6
Cascading Knapsack Inequalities: Hidden Structure in some Inventory-Production-Distribution Problems

6.1 Summary

In the last decade Mixed Integer Programming solvers have evolved enormously contributing to the widespread application of optimization in real world problems in industry. Nonetheless, it is paramount for practitioners to have basic knowledge on how these solvers work and to be able to identify model structures, so one can take full advantage of the machinery at hand. In this work we present a reformulation to a simple problem that appears as sub-problem in a vast majority of supply chain models, and we show the advantage of using suitable mathematical structures in the form of cascading knapsack inequalities to solve it. Moreover, we introduce new reformulations to some special cases, producing tighter linear relaxation and faster solution times.

6.2 Introduction

In order to solve real problems using Mixed Integer Programming it is essential for the specialist to have skills not only in modeling techniques but also in the algorithms used to solve them. Therefore, the best way to approach a problem is to propose a model close to the basic structures that are implemented in the current solvers. This is not an easy task, and frequently the most straightforward model is not the most appropriate to solve a problem in terms of computational time. Additionally, there is no simple guideline to determine the best formulation for a problem, and often this is found by experimentation.

In this work we present a reformulation of inventory balance constraints of a particular problem that leads to a formulation with a special structure identified in this thesis as Cascading Knapsack Inequalities. This structure is of great value as the knapsack inequalities have been studied since the beginning of the integer programming area [Par69, Gui72, Bal75] and nowadays all solvers have implemented very sophisticated techniques to exploit such structures [Ata05, Ash07, Bix07]. Besides, the cascading form allows us to derive tighter reformulations for some special cases.

To motivate the discussion of the reformulation using cascading knapsack inequalities, we present a simple problem that usually arises as a subproblem in a vast majority of supply chain models, namely, the inventory-production-distribution problem [Lej08, Che03, Cha94]. The problem described in this work is related to the petroleum supply planning activity at PETROBRAS. As shown by Rocha [Roc04] and Rocha et al. [Roc09] this is an important subproblem if one is to solve the petroleum supply planning problem. It is worth mentioning that this problem is also a subproblem of the practical application studied by Lejeune and Margot [Lej08] in their paper.

A petroleum company has several platforms producing crude oils that are shipped to its terminals to supply its refineries (see Figure 6.1). The daily petroleum production of each platform is a given data, being estimated by the company's Exploration and Production Department for the entire time horizon. The company wants to determine the best shipment schedule in order to satisfy the refineries demand and avoid the platforms shutdown due to maximum inventory capacity, as well as inventory shortfalls at the terminals in the planning horizon. The refineries demand is known in advance, and due to the company shipping policy, the tankers must be loaded to full capacity. The full capacity loading restriction is a common assumption in the petroleum industry due to the tanker size and cost, and it is adopted to optimize this expensive asset. Additionally, we assume that the number of shipments per time period between each pair of platform-terminal is limited to at most one, and that the number of tankers in each class of tanker is unlimited. The simplification on the number of shipments allows us to define the shipment variables as binary. However, all ideas presented in this work apply without any modification to the case of more than one shipment, i.e., integer shipment variables.

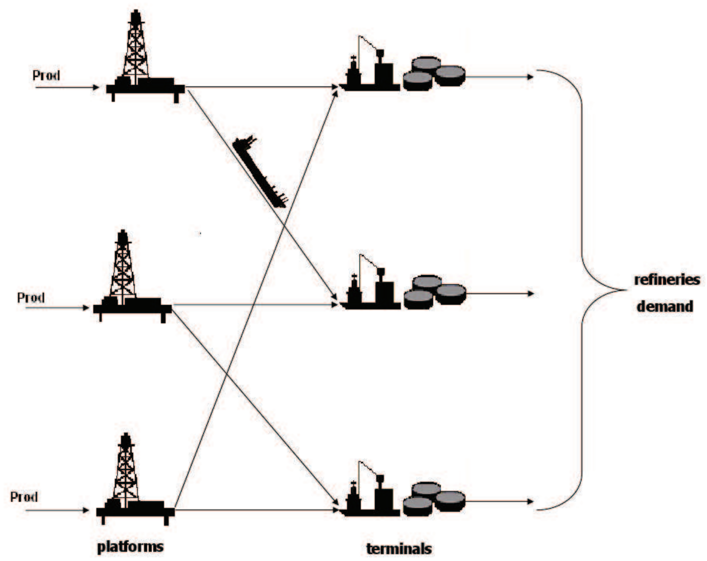

Figure 6.1: Schematic representation of the problem

The problem can be described by the following data.

Platform Data

- Number of petroleum platforms: NP
- Initial inventory of crude oil at platform p: ISP_p
- Daily production of crude oil at platform p: $PR_{p,t}$
- Maximum storage capacity at platform p and time t: $CAP_{p,t}$
- Set of terminals z that can be supplied by platform p: $S_z(p)$
- Set of classes of tankers that can offload platform p: $S_{cl}(p)$

Tanker Fleet Data

- Number of classes of tankers: NCL
- Transportation capacity of tankers in class of tanker cl: CP_{cl}
- Transportation cost per time period for tankers in class of tanker cl: C_{cl}

Terminal Data

- Number of terminals: NZ
- Initial inventory of crude oil p at terminal z: $ISZ_{z,p}$
- Maximum storage capacity at terminal z and time t: $CAP_{z,t}$
- Refineries demand of crude oil p at terminal z and time period t: $DM^{z}_{p,t}$
- Transportation time from platform p to terminal z: $VT_{p,z}$

Planning Horizon: T

(a) Initial Mathematical Formulation

The mathematical representation describing the problem presented above consists of inventory balance equations for the platforms as well as for the terminals. One of the most straightforward way to formulate these equations is to perform the material balance for consecutive times as shown below.

Nomenclature

Indices

p: Platform or Crude oil. As each platform produces only one crude oil, we can refer to platform as well as crude oil by the same index

z: Terminal

cl: Class of tanker

t: Time period

Variables

$x^{p,z}_{cl,t} \in \{0,1\}$: binary variable indicating if a tanker of class cl is assigned to offload platform p and deliver its crude oil to terminal z at time period t

$sp_{p,t} \in \mathbb{R}_+$: inventory level at platform p at time period t

$sz^{z}_{p,t} \in \mathbb{R}_+$: inventory level at terminal z of crude oil p at time period t

$$Min\sum_{p=1}^{NP}\sum_{z\in S_z(p)}\sum_{cl\in S_{cl}(p)}\sum_{t=1}^{T}2C_{cl}VT_{p,z}x_{cl,t}^{p,z} \tag{6.1}$$

s.t.

$$sp_{p,1} = ISP_p \qquad \forall p \in \{1,...,NP\} \tag{6.2}$$

$$sp_{p,t} = sp_{p,t-1} + PR_{p,t-1} - \sum_{z\in S_z(p)}\sum_{cl\in S_{cl}(p)} CP_{cl}x_{cl,t-1}^{p,z} \tag{6.3}$$

$$\forall p \in \{1,...,NP\}, \forall t \in \{2,...,T+1\}$$

$$sz_{p,1}^{z} = ISZ_{z,p} \qquad \forall z \in S_z p \qquad \forall p \in \{1,...,NP\} \tag{6.4}$$

$$sz_{p,t}^{z} = sz_{p,t-1}^{z} - DM_{p,t-1}^{z} + \sum_{cl\in S_{cl}(p)\wedge t>VT_{p,z}} CP_{cl}x_{cl,t-VT_{p,z}}^{p,z} \tag{6.5}$$

$$\forall z \in S_z p, \forall p \in \{1,...,NP\}, \forall t \in \{2,...,T+1\}$$

$$0 \leq sp_{p,t} \leq CAP_{p,t} \qquad \forall p \in \{1,...,NP\}, \forall t \in \{1,...,T+1\} \tag{6.6}$$

$$0 \leq \sum_{\substack{p=1/\\ z\in S_z(p)}}^{NP} sz_{p,t}^{z} \leq CAP_{z,t} \tag{6.7}$$

$$\forall z \in \{1,...,NZ\}, \forall t \in \{1,...,T+1\}$$

$$x_{cl,t}^{p,z} \in \{0,1\} \qquad \forall p \in \{1,...,NP\}, \forall z \in S_z(p), \forall cl \in S_{cl}(p), \forall t \in \{1,...,T\}$$

The objective function is to minimize the transportation cost from platform p to the associated terminals, as given by the set $S_z(p)$, for the planning horizon T. Note that we multiply this cost by 2 to account for the return trip, i.e., we are assuming that the tankers will go back to the same platform offloaded in the previous time period. This simplification is justified since the number of tankers is considered unlimited for each class of tanker. The equations (6.3) and (6.5) are the inventory balance at the platforms and terminals, respectively. In this work we will refer to this formulation as the Initial Formulation since subsequently reformulations will be presented having this formulation as a starting point.

(b) Inventory balance Reformulation

The previous model can be reformulated by projecting out the platforms and terminals inventory variables in the following way. First, writing down the accumulated inventory at platforms and terminals for each time period t, we obtain

$$sp_{p,t} = ISP_p + \sum_{\tau=1}^{t-1} PR_{p,\tau} - \sum_{z\in S_z(p)} \sum_{cl\in S_{cl}(p)} CP_{cl} \sum_{\tau=1}^{t-1} x_{cl,\tau}^{p,z} \tag{6.8}$$
$$\forall p \in \{1,...,P\}, \forall t \in \{2,...,T+1\}$$

$$sz_{p,t}^{z} = ISZ_{z,p} - \sum_{\tau=1}^{t-1} DM_{p,\tau}^{z} + \sum_{cl\in S_{cl}(p)} CP_{cl} \sum_{\tau=1}^{t-VT_{p,z}} x_{cl,\tau}^{p,z} \tag{6.9}$$
$$\forall z \in \{1,...,NZ\}, \forall p/z \in S_z(p), \forall t \in \{2,...,T+1\}$$

Substituting the equations (6.8) and (6.9) into (6.6) and (6.7), respectively, the initial formulation can be rewritten as

$$Min \sum_{p=1}^{NP} \sum_{z\in S_z(p)} \sum_{cl\in S_{cl}(p)} \sum_{t=1}^{T} 2C_{cl}VT_{p,z}x_{cl,t}^{p,z} \tag{6.10}$$

$s.t.$

$$\sum_{z\in S_z(p)} \sum_{cl\in S_{cl}(p)} CP_{cl} \sum_{\tau=1}^{t} x_{cl,\tau}^{p,z} \leq ISP_p + \sum_{\tau=1}^{t} PR_{p,\tau} \tag{6.11}$$
$$\forall p \in \{1,...,NP\}, \forall t \in \{1,...,T\}$$

$$\sum_{z\in S_z(p)} \sum_{cl\in S_{cl}(p)} CP_{cl} \sum_{\tau=1}^{t} x_{cl,\tau}^{p,z} \geq ISP_p + \sum_{\tau=1}^{t} PR_{p,\tau} - CAP_{p,t} \tag{6.12}$$
$$\forall p \in \{1,...,NP\}, \forall t \in \{1,...,T\}$$

$$\sum_{\substack{p=1/\\ z\in S_z(p)\wedge\\ t>VT_{p,z}}}^{NP} \sum_{cl\in S_{cl}(p)} CP_{cl} \sum_{\tau=1}^{t-VT_{p,z}} x_{cl,\tau}^{p,z} \geq \sum_{\tau=1}^{t} DM_{p,\tau}^{z} - ISZ_{z,p} \tag{6.13}$$
$$\forall z \in \{1,...,NZ\}, \forall t \in \{1,...,T\}$$

$$\sum_{\substack{p=1/\\ z\in S_z(p)\wedge\\ t\geq VT_{p,z}}}^{NP} \sum_{cl\in S_{cl}(p)} CP_{cl} \sum_{\tau=1}^{t-VT_{p,z}} x_{cl,\tau}^{p,z} \leq \sum_{\tau=1}^{t} DM_{p,\tau}^{z} + CAP_{z,t} - ISZ_{z} \tag{6.14}$$
$$\forall z \in \{1,...,NZ\}, \forall t \in \{1,...,T\}$$

$$x_{cl,t}^{p,z} \in \{0,1\} \forall p \in \{1,...,NP\}, \forall z \in S_z(p), \forall cl \in S_{cl}(p), t \in \{1,...,T\}$$

Note that the inequalities (6.11), (6.12), (6.13), and (6.14) are knapsack inequalities. Furthermore, they have a special cascading structure since the left-hand side of the inequality at time $t+1$ is equal to the left-hand side of the inequality at time t plus the corresponding term associated with the variables $x^{p,z}_{cl,t+1}$, depending on whether we are referring to the inventory balance at the platforms or at the terminals. We should also notice that this new formulation is exactly as tight as the preceding one since we have just projected the initial formulation into the binary variable space and all extreme points of the former reformulation can be obtained by lifting the extreme points of the new one. Nevertheless, we will show in the following sections that this new structure is advantageous from the perspective of solving this problem using an off-the-shelf MILP solver as well as providing a basis for deriving stronger reformulations, for some special cases. In the sequel we will study some special cases of the model in (6.10) to (6.14), and show that under some circumstances we can obtain either a closed description of the convex hull for the platforms/terminals balance constraints or a tighter formulation for their relaxation.

6.3 Simplified Case: Only one class of tanker

In this section we consider a simplified version of the problem described in section 6.2. We assume that the oil company has only one class of tanker cl servicing all platforms and terminals. In this case, we can derive a closed and polynomial formulation of the convex hull of the inventory balance at the platforms as well as at the terminals. Although this new description of the problem is much stronger than the previous ones, we should note that it does not correspond to the convex hull of the integer points to this problem since the intersection of the convex hull of subsets of constraints does not necessarily yield the convex hull of the entire problem. The model formulation under the assumption of one class of tanker cl is as follows,

$$Min \sum_{p=1}^{NP} \sum_{z \in S_z(p)} \sum_{t=1}^{T} 2C_{cl} VT_{p,z} x_{cl,t}^{p,z} \tag{6.15}$$

s.t.

$$\sum_{z \in S_z(p)} \sum_{\tau=1}^{t} CP_{cl} x_{cl,\tau}^{p,z} \leq ISP_p + \sum_{\tau=1}^{t} PR_{p,\tau} \tag{6.16}$$

$$\forall p \in \{1, ..., NP\}, \forall t \in \{1, ..., T\}$$

$$\sum_{z \in S_z(p)} \sum_{\tau=1}^{t} CP_{cl} x_{cl,\tau}^{p,z} \geq ISP_p + \sum_{\tau=1}^{t} PR_{p,\tau} - CAP_{p,t} \tag{6.17}$$

$$\forall p \in \{1, ..., NP\}, \forall t \in \{1, ..., T\}$$

$$\sum_{\substack{p=1/\\ z \in S_z(p) \wedge \\ t \geq VT_{p,z}}}^{NP} \sum_{\tau=1}^{t-VT_{p,z}} CP_{cl} x_{cl,\tau}^{p,z} \geq \sum_{\tau=1}^{t} DM_{p,\tau}^{z} - ISZ_{z,p} \tag{6.18}$$

$$\forall z \in \{1, ..., NZ\}, \forall t \in \{1, ..., T\}$$

$$\sum_{\substack{p=1/\\ z \in S_z(p) \wedge \\ t \geq VT_{p,z}}}^{NP} \sum_{\tau=1}^{t-VT_{p,z}} CP_{cl} x_{cl,\tau}^{p,z} \leq \sum_{\tau=1}^{t} DM_{z,\tau}^{z} + CAP_{z,t} - ISZ_{z,p} \tag{6.19}$$

$$\forall z \in \{1, ..., NZ\}, \forall t \in \{1, ..., T\}$$

$$x_{cl,t}^{p,z} \in \{0, 1\} \qquad \forall p \in \{1, ..., NP\}, \forall z \in S_z(p), t \in \{1, ..., T\} \tag{6.20}$$

where,

CP_{cl} is the capacity of the class of tanker cl.

$x_{cl,t}^{p,z}$ are binary variables indicating if crude oil are offloaded in platform p and shipped to the terminal z at time t.

Theorem 6.1 *The convex hull of the set of inequalities (6.16), (6.17), and (6.20) is:*

$$\sum_{z \in S_z(p)} \sum_{\tau=1}^{t} x_{cl,\tau}^{p,z} \leq \left\lfloor \frac{ISP_p + \sum_{\tau=1}^{t} PR_{p,\tau}}{CP_{cl}} \right\rfloor \tag{6.21}$$

$$\forall p \in \{1, ..., NP\}, \forall t \in \{1, ..., T\}$$

$$\sum_{z \in S_z(p)} \sum_{\tau=1}^{t} x_{cl,\tau}^{p,z} \geq \left\lceil \frac{ISP_p + \sum_{\tau=1}^{t} PR_{p,\tau} - CAP_{p,t}}{CP_{cl}} \right\rceil \tag{6.22}$$

$$\forall p \in \{1, ..., NP\}, \forall t \in \{1, ..., T\}$$

$$0 \leq x_{cl,t}^{p,z} \leq 1 \qquad \forall p \in \{1, ..., NP\}, \forall z \in S_z(p), t \in \{1, ..., T\} \tag{6.23}$$

Proof: First, by the Gomory-Chvátal procedure we know that these inequalities are valid for the convex hull of integer points of the set defined by the inequalities (6.16), (6.17), and (6.20). Also, defining the variables $w_{p,t}$ as the total number of tankers in class cl leaving the platform p from time period 1 to t, i.e., $w_{p,t} = \sum_{\tau=1}^{t} \sum_{z \in S_z(p)} x_{cl,\tau}^{p,z}$, we can rewrite (6.21), (6.22), and (6.23) as follows:

$$w_{p,1} - \sum_{z \in S_z(p)} x_{cl,1}^{p,z} = 0 \qquad \forall p \in \{1,...,NP\} \tag{6.24}$$

$$w_{p,t} - w_{p,t-1} - \sum_{z \in S_z(p)} x_{cl,t}^{p,z} = 0 \qquad \forall p \in \{1,...,NP\}, \forall t \in \{1,...,T\} \tag{6.25}$$

$$\left\lceil \frac{ISP_p + \sum_{\tau=1}^{t} PR_{p,\tau} - CAP_{p,t}}{CP_{cl}} \right\rceil \leq w_{p,t} \leq \left\lfloor \frac{ISP_p + \sum_{\tau=1}^{t} PR_{p,\tau}}{CP_{cl}} \right\rfloor \tag{6.26}$$
$$\forall p \in \{1,...,NP\}, \forall t \in \{1,...,T\}$$

$$0 \leq x_{cl,t}^{p,z} \leq 1 \qquad \forall p \in \{1,...,NP\}, \forall z \in S_z(p), t \in \{1,...,T\} \tag{6.27}$$

But this is the formulation of the Capacitated Network Flow shown in Figure 6.2, and thus all its vertices are integers [Ahu93]. ■

Observe that, under the assumption of one class of tanker to attend all platforms and terminals, an analogous theorem and proof can be stated for the terminals (inequalities (6.18), (6.19)), and (6.20). We should also mention that a similar idea was presented by Wolsey and Pochet (see [Poc06] chap. 12) in the context of the Discrete Lot Sizing with Constant Capacities.

One could wonder if the same procedure could be used to find the convex hull of integer points for cases with more than one class of tanker. In the next section we answer this question negatively. Nonetheless, we propose a polynomial method to reformulate the problem for the case with two classes of tankers that is proven to be tighter than the original reformulation.

6.4 Two classes of tankers

In this case we consider that the platforms can use two different classes of tankers to ship crude oil to the terminals. For the sake of simplicity, let A and B be the transportation capacities CP_{cl1} and CP_{cl2}, respectively. Moreover, we substitute $C_{p,t}$ and $D_{p,t}$ for $ISP_p + \sum_{\tau=1}^{t-1} PR_\tau$ and $ISP_p + \sum_{\tau=1}^{t-1} PR_\tau - CAP_{p,t}$, respectively. Additionally, we define the variables $x_t^{p,z}$ if the platform p ships crude oil to terminal z at time t using tankers with capacity A and $y_t^{p,z}$ if tankers with capacity B are used instead. In this way,

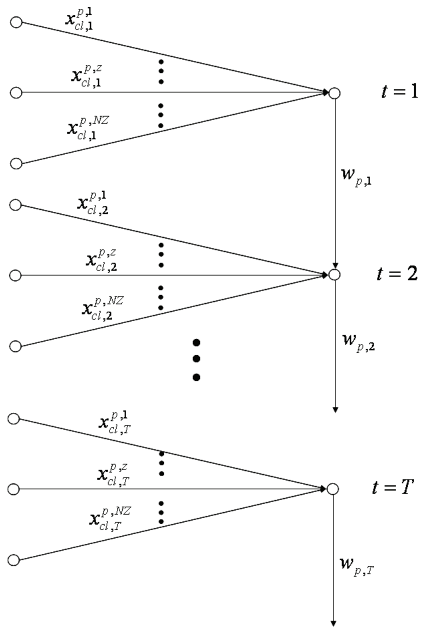

Figure 6.2: Network Flow representation of the reformulation

the sub-model associated with platforms can be rewritten as,

$$\sum_{z\in S_z(p)}\sum_{\tau=1}^{t} Ax_\tau^{p,z} + \sum_{z\in S_z(p)}\sum_{\tau=1}^{t} By_\tau^{p,z} \le C_{p,t} \tag{6.28}$$

$$\forall p \in \{1,...,NP\}, \forall t \in \{1,...,T\}$$

$$\sum_{z\in S_z(p)}\sum_{\tau=1}^{t} Ax_\tau^{p,z} + \sum_{z\in S_z(p)}\sum_{\tau=1}^{t} By_\tau^{p,z} \ge D_{p,t} \tag{6.29}$$

$$\forall p \in \{1,...,NP\}, \forall t \in \{1,...,T\}$$

$$x_t^{p,z}, y_t^{p,z} \in \{0,1\} \qquad \forall p \in \{1,...,NP\}, \forall z \in S_z(p), t \in \{1,...,T\} \tag{6.30}$$

Proposition 6.2 *The Minimization of the function*

$$\sum_{p=1}^{NP}\sum_{z\in S_z(p)}\sum_{t=1}^{T}(2C_{cl1}VT_{p,z}x_t^{p,z} + 2C_{cl2}VT_{p,z}y_t^{p,z})$$

over the constraints defined by (6.28),(6.29) and (6.30) is NP-hard.

Proof: First, we reformulate the problem by defining the variables $w_{p,t} = \sum_{z\in S_z(p)} \sum_{\tau=1}^{t} x_\tau^{p,z}$ and $v_t = \sum_{z\in S_z(p)} \sum_{\tau=1}^{t} y_\tau^{p,z}$. Thus the problem becomes,

$$w_{p,t} = \sum_{z\in S_z(p)} \sum_{\tau=1}^{t} x_\tau^{p,z} \qquad \forall p \in \{1,...,NP\}, \forall t \in \{1,...,T\} \tag{6.31}$$

$$v_{p,t} = \sum_{z\in S_z(p)} \sum_{\tau=1}^{t} y_\tau^{p,z} \qquad \forall p \in \{1,...,NP\}, \forall t \in \{1,...,T\} \tag{6.32}$$

$$Aw_{p,t} + Bv_{p,t} \leq C_{p,t} \qquad \forall p \in \{1,...,NP\}, \forall t \in \{1,...,T\} \tag{6.33}$$

$$Aw_{p,t} + Bv_{p,t} \geq D_{p,t} \qquad \forall p \in \{1,...,NP\}, \forall t \in \{1,...,T\} \tag{6.34}$$

$$x_t^{p,z}, y_t^{p,z} \in \{0,1\} \qquad \forall p \in \{1,...,NP\}, \forall z \in S_z(p), \forall t \in \{1,...,T\} \tag{6.35}$$

$$w_{p,t}, v_{p,t} \in \mathbb{Z}_+, \forall p \in \{1,...,NP\}, \forall t \in \{1,...,T\} \tag{6.36}$$

However, interpreting the variables $x_\tau^{p,z}$ and $w_{p,t}$ as associated to the flow of one class of tanker and $y_\tau^{p,z}$ and $v_{p,t}$ as the flow of the other class of tanker, this formulation corresponds to an Integer 2-Commodity Network Flow problem over time for each platform p, as depicted in Figure 6.3. In [Hal07], Hall et al. proved that this problem is NP-hard even for the case where $A = B = 1$. ■

In the sequel we capitalize on the reasoning behind the proof of the proposition (6.2) to develop a new reformulation for the problem with two classes of tankers. Observe that inequalities (6.33) and (6.34) are 2-integer knapsack inequalities. Moreover, in [Hir76], Hirschberg and Wong proposed a polynomial algorithm to solve this problem, being adapted by Agra and Constantino [Agr07] to find its convex hull of integer points in the context of lifting two-integer knapsack inequalities to obtain strong valid inequalities for integer knapsack sets. We make use of these results to state the following lemma.

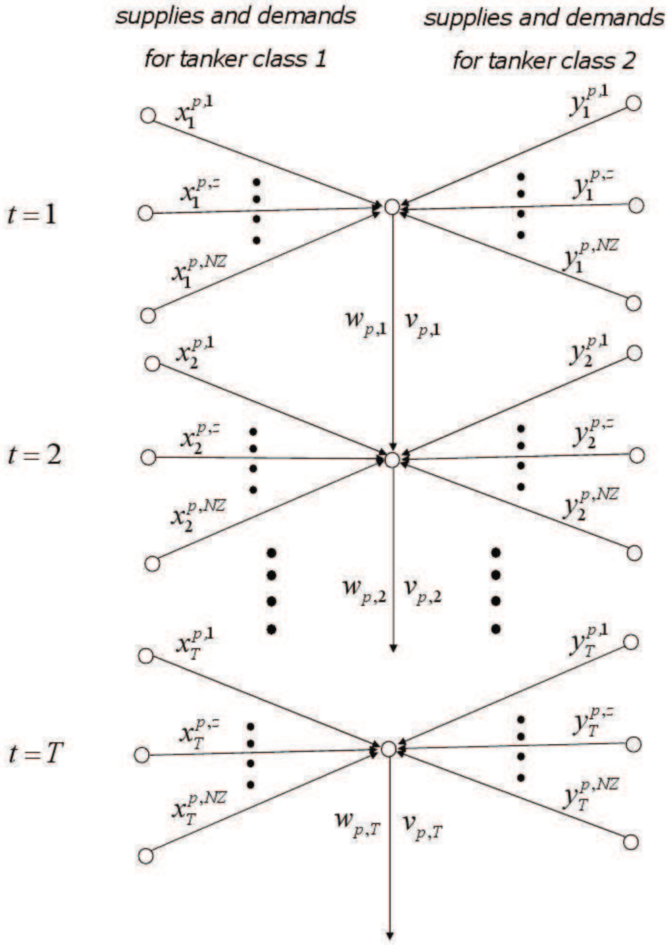

Figure 6.3: Integer 2-Commodity Network Flow representation of the reformulation

Lemma 6.3 *The convex hull of the integer points of the sub-model defined by inequalities (6.33), (6.34), and (6.36) is described by,*

$$(g_{p,t}^{i} - g_{p,t}^{i-1})w_{p,t} + (f_{p,t}^{i} - f_{p,t}^{i-1})v_{p,t} \leq g_{p,t}^{i-1}(f_{p,t}^{i-1} - f_{p,t}^{i}) + f_{p,t}^{i-1}(g_{p,t}^{i} - g_{p,t}^{i-1})$$
$$\forall i \in \{1, ..., n_t\}, \forall p \in \{1, ..., NP\}, \forall t \in \{1, ..., T\}$$

$$(j_{p,t}^{i} - j_{p,t}^{i-1})w_{p,t} + (h_{p,t}^{i} - h_{p,t}^{i-1})v_{p,t} \geq j_{p,t}^{i-1}(h_{p,t}^{i-1} - h_{p,t}^{i}) + h_{p,t}^{i-1}(j_{p,t}^{i} - j_{p,t}^{i-1})$$
$$\forall i \in \{1, ..., m_t\}, \forall p \in \{1, ..., NP\}, \forall t \in \{1, ..., T\}$$

$$w_{p,t}, v_{p,t} \in \mathbb{R}_+, \forall p \in \{1, ..., NP\}, \forall t \in \{1, ..., T\}$$

where the points $(f_{p,t}^{i}, g_{p,t}^{i})$ *and* $(h_{p,t}^{i}, j_{p,t}^{i})$ *are calculated by the algorithm (4) proposed initially by Hirschberg and Wong [Hir76] to solve the Knapsack* *problem with two integer variables and adapted here to exhibit all points of the convex hull of a given knapsack inequality with two integer variables.*

Proof: First, we need to prove that the intersection of the convex hull of the inequalities (6.33) and (6.34) for each platform p and time t is equal to the convex hull of the entire sub-model. This clearly follows since there are no linking variables between any pair of constraints for different platform p and/or time t, i.e., the subspaces where the constraints for each platform p and time t are defined are disjoint. Defining $Q_{p,t}$ as the convex hull of integer points satisfying the inequalities (6.33) and (6.34) for each platform p and time t, the convex hull of integer points for the sub-model is $\sum_{p=1}^{P}\sum_{t=1}^{T} Q_{p,t}$ in the sense of Minkowski sum [Zie98]. To conclude this proof, we need to show that the convex hull of the set of inequalities (6.33) and (6.34) is equal to the intersection of the convex hull of each inequality individually since Algorithm 4 calculates the convex of hull for each inequality separately. From Figure 6.4 this assertion is always true, provided that conditions $\lfloor \frac{C_{p,t}}{A} \rfloor - \lceil \frac{D_{p,t}}{A} \rceil \geq 0$ and $\lfloor \frac{C_{p,t}}{B} \rfloor - \lceil \frac{D_{p,t}}{B} \rceil \geq 0$ hold. But this is the case in our problem since $D_{p,t} = C_{p,t} - CAP_{p,t}$ and $\frac{CAP_p}{A} \geq 1$ (the problem is assumed to be feasible), thus $\lfloor \frac{C_{p,t}}{A} \rfloor - \lceil \frac{D_{p,t}}{A} \rceil \geq \lfloor \frac{C_{p,t}}{A} \rfloor - \lceil \frac{C_{p,t}}{A} - 1 \rceil = 0$, and hence the condition follows. The same argument can be applied to verify the condition for the B denominator.
■

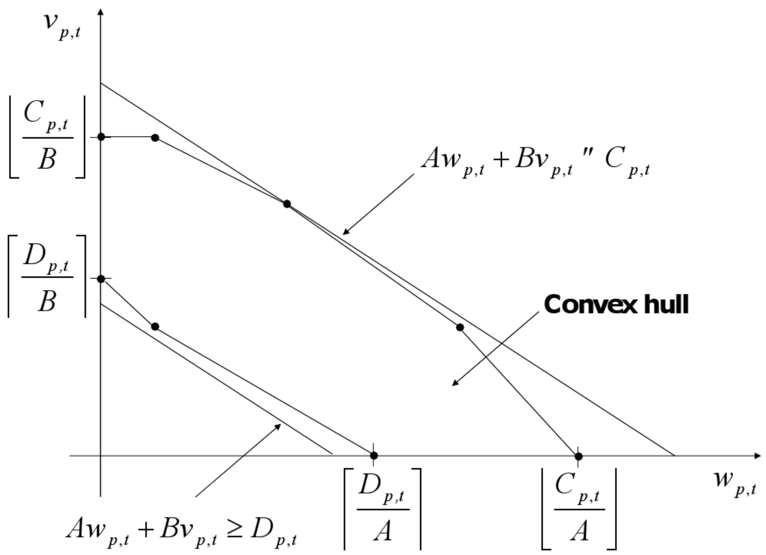

Figure 6.4: Convex hull of the pair of Knapsack inequalities

Algorithm 4 Calculate the convex hull of the knapsack with two variables

Step 0:(Initialization)
if *ineqSign is* $\leq$ **then**
  $j \leftarrow 1, (a^j, b^j) \leftarrow (\lfloor \frac{D}{A} \rfloor, 0)$
  $k \leftarrow 1, (c^k, d^k) \leftarrow (\lfloor \frac{B}{A} \rfloor, 1)$
  $l \leftarrow 1, (e^l, f^l) \leftarrow (\lceil \frac{B}{A} \rceil, 1)$
else if *ineqSign is* $\geq$ **then**
  $j \leftarrow 1, (a^j, b^j) \leftarrow (\lceil \frac{D}{A} \rceil, 0)$
  $k \leftarrow 1, (c^k, d^k) \leftarrow (\lceil \frac{B}{A} \rceil, 1)$
  $l \leftarrow 1, (e^l, f^l) \leftarrow (\lfloor \frac{B}{A} \rfloor, 1)$
end if
Step 1:
while $(a^j - c^k) \geq 0$ **do**
  if *ineqSign is* $\leq$ **then**
    $\delta_B \leftarrow D - a^j A - b^j B$
    $\delta_D \leftarrow -c^k A + d^k B$
    $\delta_F \leftarrow e^l A - f^l B$
  else if *ineqSign is* $\geq$ **then**
    $\delta_B \leftarrow a^j A + b^j B - D$
    $\delta_D \leftarrow c^k A - d^k B$
    $\delta_F \leftarrow -e^l A + f^l B$
  end if
  if $\delta_B \geq \delta_D$ **then**
    $j \leftarrow j + 1$
    $\gamma \leftarrow min\{\lfloor \frac{\delta_B}{\delta_D} \rfloor, \lfloor \frac{a^j}{c^k} \rfloor\}$
    $(a^j, b^j) \leftarrow (a^{j-1}, b^{j-1}) + \gamma(-c^k, d^k)$
  else if $\delta_B < \delta_D$ and $\delta_D \geq \delta_F$ **then**
    $k \leftarrow k + 1$
    $\gamma \leftarrow min\{\lfloor \frac{\delta_D}{\delta_F} \rfloor, \lceil \frac{\delta_D - \delta_B}{\delta_F} \rceil\}$
    $(c^k, d^k) \leftarrow (c^{k-1}, d^{k-1}) + \gamma(e^l, f^l)$
  else if $\delta_B < \delta_D$ and $\delta_D < \delta_F$ **then**
    $l \leftarrow l + 1$
    $\gamma \leftarrow \lfloor \frac{\delta_F}{\delta_D} \rfloor$
    $(e^l, f^l) \leftarrow (e^{l-1}, f^{l-1}) + \gamma(c^k, d^k)$
  end if
end while

Remark 6.4 *Algorithm 4 was initially designed to find the optimal solution of the problem* $max\, f_1x + f_2y$ *s.t.* $P_{\leq} = \{(x, y) \in \mathbb{Z}^2_+ : Ax + By \leq D$ *with* $\frac{f_2}{f_1} > \frac{B}{A}$*. It does that by calculating in sequence all extreme points of the facets of the convex hull of* $P_{\leq}$ *having slope less than or equal to* $\frac{B}{A}$*. In order to obtain the extremes points of facets having slope greater than or equal to* $\frac{B}{A}$*, it suffices to exchange* A *with* B*. Then, to find all the extreme points of the convex hull of the Knapsack with two integer variables we need to run Algorithm 4 twice and take the union of the calculated points. In the case of greater than or equal to inequalities, we only need to change the initialization and the* δ'*s update parts. Figure 6.5 illustrates how this algorithm works. This algorithm is based on the well known Euclidean algorithm to find the greatest common divisor of two integer numbers and applied to approximate a given real number within a required accuracy with a fraction of minimal denominator (see [Mes82] chap. 2 and [Sch86] chap. 6).*

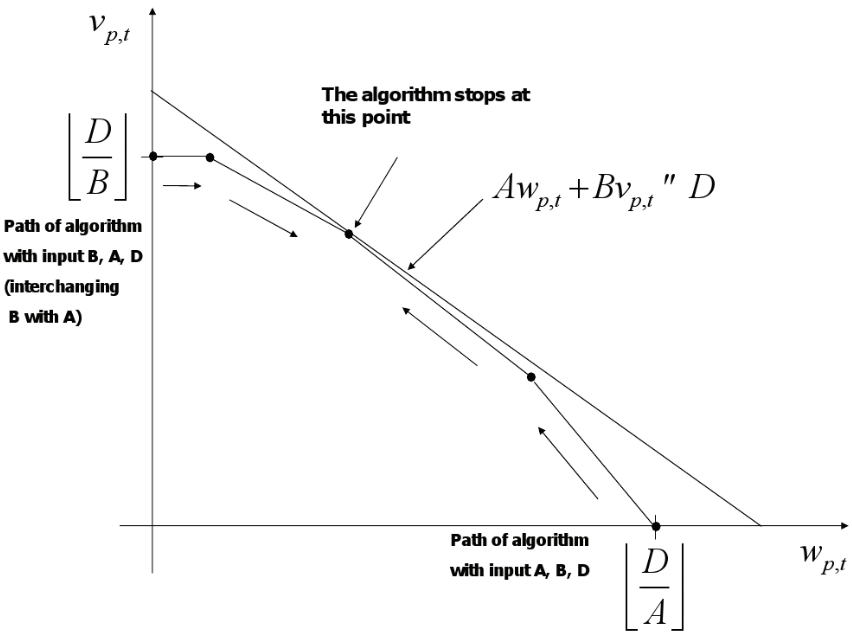

Figure 6.5: Illustration of the algorithm to calculate the convex hull of a 2 integer knapsack

(a) Hull Relaxation Formulation

A tight reformulation to the petroleum supply planning subproblem with two classes of tankers can be obtained by using the inequalities given in Lemma 6.3 and eliminating the $w's$ and $v's$ variables.

$$(g^i_{p,t} - g^{i-1}_{p,t}) \sum_{z \in S_z(p)} \sum_{\tau=1}^{t} x^{p,z}_{\tau} + (f^i_{p,t} - f^{i-1}_{p,t}) \sum_{z \in S_z(p)} \sum_{\tau=1}^{t} y^{p,z}_{\tau} \leq g^{i-1}_{p,t}(f^{i-1}_{p,t} - f^i_{p,t}) + $$
$$+ f^{i-1}_{p,t}(g^i_{p,t} - g^{i-1}_{p,t}) \quad (6.37)$$
$$\forall i \in \{1, ..., n_t\}, \forall p \in \{1, ..., NP\}, \forall t \in \{1, ..., T\}$$

$$(j^i_t - j^{i-1}_t) \sum_{z \in S_z(p)} \sum_{\tau=1}^{t} x^{p,z}_{\tau} + (h^i_t - h^{i-1}_t) \sum_{z \in S_z(p)} \sum_{\tau=1}^{t} y^{p,z}_{\tau} \geq j^{i-1}_{p,t}(h^{i-1}_{p,t} - h^i_{p,t}) + $$
$$+ h^{i-1}_{p,t}(j^i_{p,t} - j^{i-1}_{p,t}) \quad (6.38)$$
$$\forall i \in \{1, ..., m_t\}, \forall p \in \{1, ..., NP\}, \forall t \in \{1, ..., T\}$$

$$x^{p,z}_t, y^{p,z}_t \in \{0, 1\}, \forall p \in \{1, ..., NP\}, \forall z \in S_z(p), \forall t \in \{1, ..., T\} \quad (6.39)$$

It is clear that this formulation is tighter than the formulation (6.28), (6.29), and (6.30) since it is obtained by the convex hull of the knapsack inequalities with two integer variables, and hence its name.

Example:

In order to provide some insights about this new reformulation and compare it with the first reformulation proposed for this problem, we consider the following example data,

Platform Data

- Platform: p
- Initial inventory: $27 \times 10^3\ m^3$
- Daily crude oil production: $23 \times 10^3\ m^3/day$
- Maximum storage capacity: $67 \times 10^3\ m^3$

Tanker Fleet Data

- cl_1: capacity $27 \times 10^3\ m^3$
- cl_2: capacity $43 \times 10^3\ m^3$

Planning horizon: 3 *days*

For simplicity, we consider one terminal. Moreover, only the submodel representing the inventory management at platform p is studied. Thus, the first reformulation is as follows,

$$
\begin{aligned}
&27x^{p,z}_{cl1,1} + 43x^{p,z}_{cl2,1} \leq 50\\
&27x^{p,z}_{cl1,1} + 27x^{p,z}_{cl1,2} + 43x^{p,z}_{cl2,1} + 43x^{p,z}_{cl2,2} \leq 73\\
&27x^{p,z}_{cl1,1} + 27x^{p,z}_{cl1,2} + 27x^{p,z}_{cl1,3} + 43x^{p,z}_{cl2,1} + 43x^{p,z}_{cl2,2} + 43x^{p,z}_{cl2,3} \leq 96\\
&27x^{p,z}_{cl1,1} + 27x^{p,z}_{cl1,2} + 43x^{p,z}_{cl2,1} + 43x^{p,z}_{cl2,2} \geq 6\\
&27x^{p,z}_{cl1,1} + 27x^{p,z}_{cl1,2} + 27x^{p,z}_{cl1,3} + 43x^{p,z}_{cl2,1} + 43x^{p,z}_{cl2,2} + 43x^{p,z}_{cl2,3} \geq 29\\
&x^{p,z}_{cl1,1}, x^{p,z}_{cl1,2}, x^{p,z}_{cl1,3}, x^{p,z}_{cl2,1}, x^{p,z}_{cl2,2}, x^{p,z}_{cl2,3} \in \{0,1\}
\end{aligned}
$$

In the same way, using Algorithm 4, the new reformulation can be written as,

$$
\begin{aligned}
&x^{p,z}_{cl1,1} + x^{p,z}_{cl2,1} \leq 1\\
&x^{p,z}_{cl2,1} + x^{p,z}_{cl2,2} \leq 1\\
&x^{p,z}_{cl1,1} + x^{p,z}_{cl1,2} + x^{p,z}_{cl2,1} + x^{p,z}_{cl2,2} \leq 2\\
&2x^{p,z}_{cl1,1} + 2x^{p,z}_{cl1,2} + 2x^{p,z}_{cl1,3} + 3x^{p,z}_{cl2,1} + 3x^{p,z}_{cl2,2} + 3x^{p,z}_{cl2,3} \leq 6\\
&x^{p,z}_{cl1,1} + x^{p,z}_{cl1,2} + x^{p,z}_{cl2,1} + x^{p,z}_{cl2,2} \geq 1\\
&x^{p,z}_{cl1,1} + x^{p,z}_{cl1,2} + x^{p,z}_{cl1,3} + 2x^{p,z}_{cl2,1} + 2x^{p,z}_{cl2,2} + 2x^{p,z}_{cl2,3} \geq 2\\
&x^{cl1}_0, x^{cl1}_1, x^{cl1}_2, x^{cl2}_0, x^{cl2}_1, x^{cl2}_2 \in \{0,1\}
\end{aligned}
$$

It can be shown that the Hull Relaxation formulation is the intersection of the first Chvátal closure of each inequality in the first reformulation, and it is easy to verify that its inequalities dominate the ones in the first reformulation. Also it is important to note that not all facets could be obtained by simple cover cuts inequalities as shown by Balas [Bal75]. However, possibly after applying a lifting procedure we might find all facets but at a higher computational expense. We compare in Table 6.1 the convex hull of the relaxations using the two reformulation to shed some light on how different they can be. The extremes points in Table 6.1 were obtained using PORTA [Por99].

Table 6.1: Extreme points of the Relaxations

Inventory Balance Reformulation

Points	$x^{p,z}_{cl1,1}$	$x^{p,z}_{cl1,2}$	$x^{p,z}_{cl1,3}$	$x^{p,z}_{cl2,1}$	$x^{p,z}_{cl2,2}$	$x^{p,z}_{cl2,3}$
(1)	0	0	0	0	6/43	23/43
(2)	0	0	0	0	29/43	0
(3)	0	0	0	6/43	0	23/43
(4)	0	0	0	29/43	0	0
(5)	0	0	23/27	0	6/43	0
(6)	0	0	23/27	6/43	0	0
(7)	0	2/9	0	0	0	23/43
(8)	0	2/9	23/27	0	0	0
(9)	2/9	0	0	0	0	23/43
(10)	2/9	0	23/27	0	0	0
(11)	0	0	0	0	6/43	1
(12)	0	0	0	0	1	0
(13)	0	0	0	6/43	0	1
(14)	0	0	0	30/43	1	0
(15)	0	0	0	30/43	1	23/43
(16)	0	0	0	1	0	0
(17)	0	0	0	1	30/43	0
(18)	0	0	0	1	30/43	23/43
(19)	0	0	23/27	30/43	1	0
(20)	0	0	23/27	1	30/43	0
(21)	0	0	1	0	6/43	0
(22)	0	0	1	6/43	0	0
(23)	0	2/9	0	0	0	1
(24)	0	2/9	1	0	0	0
(25)	0	1	0	0	0	2/43
(26)	0	1	0	0	2/43	0
(27)	0	1	0	2/43	0	0
(28)	0	1	2/27	0	0	0
(29)	2/27	1	0	0	0	0
(30)	2/9	0	0	0	0	1
(31)	2/9	0	1	0	0	0
(32)	7/27	0	0	1	0	0
(33)	7/27	0	0	1	23/43	0
(34)	7/27	0	0	1	23/43	23/43
(35)	7/27	0	23/27	1	23/43	0
(36)	7/27	23/27	0	1	0	0
(37)	7/27	23/27	0	1	0	23/43
(38)	7/27	23/27	23/27	1	0	0
(39)	1	0	0	0	0	2/43
(40)	1	0	0	0	2/43	0
(41)	1	0	0	2/43	0	0
(42)	1	0	0	23/43	0	0
(43)	1	0	0	23/43	23/43	0
(44)	1	0	0	23/43	23/43	23/43
(45)	1	0	2/27	0	0	0
(46)	1	0	23/27	23/43	23/43	0
(47)	1	2/27	0	0	0	0
(48)	1	23/27	0	23/43	0	0
(49)	1	23/27	0	23/43	0	23/43

Points	$x^{p,z}_{cl1,1}$	$x^{p,z}_{cl1,2}$	$x^{p,z}_{cl1,3}$	$x^{p,z}_{cl2,1}$	$x^{p,z}_{cl2,2}$	$x^{p,z}_{cl2,3}$
(51)	0	0	0	0	1	1
(52)	0	0	0	10/43	1	1
(53)	0	0	0	1	0	1
(54)	0	0	0	1	10/43	1
(55)	0	0	10/27	0	1	1
(56)	0	0	10/27	1	0	1
(57)	0	0	1	0	6/43	1
(58)	0	0	1	0	26/43	1
(59)	0	0	1	0	1	0
(60)	0	0	1	0	1	26/43
(61)	0	0	1	6/43	0	1
(62)	0	0	1	26/43	0	1
(63)	0	0	1	26/43	1	0
(64)	0	0	1	1	0	0
(65)	0	0	1	1	0	26/43
(66)	0	0	1	1	26/43	0
(67)	0	2/9	1	0	0	1
(68)	0	10/27	0	0	1	1
(69)	0	10/27	0	1	0	1
(70)	0	26/27	1	0	0	1
(71)	0	26/27	1	0	1	0
(72)	0	26/27	1	1	0	0
(73)	0	1	0	0	0	1
(74)	0	1	0	0	26/43	1
(75)	0	1	0	0	1	0
(76)	0	1	0	0	1	26/43
(77)	0	1	0	3/43	1	0
(78)	0	1	0	3/43	1	23/43
(79)	0	1	0	26/43	0	1
(80)	0	1	0	1	0	0
(81)	0	1	0	1	0	26/43
(82)	0	1	0	1	3/43	0
(83)	0	1	0	1	3/43	23/43
(84)	0	1	23/27	3/43	1	0
(85)	0	1	23/27	1	3/43	0
(86)	0	1	26/27	0	0	1
(87)	0	1	26/27	0	1	0
(88)	0	1	26/27	1	0	0
(89)	0	1	1	0	0	0
(90)	0	1	1	0	0	42/43
(91)	0	1	1	0	42/43	0
(92)	0	1	1	42/43	0	0
(93)	1/9	1	0	0	1	0
(94)	1/9	1	0	0	1	23/43
(95)	1/9	1	0	1	0	0
(96)	1/9	1	0	1	0	23/43
(97)	1/9	1	23/27	0	1	0
(98)	1/9	1	23/27	1	0	0
(99)	2/9	0	1	0	0	1

Points	$x^{p,z}_{cl1,1}$	$x^{p,z}_{cl1,2}$	$x^{p,z}_{cl1,3}$	$x^{p,z}_{cl2,1}$	$x^{p,z}_{cl2,2}$	$x^{p,z}_{cl2,3}$
(100)	7/27	0	0	1	0	1
(101)	7/27	0	0	1	3/43	1
(102)	7/27	0	1/9	1	0	1
(103)	7/27	0	1	1	0	0
(104)	7/27	0	1	1	0	19/43
(105)	7/27	0	1	1	19/43	0
(106)	7/27	1/9	0	1	0	1
(107)	7/27	19/27	1	1	0	0
(108)	10/27	0	0	0	1	1
(109)	26/27	0	1	0	0	1
(110)	26/27	0	1	0	1	0
(111)	26/27	1	0	0	0	1
(112)	1	0	0	0	0	1
(113)	1	0	0	0	26/43	1
(114)	1	0	0	0	1	0
(115)	1	0	0	0	1	26/43
(116)	1	0	0	3/43	1	0
(117)	1	0	0	3/43	1	23/43
(118)	1	0	0	23/43	0	1
(119)	1	0	0	23/43	3/43	1
(120)	1	0	1/9	23/43	0	1
(121)	1	0	23/27	3/43	1	0
(122)	1	0	26/27	0	0	1
(123)	1	0	26/27	0	1	0
(124)	1	0	1	0	0	0
(125)	1	0	1	0	0	42/43
(126)	1	0	1	0	42/43	0
(127)	1	0	1	23/43	0	0
(128)	1	0	1	23/43	0	19/43
(129)	1	0	1	23/43	19/43	0
(130)	1	1/9	0	0	1	0
(131)	1	1/9	0	0	1	23/43
(132)	1	1/9	0	23/43	0	1
(133)	1	1/9	23/27	0	1	0
(134)	1	19/27	1	23/43	0	0
(135)	1	26/27	0	0	0	1
(136)	1	1	0	0	0	0
(137)	1	1	0	0	0	42/43
(138)	1	1	0	0	19/43	0
(139)	1	1	0	0	19/43	23/43
(140)	1	1	0	19/43	0	0
(141)	1	1	0	19/43	0	23/43
(142)	1	1	23/27	0	19/43	0
(143)	1	1	23/27	19/43	0	0
(144)	1	1	1	0	0	0
(145)	1	1	1	0	0	15/43
(146)	1	1	1	0	15/43	0
(147)	1	1	1	15/43	0	0

Hull Relaxation Reformulation

Points	$x^{p,z}_{cl1,1}$	$x^{p,z}_{cl1,2}$	$x^{p,z}_{cl1,3}$	$x^{p,z}_{cl2,1}$	$x^{p,z}_{cl2,2}$	$x^{p,z}_{cl2,3}$
(1)	0	0	0	0	1	0
(2)	0	0	0	1	0	0
(3)	0	1	0	0	0	1/2
(4)	0	1	0	0	1/2	0
(5)	0	1	0	1/2	0	0
(6)	1/2	0	1	1/2	1/2	0
(7)	1/2	1	0	1/2	1/2	0
(8)	1	0	0	0	0	1/2
(9)	1	0	0	0	1/2	0
(10)	0	0	0	0	1	1
(11)	0	0	0	1	0	1
(12)	0	0	1	0	1	0
(13)	0	0	1	0	1	1/3
(14)	0	0	1	1	0	0
(15)	0	0	1	1	0	1/3
(16)	0	1/2	1	0	1	0
(17)	0	1/2	1	1	0	0
(18)	0	1	0	0	0	1
(19)	0	1	0	0	1/3	1
(20)	0	1	0	0	1	0
(21)	0	1	0	0	1	1/3
(22)	0	1	0	1/3	0	1
(23)	0	1	0	1	0	0
(24)	0	1	0	1	0	1/3
(25)	0	1	1/2	0	0	1
(26)	0	1	1/2	0	1	0
(27)	0	1	1/2	1	0	0
(28)	0	1	1	0	0	0
(29)	0	1	1	0	0	2/3
(30)	0	1	1	0	2/3	0
(31)	0	1	1	2/3	0	0
(32)	1/2	0	1	0	1	0
(33)	1/2	1	0	0	0	1
(34)	1/2	1	0	0	1	0
(35)	1	0	0	0	0	1
(36)	1	0	0	0	1/3	1
(37)	1	0	0	0	1	0
(38)	1	0	0	0	1	1/3
(39)	1	0	1/2	0	0	1
(40)	1	0	1/2	0	1	0
(41)	1	0	1	0	0	0
(42)	1	0	1	0	0	2/3
(43)	1	0	1	0	2/3	0
(44)	1	1/2	0	0	0	1
(45)	1	1/2	0	0	1	0
(46)	1	1	0	0	0	0
(47)	1	1	0	0	0	2/3
(48)	1	1	0	0	2/3	0
(49)	1	1	1	0	0	0

Notice that the Inventory Balance Reformulation has 147 extreme points. In contrast, the Hull Relaxation Reformulation has only 49. This result is somewhat surprising, because usually the number of extreme points in tighter reformulation tends to stay approximately the same, or increases due to the number of faces introduced. However, this was not the case here. Besides, the reduction in the number of extreme points in the Hull Relaxation Reformulation was also observed in all tests we conducted that are not reported in this thesis.

Another interesting fact is the integer extreme points. The Hull Relaxation Reformulation has exactly the same integer points as the Inventory Balance Reformulation, i.e., 15 integer extreme points, illustrating that in this example 98 fractional extreme points were cut off from the Inventory Reformulation and no integer extreme points were added by making use of a tighter formulation.

6.5 Computational Study

To evaluate the effectiveness of each formulation, we implemented them in Visual C++ on Windows using ILOG Concert Technology and solved them using CPLEX 11.0 [Ilo07] with default parameters as provided by the Vendor, and where the presolve was used in all cases. All instances were run on a personal computer Pentium 4 3.2GHz and 4Gb of RAM. The first implementation is the initial formulation and we refer to it in this work as **InitModel**. The second implementation is the one using the inventory balance reformulation presented in section 6.2(b), and it is named **InvRef**. The third implementation corresponds to the model applying the tighter reformulations for the platforms/terminals submodels with one and two classes of tankers. Basically, this last implementation checks if the platforms and/or the terminals have one or two classes of tankers associated to them and generates automatically the best reformulation for that submodel. This last implementation is referred to as **HullRel**.

We randomly generated 75 instances consisting of 9 platforms, 2 terminals and up to 5 classes of tankers. For the sake of simplification, we assume that all platforms produce crude oils with approximately the same quality, allowing us to satisfy the demand of a crude oil with another one. This simplification is a fact that occur in real world practice and allows us to aggregate the inventories at terminals as well as the refineries demand. The objective in all instances is to minimize the transportation cost over a horizon of 30 days. In the first 25 instances, we consider just one class of tanker for all platforms. In these cases, we attempt to solve the instances to optimality, i.e., 0% optimality gap. In the instances from 26 to 50, each platform has also one class of tanker. However, they can be different from platform to platform. This implies that the reformulations presented in sections 6.3 and 6.4 cannot be applied to the terminals since, in general, they will be supplied by more than two classes of tankers. For these instances, we set a 0.1% optimality gap since they are harder to solve and we want to further evidence the differences between formulations. Finally, for instances numbered from 51 to 75, the platforms can ship crude oil to the terminals with two different classes of tankers. These are the hardest instances, and as none of the models could solve them to optimality for a time limit of 20 minutes, we set a 1% optimality gap to accept the best solution as optimal. Data for all instances are available from the authors upon request.

Table 6.2: Comparison of LP relaxation

instance	Best Solution	**InitModel & InvRef**			**HullRel**			$\frac{gap_2}{gap_1}$ (%)
		LP value	abs gap_1	gap_1	LP value	abs gap_2	gap_2	
1	66880	65687	1193	1.8	66474	406	0.6	33.3
2	78672	77351.1	1320.9	1.7	78505.1	166.9	0.2	11.8
3	167200	165210	1990	1.2	167200	0	0.0	0.0
4	102376	100491	1885	1.8	101590	786	0.8	44.4
5	152760	151009	1751	1.1	151840	920	0.6	54.5
6	117920	113614	4306	3.7	116294	1626	1.4	37.8
7	123280	120234	3046	2.5	122717	563	0.5	20.0
8	226480	221116	5364	2.4	225199	1281	0.6	25.0
9	187740	182883	4857	2.6	186727	1013	0.5	19.2
10	282472	278470	4002	1.4	281400	1072	0.4	28.6
11	108770	105894	2876	2.6	107630	1140	1.0	38.5
12	104880	103300	1580	1.5	104220	660	0.6	40.0
13	100426	99003.1	1422.9	1.4	99561.8	864.2	0.9	64.3
14	67646	66528.5	1117.5	1.7	67020.2	625.8	0.9	52.9
15	117384	115830	1554	1.3	116750	634	0.5	38.5
16	143338	141841	1497	1.0	142824	514	0.4	40.0
17	178868	176753	2115	1.2	178868	0	0.0	0.0
18	227856	226753	1103	0.5	227004	852	0.4	80.0
19	117710	115438	2272	1.9	117069	641	0.5	26.3
20	118560	116800	1760	1.5	117600	960	0.8	53.3
21	112346	110066	2280	2.0	111527	819	0.7	35.0
22	51708	49825.1	1882.9	3.6	51376.1	331.9	0.6	16.7
23	70030	68495.3	1534.7	2.2	69784.2	245.8	0.4	18.2
24	96254	94659.7	1594.3	1.7	95874.1	379.9	0.4	23.5
25	108996	107840	1156	1.1	108721	275	0.3	27.3
26	55314	54165.3	1148.7	2.1	55071.6	242.4	0.4	19.0
27	83540	81541.3	1998.7	2.4	83406.3	133.7	0.2	8.3
28	148668	146703	1965	1.3	148668	0	0.0	0.0
29	98254	96970.1	1283.9	1.3	97968.6	285.4	0.3	23.1
30	133718	132906	812	0.6	133590	128	0.1	16.7
31	117384	114404	2980	2.5	116842	542	0.5	20.0
32	121042	120340	702	0.6	120767	275	0.2	33.3
33	298556	286628	11928	4.0	293247	5309	1.8	45.0
34	220736	218189	2547	1.2	220066	670	0.3	25.0
35	275380	271362	4018	1.5	274893	487	0.2	13.3
36	120286	119005	1281	1.1	119778	508	0.4	36.4
37	86296	85530.3	765.7	0.9	86156	140	0.2	22.2

Table 6.2: (Continued)

instance	Best Solution	InitModel & InvRef			HullRel			$\frac{gap_2}{gap_1}$(%)
		LP value	abs gap_1	gap_1	LP value	abs gap_2	gap_2	
38	118084	117439	645	0.5	117979	105	0.1	20.0
39	67810	67238.3	571.7	0.8	67728.7	81.3	0.1	12.5
40	110058	108338	1720	1.6	109497	561	0.5	31.3
41	135022	133629	1393	1.0	134568	454	0.3	30.0
42	198060	196180	1880	0.9	197619	441	0.2	22.2
43	186144	183404	2740	1.5	185355	789	0.4	26.7
44	135900	133918	1982	1.5	135089	811	0.6	40.0
45	107044	106120	924	0.9	106756	288	0.3	33.3
46	121298	120009	1289	1.1	121003	295	0.2	18.2
47	79292	77516.3	1775.7	2.2	78594.2	697.8	0.9	40.9
48	107976	106622	1354	1.3	107886	90	0.1	7.7
49	144550	141641	2909	2.0	143592	958	0.7	35.0
50	102472	100036	2436	2.4	102285	187	0.2	8.3
51	50486	49323.5	1162.5	2.3	50168.6	317.4	0.6	26.1
52	67522	65812.9	1709.1	2.5	66981.7	540.3	0.8	31.6
53	123850	121625	2225.0	1.8	123136	714.0	0.6	32.1
54	82906	81053	1853	2.2	82077.4	828.6	1.0	45.5
55	117606	95789.1	21816.9	18.6	96834.4	20771.6	17.7	95.2
56	99018	75697	23321	23.6	77811.7	21206.3	21.4	90.7
57	89020	87313.2	1706.8	1.9	88508.1	511.9	0.6	31.6
58	205196	195439	9757	4.8	199831	5365	2.6	54.2
59	188028	175264	12764	6.8	180467	7561	4.0	58.8
60	200524	193213	7311.0	3.6	197915	2609.0	1.3	35.7
61	93886	91788.6	2097.4	2.2	93305.8	580.2	0.6	27.3
62	65870	63683.4	2186.6	3.3	65257.1	612.9	0.9	27.3
63	75904	74329.6	1574.4	2.1	75145.2	758.8	1.0	47.6
64	53158	52116.8	1041.2	2.0	52877.2	280.8	0.5	25.0
65	80162	77877.3	2284.7	2.9	79208.7	953.3	1.2	41.4
66	115182	112243	2939.0	2.6	113996	1186.0	1.0	40.4
67	176536	172571	3965	2.2	176117	419	0.2	9.1
68	170230	160974	9256	5.4	164380	5850	3.4	63.0
69	101072	99192.9	1879.1	1.9	100861	211	0.2	10.5
70	72762	68900.2	3861.8	5.3	71483.9	1278.1	1.8	34.0
71	99322	97787.3	1534.7	1.5	98664.7	657.3	0.7	46.7
72	54724	53430.3	1293.7	2.4	54223.9	500.1	0.9	37.5
73	65418	63027.6	2390.4	3.7	64860.1	557.9	0.9	24.3
74	81666	80187	1479.0	1.8	81009	657.0	0.8	44.4
75	139990	137795	2195	1.6	138953	1037	0.7	43.8

Table 6.3: Results for one class of tanker for all platforms and terminals

instance	bin. var.	InitModel				InvRef			HullRel		
		constr.	cont. var.	Best Obj.	time(s)	constr.	Best Obj.	time(s)	constr.	Best Obj.	time(s)
1	360	330	341	66880	720.00	660	66880	0.08	660	66880	0.08
2	360	330	341	78672	720.00	660	78672	0.09	660	78672	0.09
3	540	330	341	167200	720.00	660	167200	0.09	660	167200	0.09
4	330	330	341	–	720.00	660	102376	0.09	660	102376	0.09
5	540	330	341	–	720.00	660	152760	0.14	660	152760	0.13
6	540	330	341	–	720.00	660	117920	0.13	660	117920	0.11
7	450	330	341	123280	720.00	660	123280	0.11	660	123280	0.09
8	540	330	341	–	720.00	660	226480	0.13	660	226480	0.13
9	480	330	341	–	720.00	660	187740	0.14	660	187740	0.14
10	540	330	341	282472	720.00	660	282472	0.13	660	282472	0.13
11	480	330	341	–	720.00	660	108770	0.11	660	108770	0.11
12	510	330	341	–	720.00	660	104880	0.11	660	104880	0.11
13	480	330	341	–	720.00	660	100426	0.13	660	100426	0.11
14	450	330	341	67646	720.00	660	67646	0.11	660	67646	0.11
15	510	330	341	117652	720.00	660	117384	0.11	660	117384	0.09
16	540	330	341	143338	720.00	660	143338	0.13	660	143338	0.11
17	480	330	341	–	720.00	660	178868	0.13	660	178868	0.11
18	480	330	341	–	720.00	660	227856	0.11	660	227856	0.09
19	450	330	341	117710	720.00	660	117710	0.13	660	117710	0.13
20	540	330	341	118560	720.00	660	118560	0.11	660	118560	0.11
21	510	330	341	–	720.00	660	112346	0.14	660	112346	0.14
22	390	330	341	–	720.00	660	51708	0.11	660	51708	0.11
23	360	330	341	–	720.00	660	70030	0.09	660	70030	0.09
24	540	330	341	96254	720.00	660	96254	0.13	660	96254	0.11
25	480	330	341	108996	720.00	660	108996	0.13	660	108996	0.11

Table 6.4: Results for one class of tanker for each platform, however it can be different from platform to platform

instance	bin. var.	InitModel				InvRef			HullRel		
		constr.	cont. var.	Best Obj.	time(s)	constr.	Best Obj.	time(s)	constr.	Best Obj.	time(s)
26	360	330	341	55314	720.00	660	55314	1.06	660	55314	1.02
27	360	330	341	83540	720.00	660	83540	0.13	660	83540	0.13
28	540	330	341	148668	720.00	660	148668	0.16	660	148668	0.16
29	480	330	341	98254	720.00	660	98254	720.00	660	98254	720.00
30	540	330	341	–	720.00	660	133718	33.09	660	133718	33.03
31	600	330	341	–	720.00	660	117384	3.08	660	117384	3.08
32	540	330	341	121042	720.00	660	121042	720.00	660	121042	720.00
33	540	330	341	–	720.00	660	298556	720.00	660	298556	720.00
34	480	330	341	220736	720.00	660	220736	2.45	660	220736	2.44
35	510	330	341	275380	720.00	660	275380	720.00	660	275380	720.00
36	480	330	341	120286	720.00	660	120286	720.00	660	120286	720.00
37	510	330	341	86296	720.00	660	86296	1.70	660	86296	1.67
38	480	330	341	118084	720.00	660	118084	0.69	660	118084	0.69
39	450	330	341	67810	720.00	660	67810	1.13	660	67810	1.13
40	510	330	341	110088	720.00	660	110058	720.00	660	110058	720.00
41	540	330	341	–	720.00	660	135022	0.91	660	135022	0.88
42	480	330	341	–	720.00	660	198060	2.22	660	198060	2.19
43	540	330	341	186144	720.00	660	186144	720.00	660	186144	720.00
44	450	330	341	135900	720.00	660	135900	1.61	660	135900	1.58
45	540	330	341	107044	720.00	660	107044	2.16	660	107044	2.09
46	480	330	341	121616	720.00	660	121298	720.00	660	121298	720.00
47	390	330	341	79292	720.00	660	79292	720.00	660	79292	720.00
48	540	330	341	107976	720.00	660	107976	1.83	660	107976	1.80
49	450	330	341	146054	720.00	660	144550	720.00	660	144550	720.00
50	540	330	341	102472	720.00	660	102472	1.86	660	102472	1.86

Table 6.5: Results for two classes of tankers

instance	bin. var.	**InitModel**				**InvRef**			**HullRel**		
		constr.	cont. var.	Best Obj.	time(s)	constr.	Best Obj.	time(s)	constr.	Best Obj.	time(s)
51	720	330	341	50486	720.00	660	50486	2.89	1023	50486	1.24
52	720	330	341	68026	720.00	660	67586	1.69	1294	67522	1.58
53	720	330	341	–	720.00	660	124186	4.28	1427	123850	2.13
54	660	330	341	–	720.00	660	82906	145.67	1287	83184	5.13
55	1080	330	341	–	720.00	660	120692	720.00	1410	117606	720.00
56	1140	330	341	–	720.00	660	115428	720.00	1267	99018	720.00
57	900	330	341	89580	663.80	660	89100	1.73	1283	89020	1.55
58	1080	330	341	–	720.00	660	205612	720.00	1400	205196	156.53
59	960	330	341	–	720.00	660	188724	31.56	1264	188028	3.53
60	900	330	341	–	720.00	660	200524	134.39	1337	200732	31.64
61	960	330	341	–	720.00	660	93886	35.72	1388	94136	5.00
62	1020	330	341	–	720.00	660	65870	260.78	1082	66038	28.95
63	960	330	341	–	720.00	660	76144	12.00	1228	75904	9.63
64	900	330	341	53158	23.89	660	53344	1.42	1047	53314	1.06
65	1020	330	341	–	720.00	660	80162	20.88	1304	80216	4.53
66	960	330	341	–	720.00	660	115348	33.09	1459	115182	15.74
67	870	330	341	–	720.00	660	176536	8.89	1446	176760	1.20
68	960	330	341	–	720.00	660	171290	43.16	1399	170230	16.64
69	900	330	341	102778	720.00	660	101072	1.44	1365	101594	1.39
70	1080	330	341	–	720.00	660	72762	49.27	1268	72762	6.94
71	960	330	341	99370	338.53	660	99322	2.59	1365	99628	1.09
72	780	330	341	54844	720.00	660	54724	5.59	1061	54724	3.22
73	720	330	341	–	720.00	660	65418	37.34	1087	65440	4.38
74	1080	330	341	–	720.00	660	81666	22.49	1305	81678	6.30
75	960	330	341	140226	656.83	660	140218	20.69	1475	139990	8.98

(a) Discussion of Results

Table 6.2 shows a comparison for the LP relaxation at the root node of the search tree against the best known solution for each instance. First, as mentioned previously, the relaxation of the **InvRef** reformulation is exactly the same as the **InitModel** formulation. Second, the initial gaps for all formulations are surprisingly small. In fact, looking at the objective function of the problem, i.e., minimization of the transportation costs, we should expect this to happen since a relaxation solution that uses a fraction of a tanker in different times summing up to one, has the same cost of an integer solution using an entire tanker at some point in time. We believe that the initial gaps for instances 55 and 56 are large, not due to the linear relaxation value, but because the best known solutions are likely to be far from the optimum. This brings up an important theoretical question on how to compare different formulations of a given problem since the initial gap can be made as small or as big as one wants just by changing the objective function. In this work we adopt the relation between gaps of formulations. In this case, we see that the **HullRel** is on average three times better than the other two formulations.

Regarding the solution results in Tables 6.3, 6.4 and 6.5, the differences between the **InitModel** and the reformulations **InvRef** and **HullRel** are much more pronounced. With the **InitModel**, feasible solutions were found in 41 out of 75 instances, while with both reformulations **InvRef** and **HullRel**, feasible solutions were found in all instances confirming the importance of a good formulation to finding feasible solutions to a problem. Concerning the structural differences of formulations, we notice that the number of binary variables are the same for all models and the continuous variables are not present in the **InvRef** and **HullRel**. Besides, the number of constraints for the **InvRef** is twice the number for the **InitModel**, whereas the number of constraints for the **HullRel** compared to the **InitModel** is at least twice as large depending on the number of classes of tankers that can offload the platforms.

Tables 6.3 and 6.4 present the results for one class of tanker. As we can see, even for the easiest instances in Table 6.3, the **InitModel** was only capable of finding feasible solutions to 12 out of 25 instances, whereas the **InvRef** and **HullRel** models were solved in all instances at the root node of the Branch-and-Bound tree. We note that the instances in Table 6.4, where the tankers can be different from platform to platform, are harder to solve. Moreover, we cannot apply the convex hull reformulation to the terminals, as mentioned earlier. In this case, although the **InvRef** and **HullRel** could not close the 0.1% gap in 20 minutes time limit in all instances, they are by far more efficient than the **InitModel**, solving 15 out of 25 instances. We can observe the fact that for one class of tanker, i.e., instances 1 to 50, the results for **InvRef** and **HullRel** are identical. This was expected because the coefficient reduction algorithm applied by CPLEX during the presolve steps turns the **InvRef** precisely into the reformulation presented in section 6.3.

Table 6.5 shows results for two classes of tankers for different formulations. One interesting remark is that with a time limit of 20 minutes, the **InitModel** was only solved in 4 out of 25 instances, while the **InvRef** was solved in 22 out of 25 instances and the **HullRel** was solved in 23 instances. Regarding the computation time, we verify that although the **HullRel** is the largest reformulation in terms of number of constraints, it is also the fastest one, showing that it pays off having a tighter formulation in order to expedite the solution of the problem. On average, the **HullRel** is 4.7 times faster than the **InvRef** for instances with two classes of tankers.

From these results, we can safely assert the following conclusions:

– The reformulations **InvRef** and **HullRel** are by far superior to the **InitModel** if one is to solve problems with the characteristics of the one presented in this chapter, i.e., full load capacity transportation policy.

– For subsets of constraints, where the two classes of tankers assumption holds, the **HullRel** is the reformulation of choice due to its computational efficiency.

6.6 Conclusions

In this chapter we have presented and tested reformulations to a sub-problem of the petroleum supply planning problem that arises frequently in petroleum supply chain models. Through an inventory reformulation, the special structure of Cascading Knapsack Inequalities, hidden in the Initial Formulation, was identified. This allowed us to use an off-the-shelf MIP solver to solve instances that were out of reach with the Initial Model. Furthermore, capitalizing on this special structure, we have proposed tighter reformulations for some special cases of this problem, reducing further the solution times for a large number of instances. Finally, we have pointed out that although we have nowadays sophisticated MIP solvers capable of solving problems never envisaged before, it is still paramount for the users of MIP solvers to have an understanding not only on the modeling but also on how the MIP solvers work if they are to approach more challenging problems.