u\cdot(-\nu\wedge\nu_v)\\ &= -b_{12}[\nu_u,\nu,\nu_u] + b_{22}[\nu_u,\nu,\nu_v]\\ &= b_{22}\rho.\end{aligned}$$

From those two equations, we can conclude that $b_{11} = b_{22}$ when we are working in the non-convex case.
To prove that the Shape operator S is self-adjoint, it is enough to show that

$$\xi_u\cdot\nu_v = \nu_u\cdot\xi_v.$$

It is obvious, since $b_{11} = b_{22}$, where $b_{11} = -\nu_v\cdot\xi_u$ and $b_{22} = -\nu_u\cdot\xi_v$.

Therefore,

$$2\nu_{uv}\cdot\xi = (\nu_{uv}+\nu_{vu})\cdot\xi = \nu_{uv}\cdot\xi + \nu_{vu}\cdot\xi = (b_{11}+b_{22})\rho = 2\rho\mathcal{H}$$

Hence, we can conclude that

$$\nu_{uv} = \rho\mathcal{H}\nu.$$

Furthermore,

$$\triangle_h\nu = -2\mathcal{H}\nu,$$

since $\triangle_h\nu := -\dfrac{\nu_{uv}}{2\rho}$.
Thus, we have a relationship between the mean affine curvature $\mathcal{H}$ and the Laplacian of the affine conormal vector.

■

Now we want to compute the rest of the coefficients of the matrix B. From Theorem 3.4.2, we have that $\nu_{vu}\cdot\xi = -\nu_v\cdot\xi_u = b_{11}\rho$ and $\nu_{uv}\cdot\xi = -\nu_u\cdot\xi_v = b_{22}\rho$.

From the definition of affine normal vector (3.4), $\nu_v\cdot\xi = 0$, differentiating with respect to v, we have

$$\nu_{vv}\cdot\xi + \nu_v\cdot\xi_v = 0.$$

Hence,

$$\begin{aligned}
\nu_{vv} \cdot \xi &= -\nu_v \cdot \xi_v \\
&= -\nu_v \cdot (b_{12}\Psi_u + b_{22}\Psi_v) \\
&= -b_{12}\nu_v \cdot (\nu \wedge \nu_u) - b_{22}\nu_v \cdot (-\nu \wedge \nu_v) \\
&= -b_{12}[\nu_v, \nu, \nu_u] + b_{22}[\nu_v, \nu, \nu_v] \\
&= b_{12}\rho.
\end{aligned}$$

From Definition (3.4), we have that $\nu_u \cdot \xi = 0$, differentiating with respect to u, we have

$$\nu_{uu} \cdot \xi + \nu_u \cdot \xi_u = 0.$$

Thus,

$$\begin{aligned}
\nu_{uu} \cdot \xi &= -\nu_u \cdot \xi_u \\
&= -\nu_u \cdot (b_{11}\Psi_u + b_{21}\Psi_v) \\
&= -b_{11}\nu_u \cdot (\nu \wedge \nu_u) - b_{21}\nu_u \cdot (-\nu \wedge \nu_v) \\
&= -b_{11}[\nu_u, \nu, \nu_u] + b_{21}[\nu_u, \nu, \nu_v] \\
&= b_{21}\rho.
\end{aligned}$$

The following equations give us other formula to compute the coefficients of the Shape Operator.
From (3.15). We have that, $\nu_v = \xi \wedge \Psi_v$ and $\nu_u = \Psi_u \wedge \xi$.

$$\begin{aligned}
b_{11} &= -\frac{1}{\rho}\nu_v \cdot \xi_u = -\frac{1}{\rho}[\xi, \Psi_v, \xi_u] = \frac{1}{\rho}[\xi_u, \Psi_v, \xi] \\
b_{12} &= -\frac{1}{\rho}\nu_v \cdot \xi_v = -\frac{1}{\rho}[\xi, \Psi_v, \xi_v] = \frac{1}{\rho}[\xi_v, \Psi_v, \xi] \\
b_{21} &= -\frac{1}{\rho}\nu_u \cdot \xi_u = -\frac{1}{\rho}[\Psi_u, \xi, \xi_u] = \frac{1}{\rho}[\Psi_u, \xi_u, \xi] \\
b_{22} &= -\frac{1}{\rho}\nu_u \cdot \xi_v = -\frac{1}{\rho}[\Psi_u, \xi, \xi_v] = \frac{1}{\rho}[\Psi_u, \xi_v, \xi].
\end{aligned}$$

Finally, we calculated the coefficients of the matrix B.

Furthermore, from Theorem 3.4.2, we know that $\nu \parallel \nu_{uv}$. Where by $\parallel$ we indicate parallel to. Hence, from Lelieuvre Formula, we know that $\Psi_u = \nu \wedge \nu_u$, differentiating the above equation with respect to v, we have

$$\Psi_{uv} = (\nu_v \wedge \nu_u) + (\nu \wedge \nu_{uv}) = \nu_v \wedge \nu_u = \rho\frac{1}{\rho}\nu_v \wedge \nu_u = \rho\xi.$$

Thus, we obtained the next formula

$$\Psi_{uv} = \rho\xi.$$

3.5
Improper affine spheres and affine maximal maps

In this section, we introduce the notion of affine minimal surface and affine minimal maps. We also define an improper affine sphere.

3.5.1
Affine maximal maps

Surfaces whose affine mean curvature $\mathcal{H}$ is identically zero are called affine-minimal surfaces, because the equation $\mathcal{H} = 0$ is the Euler-Lagrange equation characterizing surfaces whose affine area is stationary (critical) with respect to smooth deformations with compact interior support. In the convex case, we prefer to designate them as maximal, rather than minimal (as in the case of euclidean geometry), because the second variation of the affine area for such surfaces is always negative. E. Calabi was the first person who used this notation in [11].

In other words, we study surfaces with zero affine mean curvature that are called affine minimal surfaces and for convex surfaces, they are also called affine maximal surfaces.

Definition 3.5.1 *Surfaces with vanishing affine mean curvature are called affine minimal surfaces.*

Definition 3.5.2 *An immersion $\Psi : \Omega \to \mathbb{R}^3$ with constant affine normal ξ is called an improper affine sphere.*

Remark 1 *An improper affine sphere is an affine minimal surface.*

Remark 2 *Minimal affine surfaces are equivalent to ν is harmonic. In fact, in the convex case from Theorem (3.3.2) we have $\nu_{uu} + \nu_{vv} = 0$ if and only if $\mathcal{H} = 0$.*
In the indefinite case from Theorem (3.4.2) we also have $\nu_{uv} = 0$ if and only if $\mathcal{H} = 0$.

Conversely, assume that the conormal affine vector $\nu : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ is harmonic and $Im(\nu) \subset$ plane, we shall prove that Ψ is an improper affine sphere.

Without loss of generality we can suppose that the plane is $z = 1$. If $Im(\nu) \subset$ plane, then the paramatrization for the conormal vector is $\nu(u,v) = (v_1(u,v), v_2(u,v), 1)$, we can easily verify that the normal vector $\xi = (0,0,1)$ satisfies the equation (3.4). Note that (3.4) is equivalent to following system of equations

$$\begin{cases} \nu \cdot \xi = 1, \\ \nu \cdot \xi_u = 0, \\ \nu \cdot \xi_v = 0. \end{cases}$$

Thus, we can conclude that ξ is constant. It means that Ψ is an improper affine sphere.

Definition 3.5.3 *Let Ω be a regular surface. We say that a map $\Psi : \Omega \to \mathbb{R}^3$ is an affine maximal map if there exists a harmonic vector field ν such that $[\nu, \nu_u, \nu_v] \neq 0$ at $\Omega \backslash S_\Psi$, where S_Ψ is the set of singular curves and Ψ is given as in (3.13). In other words, Ψ is an affine minimal map if it is an affine minimal surface and it admits certain singularities at points where $[\nu, \nu_u, \nu_v] = 0$.*

The singular set S_Ψ of an affine maximal map Ψ is the set of points where $\rho = [\nu, \nu_u, \nu_v]$ vanishes.

Observe that in this case the affine normal ξ may not be well-defined on S_Ψ. Of course, when $S_\Psi = \emptyset$ we have an affine minimal surface, which is an improper affine sphere if ξ is constant.

4
Euclidean and Affine Minimal Surfaces

In this chapter, we shall see when an euclidean minimal surface is at the same time euclidean and affine minimal surface. The precise relation between them turns out to be quite subtle, so we begin by obtaining and solving a system of a partial differential equation describing the metric function of the euclidean minimal surface. Later, we obtain a single-parameter deformation of all minimal surfaces preserving the planar curvature line condition. Finally, we state the classification, parametrization, and deformation of all minimal surfaces with planar curvature lines. See [8] and [10] for more details.

4.1
Minimal surfaces with planar curvature lines

In this section, we shall use the notation of the Chapter 2 we attain an additional partial differential equation regarding w from the planar curvature line condition, allowing us to solve for w. See [8].

Lemma 4.1.1 *For non-planar umbilic-free minimal surfaces with isothermal coordinates (u, v) that are curvature lines, the following statements are equivalent*

1. *u-curvature lines are planar,*

2. *v-curvature lines are planar,*

3. *$w_{uv} + w_u w_v = 0$.*

Proof. The u-curvature lines are planar if and only if

$$[X_u, X_{uu}, X_{uuu}] = 0.$$

However, from (2.9),

$$X_{uuu} = (w_{uu} + w_u^2 - w_v^2 - e^{-2w})X_u + (-2w_u w_v - w_{uv})X_v - w_u N.$$

Therefore, a u-curvature line is planar if and only if

$$\begin{aligned}
[X_u, X_{uu}, X_{uuu}] &= [X_u, w_u X_u - w_v X_v - N, (-2w_u w_v - w_{uv})X_v - w_u N] \\
&= [X_u, -w_v X_v - N, (-2w_u w_v - w_{uv})X_v - w_u N] \\
&= [X_u, -w_v X_v, (-2w_u w_v - w_{uv})X_v - w_u N]+ \\
&\quad + [X_u, -N, (-2w_u w_v - w_{uv})X_v - w_u N] \\
&= [X_u, -w_v X_v, -w_u N] - [X_u, N, (-2w_u w_v - w_{uv})X_v] \\
&= w_u w_v [X_u, X_v, N] - [N, (-2w_u w_v - w_{uv})X_v, X_u] \\
&= w_u w_v e^{-2w} + (-2w_u w_v - w_{uv})[N, X_u, X_v] \\
&= -e^{-2w}(w_u w_v + w_{uv}) = 0.
\end{aligned}$$

Similarly, (2.9) implies that a v-curvature line is planar if and only if $w_{uv} + w_u w_v = 0$.

■

Example 4.1.1 *The Enneper surface (Example 2.3.1) satisfies the planar curvature line condition.*

It is easy to see that $w_{uv} + w_u w_v = 0$, since $w = \log(1 + u^2 + v^2)$. Where its derivatives are

$$w_{uv} = \frac{-4uv}{(1 + u^2 + v^2)^2} = -w_u w_v.$$

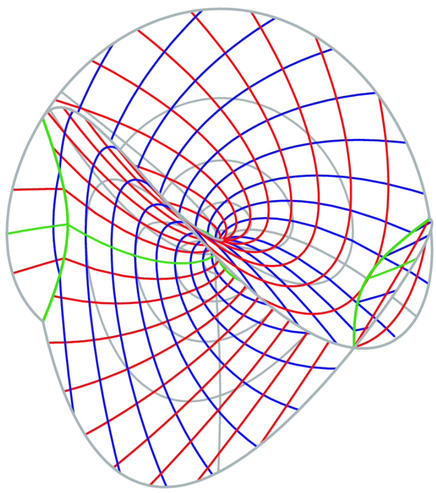

Figure 4.1: Curvature lines on the Enneper surface [14].

Example 4.1.2 *We can also see that the catenoid (Example 2.3.2) satisfies the planar curvature line condition.*

In fact, recall that $w = \log(a \cosh v)$. Its derivatives are

$$w_u = 0, \quad w_v = \frac{\sinh v}{\cosh v} \quad \text{and} \quad w_{uv} = 0.$$

Thus, it is obvious that $w_{uv} + w_u w_v = 0$.

4.2 Relationship betweeen affine and euclidean minimal surfaces

In this section, we shall prove that the conjugate of an euclidean minimal surface with planar curvature lines is an affine minimal surface. For this, we need first to introduce the notion of conjugate surface. See [8] for more details.

4.2.1 Conjugate minimal surfaces

A minimal surface is related to another minimal surface, known as its conjugate minimal surface, in an interesting and important way. Recall from complex analysis, that if $f(u,v) = f^1(x,y) + i f^2(x,y)$ is an analytic function, then the Cauchy-Riemann equations hold for f. That is,

$$f^1_u = f^2_v \qquad \text{and} \qquad f^1_v = -f^2_u.$$

In such a case, f^2 is called the harmonic conjugate of f^1. Also, if f is analytic, then

$$f'(z) = \frac{1}{2}(f^1_u + i f^2_u).$$

Definition 4.2.1 *Let X and Y be isothermal parametrizations of minimal surfaces such that their component functions are pairwise harmonic conjugates. That is,*

$$X_u = Y_v \qquad \textit{and} \qquad X_v = -Y_u. \tag{4.1}$$

In such a case, X and Y are called conjugate minimal surfaces.

Any two conjugate minimal surfaces can be joined through a one-parameter family of minimal surfaces by the equation

$$Z = (\cos t)X + (\sin t)Y,$$

where $t \in \mathbb{R}^3$. Note that when $t = 0$ we have the minimal surface parametrized by X, and when $t = \frac{\pi}{2}$ we have the minimal surface parametrized by Y. So for $0 \leq t \leq \dfrac{\pi}{2}$, we have a continuous parameter of minimal surfaces known as **associated surfaces**.

Theorem 4.2.1 *Let X be an euclidean minimal surface. Then the curvature lines of X corresponds to the asymptotic lines of its conjugate surface Y.*

Proof. We know that a line of curvature satisfies the next equation

$$(fE - eF)(u')^2 + (gE - eG)u'v' + (gF - fG)(v')^2 = 0. \tag{4.2}$$

Since $F = 0, \quad E = G$ and $g = -e$ the above equation becomes

$$fE(u')^2 + 2gEu'v' - fE(v')^2 = 0.$$

It is equivalently to,

$$f(u')^2 + 2gu'v' - f(v')^2 = 0. \tag{4.3}$$

Now, as we have seen in Chapter 3, a curve $X \circ \gamma$ is asymptotic if and only if

$$L\dot{u}^2 + 2M\dot{u}\dot{v} + N\dot{v}^2 = 0,$$

where $L = [X_u, X_v, X_{uu}], \quad M = [X_u, X_v, X_{uv}], \quad N = [X_u, X_v, X_{vv}]$ and

$$l_{ij} = \langle \mathbf{N}, X_{ij} \rangle = \left\langle \frac{X_u \times X_v}{\|X_u \times X_v\|}, X_{ij} \right\rangle = \frac{[X_u, X_v, X_{ij}]}{\|X_u \times X_v\|}, \quad \text{for} \quad i, j = 1, 2.$$

where $e = l_{11}, \quad f = l_{12}$ and $g = l_{22}$. Hence, $e = \|X_u \times X_v\|L, \quad f = \|X_u \times X_v\|M$ and $g = \|X_u \times X_v\|N$. We know that, if X and Y are conjugate minimal surfaces, then

$$X_u = Y_v \qquad \text{and} \qquad X_v = -Y_u. \tag{4.4}$$

Hence, their derivatives are

$$X_{uu} = Y_{vu}, \qquad X_{vv} = -Y_{uv} \qquad \text{and} \qquad X_{uv} = Y_{vv}, \qquad X_{vu} = -Y_{uu}.$$

Recall that,

$$\begin{aligned}
e &= \langle \mathbf{N}, X_{uu} \rangle = \langle \mathbf{N}, Y_{uv} \rangle = \bar{f} = \|X_u \times X_v\|M, \\
f &= \langle \mathbf{N}, X_{uv} \rangle = \langle \mathbf{N}, Y_{vv} \rangle = \bar{g} = \|X_u \times X_v\|N, \\
g &= \langle \mathbf{N}, X_{vv} \rangle = \langle \mathbf{N}, -Y_{uv} \rangle = -\bar{f} = -\|X_u \times X_v\|M.
\end{aligned}$$

Hence, using equation (4.3) and replacing the above values we have that

$$\begin{aligned}
\bar{g}(u')^2 - 2\bar{f}u'v' - \bar{g}(v')^2 &= 0,\\
-\bar{e}(u')^2 - 2\bar{f}u'v' - \bar{g}(v')^2 &= 0,\\
L(u')^2 + 2Mu'v' + N(v')^2 &= 0.
\end{aligned}$$

■

Theorem 4.2.2 *The conjugate of an euclidean minimal surface with planar curvature lines is an euclidean and affine minimal surface.*
Conversely, the conjugate of an euclidean and affine minimal surface is an euclidean minimal surface with planar curvature lines.

Proof. Recall from Chapter 3 we have that

$$\nu = |K|^{-1/4}\mathbf{N} = |k_1k_2|^{-1/4}\mathbf{N} = e^w\mathbf{N}. \tag{4.5}$$

Its derivative with respect to u is

$$\nu_u = w_u e^w\mathbf{N} + e^w\mathbf{N}_u = e^w w_u\mathbf{N} + e^{-2w}X_u e^w = e^w w_u\mathbf{N} + e^{-w}X_u.$$

Now, using Theorem 3.4.2 we can compute ν_{uv} and prove that it is equal to zero to show that the surface is an affine minimal surface. That is

$$\begin{aligned}
\nu_{uv} &= w_v e^w w_u\mathbf{N} + e^w w_{uv}\mathbf{N} + e^w w_u\mathbf{N}_v - w_v e^{-w}X_u + e^{-w}X_{uv}\\
&= e^w\mathbf{N}(w_uw_v + w_{uv}) + e^w w_u(-e^{-2w}X_v) - e^{-w}w_vX_u + e^{-w}(w_vX_u + w_uX_v)\\
&= e^w\mathbf{N}(w_uw_v + w_{uv})\\
&= \mathbf{N}(e^w)_{uv} = 0.
\end{aligned}$$

Since, $w_uw_v + w_{uv} = 0$. We obtained the desired result.

■

Example 4.2.1 *The helicoid is an affine minimal surface.*

We can easily verified that the helicoid

$$Y(u,v) = (a\sinh v\cos u, a\sinh v\sin u, au), \qquad 0 < u < 2\pi, \quad -\infty < v < \infty.$$

is the conjugate surface of the catenoid (Example 2.3.2). Since

$$X_u = (a\cosh v\cos u, a\cosh v\sin u, 0) = -Y_v,$$

$$X_v = (a\sinh v\sin u, -a\sinh v\cos u, -a) = Y_u,$$

the normal vector is the same as the catenoid

$$\mathbf{N} = \left(\frac{\cos u}{\cosh v}, \frac{\sin u}{\cosh v}, -\frac{\sinh v}{\cosh v} \right)$$

and the coefficients of the second fundamental form of the helicoid are $\bar{e} = \bar{g} = 0$ and $\bar{f} = e = -a$.

From Theorem 4.2.1, we can conclude that the curvature lines of the catenoid are the asymptotic lines of the helicoid (conjugate surface). Recall that the conormal vector is given by the following formula

$$\nu = |K|^{-1/4}\mathbf{N} = |k_1 k_2|^{-1/4}\mathbf{N} = e^w \mathbf{N} = (a\cos u, a\sin u, -a\sinh v).$$

Its derivatives are

$$\nu_u = (-a\sin u, a\cos u, 0) \qquad \text{and} \qquad \nu_{uv} = (0, 0, 0).$$

Hence, we can conclude that helicoid is an affine minimal surface.

Similarly, we can prove the conjugate Enneper surface is an affine minimal surface.

Example 4.2.2 *The conjugate of Enneper surface is an affine minimal surface.*

We can easily verify that the conjugate of the Enneper surface with planar curvature lines is again an Enneper surface. The parametrization of this surface is

$$Y(u, v) = \left(v + \frac{v^3}{3} - u^2 v, u + \frac{u^3}{3} - uv^2, 2uv \right).$$

Its derivatives are

$$Y_u = (-2uv, 1 + u^2 - v^2, 2v) \quad \text{and} \quad Y_v = (1 + v^2 - u^2, -2uv, 2u)$$

and the normal vector is

$$\mathbf{N} = \frac{(-2u, -2v, 1 - u^2 - v^2)}{1 + u^2 + v^2}.$$

The coefficients of the first fundamental form are $\tilde{E} = \tilde{G} = E = G = (1 + u^2 + v^2)^2$ and $\tilde{G} = -F = 0$. Furthermore, the coefficients of the second fundamental form are $\tilde{e} = -f = 0$, $\tilde{g} = f = 0$ and $\tilde{f} = -g = 2$. Finally, the

Gaussian curvature is

$$K = \frac{eg - f^2}{EG - F^2} = \frac{-4}{(1 + v^2 + u^2)^4}$$

Now we compute the conormal affine vector

$$\begin{aligned} \nu = |k_1 k_2|^{-1/4}\mathbf{N} &= \left(\frac{(1+u^2+v^2)^4}{-4}\right)^{1/4}\left(\frac{(-2u, -2v, 1-u^2-v^2)}{1+u^2+v^2}\right) \\ &= \frac{(-2u, -2v, 1-u^2-v^2)}{(-4)^{1/4}}. \end{aligned}$$

and its partial derivatives are

$$\nu_u = \frac{(-2, 0, -2u)}{(-4)^{1/4}} \quad \text{and} \qquad \nu_{uv} = 0.$$

Hence, we can conclude that the conjugate Enneper surface is an affine minimal surface.

Now, we will show that the unique affine minimal surface with planar asymptotic lines is the hyperbolic paraboloid.
Recall that in the indefinite case Ψ is an affine minimal surface if and only if $\nu_{uv} = 0$, where its solution is given by $\nu(u, v) = \alpha(u) + \beta(v)$, where α and $\beta \in \mathbb{R}^3$.

When Ψ is an improper affine sphere we can assume that the conormal vector is $\nu(u, v) = (\alpha_1(u) + \beta_1(v), \alpha_2(u) + \beta_2(v), 1)$.

Lemma 4.2.3 *Ψ is an improper affine sphere with planar asymptotic lines, if and only if, α and β are contained in a straight line.*

Proof. We know that a surface Ψ has planar asymptotic lines when $[\Psi_u, \Psi_{uu}, \Psi_{uuu}] = 0$. Using Lelieuvre formula we have that $\Psi_u = \nu \wedge \nu_u$ and $\Psi_{uu} = \nu \wedge \nu_{uu}$ and $\Psi_{uuu} = \nu \wedge \nu_{uuu} + \nu_u \wedge \nu_{uu}$, then

$$\Psi_u \wedge \Psi_{uu} = (\nu \wedge \nu_u) \wedge (\nu \wedge \nu_{uu}) = [\nu, \nu_u, \nu_{uu}]\nu$$

and

$$[\Psi_u, \Psi_{uu}, \Psi_{uuu}] = (\Psi_u \wedge \Psi_{uu}) \cdot \nu_{uuu} = [\nu, \nu_u, \nu_{uu}]\nu \cdot (\nu \wedge \nu_{uuu} + \nu_u \wedge \nu_{uu}) = [\nu, \nu_u, \nu_{uu}]^2 = 0.$$

Similarly,

$$[\Psi_v, \Psi_{vv}, \Psi_{vvv}] = [\nu, \nu_v, \nu_{vv}]^2 = 0.$$

On the other hand,

$$[\nu, \nu_u, \nu_{uu}] = \begin{bmatrix} \alpha_1(u) + \beta_1(v) & \alpha_2(u) + \beta_2(v) & 1 \\ \alpha_1'(u) & \alpha_2'(u) & 0 \\ \alpha_1''(u) & \alpha_1''2(u) & 0 \end{bmatrix}$$
$$= [\alpha', \alpha''].$$

Thus, $[\alpha', \alpha''] = 0$. It means that α is contained in a straight line. Similarly, we can easily prove that β is contained in a straight line.

■

Lemma 4.2.4 *If Ψ is an affine minimal surface with planar asymptotic lines, then the conormal affine vector ν is contained in a plane.*
Moreover, we can conclude that Ψ is an improper affine sphere.

Proof. Recall that Ψ has planar asymptotic lines if and only if $[\nu, \nu_u, \nu_{uu}] = 0$. And the above equation is equivalent to $[\alpha + \beta, \alpha', \alpha''] = 0$. Let $P(u)$ be the osculating plane of the curve α, which is generated by its derivatives α' and α'' and $Q(v)$ the osculating plane of the curve β which is generated by β' and β''. Fixing u, it is $\lambda(u_0, v) = \alpha(u_0) + \beta(v)$, we can see that β' is contained in $P(u)$. Similarly β'' is also contained in $P(u)$. Hence, $Q(v)$ is contained in the osculating plane $P(u)$. In the same way we can also have that $P(u)$ is contained in the osculating plane $Q(v)$. So, we have that $P(u) = Q(v)$.

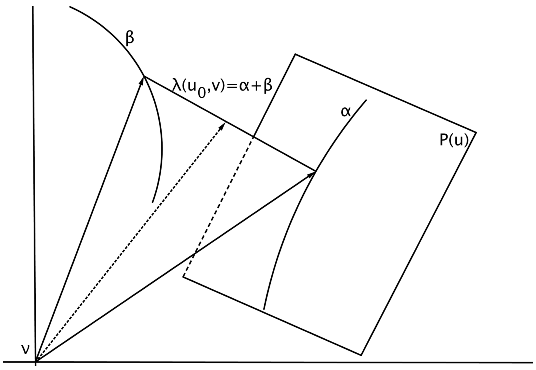

Figure 4.2: Geometrical interpretation of the Lema 4.2.4.

■

Lemma 4.2.5 *The regular surface S is an improper affine sphere with planar asymptotic lines, if and only if, S is an hyperbolic paraboloid.*

Proof. Recall that we can assume that the conormal vector is

$$\nu(u, v) = (\alpha_1(u) + \beta_1(v), \alpha_2(u) + \beta_2(v), 1).$$

From the above Lemma we have $\alpha(u) = \alpha_0 + u\bar{U_0}$ and $\beta(v) = \beta_0 + v\bar{V_0}$. Now, we can reparametrize the normal vector as follows

$$\nu(u,v) = (a_1 + b_2u + c_1v, a_2 + b_2u + c_2v, 1).$$

Its derivatives are

$$\nu_u = (b_1, b_2, 0) \qquad \text{and} \qquad \nu_v = (c_1, c_2, 0).$$

From Leliuvre Formula we have

$$\Psi_u = \nu \wedge \nu_u = (-b_2, b_1, a_1b_2 - a_2b_1, (b_2c_1 - b_1c_2)v)$$

and

$$\Psi_v = -\nu \wedge \nu_v = (c_2, -c_1, a_2c_1 - a_1c_2, (b_2c_1 - b_1c_2)v).$$

We can conclude that the improper affine sphere is

$$\Psi(u,v) = (-b_2u + c_2v, b_1u - c_1v, (a_1b_2 - b_1a_2)u + (a_2c_1 - a_1c_2)v + (b_2c_1 - b_1c_2)uv).$$

Recall the most common parametrization of the hyperbolic paraboloid is $f(u,v) = (u,v,uv)$. We can apply an affine transformation A such that

$$\Psi(u,v) = \begin{bmatrix} -b_2 & c_2 & 0 \\ b_1 & -c_1 & 0 \\ a_1b_2 - b_1a_2 & a_2c_1 - a_1c_2 & b_2c_1 - b_1c_2 \end{bmatrix} \begin{bmatrix} u \\ v \\ uv \end{bmatrix}$$

Thus, we proved the statement.

■

Theorem 4.2.6 *An affine minimal surface with planar asymptotic lines is the hyperbolic paraboloid.*

4.3 Finding the solution of the minimal surfaces with planar curvature lines

Finding non-planar umbilic-free minimal surfaces with planar curvature lines is equivalent to finding solutions to the following system of partial differential equations:

$$\begin{cases} \Delta w - e^{-2w} = 0 & \textbf{(minimal condition)} \\ w_{uv} + w_u w_v = 0 & \textbf{(planar curvature line condition).} \end{cases} \tag{4.6}$$

In general, solving systems of partial differential equations may be difficult. However, the next Lemma shows that (4.6) can be reduced to a system of ordinary differential equations.

Lemma 4.3.1 *The solution $w : \Omega \to \mathbb{R}$ of (4.6) is precisely given by*

$$e^{w(u,v)} = \frac{1 + f(u)^2 + g(v)^2}{f_u(u) + g_v(v)} \tag{4.7}$$

where $f(u)$ and $g(v)$ are real-valued meromorphic functions satisfying the following system of ordinary differential equations

$$\begin{cases} (f_u(u))^2 = (c-d)f(u)^2 + c \\ f_{uu}(u) = (c-d)f(u) \\ (g_v(v))^2 = (d-c)g(v)^2 + d \\ g_{vv}(v) = (d-c)g(v). \end{cases} \tag{4.8}$$

for some real constants c and d such that $c^2 + d^2 \neq 0$. Moreover, f and g can be recovered from w by

$$\begin{cases} w_u = e^{-w} f(u) \\ w_v = e^{-w} g(v) \end{cases} \tag{4.9}$$

Proof. Using the second equation on (4.6). Hence,

$$w_{uv} = -w_u w_v \Longrightarrow w_u = e^{-w} f(u).$$

Similarly computation shows $w_v = e^{-w} g(v)$. Then (4.9) is proved for some constants of integration $f(u)$ and $g(v)$. Using these definitions of f and g, it is straightforward to check that (4.7) holds using (4.6). Now, from the fact that $w_u e^{-w} = e^{-2w} f$,

$$\frac{f_{uu}}{1 + f^2 + g^2} - \frac{f(f_u^2 - g_v^2)}{1 + f^2 + g^2} = 0 \tag{4.10}$$

and multiplying both sides by $2f_u$ and integrating with respect to u tells us that

$$f_u^2 = g_v^2 + D(v)(1 + f^2 + g^2) \tag{4.11}$$

for some constant of integration $D(v)$. Substituting (4.11) into (4.10), we get $f_{uu}(u) = D(v)f(u)$ implying that $D(v) = \hat{c}$ for some constant $\hat{c}$. Hence,

$$f_{uu} = \hat{c}f,$$

and again multiplying both sides by $2f_u$ and integrating with respect to u implies that

$$f_u^2 = \hat{c}f^2 + c$$

for some constant c. Similarly, from the fact that $w_v e^{-w} = e^{-2w}g$, we can show that

$$\begin{cases} g_{vv} = \hat{d}g \\ g_v^2 = \hat{d}g^2 + d \end{cases} \tag{4.12}$$

for some constants d and $\hat{d}$. Substituting these differential equations into (4.10) shows that $-\hat{d} = \hat{c} = c - d$

■

To find the explicit solution for f, we should consider the initial conditions of $f(u)$ and $g(v)$ satisfying (4.8). We assume $f(0) = g(0) = 0$ for simplicity, therefore, we first identify the conditions for $f(u)$ and $g(v)$ having a zero and prove that both $f(u)$ and $g(v)$ has a zero, using the next couple of lemmas.

Lemma 4.3.2 *$f(u)$ (respectively $g(v)$) satisfying Lemma 4.3.1) has a zero if and only if $c \geq 0$ (respectively $d \geq 0$).*

Proof. If $c = d$, then the statement is a result of direct computation. Now, assume $c \neq d$. First, to see that $c \geq 0$ implies that $f(u)$ has a zero, from (4.8), we get

$$f(u) = C_1 e^{\sqrt{c-d}u} + C_2 e^{-\sqrt{c-d}u}$$

for some complex constants C_1 and C_2 such that $c = -4C_1C_2(c-d)$. If $c = 0$, then either $f(u) \equiv 0$ or (4.8) implies $d < 0$, a contradiction to (4.8). Now assume $c > 0$. Then, C_1 and C_2 are non-zero, and we may let $C_1 = \dfrac{1}{2}\sqrt{\dfrac{c}{c-d}} = -C_2$ so that f is real-valued and $f(0) = 0$.

On the other hand, if $f(u_0) = 0$ for some u_0, then $(f_u(u_0))^2 = c$, implying that $c \geq 0$. Therefore, $f(u)$ has a zero if and only if $c \geq 0$. The case for $g(v)$ is proven similarly using (4.8).

■

Lemma 4.3.3 *Let $f(u)$ and $g(v)$ be functions satisfying (4.8). Then both $f(u)$ and $g(v)$ have a zero.*

Proof. Suppose by contradiction. It is that f does not have a zero. Then by the above lemma, $c < 0$. From (4.8), we see that $c < 0$ implies $c - d > 0$. However, (4.8) implies that if $c - d > 0$, then $d > 0$, a contradiction since $c < 0$ and $c > d$. Therefore, f must always have a zero. Similarly, g must have a zero. ∎

By shifting parameters u and v, we may assume $f(0) = g(0) = 0$. Using these initial conditions, we may solve (4.8) to get the following:

$$f(u) = \begin{cases} \pm\dfrac{\alpha}{\sqrt{\alpha^2-\beta^2}}\sinh(\sqrt{\alpha^2-\beta^2}u), & \text{if} \quad \alpha \neq \beta \\ \pm\alpha u, & \text{if} \quad \alpha = \beta \end{cases}, \tag{4.13}$$

$$g(v) = \begin{cases} \pm\dfrac{\beta}{\sqrt{\beta^2-\alpha^2}}\sinh(\sqrt{\beta^2-\alpha^2}v), & \text{if} \quad \alpha \neq \beta \\ \pm\beta v, & \text{if} \quad \alpha = \beta \end{cases}, \tag{4.14}$$

where $\alpha^2 = c$ and $\beta^2 = d$. It should be noted that by letting $u \mapsto -u$ and $v \mapsto -v$, we may drop the plus or minus condition of (4.13) and (4.14). Finally, we arrive at the following result.

Proposition 4.3.4 *For non-planar minimal surface $X(u,v)$ with planar curvature lines, the real-analytic solution $w : \mathbb{R}^2 \to \mathbb{R}$ of (4.6) is precisely given by*

$$e^{w(u,v)} = \frac{1 + f(u)^2 + g(v)^2}{f_u(u) + g_v(v)} \tag{4.15}$$

with

$$f(u) = \begin{cases} \dfrac{\alpha}{\sqrt{\alpha^2-\beta^2}}\sinh(\sqrt{\alpha^2-\beta^2}u), & \textit{if} \quad \alpha \neq \beta \\ \alpha u, & \textit{if} \quad \alpha = \beta \end{cases}, \tag{4.16}$$

$$g(v) = \begin{cases} \dfrac{\beta}{\sqrt{\beta^2-\alpha^2}}\sinh(\sqrt{\beta^2-\alpha^2}v), & \textit{if} \quad \alpha \neq \beta \\ \beta v, & \textit{if} \quad \alpha = \beta \end{cases}, \tag{4.17}$$

where $\alpha + \beta > 0$.

Proof. To see that w is a real number, we only need to show that $f_u(u)+g_v(v) > 0$ for any $(u,v) \in \sum$. If $\alpha = \beta$, then $f_u + g_v = \alpha + \beta > 0$. Without loss of generality, assume $\alpha > \beta$, since $\alpha + \beta > 0$,$\alpha > |\beta|$. From (4.16) and (4.17), we have that

$$
\begin{aligned}
f_u(u) &= \alpha \cosh(\sqrt{\alpha^2 - \beta^2} u) \geq \alpha, \\
g_v(v) &= \beta \cosh(\sqrt{\beta^2 - \alpha^2} v) = \beta \cos(\sqrt{\alpha^2 - \beta^2} v) \geq -|\beta|.
\end{aligned}
$$

since $\cosh(iz) = \cos(z)$. Therefore, $f_u + g_v \geq \alpha - |\beta| > 0$. The case for $\alpha < \beta$ can be proved similarly. Finally, the real-analyticity of $f(u)$ and $g(v)$ tells us that the domain of $w(u, v)$ can be extended to $\mathbb{R}^2$ globally.

■

Since the u-direction and v-direction of $w(u, v)$ depend only on $f(u)$ and $g(v)$ respectively, by choosing different values for α and β, we may analytically understand how the surfaces behave in either direction. The following theorem and Figure 4.3 explains the relationship between different values of α and β and the surface generated by the corresponding $w(u, v)$. Note that in the Figure 4.3, the subscript $u \leftrightarrow v$ denotes that the role of u and v are switched.

Theorem 4.3.5 *Let $X(u, v)$ be a non-planar minimal surface in $\mathbb{R}^3$ with isothermic coordinates (u, v) such that $ds^2 = e^{2w}(du^2 + dv^2)$. Then X has planar curvature lines if and only if $w(u, v)$ satisfies the Proposition 4.3.4. Furthermore, for different values of α and β, the metric function of $X(u, v)$ have the following properties, based on Figure (4.3).*

- ①,①′ *are not periodic in the u-direction but periodic in the v-direction.*
- ② *is not periodic in the u-direction but constant in the v-direction.*
- ③ *is not periodic in both the u-direction and v-direction.*

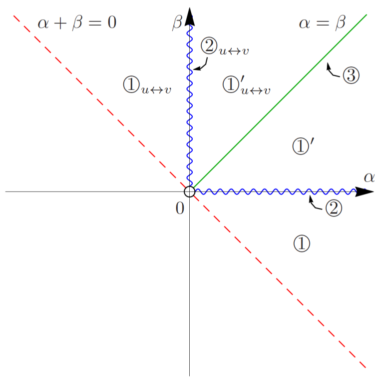

Figure 4.3: Classification diagram for non-planar minimal surfaces with planar curvature lines [8].

Theorem 4.3.6 (Weierstrass Representation Formula) *Any minimal surface $X : \Omega \subset \mathbb{C} \to \mathbb{R}^3$ can be locally represented as*

$$X(z) = Re \int (1 - h^2, i(1 + h^2), 2h)\eta dz$$

over a simply-connected domain Ω on which h is meromorphic, while η and $h^2\eta$ are holomorphic.

Using the Weierstrass data $(h, \eta dz)$, we can classify different types of minimal surfaces with planar curvature lines, stated here along with their respective Weierstrass data. See [8] for more details.

4.4 Normal vector and the parametrizations of the minimal surfaces with planar curvature lines

In this section we use Theorem 4.3.5 to identify the non-planar minimal surfaces with planar curvature lines. Hence, we are going to find their parametrization and obtain a deformation between them.

4.4.1 Normal vector

Note that if we normalize $\vec{v_1}$ and $\vec{v_2}$ the axial directions of the surface, we can calculate the unit normal vector of the surface as we can see now.

Proposition 4.4.1 (Normal vector) *Let $f(u)$ and $g(v)$ be as in Proposition (4.3.4) If $\alpha, \beta \neq 0$, then the unit normal vector $\boldsymbol{N}(u,v)$ is given by*

$$\boldsymbol{N}(u,v) = \left(\frac{1}{\alpha}w_u, \frac{1}{\beta}w_v, \sqrt{1 - \frac{1}{\alpha^2}w_u^2 - \frac{1}{\beta^2}w_v^2}\right). \tag{4.18}$$

Now we calculate the Weierstrass data using the normal vector. Where h is a meromorphic function and also it is the normal vector function under stereographic projection

$$h^{(\alpha,\beta)}(u,v) = \frac{1}{1-\mathbf{N}_3}(\mathbf{N}_1 + i\mathbf{N}_2) = \frac{\sqrt{\alpha^2-\beta^2}}{\alpha-\beta}\tanh\left(\frac{\sqrt{\alpha^2-\beta^2}}{2}(u+iv)\right)$$

Since $Q = -\frac{1}{2}(h_u + ih_v)\eta = -\frac{1}{2}$, we also have

$$\eta^{(\alpha,\beta)}(u,v) = \frac{1}{h_u + ih_v} = \frac{1}{\alpha+\beta}\cosh^2\left(\frac{\sqrt{\alpha^2-\beta^2}}{2}(u+iv)\right)$$

for $\alpha+\beta > 0$. Let be $\alpha = r\cos\theta$ and $\beta = r\sin\theta$, then it is easy to see that r is a homothety factor. Therefore, we assume $r = 1$, and rewrite $h^{(\alpha,\beta)}(u,v)$

and $\eta^{(\alpha,\beta)}(u,v)$ as follow

$$h^{\theta}(u,v) = \begin{cases} \dfrac{\sqrt{\cos(2\theta)}}{\cos\theta - \sin\theta}\tanh\left(\dfrac{\sqrt{\cos(2\theta)}}{2}(u+iv)\right), \text{if} \quad \theta \neq \dfrac{\pi}{4} \\ \dfrac{u+iv}{\sqrt{2}}, \quad \text{otherwise.} \end{cases} \tag{4.19}$$

$$\eta^{\theta}(u,v) = \begin{cases} \dfrac{1}{\cos\theta + \sin\theta}\cosh^2\left(\dfrac{\sqrt{\cos(2\theta)}}{2}(u+iv)\right), \text{if} \quad \theta \neq \dfrac{\pi}{4} \\ \dfrac{1}{\sqrt{2}}, \quad \text{otherwise.} \end{cases} \tag{4.20}$$

where $\theta \in \left(-\frac{\pi}{4}, \frac{3\pi}{4}\right)$.

Note that, $(h^{\theta})^2\eta^{\theta}$ is holomorphic, since h^{θ} is meromorphic and η^{θ} is an holomorphic function. Thus, we can use the Weierstrass representation Theorem 4.3.6 to obtain the following parametrizations for minimal surfaces with planar curvature lines.

Proposition 4.4.2 *Let $X(u,v)$ be a non-planar minimal surface with planar curvature lines in $\mathbb{R}^3$. Then $X(u,v)$ has the following parametrization*

$$X^{\theta} = \begin{cases} \begin{pmatrix} \dfrac{u\cos\theta\sqrt{\cos 2\theta} - \sin\theta\sinh(u\sqrt{\cos 2\theta})\cos(v\sqrt{\cos 2\theta}))}{(\cos 2\theta)^{3/2}} \\ \dfrac{v\sin\theta\sqrt{\cos 2\theta} - \cos\theta\cosh(u\sqrt{\cos 2\theta})\sin(v\sqrt{\cos 2\theta})}{(\cos 2\theta)^{3/2}} \\ \dfrac{\cosh(u\sqrt{\cos 2\theta})\cos(v\sqrt{\cos 2\theta})}{\sqrt{\cos 2\theta}} \end{pmatrix}^{T} & \textit{if} \quad \theta \neq \dfrac{\pi}{4} \\ \left(-\dfrac{u(-6+u^2-3v^2)}{6\sqrt{2}}, \dfrac{v(-6-3u^2+v^2)}{6\sqrt{2}}, \dfrac{u^2-v^2}{2}\right), & \textit{otherwise.} \end{cases}$$

for some $\theta \in \left(-\frac{\pi}{4}, \frac{3\pi}{4}\right)$ on its domain up to isometries and homotheties.

4.5 Continuous deformation of minimal surfaces with planar curvature lines

In this section, we show that the parameter θ defines a locally continuous deformation between non-planar minimal surfaces preserving the planar curvature line condition. Furthermore, by introducing a homothety factor depending on θ, we can extend the deformation including the plane.

We begin this section showing that the deformation of minimal surfaces in $\mathbb{R}^3$ with planar curvature lines is continuous. The continuity is obvious at any $\theta \neq \dfrac{\pi}{4}$. Thus, we only need to check when $\theta = \dfrac{\pi}{4}$.
From the Weierstrass data, it is easy to check that at any point (u, v),

$$\lim_{\theta \to \frac{\pi}{4}} h^{\theta}(u,v) = h^{\frac{\pi}{4}}(u,v) \qquad \text{and} \qquad \lim_{\theta \to \frac{\pi}{4}} \eta^{\theta}(u,v) = \eta^{\frac{\pi}{4}}(u,v).$$

Note also that, each component of the parametrization in Proposition 4.4.2 is also continuous at $\theta = \frac{\pi}{4}$ at any point (u, v).

$$\lim_{\theta \to \frac{\pi}{4}} X^{\theta}(u,v) = X^{\frac{\pi}{4}}(u,v).$$

Hence, $X^{\theta}(u, v)$ is a continuous deformation.

Now, we would like to extend our parametrization to include the plane. To do so, we will define the homotethy R^{θ} as follow

$$R^{\theta} = \left(1 - \sin\left(\theta + \frac{\pi}{4}\right)\right) |\cos 2\theta| + \sin\left(\theta + \frac{\pi}{4}\right). \tag{4.21}$$

For $\theta \in \left[-\frac{\pi}{4}, \frac{3\pi}{4}\right]$. Note that $R^{\theta} > 0$ and $R^{\theta} = 0$ if and only if $\theta = \frac{\pi}{4}$ and $\theta = \frac{3\pi}{4}$. Now we define

$$\bar{X}^{\theta} = R^{\theta} X^{\theta}(u,v).$$

It is a straightforward computation that

$$\lim_{\theta \searrow -\frac{\pi}{4}} \bar{X}^{\theta}(u,v) = \frac{3}{\sqrt{2}}(u, -v, 0) = \lim_{\theta \nearrow \frac{3\pi}{4}} \bar{X}^{\theta}(u,v).$$

Therefore, the extension of X^{θ} is $\bar{X}^{\theta}$. It is defined as follows

$$\bar{X}^{\theta}(u,v) \quad = \quad \begin{cases} \dfrac{3}{\sqrt{2}}(u, -v, 0), & \text{if} \quad \theta = -\dfrac{\pi}{4}, \dfrac{3\pi}{4}, \\ R^{\theta} X^{\theta}(u,v), & \text{if} \quad \theta \in \left(-\dfrac{\pi}{4}, \dfrac{3\pi}{4}\right). \end{cases} \tag{4.22}$$

It again is a continuous deformation for $\theta \in \left[-\dfrac{\pi}{4}, \dfrac{3\pi}{4}\right]$.

Hence, we obtained a classification and deformation of minimal surfaces with planar curvature lines. Therefore, we state this result as a theorem.

Theorem 4.5.1 *If $\bar{X}^{\theta}(u, v)$ is a minimal surface with planar curvature lines* *in* $\mathbb{R}^3$*, then the surface is given by the following parametrization on its domain*

$$\bar{X}^\theta(u,v) = \begin{cases} R^\theta \begin{pmatrix} \dfrac{u\cos\theta\sqrt{\cos 2\theta} - \sin\theta\sinh(u\sqrt{\cos 2\theta})\cos(v\sqrt{\cos 2\theta}))}{(\cos 2\theta)^{3/2}} \\ \dfrac{v\sin\theta\sqrt{\cos 2\theta} - \cos\theta\cosh(u\sqrt{\cos 2\theta})\sin(v\sqrt{\cos 2\theta})}{(\cos 2\theta)^{3/2}} \\ \dfrac{\cosh(u\sqrt{\cos 2\theta})\cos(v\sqrt{\cos 2\theta})}{\cos 2\theta} \end{pmatrix}^T & \text{if} \quad \theta \neq -\dfrac{\pi}{4}, \dfrac{\pi}{4}, \dfrac{3\pi}{4}. \\ \left(-\dfrac{u(-6+u^2-3v^2)}{6\sqrt{2}}, \dfrac{v(-6-3u^2+v^2)}{6\sqrt{2}}, \dfrac{u^2-v^2}{2}\right), & \text{if} \quad \theta = \dfrac{\pi}{4}. \\ \dfrac{3}{\sqrt{2}}(u,-v,0), & \text{if} \quad \theta = -\dfrac{\pi}{4}, \dfrac{3\pi}{4}. \end{cases}$$

up to isometries and homotheties of $\mathbb{R}^3$ *for some* $\theta \in \left[-\frac{\pi}{4}, \frac{3\pi}{4}\right]$*, where* $R^\theta = \left(1 - \sin\left(\theta + \frac{\pi}{4}\right)\right)|\cos 2\theta| + \sin\left(\theta + \frac{\pi}{4}\right)$*. In fact it must be a piece of one, and only one, of the following*

- *plane* $\left(\theta = -\frac{\pi}{4}, \frac{3\pi}{4}\right)$,
- *catenoid* $\left(\theta = 0, \frac{\pi}{2}\right)$,
- *Enneper surface* $\left(\theta = \frac{\pi}{4}\right)$*, or*
- *a surface in the Bonnet family* $\theta \in \left(-\frac{\pi}{4}, \frac{3\pi}{4}\right) \setminus \left\{0, \frac{\pi}{4}, \frac{\pi}{2}\right\}$.

Moreover, the deformation $\bar{X}(u,v)$ *depending on the parameter* θ *is continuous (see Figure 4.5).*

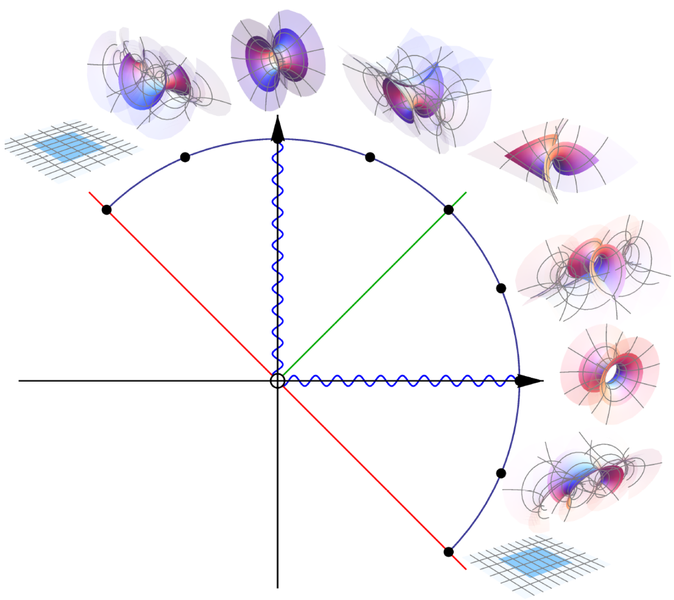

Figure 4.4: Deformation of minimal surfaces with planar curvature lines [8].

Furthermore, considering the conjugate of minimal surfaces with planar curvature lines, we get the following classification and deformation of minimal surfaces that are also affine minimal maps.

Theorem 4.5.2 *If $\hat{X}^\theta(u,v)$ is a minimal surface that is also an affine minimal surface in $\mathbb{R}^3$, then the surface is given by the following parametrization on its domain*

$$\hat{X}^\theta(u,v) = \begin{cases} R^\theta \begin{pmatrix} \dfrac{-v\cos\theta\sqrt{\cos 2\theta} + \sin\theta\cosh(u\sqrt{\cos 2\theta})\sin(v\sqrt{\cos 2\theta}))}{(\cos 2\theta)^{3/2}} \\ \dfrac{u\sin\theta\sqrt{\cos 2\theta} - \cos\theta\sinh(u\sqrt{\cos 2\theta})\cos(v\sqrt{\cos 2\theta})}{(\cos 2\theta)^{3/2}} \\ \dfrac{-\sinh(u\sqrt{\cos 2\theta})\sin(v\sqrt{\cos 2\theta})}{\cos 2\theta} \end{pmatrix}^T & \text{if } \theta \neq -\dfrac{\pi}{4}, \dfrac{\pi}{4}, \dfrac{3\pi}{4}. \\ \left(-\dfrac{v(6-3u^2+v^2)}{6\sqrt{2}}, \dfrac{-u(6+u^2-3v^2)}{6\sqrt{2}}, \dfrac{-uv}{2}\right), & \text{if } \theta = \dfrac{\pi}{4}. \\ \dfrac{3}{\sqrt{2}}(-v,-u,0), & \text{if } \theta = -\dfrac{\pi}{4}, \dfrac{3\pi}{4}. \end{cases}$$

In fact it must be a piece of one, and only one, of the following

- *plane* $\left(\theta = -\dfrac{\pi}{4}, \dfrac{3\pi}{4}\right)$,
- *helicoid* $\left(\theta = 0, \dfrac{\pi}{2}\right)$,
- *Enneper surface* $\left(\theta = \dfrac{\pi}{4}\right)$, *or*
- *a surface in the Thomsen family* $\theta \in \left(-\dfrac{\pi}{4}, \dfrac{3\pi}{4}\right) \setminus \left\{0, \dfrac{\pi}{4}, \dfrac{\pi}{2}\right\}$.

up to isometries and homotheties of $\mathbb{R}^3$ for some $\theta \in \left[-\dfrac{\pi}{4}, \dfrac{3\pi}{4}\right]$ Moreover, the deformation $\hat{X}(u,v)$ depending on the parameter θ is continuous (see Figure 4.5).

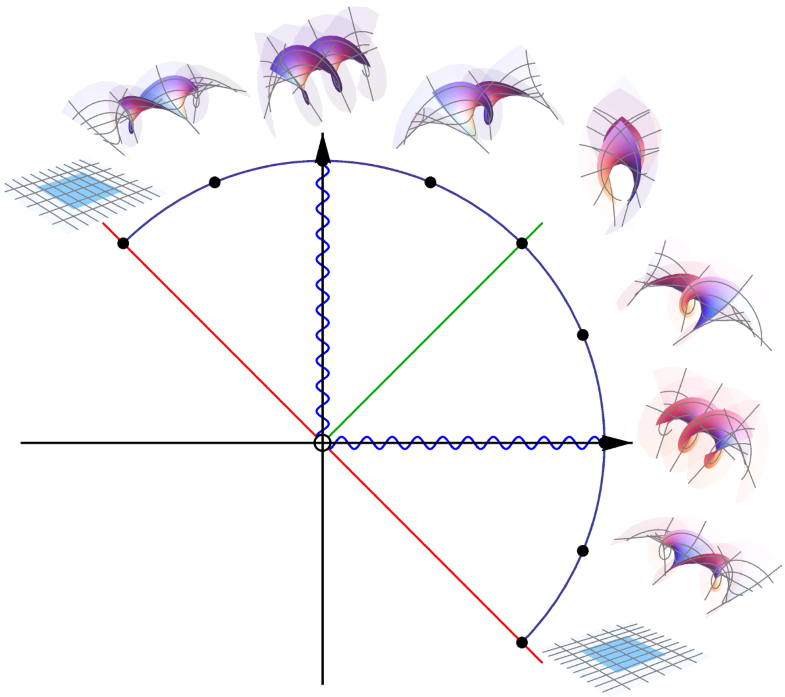

Figure 4.5: Deformation of minimal surfaces that are also affine minimal maps [8].

5
Affine Maximal Surfaces with Singularities in the convex case

In this last Chapter, we will take the solution of the affine Cauchy problem and give the conditions to the existence and uniqueness of affine maximal maps with the desired singularities. In particular, we also characterize when an analytic curve of $\mathbb{R}^3$ is the singular curve of some affine maximal maps and improper affine sphere with prescribed cuspidal edges and swallowtails. See [7] for more details.

5.1
Affine maximal surfaces constructed from a curve

From now, we consider an affine maximal map $\Psi : \Omega \to \mathbb{R}^3$ with affine conormal $\nu : \Omega \to \mathbb{R}^3$ and a regular analytic curve $\gamma : I \to \Omega$, for an interval I.
We can denote $\alpha = \Psi \circ \gamma$ and $U = \nu \circ \gamma$, with parameter $s \in I$. Thus, by the Inverse Function Theorem, there exists a conformal parameter $z = s + it$ and we can parametrize a piece of the affine maximal map by $\Psi : \Omega \subset \mathbb{C} \to \mathbb{R}^3$ with $I \subset \Omega$,

$$\Psi(s,0) = \alpha(s), \qquad \nu(s,0) = U(s). \tag{5.1}$$

Proposition 5.1.1 *The analytic maps α, η and U satisfy the condition $\eta \wedge U = -\alpha$. Moreover, ν is given by*

$$\nu(z) = Re\left(U(z) - i\int_{s_0}^{z} \eta(\zeta)d\zeta\right), \qquad z \in \Omega \subset \mathbb{C}, \quad s_0 \in I. \tag{5.2}$$

Proof. We claim that $\nu(z) = \mathrm{Re}\left(U(z) - i\int_{s_0}^{z} \eta(\zeta)d\zeta\right)$ defined as follows is the solution of the following PDE,

$$\begin{cases} \Delta\nu = 0 \quad \text{in} \quad \Omega, \\ \nu(s,0) = U(s), \quad \text{on} \quad \Omega, \\ \nu_t(s,0) = \eta(s), \quad \text{on} \quad \partial\Omega. \end{cases} \tag{5.3}$$

When $\eta(s) = 0$ is trivial. Now, we only need to prove when $\eta(s) \neq 0$.
It is easy to see that $\nu(z)$ is harmonic, since is the real part of an holomorphic function.

Now we need to prove that $\nu(z)$ satisfies the boundary conditions. Observe that

$$\begin{aligned}\nu(s,0) &= \text{Re }\left(U(s+i0) - i\int_{s_0}^{s+it}\eta(\zeta)d\zeta\right)\\ &= \text{Re }\left(U(s) - i\int_{s_0}^{s}\eta(\zeta)d\zeta\right)\\ &= U(s).\end{aligned}$$

It follows from the fact that $\int_{s_0}^{s}\eta(\zeta)d\zeta$ is real then

$$\text{Re }\left(-i\int_{s_0}^{s}\eta(\zeta)d\zeta\right) = 0.$$

From the Cauchy- Riemann equations

$$U(z) = u(s,t) + iv(s,t) = \text{Re } U(s,t) + i \text{ Im } U(s,t).$$

$$u_s = v_t \qquad \text{and} \quad u_t = -v_s.$$

That is,

$$(\text{Re } U(s,t))_s = (\text{ Im } U(s,t))_t \qquad \text{and} \quad (\text{Re } U(s,t))_t = -(\text{ Im}U(s,t))_s$$

$$\begin{aligned}\nu_t(z) &= (\text{Re } U(s,t))_t - \left(\text{Re }\left(i\int_{s_0}^{s+it}\eta(\zeta)d\zeta\right)\right)_t\\ \nu_t(s,t) &= -(\text{Im } U(s,t))_s - \text{Re }(i^2\eta(s+it))\end{aligned}$$

In $t=0$, we have that

$$\nu_t(s,0) = -(\text{Im } U(s))_s + \eta(s)$$

where Im $(U_s(s)) = 0$, since, $U_s(s)$ is real. Hence, we can conclude that

$$\nu_t(s,0) = \eta(s).$$

That is, we proved that $\nu(z)$ given by

$$\nu(z) = \text{Re }\left(U(z) - i\int_{s_0}^{z}\eta(\zeta)d\zeta\right), \qquad z\in\Omega\subset\mathbb{C}, \quad s_0\in I. \tag{5.4}$$

is the solution of (5.3).

■

We conclude that, the map Ψ can be recovered by the following formula

$$\Psi = \alpha(s_0) + 2\text{Re}\int_{s_0}^{z} i\nu\times\nu_z, \qquad z\in\Omega\subset\mathbb{C}, \quad s_0\in I. \tag{5.5}$$

with $U(z)$ and $\eta(z)$ the holomorphic extensions of $U(s) = \nu(s, 0)$ and $\eta(s) = \nu_t(s, 0)$ to a neighborhood Ω of I.

5.2 Singular curves of affine maximal maps

From Lelieuvre Formula we have that, along α

$$\eta \wedge U = -\alpha \tag{5.6}$$

and $s_0 \in I$ is a singular point of Ψ if

$$\rho(s_0) = U \cdot \alpha'' = -U' \cdot \alpha' = 0, \tag{5.7}$$

where by prime we indicate derivative with respect to the variable s.

Now, we try to analyze the solution of the above problem when $\alpha : I \to \mathbb{R}^3$ is a singular curve of Ψ. This case is interesting since the data U and η are just determined by α and two analytic functions λ and ϕ. In fact, from (5.6) and (5.7) we have

$$\begin{aligned} \alpha' \wedge \alpha'' &= -(\eta \wedge U) \wedge \alpha'' \\ &= -(\eta \cdot \alpha'')U + (U \cdot \alpha'')\eta, \end{aligned}$$

where, $U \cdot \alpha'' = 0$, since s_0 is a singular point of Ψ and $(u \wedge v) \wedge w = (u \cdot w)v - (v \cdot w)u$, where by $\cdot$ we indicate the inner product.
Hence, we have in that case

$$U = \frac{\alpha' \wedge \alpha''}{-\eta \cdot \alpha''}.$$

Setting $\lambda = -\eta \cdot \alpha''$, we get the following formula to find the data U and it is only determinated by α and a analytic function λ, where $\lambda : I \to \mathbb{R}$.

$$U = \frac{\alpha' \wedge \alpha''}{\lambda}. \tag{5.8}$$

Now, we need to prove that

$$\eta = \phi\alpha' \wedge \alpha'' - \frac{\lambda}{|\alpha' \wedge \alpha''|^2}(\alpha' \wedge \alpha'') \wedge \alpha'. \tag{5.9}$$

Using equation (5.8), we have that

$$U\lambda = \alpha' \wedge \alpha''.$$

Replacing U in the above equation we have

$$\eta = \phi U\lambda - \frac{\lambda}{\lambda^2|U|^2}(U\lambda) \wedge \alpha'.$$

to prove our assumption it is sufficient to prove that

$$\left(\eta + \frac{U \wedge \alpha'}{|U|^2}\right) \parallel U.$$

where by $\parallel$ we indicate parallel to. Now we need to verify that

$$\left(\eta + \frac{U \wedge \alpha'}{|U|^2}\right) \wedge U = 0.$$

We know that

$$(\eta \wedge U) + \left(\left(\frac{U \wedge \alpha'}{|U|^2}\right) \wedge U\right) = -\alpha' + \left(\left(\frac{U \wedge \alpha'}{|U|^2}\right) \wedge U\right).$$

If, $\hat{U} = \dfrac{U}{|U|}$. We get that

$$\begin{aligned}\left(\frac{U \wedge \alpha'}{|U|^2}\right) \wedge U &= (\hat{U} \wedge \alpha') \wedge \hat{U} \\ &= (\hat{U} \cdot \hat{U})\alpha' - (\alpha' \cdot \hat{U})\hat{U} \\ &= \alpha'.\end{aligned}$$

Since, we have that $\alpha' \cdot \hat{U} = 0$ from the equation (5.7). Hence, we have that

$$(\eta \wedge U) + \left(\left(\frac{U \wedge \alpha'}{|U|^2}\right) \wedge U\right) = 0.$$

Thus, equation (5.9) holds for some analytic function $\phi : I \to \mathbb{R}^3$.

Now, we can obtain some results about singular curves of affine maximal maps.

5.3 Affine maximal surface with singularities

In this section we will take the solution of the affine Cauchy problem and give the conditions to the existence and uniqueness of affine maximal maps with cuspidal edges or swallowtail singularities. In particular, we also characterize when an analytic curve of $\mathbb{R}^3$ is the singular curve of some affine maximal map with the desired singularities. See [1] and [7].

We know that $z_0 \in \Omega$ is a non-degenerate singular point of the map, if and only if,

$$\rho(z_0) = 0, \qquad d\rho\,|_{z_0} \neq 0.$$

In this case, either $\Psi(z_0)$ is an isolated singularity or the singular set of Ψ around z_0 locally becomes a regular curve $\gamma : I \subset \mathbb{R} \to \Omega$ and we have the following criterion due to [7] for the singular curve $\alpha = \Psi \circ \gamma$.

Theorem 5.3.1 *If v is a vector field along γ, with $v(s) \neq 0$ in the kernel of $d\Psi_{\gamma(s)}$ for any $s \in I$, then the following hold.*

- *$\gamma(0) = z_0$, is a cuspidal edge if and only if $\det(\gamma'(0), v(0)) \neq 0$, where $\det$ denotes the determinant of 2×2 matrices and prime indicates differentiation with respect to s.*
- *$\gamma(0) = z_0$ is a swallowtail if and only if $\det(\gamma'(0), v(0)) = 0$ and*

$$\left.\frac{d}{ds}\right|_{s=0} \det\left(\gamma'(s), v(s)\right) \neq 0.$$

Theorem 5.3.2 *Let $\alpha : I \to \mathbb{R}^3$ be an analytic curve with non-vanishing curvature on I. Then, for any analytic functions λ, $\phi : I \to \mathbb{R}^3 \quad \lambda > 0$, there is a unique affine maximal map Ψ with U and η given by (5.8) and (5.9), respectively.*
Moreover, α is a singular curve of Ψ and $\alpha(s)$ is a cuspidal edge for all $s \in I$.

Proof. From the hypothesis, we can define the affine maximal map Ψ as in (5.5), with the affine conormal ν given by (5.4). Now, recall that ν is harmonic, hence from (5.8) and (5.9) we get that, along α.

$$[\nu, \nu_s, \nu_t] = [U, U', \eta] = -\langle \alpha', U' \rangle = 0, \tag{5.10}$$

due to (5.7) we have the last part of the above equation. The partial derivative of the above determinant with respect to t is

$$\begin{aligned} [\nu, \nu_s, \nu_t]_t &= [\nu_t, \nu_s, \nu_t] + [\nu, \nu_{st}, \nu_t] + [\nu, \nu_s, \nu_{tt}] \\ &= [\nu, \nu_{st}, \nu_t] - [\nu, \nu_s, \nu_{ss}] \\ &= [U, \eta', \eta] - [U, U', U'']. \end{aligned}$$

From (5.9) we have that,

$$\eta' = \phi\lambda U' + (\phi'\lambda + \phi\lambda')U - \frac{1}{|U|^2}(U' \wedge \alpha' + U \wedge \alpha'') + \frac{2(U \wedge \alpha')}{|U|^3}.$$

Note also that,

$$
\begin{aligned}
[U,\eta',\eta] &= -\alpha'\cdot\eta' \\
&= -\alpha'\cdot\left(\phi\lambda U' + (\phi'\lambda+\phi\lambda')U - \frac{1}{|U|^2}(U'\wedge\alpha' + U\wedge\alpha'') + \frac{2(U\wedge\alpha')}{|U|^3}\right) \\
&= \alpha'\cdot\left(\frac{U\wedge\alpha''}{|U|^2}\right) \\
&= \frac{1}{|U|^2}U\cdot(\alpha''\wedge\alpha') \\
&= \frac{-\lambda}{|U|^2}U\cdot U \\
&= -\lambda.
\end{aligned}
$$

Here, we have used (5.8) and (5.9), to conclude that $\alpha'\cdot U = \dfrac{[\alpha',\alpha',\alpha'']}{\lambda} = 0$, $\alpha'\cdot U' = 0$, $[\alpha',U',\alpha'] = 0$, and $\alpha'\cdot(U\wedge\alpha') = [\alpha',U,\alpha'] = 0$.

Now, we compute the first and second derivative of U

$$
U' = \frac{\alpha'\wedge\alpha'''}{\lambda} - \frac{\lambda'(\alpha'\wedge\alpha'')}{\lambda^2}.
$$

and

$$
\begin{aligned}
U'' &= \frac{(\alpha'\wedge\alpha''')'}{\lambda} - \frac{2\lambda'(\alpha'\wedge\alpha''')}{\lambda^2} \\
&= \frac{(\alpha'\wedge\alpha^{(4)}) + \alpha''\wedge\alpha'''}{\lambda} - \frac{2\lambda'(\alpha'\wedge\alpha''')}{\lambda^2}.
\end{aligned}
$$

The determinant of the vector U, U' and U'' is

$$
\begin{aligned}
[U,U',U''] &= \left[\frac{\alpha'\wedge\alpha''}{\lambda}, \frac{\alpha'\wedge\alpha'''}{\lambda} - \frac{\lambda'(\alpha'\wedge\alpha'')}{\lambda^2}, \frac{(\alpha'\wedge\alpha^{(4)}) + \alpha''\wedge\alpha'''}{\lambda} - \frac{2\lambda'(\alpha'\wedge\alpha''')}{\lambda^2}\right] \\
&= \left[\frac{\alpha'\wedge\alpha''}{\lambda}, \frac{\alpha'\wedge\alpha'''}{\lambda}, \frac{(\alpha'\wedge\alpha^{(4)}) + \alpha''\wedge\alpha'''}{\lambda}\right] - \\
&\quad - \left[\frac{\alpha'\wedge\alpha''}{\lambda}, \frac{\lambda'(\alpha'\wedge\alpha'')}{\lambda^2}, \frac{(\alpha'\wedge\alpha^{(4)}) + \alpha''\wedge\alpha'''}{\lambda}\right] - \\
&\quad - \left[\frac{\alpha'\wedge\alpha''}{\lambda}, \frac{\alpha'\wedge\alpha'''}{\lambda}, \frac{2\lambda'(\alpha'\wedge\alpha''')}{\lambda^2}\right] + \\
&\quad + \left[\frac{\alpha'\wedge\alpha''}{\lambda}, \frac{\lambda'(\alpha'\wedge\alpha'')}{\lambda^2}, \frac{2\lambda'(\alpha'\wedge\alpha''')}{\lambda^2}\right]. \\
&= \left[\frac{\alpha'\wedge\alpha''}{\lambda}, \frac{\alpha'\wedge\alpha'''}{\lambda}, \frac{(\alpha'\wedge\alpha^{(4)}) + \alpha''\wedge\alpha'''}{\lambda}\right] \\
&= \left[\frac{\alpha'\wedge\alpha''}{\lambda}, \frac{\alpha'\wedge\alpha'''}{\lambda}, \frac{\alpha'\wedge\alpha^{(4)}}{\lambda}\right] + \left[\frac{\alpha'\wedge\alpha''}{\lambda}, \frac{\alpha'\wedge\alpha'''}{\lambda}, \frac{\alpha''\wedge\alpha'''}{\lambda}\right].
\end{aligned}
$$

Remember that, $(A \wedge B) \wedge (A \wedge C) = [A, B, C]A$, then we have that

$$\begin{aligned}\left[\frac{\alpha' \wedge \alpha''}{\lambda}, \frac{\alpha' \wedge \alpha'''}{\lambda}, \frac{\alpha' \wedge \alpha^{(4)}}{\lambda}\right] &= \frac{1}{\lambda^3}\left[(\alpha' \wedge \alpha'') \wedge (\alpha' \wedge \alpha''')\right] \cdot (\alpha' \wedge \alpha^{(4)}) \\ &= \frac{1}{\lambda^3}\left[\alpha', \alpha'', \alpha'''\right] \alpha' \cdot (\alpha' \wedge \alpha^{(4)}) \\ &= \frac{1}{\lambda^3}\left[\alpha', \alpha'', \alpha'''\right]\left[\alpha', \alpha', \alpha^{(4)}\right] = 0.\end{aligned}$$

Similarly, we can compute

$$\begin{aligned}\left[\frac{\alpha' \wedge \alpha''}{\lambda}, \frac{\alpha' \wedge \alpha'''}{\lambda}, \frac{\alpha'' \wedge \alpha'''}{\lambda}\right] &= \frac{1}{\lambda^3}\left[(\alpha' \wedge \alpha'') \wedge (\alpha' \wedge \alpha''')\right] \cdot (\alpha'' \wedge \alpha''') \\ &= \frac{1}{\lambda^3}\left[\alpha', \alpha'', \alpha'''\right] \alpha' \cdot (\alpha'' \wedge \alpha''') \\ &= \frac{1}{\lambda^3}\left[\alpha', \alpha'', \alpha'''\right]\left[\alpha', \alpha'', \alpha'''\right].\end{aligned}$$

Hence, we conclude that

$$[U, U', U''] = \frac{1}{\lambda^3}[\alpha', \alpha'', \alpha''']^2.$$

Furthermore,

$$[\nu, \nu_s, \nu_t]_t = -\lambda - \frac{1}{\lambda^3}[\alpha', \alpha'', \alpha''']^2 < 0.$$

Consequently, $[\nu, \nu_z, \nu_{\bar{z}}]$ does not vanish identically and the points of α are the unique singular points in a neighborhood of it.

Along the α we have

$$\Psi_s = \alpha' \qquad \text{and} \qquad \Psi_t = -\frac{[\alpha', \alpha'', \alpha''']}{\lambda^2}\alpha'. \tag{5.11}$$

Note also that,

$$\begin{aligned} d\Psi \cdot X = \begin{pmatrix} | & | \\ \Psi_s & \Psi_t \\ | & | \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ = \begin{pmatrix} \Psi_s^1 & \Psi_t^1 \\ \Psi_s^2 & \Psi_t^2 \\ \Psi_s^3 & \Psi_t^3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\end{aligned}$$

so,

$$\alpha' x - -\frac{[\alpha', \alpha'', \alpha''']}{\lambda^2}\alpha' y = 0$$

if $v = (x, y)$, we can conclude that the kernel of $d\Psi$ at $\gamma(s) = (s, 0)$ is spanned by

$$v = ([\alpha', \alpha'', \alpha'''], \lambda^2).$$

Hence,

$$\begin{aligned} det(\gamma', v) &= \begin{pmatrix} 1 & 0 \\ [\alpha', \alpha'', \alpha'''] & \lambda^2 \end{pmatrix} \\ &= \lambda^2 \neq 0. \end{aligned}$$

and we can conclude that $\alpha(s)$ is a cuspidal edge for all $s \in I$. Where we are using the techniques developed in [10].

■

Theorem 5.3.3 *Let $\alpha : \mathbb{R} \to \mathbb{R}^3$ be an analytic curve with non-vanishing curvature on $I - \{0\}$ and such that $0 \in I$ is a zero of $\alpha', \alpha' \wedge \alpha''$ and $[\alpha', \alpha'', \alpha''']$ of order 1, 2 and 3 respectively. Then, for any analytic functions $\lambda, \phi : I \to \mathbb{R}$, $\lambda > 0$ on $I - \{0\}$ and with a zero of order 2 in 0, there is a unique affine maximal map Ψ with U and η given by (5.8) and (5.9), respectively. Moreover, α is a singular curve of Ψ and $\alpha(0)$ is a swallowtail.*

Proof. We are following the arguments of the above proof, from (5.8), (5.9) and (5.11). Note also that U, η and Ψ_t are well defined by the hypothesis.

Hence, from (5.8) and (5.11) we have

$$U = \frac{\alpha' \wedge \alpha''}{\lambda}.$$

$$\Psi_s = \alpha' \qquad \Psi_t = -\frac{[\alpha', \alpha'', \alpha''']}{\lambda^2}\alpha'.$$

It is easy to see that these equations are well defined if $0 \in I$ is a zero of $\alpha', \alpha' \wedge \alpha''$ and $[\alpha', \alpha'', \alpha''']$ of order 1, 2, 3 and $\lambda > 0$ with a zero of order 2 in 0. Since in the data U its numerator has order 2 then its denominator would have also order 2. On the other hand, if α' has order 1 then the numerator of Ψ_t should have order 4 since its denominator has order 4, then $[\alpha', \alpha'', \alpha''']$ has order 3.

However, in this case, the kernel of $d\Psi$ at $\gamma(s) = (s, 0)$ is spanned by

$$v = \left(1, \frac{\lambda^2}{[\alpha', \alpha'', \alpha''']}\right)$$

and $\alpha(0)$ is a swallowtail, since 0 is a zero of orden 1 of

$$\det(\gamma', v) = \frac{\lambda^2}{[\alpha', \alpha'', \alpha''']}.$$

■

5.4
Improper affine spheres with singularities

When we fixed λ and η, from (5.7) and Theorem (3.4.1), if we take $\lambda = [\alpha', \alpha'', \xi_0]$ and $\eta = -\xi_0 \wedge \alpha'$, with $\xi_0 = (0, 0, 1)$, then we can deduce the following results for definite improper affine spheres with singular curves.

Corollary 5.4.1 *Let $\alpha : I \to \mathbb{R}^3$ be an analytic curve with $[\alpha', \alpha'', \xi_0] \neq 0$ on I. Then, there is a unique definite improper affine map containing $\alpha(I)$ in its singular set.*
Moreover, $\alpha(s)$ is a cuspidal edge for all $s \in I$.

Corollary 5.4.2 *Let $\alpha : I \to \mathbb{R}^3$ be an analytic curve with $[\alpha', \alpha'', \xi_0] \neq 0$ on $I - \{0\}$ and such that $0 \in I$ is a zero of $\alpha', \alpha' \wedge \alpha''$, $[\alpha', \alpha'', \xi_0]$ and $[\alpha', \alpha'', \alpha''']$ of order 1, 2, 2 and 3 respectively. Then, there is a unique definite improper affine map containing $\alpha(I)$ in its singular set and $\alpha(0)$ is a swallowtail.*

5.5
Examples of affine maximal surfaces

In this section, we give some examples of affine maximal maps with cuspidal edges.

Example 5.5.1 *Take the curve $\alpha : \mathbb{R} \to \mathbb{R}^3$ and the functions $\lambda, \phi : \mathbb{R} \to \mathbb{R}$ given by*

$$\alpha(s) = (\cos(s), \sin(s), 0), \qquad \lambda, \phi = 1, \quad s \in \mathbb{R}.$$

From Theorem 5.3.2,

$$\alpha'(s) = (-\sin(s), \cos(s), 0), \qquad \alpha''(s) = (-\cos(s), -\sin(s), 0).$$

The data U is

$$U(s) = \frac{\alpha' \wedge \alpha''}{\lambda} = (0, 0, 1).$$

And, similarly with straight-forward calculations η is,

$$\eta(s) = \phi\alpha' \wedge \alpha'' - \frac{\lambda}{|\alpha' \wedge \alpha''|^2}(\alpha' \wedge \alpha'') \wedge \alpha'.$$
$$= (\cos(s), \sin(s), 1)$$

It provides the harmonic affine conormal $\nu : \mathbb{R}^2 \to \mathbb{R}^3$,

$$\nu(z) = Re\left(U(z) - i\int_{s_0}^{z} \eta(\zeta)d\zeta\right), \qquad z \in \Omega \subset \mathbb{C}$$
$$= Re\left((0,0,1) - i\int_{s_0}^{z} (\cos\zeta, \sin\zeta, 1)d\zeta\right),$$
$$= Re\left((0,0,1) - i(\sin(z) - \sin(s_0), -\cos(z) - \cos(s_0), z - s_0)\right)$$

if $z = s + it$,

$$\sin(s+it) = \sin(s)\cos(it) + \cos(s)\sin(it) = \sin(s)\cosh(t) + i\cos(s)\sinh(t).$$

Similarly,

$$\cos(z) = \cos(s)\cosh(t) - i\sin(s)\sinh(t).$$

Therefore,

$$\nu(s,t) = (\cos(s)\sinh(t), \sin(s)\sinh(t), 1 + t).$$

Its derivatives are

$$\nu_t(s,t) = (\cos(s)\cosh(t), \sin(s)\cosh(t), 1),$$

$$\nu_s(s,t) = (-\sin(s)\sinh(t), \cos(s)\sinh(t), 0).$$

Furthermore, using Lelievre formula, i.e.

$$\Psi_s = \nu \wedge \nu_t = (\sin s \sinh t - (1+t)\sin s \cosh t, -\cos s \sinh t + (1+t)\cos s \cosh t, 0),$$

$$\Psi_t = \nu_s \wedge \nu = ((1+t)\cos s \sinh t, (1+t)\sin s \sinh t, -\sinh^2 t).$$

and the affine maximal map $\Psi : \mathbb{R}^2 \to \mathbb{R}^3$ with coordinates,

$$\Psi_1(s,t) = (1+t)\cos s \cosh t - \cos s \sinh t,$$

$$\Psi_2(s,t) = (1+t)\sin s \cosh t - \sin s \sinh t,$$

$$\Psi_3(s,t) = \frac{t}{2} - \frac{1}{4}\sinh 2t.$$

Thus, around $t = 0$, the singular set of Ψ is the circle $\alpha(\mathbb{R}) = \Psi(\mathbb{R} \times \{0\})$ and the singularities are cuspidal edges, (see Figure 5.1).

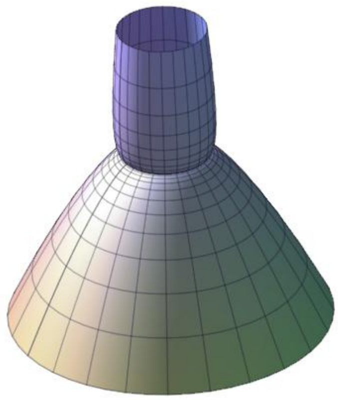

Figure 5.1: Affine maximal map with cuspidal edges.

Example 5.5.2 *Similarly, we can obtain an affine maximal map* $\Psi : \mathbb{R}^2 \to \mathbb{R}^3$ *with*

$$\alpha(s) = (\cos(s), \sin(s), s), \qquad \lambda(s) = \phi(s) = 1, \quad s \in \mathbb{R}.$$

We have that

$$\alpha'(s) = (-\sin(s), \cos(s), 1), \qquad \alpha''(s) = (-\cos(s), -\sin(s), 0).$$

Then,

$$U = (\sin(s), -\cos(s), 1).$$

and

$$\begin{aligned}\eta(s) &= \phi\alpha' \wedge \alpha'' - \frac{\lambda}{|\alpha' \wedge \alpha''|^2}(\alpha' \wedge \alpha'') \wedge \alpha'. \\ &= (\sin(s), -\cos(s), 1) - \frac{1}{2}(-2\cos(s), -2\sin(s), 0) \\ &= (\sin(s) + \cos(s), \sin(s) - \cos(s), 1).\end{aligned}$$

The harmonic affine conormal $\nu : \mathbb{R}^2 \to \mathbb{R}^3$ is

$$\begin{aligned}\nu(z) &= Re\left(U(z) - i\int_{s_0}^{z} \eta(\zeta)d\zeta\right), \qquad z \in \Omega \subset \mathbb{C} \\ &= Re\left((\sin(s), -\cos(s), 1) - i\int_{s_0}^{z}(\sin(z) + \cos(z), \sin(z) - \cos(z), 1)d\zeta\right), \\ &= Re((\sin(z) + i\cos(z) - i\sin(z) - i\cos(s_0) + i\sin(s_0), -\cos(z) + i\cos(z) + \\ &\quad - i\sin(z) - i\cos(s_0) - i\sin(s_0), 1 - iz + is_0)).\end{aligned}$$

If $z = s + it$, we have that,

$$\sin(z) = \sin(s)\cos(it) + \cos(s)\sin(it) = \sin(s)\cosh(t) + i\cos(s)\sinh(t).$$

Similarly,

$$\cos(z) = \cos(s)\cosh(t) - i\sin(s)\sinh(t).$$

The conormal vector is

$$\nu = (\sin s\cosh t{+}\sin s\sinh t{+}\cos s\sinh t, -\cos s\cosh t{+}\sin s\sinh t{-}\cos s\sinh t, 1{+}t),$$

where its derivatives are

$$\nu_s = (\cos s\cosh t{+}\cos s\sinh t{-}\sin s\sinh t, \sin s\cosh t{+}\sin s\sinh t{+}\cos s\sinh t, 0),$$

$$\nu_t = (\sin s\sinh t{+}\sin s\cosh t{+}\cos s\cosh t, -\cos s\sinh t{+}\sin s\cosh t{-}\cos s\cosh t, 1).$$

Therefore, using Lelievre formula we can obtain $\Psi_s = (\Psi_s^1, \Psi_s^2, \Psi_s^3) = \nu \wedge \nu_t$, where each coordinate is

$$\begin{aligned}
\Psi_s^1 &= -t\sin s\cosh t + t\cos s\cosh t + t\cos s\sinh t - \sin s\cosh t + \sin s\sinh t,\\
\Psi_s^2 &= t\sin s\cosh t + t\sin s\sinh t + t\cos s\cosh t + \cos s\cosh t - \cos s\sinh t,\\
\Psi_s^3 &= 1.
\end{aligned}$$

In the same way, we can obtain, $\Psi_t = (\Psi_t^1, \Psi_t^2, \Psi_t^3) = \nu_s \wedge \nu$, with coordinates

$$\begin{aligned}
\Psi_t^1 &= (1+t)(\sin s\cosh t + \sin s\sinh t + \cos s\sinh t),\\
\Psi_t^2 &= (t+1)(-\cos s\cosh t + \sin s\sinh t - \cos s\sinh t),\\
\Psi_t^3 &= -3\cosh^2 t - 2\cosh t\sinh t + 2.
\end{aligned}$$

Integrating Ψ_s with respect to s, we get that

$$\begin{aligned}
\Psi^1(s,t) &= t\sin s\cosh t + t\sin s\sinh t + t\cos s\cosh t + \cos s\cosh t - \cos s\sinh t,\\
\Psi^2(s,t) &= t\sin s\cosh t - t\cos s\cosh t - t\cos s\sinh t + \sin s\cosh t - \sin s\sinh t,\\
\Psi^3(s,t) &= s.
\end{aligned}$$

Similarly, integrating Ψ_t with respect to t

$$\begin{aligned}
\Psi^1(s,t) &= t\sin s\cosh t + t\sin s\sinh t + t\cos s\cosh t + \cos s\cosh t - \cos s\sinh t,\\
\Psi^2(s,t) &= t\sin s\cosh t - t\cos s\cosh t - t\cos s\sinh t + \sin s\cosh t - \sin s\sinh t,\\
\Psi^3(s,t) &= -\frac{3}{2}\cosh t\sinh t + \frac{1}{2}t - \cosh^2 t.
\end{aligned}$$

Hence, the affine maximal map $\Psi : \mathbb{R}^2 \to \mathbb{R}^3$ with coordinates is,

$$\begin{aligned}
\Psi^1(s,t) &= t\sin s\cosh t + t\sin s\sinh t + t\cos s\cosh t + \cos s\cosh t - \cos s\sinh t,\\
\Psi^2(s,t) &= t\sin s\cosh t - t\cos s\cosh t - t\cos s\sinh t + \sin s\cosh t - \sin s\sinh t,\\
\Psi^3(s,t) &= -\frac{3}{2}\cosh t\sinh t + \frac{1}{2}t - \cosh^2 t + s.
\end{aligned}$$

That is, with the helix $\alpha(\mathbb{R}) = \Psi(\mathbb{R}\times\{0\})$ in its singular set.(see Figure 5.2)

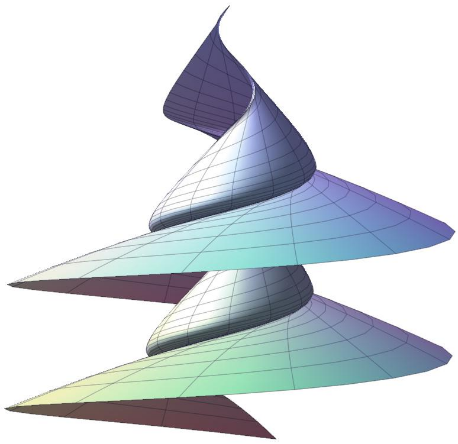

Figure 5.2: Affine maximal map with cuspidal edges.

Example 5.5.3 *The curve* $\alpha : \mathbb{R} \to \mathbb{R}^3$ *defined by*

$$\alpha(s) = \left(\cos(s) + \frac{1}{2}\cos(2s), -\sin(s) + \frac{1}{2}\sin(2s), \frac{1}{6}\cos(3s)\right).$$

has derivatives

$$\alpha'(s) = \left(-\sin(s) - \sin(2s), -\cos(s) + \cos(2s), -\frac{1}{2}\sin(3s)\right),$$

$$\alpha''(s) = \left(-\cos(s) - 2\cos(2s), \sin(s) - 2\sin(2s), -\frac{3}{2}\cos(3s)\right).$$

The data U and η are,

$$U(s) = \frac{\alpha' \wedge \alpha''}{\lambda} = \left(\frac{1}{2}\cos(2s) - \cos(s), \frac{1}{2}\sin(2s) + \sin(s), 1\right).$$

$$\eta = -\xi_0 \wedge \alpha' = (\cos(2s) - \cos(s), \sin(s) - \sin(2s), 0).$$

Therefore,

$$[\alpha', \alpha'', \alpha'''] = \sin(3s) - \frac{1}{2}\sin(6s).$$

with the same 2π -periodic zeros, $\frac{2}{3}\pi$, $\frac{4}{3}\pi$ and 2π, that the function $\lambda : \mathbb{R} \to \mathbb{R}$ given by

$$\lambda(s) = 1 - \cos(3s).$$

Note also that, $\lambda = [\alpha', \alpha'', \xi_0]$, where $\xi_0 = (0, 0, 1)$.

Those provide the harmonic affine conormal $\nu = (\nu^1, \nu^2, \nu^3)$, where each coordinate is

$$\begin{aligned}
\nu^1(s,t) &= \frac{1}{2}\cos(2s)\cosh(2t) - \cos(s)\sinh(t) + \cosh(t)(-\cos(s) + \cos(2s)\sinh(t)),\\
\nu^2(s,t) &= \nu^2 = \frac{1}{2}\cosh(2t)\sin(2s) + \sin(s)(\cosh(t) + \sinh(t) + \cos(s)\sinh(2t),\\
\nu^3(s,t) &= 1.
\end{aligned}$$

From Theorem 5.3.3, we can obtain an affine maximal map $\Psi : \mathbb{R}^2 \to \mathbb{R}^3$ with α as a singular curve with three swallowtails connected by three arcs with cuspidal edges, (see Figure 5.3).

$$\begin{aligned}
\Psi^1(s,t) &= \frac{1}{2}e^t(2\cos(s) + e^t\cos(2s)),\\
\Psi^2(s,t) &= e^{-t}(-1 + e^t\cos(s))\sin(s),\\
\Psi^3(s,t) &= \frac{1}{24}e^{2t}(12 - 3e^{2t} + 4e^t\cos(3s)).
\end{aligned}$$

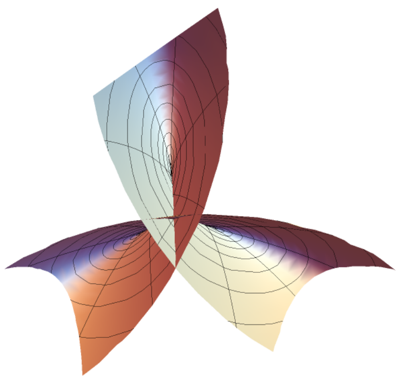

Figure 5.3: Affine maximal map with 3 swallowtails.

Bibliography

[1] J. A. Aledo, A. Martínez and F. Milán. **The affine Cauchy problem**, J. Math. Anal. Appl. 351 (2009), 70-83.

[2] J. A. Aledo, A. Martínez and F. Milán. **Affine maximal surfaces with singularities**, Results Math. 56 (2009), 91-107.

[3] E. Calabi, **Affine differential geometry and holomorphic curves**, Lect. Notes Math. 1422 (1990), 15-21.

[4] W. Blaschke, **Vorlesungen über Differentialgeometrie II**, Affine Differentialgeometrie. Springer, 1923.

[5] Manfredo do Carmo, **Differential Geometry of Curves and Surfaces**, NJ. Prentice Hall, 1976.

[6] K. Nomizu and T. Sasaki, **Affine Differential Geometry**, Cambridge University Press, 1994.

[7] F. Milán, **Singular curves of affine maximal maps**, Fundamental Journal of Mathematics and Mathematical Sciences, Vol. 1, Issue 1, 2014, 57-68.

[8] J. Cho, and Y. Ogata, **Deformation of minimal surfaces with planar curvature lines**, J. Geom. (2016). doi:10.1007/s00022-016-0352-0. Vol. 108, Issue 2, pp 463–479.

[9] U. Abresch. **Constant mean curvature tori in terms of elliptic functions**, J. Reine Angew. Math., 374:169-192, 1987.

[10] M. Kokubu, W. Rossman, K. Saji, M. Umehara and K. Yamada, **Singularities of flat fronts in hyperbolic space**, . Pacific J. Math. 221 (2005), 303-351.

[11] E. Calabi, **Convex Affine Maximal Surfaces**, Results. Math. (1988) 13: 199. doi:10.1007/BF03323241.

[12] L. Verstraelen, L. Vrancken, **Affine Variation Formulas and Affine Minimal Surfaces**, Results. Math. (1988) 13: 199. doi:10.1007/BF03323241.

[13] L. Bers, **Riemann Surfaces**, New York University, Institute of Mathematical Sciences, New York, 1957–1958, pp. 15–35.

[14] Vesna Velickovic, **Visualization of Enneper's Surface by Line Graphics**, Filomat 31:2 (2017), 387–405 DOI 10.2298/FIL1702387V.