

III

intuitionistic PUC-Logic

In human speech, one may change his opinion about the truth of a given counterfactual sentence if his knowledge about the subject grows. For example:

- (Pedro) I found Ana sad during the party.
- (Jonas) If her boyfriend had come to party, then Ana would have been happy during the party.

A day after this speech, Jonas discovered that Ana was sad during the party, because his boyfriend betrayed her. So, he may think that his counterfactual sentence is no longer valid.

The intuitionistic approach is a traditional way to deal with knowledge growth. We discuss one alternative approach over the counterfactual logic to express this property of the human speech, presenting the intuitionistic reformulation of the PUC-logic, called iPUC-Logic for short.

We avoid the repetition in this chapter of definitions and lemmas of PUC-Logic that are the same for iPUC-Logic.

Definition 61 *Given a set of worlds $\mathcal{W}$, a nested sets function $\$$ over $\mathcal{W}$ and a truth evaluation function $\mathcal{V}$ for each atomic formula, we define a relation of accessibility from a world u to a world v , denoted by $u \succ v$, as a reflexive and transitive relation such that $\$(v) = \(u) and, if $u \in \mathcal{V}(\alpha)$ and $u \succ v$, then $v \in \mathcal{V}(\alpha)$. Given a world u , the set of worlds v , such that $u \succ v$, is denoted by $\mathcal{A}(u)$.*

The restriction $\$(v) = \(u) means that the proximity notions are preserved by the accessible worlds. This restriction was meant to preserve lemma 63. But since we are interested in knowledge growth, it is not an artificial restriction. In the example above, the perception of Jonas about how things works did not changed much from the additional knowledge. It means that, for him, the notions of similarity were just the same as in the day before.

Definition 62 Given a variable assignment function σ , the relation $\models$ of satisfaction between wff, labels, models and templates is given by:

2. $\langle \mathcal{W}, \$, \mathcal{V}, \chi \rangle \models \neg (\alpha^\Sigma)$ iff: $\neg (\alpha^\Sigma) \in \mathbf{F}_n$ and, $\forall \lambda \in \mathcal{A}(\chi) : \langle \mathcal{W}, \$, \mathcal{V}, \lambda \rangle \not\models \alpha^\Sigma$;
5. $\langle \mathcal{W}, \$, \mathcal{V}, \chi \rangle \models \alpha^\Sigma \rightarrow \beta^\Omega$ iff: $\alpha^\Sigma \rightarrow \beta^\Omega \in \mathbf{F}_n$ and, $\forall \lambda \in \mathcal{A}(\chi)$, if $\langle \mathcal{W}, \$, \mathcal{V}, \lambda \rangle \models \alpha^\Sigma$, then $\langle \mathcal{W}, \$, \mathcal{V}, \lambda \rangle \models \beta^\Omega$;
14. $\langle \mathcal{W}, \$, \mathcal{V}, \chi, N \rangle \models \neg (\alpha^\Sigma)$ iff: $\neg (\alpha^\Sigma) \in \mathbf{F}_w$ and, $\forall \lambda \in \mathcal{A}(\chi) :$
 $\langle \mathcal{W}, \$, \mathcal{V}, \lambda, N \rangle \not\models \alpha^\Sigma$;
17. $\langle \mathcal{W}, \$, \mathcal{V}, \chi, N \rangle \models \alpha^\Sigma \rightarrow \beta^\Omega$ iff: $\alpha^\Sigma \rightarrow \beta^\Omega \in \mathbf{F}_w$ and $\forall \lambda \in \mathcal{A}(\chi) :$
 $\langle \mathcal{W}, \$, \mathcal{V}, \lambda, N \rangle \models \alpha^\Sigma$, then $\langle \mathcal{W}, \$, \mathcal{V}, \lambda, N \rangle \models \beta^\Omega$.

The iPUC Natural Deduction System is obtained from the PUC-ND by removing the rule 7 (classical absurd rule).

III.1 iPUC Soundness and Completeness

Lemma 63 Given Δ without existential quantifiers, if $(\alpha^\Sigma \rightarrow \beta^\Omega)^{\bar{\Delta}}$ is wff, then it implies $\alpha^{\Sigma, \bar{\Delta}} \rightarrow \beta^{\Omega, \bar{\Delta}}$.

Proof: We proceed by induction on the size of Δ :

If Δ is empty, then the implication is true;

(base) If Δ contains only one label, it must be a neighbourhood label:

- $(\alpha^\Sigma \rightarrow \beta^\Omega)^*$ means, by definition, that $\forall N \in \$(\chi) : \langle \mathcal{W}, \$, \mathcal{V}, \chi, N \rangle \models \alpha^\Sigma \rightarrow \beta^\Omega$. Then we know that $\forall N \in \$(\chi) : \forall \lambda \in \mathcal{A}(\chi) :$ if $\langle \mathcal{W}, \$, \mathcal{V}, \lambda, N \rangle \models \alpha^\Sigma$, then $\langle \mathcal{W}, \$, \mathcal{V}, \lambda, N \rangle \models \beta^\Omega$. Given an arbitrary $\lambda \in \mathcal{A}(\chi)$, if $\langle \mathcal{W}, \$, \mathcal{V}, \lambda \rangle \models \alpha^{\Sigma, *}$, then $\forall M \in \$(\lambda) : \langle \mathcal{W}, \$, \mathcal{V}, \lambda, M \rangle \models \alpha^\Sigma$. Since $\$(\lambda) = \(χ) , by the definition of the accessibility relation, we have $\forall M \in \$(\chi) : \langle \mathcal{W}, \$, \mathcal{V}, \lambda, M \rangle \models \alpha^\Sigma$. So, by a conclusion above, we know that $\forall M \in \$(\chi) : \langle \mathcal{W}, \$, \mathcal{V}, \lambda, M \rangle \models \beta^\Omega$ and $\forall M \in \$(\lambda) : \langle \mathcal{W}, \$, \mathcal{V}, \lambda, M \rangle \models \beta^\Omega$, then $\langle \mathcal{W}, \$, \mathcal{V}, \lambda \rangle \models \beta^{\Omega, *}$. In other words, $\alpha^{\Sigma, *} \rightarrow \beta^{\Omega, *}$;
- $(\alpha^\Sigma \rightarrow \beta^\Omega)^N$ means, by definition, that $\sigma(N) \in \$(\chi)$ and $\langle \mathcal{W}, \$, \mathcal{V}, \chi, \sigma(N) \rangle \models \alpha^\Sigma \rightarrow \beta^\Omega$. Then we know that $\forall \lambda \in \mathcal{A}(\chi) :$ if $\langle \mathcal{W}, \$, \mathcal{V}, \lambda, \sigma(N) \rangle \models \alpha^\Sigma$, then $\langle \mathcal{W}, \$, \mathcal{V}, \lambda, \sigma(N) \rangle \models \beta^\Omega$. Given an arbitrary $\lambda \in \mathcal{A}(\chi)$, if $\langle \mathcal{W}, \$, \mathcal{V}, \lambda \rangle \models \alpha^{\Sigma, N}$, then $\sigma(N) \in \$(\lambda)$ and $\langle \mathcal{W}, \$, \mathcal{V}, \lambda, \sigma(N) \rangle \models \alpha^\Sigma$. Since $\$(\lambda) = \(χ) , we know that $\sigma(N) \in \$(\chi)$ and by a conclusion above, we know that $\langle \mathcal{W}, \$, \mathcal{V}, \lambda, \sigma(N) \rangle \models \beta^\Omega$, so $\langle \mathcal{W}, \$, \mathcal{V}, \lambda \rangle \models \beta^{\Omega, N}$. In other words, $\alpha^{\Sigma, N} \rightarrow \beta^{\Omega, N}$;

(base) If Δ contains two labels, it may be $\{\otimes, *\}$, $\{N, *\}$, $\{\otimes, u\}$ or $\{N, u\}$. But we just need to look at the distributivity for the $*$ label and for world variables, because we have already seen the distributivity of the $\rightarrow$ connective for the label $\otimes$ and for any neighbourhood variable.

- $(\alpha^\Sigma \rightarrow \beta^\Omega)^{*,\otimes}$ means, by definition, that $\forall N \in \$(\chi) : \forall w \in N : \langle \mathcal{W}, \$, \mathcal{V}, w \rangle \models \alpha^\Sigma \rightarrow \beta^\Omega$. Then we know that $\forall N \in \$(\chi) : \forall w \in N : \forall \lambda \in \mathcal{A}(w) : \text{if } \langle \mathcal{W}, \$, \mathcal{V}, \lambda \rangle \models \alpha^\Sigma \text{ then } \langle \mathcal{W}, \$, \mathcal{V}, \lambda \rangle \models \beta^\Omega$. Given an arbitrary $\lambda \in \mathcal{A}(\chi)$, if we have $\langle \mathcal{W}, \$, \mathcal{V}, \lambda \rangle \models \alpha^{\Sigma,*,\otimes}$, then $\forall M \in \$(\lambda) : \forall z \in M : \langle \mathcal{W}, \$, \mathcal{V}, z \rangle \models \alpha^\Sigma$. Since $\$(\lambda) = \(χ) , $\forall M \in \$(\chi) : \forall z \in M : \langle \mathcal{W}, \$, \mathcal{V}, z \rangle \models \alpha^\Sigma$. From $z \in \mathcal{A}(z)$ and a conclusion above, we know that $\forall M \in \$(\chi) : \forall z \in M : \langle \mathcal{W}, \$, \mathcal{V}, z \rangle \models \beta^\Omega$ and $\forall M \in \$(\lambda) : \forall z \in M : \langle \mathcal{W}, \$, \mathcal{V}, z \rangle \models \beta^\Omega$, so $\langle \mathcal{W}, \$, \mathcal{V}, \lambda \rangle \models \beta^{\Omega,*,\otimes}$. It means that $\alpha^{\Sigma,*,\otimes} \rightarrow \beta^{\Omega,*,\otimes}$;
- The proofs of $(\alpha^\Sigma \rightarrow \beta^\Omega)^{*,N}$, $(\alpha^\Sigma \rightarrow \beta^\Omega)^{u,\otimes}$ and $(\alpha^\Sigma \rightarrow \beta^\Omega)^{u,N}$ are analogous.

(induction) If $\mathcal{C}(\alpha^\Sigma \rightarrow \beta^\Omega) = \ddagger$ and $s(\Delta) = n + 1$, then the scope must be a neighbourhood label and $\alpha^{\Sigma,\bar{\Delta}} \rightarrow \beta^{\Omega,\bar{\Delta}}$ may be written as $\alpha^{\Sigma,\phi,\bar{\Delta}'} \rightarrow \beta^{\Omega,\phi,\bar{\Delta}'}$, where $s(\Delta') = n$. Then, by the induction hypothesis, $\alpha^{\Sigma,\phi,\bar{\Delta}'} \rightarrow \beta^{\Omega,\phi,\bar{\Delta}'}$ = $(\alpha^{\Sigma,\phi} \rightarrow \beta^{\Omega,\phi})^{\bar{\Delta}'}$. From the base assertions, $(\alpha^{\Sigma,\phi} \rightarrow \beta^{\Omega,\phi})^{\bar{\Delta}'} = ((\alpha^\Sigma \rightarrow \beta^\Omega)^\phi)^{\bar{\Delta}'} = (\alpha^\Sigma \rightarrow \beta^\Omega)^{\phi,\bar{\Delta}'} = (\alpha^\Sigma \rightarrow \beta^\Omega)^{\bar{\Delta}}$;

(induction) If $\mathcal{C}(\alpha^\Sigma \rightarrow \beta^\Omega) = \dagger$ and $s(\Delta) = n + 2$, then the scope must be a world label and $\alpha^{\Sigma,\bar{\Delta}} \rightarrow \beta^{\Omega,\bar{\Delta}}$ may be written as $\alpha^{\Sigma,\phi,\Theta,\bar{\Delta}'} \rightarrow \beta^{\Omega,\phi,\Theta,\bar{\Delta}'}$, where $s(\Delta') = n$. Then, by the induction hypothesis, $\alpha^{\Sigma,\phi,\Theta,\bar{\Delta}'} \rightarrow \beta^{\Omega,\phi,\Theta,\bar{\Delta}'}$ = $(\alpha^{\Sigma,\phi,\Theta} \rightarrow \beta^{\Omega,\phi,\Theta})^{\bar{\Delta}'}$. By the base, $(\alpha^{\Sigma,\phi,\Theta} \rightarrow \beta^{\Omega,\phi,\Theta})^{\bar{\Delta}'} = ((\alpha^\Sigma \rightarrow \beta^\Omega)^{\phi,\Theta})^{\bar{\Delta}'} = (\alpha^\Sigma \rightarrow \beta^\Omega)^{\phi,\Theta,\bar{\Delta}'} = (\alpha^\Sigma \rightarrow \beta^\Omega)^{\bar{\Delta}}$. ■

Lemma 64 *iPUC-ND without the rules 5, 11, 18, 20, 27, 28 and 29 preserves resolution.*

Proof: Consider $\mathcal{M} = \langle \mathcal{W}, \$, \mathcal{V}, \chi \rangle$. We present the proof for rule 12, because the proof for the other rules are equal as in lemma 26.

(12) If $\mathcal{M} \models^\Delta \alpha^\Sigma \rightarrow \beta^\Omega$, then $\mathcal{M} \models (\alpha^\Sigma \rightarrow \beta^\Omega)^{\bar{\Delta}}$, then, by lemma 63, $\mathcal{M} \models \alpha^{\Sigma,\bar{\Delta}} \rightarrow \beta^{\Omega,\bar{\Delta}}$. Then, by definition, $\forall \lambda \in \mathcal{A}(\chi) : \text{if } \langle \mathcal{W}, \$, \mathcal{V}, \lambda \rangle \models \alpha^{\Sigma,\bar{\Delta}}$, then $\langle \mathcal{W}, \$, \mathcal{V}, \lambda \rangle \models \beta^{\Omega,\bar{\Delta}}$. Since $\chi \in \mathcal{A}(\chi)$, then, if $\langle \mathcal{W}, \$, \mathcal{V}, \chi \rangle \models \alpha^{\Sigma,\bar{\Delta}}$, then $\langle \mathcal{W}, \$, \mathcal{V}, \chi \rangle \models \beta^{\Omega,\bar{\Delta}}$. But we already know that $\langle \mathcal{W}, \$, \mathcal{V}, \chi \rangle \models \alpha^{\Sigma,\bar{\Delta}}$ and we can conclude $\mathcal{M} \models^\Delta \beta^\Omega$. ■

Lemma 65 *If $\langle \mathcal{W}, \$, \mathcal{V}, \chi \rangle \models \alpha^\Sigma$, then $\forall \lambda \in \mathcal{A}(\chi) : \langle \mathcal{W}, \$, \mathcal{V}, \lambda \rangle \models \alpha^\Sigma$*

Proof: If $\langle \mathcal{W}, \$, \mathcal{V}, \chi \rangle \models \alpha$, α atomic, then, by the definition of the accessibility relation, then $\forall \lambda \in \mathcal{A}(\chi) : \langle \mathcal{W}, \$, \mathcal{V}, \lambda \rangle \models \alpha$. If $\langle \mathcal{W}, \$, \mathcal{V}, \chi \rangle \models \alpha^\Sigma$, Σ non empty, then $\forall \lambda \in \mathcal{A}(\chi) : \langle \mathcal{W}, \$, \mathcal{V}, \lambda \rangle \models \alpha^\Sigma$ because $\forall \lambda \in \mathcal{A}(\chi) : \$(\chi) = \$(\lambda)$. The systems of neighbourhoods being the same makes equal the evaluation of the formula. So, if $\langle \mathcal{W}, \$, \mathcal{V}, \chi \rangle \models \alpha^\Sigma$, then $\forall \lambda \in \mathcal{A}(\chi) : \langle \mathcal{W}, \$, \mathcal{V}, \lambda \rangle \models \alpha^\Sigma$ because the evaluation of all subformulas of α^Σ are the same at all worlds of $\mathcal{A}(\chi)$, including χ itself. ■

Lemma 66 *iPUC-ND preserves resolution.*

Proof: During the proof $\mathcal{M} = \langle \mathcal{W}, \$, \mathcal{V}, \chi \rangle$. We present the proof for each remaining rule of the iPUC-ND inside an induction. Base argument:

5. If $\mathcal{M} \models^\Delta \alpha^\Sigma \vee \beta^\Omega$, then $\mathcal{M} \models (\alpha^\Sigma \vee \beta^\Omega)^\Delta$, then, by lemma 23, $\mathcal{M} \models \alpha^{\Sigma, \Delta} \vee \beta^{\Omega, \Delta}$, then, by definition, $\mathcal{M} \models \alpha^{\Sigma, \Delta}$ or $\mathcal{M} \models \beta^{\Omega, \Delta}$. This means, by definition, that $\mathcal{M} \models^\Delta \alpha^\Sigma$ or $\mathcal{M} \models^\Delta \beta^\Omega$. So, if Π_1 and Π_2 only contains the rules from lemma 64, $\mathcal{M} \models^\Theta \gamma^\Lambda$ in both cases, because of the preservation of the resolution relation. And, for that conclusion, the hypothesis are no longer necessary and may be discharged;
11. If Π only contains the rules of lemma 64, then, from the hypothesis that $\mathcal{M} \models^\Delta \alpha^\Sigma$, the derivation gives us $\mathcal{M} \models^\Delta \beta^\Omega$. If $\mathcal{C}(\beta^\Omega) = \dagger$, then, by the fitting relation and lemma 19, we know that $s(\Delta)$ is even. If we take some model $\mathcal{H} = \langle \mathcal{W}, \$, \mathcal{V}, z \rangle$, such that $\mathcal{M} \multimap_{s(\Delta)} \mathcal{H}$ and $\mathcal{H} \models \beta^\Omega$, then, by lemma 65, $\forall w \in \mathcal{A}(z) : \langle \mathcal{W}, \$, \mathcal{V}, w \rangle \models \beta^\Omega$ and, by definition, $\mathcal{H} \models \alpha^\Sigma \rightarrow \beta^\Omega$. So, by definition, $\beta^\Omega \models_{\mathcal{M}:s(\Delta)} \alpha^\Sigma \rightarrow \beta^\Omega$, which means, by lemma 21, that $\mathcal{M} \models^\Delta \alpha^\Sigma \rightarrow \beta^\Omega$. If $\mathcal{C}(\beta^\Omega) = \ddagger$, then, by the fitting relation and lemma 19, we know that $s(\Delta)$ is odd. If we take some template $\mathcal{T} = \langle \mathcal{W}, \$, \mathcal{V}, z, L \rangle$, such that $\mathcal{M} \multimap_{s(\Delta)} \mathcal{T}$ and $\mathcal{T} \models \beta^\Omega$, then, by lemma 65, $\forall w \in \mathcal{A}(z) : \langle \mathcal{W}, \$, \mathcal{V}, w, L \rangle \models \beta^\Omega$ and, by definition, $\mathcal{T} \models \alpha^\Sigma \rightarrow \beta^\Omega$. So, by definition, $\beta^\Omega \models_{\mathcal{M}:s(\Delta)} \alpha^\Sigma \rightarrow \beta^\Omega$, which means, by lemma 21, that $\mathcal{M} \models^\Delta \alpha^\Sigma \rightarrow \beta^\Omega$. So the hypothesis is unnecessary and may be discharged;
18. If $\mathcal{M} \models^{\Delta, \bullet} \alpha^\Sigma$, then, by the rule 14, $\mathcal{M} \models^\Delta \alpha^{\Sigma, \bullet}$. From $\mathcal{C}(\alpha^{\Sigma, \bullet}) = \ddagger$, the fitting relation and lemma 19, we know that $s(\Delta)$ is odd. If we take some template $\mathcal{T} = \langle \mathcal{W}, \$, \mathcal{V}, z, N \rangle$, such that $\mathcal{M} \multimap_{s(\Delta)} \mathcal{T}$ and $\mathcal{T} \models \alpha^{\Sigma, \bullet}$, then, $N \in \$(\mathcal{V}(z))$ and $\exists w \in N : \langle \mathcal{W}, \$, \mathcal{V}, w \rangle \models \alpha^\Sigma$. Since the variable u occurs nowhere else in the derivation, u can be taken as a denotation of the given existential and we conclude that $\langle \mathcal{W}, \$, \mathcal{V}, u \rangle \models \alpha^\Sigma$, what

means that $\mathcal{T} \models \alpha^{\Sigma, u}$. So, by definition, $\alpha^{\Sigma, \bullet} \models_{\mathcal{M}:s(\Delta)} \alpha^{\Sigma, u}$, which means, by lemma 21, that $\mathcal{M} \models^{\Delta} \alpha^{\Sigma, u}$. We conclude, using the rule 13, that $\mathcal{M} \models^{\Delta, u} \alpha^{\Sigma}$. If Π only contains rules of the lemma 64, then we can conclude $\mathcal{M} \models^{\Theta} \beta^{\Omega}$. Then we can discharge the hypothesis because we know that any denotation of the existential may provide the same conclusion;

20. If $\mathcal{M} \models^{\Delta, \odot} \alpha^{\Sigma}$, then, by the rule 14, $\mathcal{M} \models^{\Delta} \alpha^{\Sigma, \odot}$. From $\mathcal{C}(\alpha^{\Sigma, \odot}) = \dagger$, the fitting relation and lemma 19, we know that $s(\Delta)$ is even. If we take some model $\mathcal{H} = \langle \mathcal{W}, \$, \mathcal{V}, z \rangle$, such that $\mathcal{M} \multimap_{s(\Delta)} \mathcal{H}$ and $\mathcal{H} \models \alpha^{\Sigma, \odot}$, then $\exists M \in \$ (z) : \langle \mathcal{W}, \$, \mathcal{V}, z, M \rangle \models \alpha^{\Sigma}$. Since the variable N occurs nowhere else in the derivation, N can be taken as a denotation of the given existential and we conclude that $\langle \mathcal{W}, \$, \mathcal{V}, z, N \rangle \models \alpha^{\Sigma}$, what means that $\mathcal{H} \models \alpha^{\Sigma, N}$. So, by definition, $\alpha^{\Sigma, \odot} \models_{\mathcal{M}:s(\Delta)} \alpha^{\Sigma, N}$, which means, by lemma 21, that $\mathcal{M} \models^{\Delta} \alpha^{\Sigma, N}$. We conclude, using the rule 13, that $\mathcal{M} \models^{\Delta, N} \alpha^{\Sigma}$. If Π only contains rules of the lemma 64, then we can conclude $\mathcal{M} \models^{\Theta} \beta^{\Omega}$. Then we can discharge the hypothesis because we know that any denotation of the existential may provide the same conclusion.

Inductive case: the same argument as in lemma 28. ■

Theorem 67 $\Gamma \vdash \alpha^{\Sigma}$ implies $\Gamma \models \alpha^{\Sigma}$ (*Soundness*).

Proof: The fitting restriction of the rules of iPUC-ND ensures that α^{Σ} has neighbourhood characteristic because it appears in the empty context. The same conclusion follows for every formula of Γ . The derivability assures that there is a derivation that concludes α^{Σ} and takes as open hypothesis a subset of Γ , which we call Γ' . If we take a model $\mathcal{M}$ that satisfies every formula of Γ , then it also satisfies every formula of Γ' . So, $\mathcal{M} \models \gamma^{\Theta}$, for every $\gamma^{\Theta} \in \Gamma'$. But this means, by definition, that for every wff of Γ' the resolution relation holds with the empty context. Then, from lemma 66, we know that $\mathcal{M} \models \alpha^{\Sigma}$. So, every model that satisfies every formula of Γ also satisfies α^{Σ} and, by definition, $\Gamma \models \alpha^{\Sigma}$. ■

We use prime theories to prove completeness. The reader can see the intuitionistic logic case of this way of proving completeness in [17].

Lemma 68 Given $\Gamma \subset \mathcal{S}_n$ ($\Gamma \subset \mathcal{S}_w$), if there is a model (template) that satisfies every formula of Γ , then Γ is n -consistent.

Proof: This proof is analogous to the classical case, with the addition that here we use theorem 67 instead of theorem 31. ■

Lemma 69 *Given $\Gamma \subset \mathbf{S}_n$ ($\Gamma \subset \mathbf{S}_w$), if $\Gamma \cup \{\phi^\ominus\} \vdash \perp_n$, then $\Gamma \vdash \neg\phi^\ominus$.*

Proof: The assumption implies that there is a derivation $\mathcal{D}$ with hypothesis in $\Gamma \cup \{\phi^\ominus\}$ and conclusion $\perp_n$. If we apply the rule $\rightarrow$ -introduction and eliminate all occurrences of $\phi^\ominus$ as hypothesis, then we obtain a derivation with hypothesis in Γ and conclusion $\neg\phi^\ominus$. The same argument holds for $\Gamma \subset \mathbf{S}_w$. ■

Definition 70 $\Gamma \subset \mathbf{S}_n$ ($\Gamma \subset \mathbf{S}_w$) is a prime n -theory (prime w -theory) iff (i) $\Gamma \vdash \alpha^\Sigma$ ($\Gamma \vdash_N^N \alpha^\Sigma$) implies $\alpha^\Sigma \in \Gamma$ and (ii) if $\alpha^\Sigma \vee \beta^\Omega \in \Gamma$ implies $\alpha^\Sigma \in \Gamma$ or $\beta^\Omega \in \Gamma$.

Lemma 71 *Given $\Gamma \cup \{\alpha^\Sigma\} \subset \mathbf{S}_n$ ($\Gamma \cup \{\alpha^\Sigma\} \subset \mathbf{S}_w$), if $\Gamma \not\vdash \alpha^\Sigma$ ($\Gamma \not\vdash_N^N \alpha^\Sigma$), then there is a prime n -theory (w -theory) Γ' , such that $\Gamma \subset \Gamma'$ and $\Gamma' \not\vdash \alpha^\Sigma$.*

Proof: According to lemma 37, we may have a list $\varphi_0, \varphi_1, \dots$ of all wff in $\mathbf{S}_n$. We build a non-decreasing sequence of sets Γ_i such that the union is a prime theory. We put $\Gamma_0 = \Gamma$. Then we take the first disjunctive sentence that has not been treated and $\Gamma_n \vdash \varphi_0 \vee \varphi_1$: $\Gamma_{n+1} = \Gamma_n \cup \{\varphi_0\}$ if $\Gamma_n \cup \{\varphi_0\} \not\vdash \alpha^\Sigma$, $\Gamma_{n+1} = \Gamma_n \cup \{\varphi_1\}$ otherwise. It cannot be the case that $\Gamma_n \cup \{\varphi_0\} \vdash \alpha^\Sigma$ and $\Gamma_n \cup \{\varphi_1\} \vdash \alpha^\Sigma$, because, in this case, $\Gamma_n \vdash \alpha^\Sigma$ by $\vee$ -elimination. So, $\Gamma' = \bigcup \{\Gamma_n \mid n \geq 0\}$.

(a) $\Gamma' \not\vdash \alpha^\Sigma$: by induction, since $\Gamma_0 \not\vdash \alpha^\Sigma$ and $\Gamma_{n+1} \not\vdash \alpha^\Sigma$ by the definition of the induction step. (b) Γ' is a prime theory: (i) if $\psi_1 \vee \psi_2 \in \Gamma'$, then take the least number k such that $\Gamma_k \vdash \psi_1 \vee \psi_2$. So, $\psi_1 \vee \psi_2$ cannot have been treated at a stage before k and $\Gamma_h \vdash \psi_1 \vee \psi_2$, for $h \geq k$. At some point $\psi_1 \vee \psi_2$ must be treated at a stage $h \geq k$. Then, $\psi_1 \in \Gamma_{h+1}$ or $\psi_2 \in \Gamma_{h+1}$ and, by definition, $\psi_1 \in \Gamma'$ or $\psi_2 \in \Gamma'$. (ii) if $\Gamma' \vdash \psi$, then $\Gamma' \vdash \psi \vee \psi$, then by (i) $\psi \in \Gamma'$.

The same argument holds for wsents. ■

Definition 72 *Given a prime n -theory Γ and a prime w -theory Λ , we say that Γ accepts Λ ($\Gamma \propto \Lambda$) if $\alpha^\Sigma \in \Lambda$ implies $\alpha^{\Sigma, \odot} \in \Gamma$. If $\alpha^\Sigma \in \Gamma$ implies $\alpha^{\Sigma, \bullet} \in \Lambda$, then $\Lambda \propto \Gamma$.*

Definition 73 *Given prime w -theories Γ and Λ , we say that Γ subordinates Λ ($\Lambda \sqsubset \Gamma$) iff $\alpha^{\Sigma, \bullet} \in \Lambda$ implies $\alpha^{\Sigma, \bullet} \in \Gamma$ and $\alpha^{\Sigma, *}$ $\in \Gamma$ implies $\alpha^{\Sigma, *} \in \Lambda$.*

Lemma 74 *If $\Gamma \subset \mathbf{S}_n$ is n -consistent, then there is a model $\mathcal{M}$, such that $\mathcal{M} \models \alpha^\Sigma$, for every $\alpha^\Sigma \in \Gamma$.*

Proof: By definition, $\Gamma \not\vdash \perp_n$ and, by lemma 71, Γ is contained in a prime n -theory Γ' , such that $\Gamma' \not\vdash \perp_n$. We take every prime n -theory Ψ as a representation of one world of $\mathcal{W}$, denoted by χ_Ψ . Every prime w -theory will be seen as a set of worlds that may be a neighbourhood. We take α as the nested neighbourhood function $\$$ and $\sqsubset$ as the total order among neighbourhoods. We take the subset relation among prime n -theories as the accessibility relation among worlds. To build the truth evaluation function $\mathcal{V}$, we require, for every prime n -theory Ψ and for every α atomic: (a) $\chi_\Psi \in \mathcal{V}(\alpha)$ if $\alpha \in \Psi$; (b) $\chi_\Psi \notin \mathcal{V}(\alpha)$ if $\alpha \notin \Psi$. If we take $\mathcal{M} = \langle \mathcal{W}, \$, \mathcal{V}, \chi_{\Gamma'} \rangle$, then, for every wff $\alpha^\Sigma \in \Gamma'$, $\mathcal{M} \models \alpha^\Sigma$. We proceed by induction on the structure of α^Σ :

(Base) If α^Σ is atomic, $\mathcal{M} \models \alpha^\Sigma$ iff $\alpha^\Sigma \in \Gamma'$, by the definition of $\mathcal{V}$;

$\alpha^\Sigma = \beta^\Omega \wedge \gamma^\Theta$. $\mathcal{M} \models \alpha^\Sigma$ iff $\mathcal{M} \models \beta^\Omega$ and $\mathcal{M} \models \gamma^\Theta$ iff (induction hypothesis) $\beta^\Omega \in \Gamma'$ and $\gamma^\Theta \in \Gamma'$. We conclude that $\alpha^\Sigma \in \hat{\Gamma}$ by a single application of the $\wedge$ -introduction rule and the fact that a prime theory is closed by derivability. Conversely $\alpha^\Sigma \in \Gamma'$ iff $\beta^\Omega \in \Gamma'$ and $\gamma^\Theta \in \Gamma'$ by $\wedge$ -elimination and the fact that a prime theory is closed by derivability. The rest follows by the induction hypothesis;

$\alpha^\Sigma = \beta^\Omega \vee \gamma^\Theta$. By the definition of prime theory, $\beta^\Omega \in \Gamma'$ or $\gamma^\Theta \in \Gamma'$ and we proceed by induction.

$\alpha^\Sigma = \beta^\Omega \rightarrow \gamma^\Theta$. $\mathcal{M} \models \alpha^\Sigma$ iff $\exists \chi_\Psi \in \mathcal{A}(\chi_{\Gamma'})$ such that $\langle \mathcal{W}, \$, \mathcal{V}, \chi_\Psi \rangle \models \beta^\Omega$ and $\langle \mathcal{W}, \$, \mathcal{V}, \chi_\Psi \rangle \not\models \gamma^\Theta$ iff (induction hypothesis) $\beta^\Omega \in \Psi$ and $\gamma^\Theta \notin \Psi$. So, by the fact that a prime theory is closed by derivability and by $\rightarrow$ -elimination, we conclude that $\beta^\Omega \rightarrow \gamma^\Theta \notin \Psi$. By the definition of accessibility among worlds, $\Gamma' \subset \Psi$ and $\beta^\Omega \rightarrow \gamma^\Theta \notin \Gamma'$. Conversely, $\beta^\Omega \rightarrow \gamma^\Theta \notin \Gamma'$ implies that, if $\beta^\Omega \in \Gamma'$, then $\gamma^\Theta \notin \Gamma'$, because, on the contrary, for every prime n -theory Ψ , such that $\Gamma' \subset \Psi$, $\gamma^\Theta \in \Psi$ and by i.h. $\langle \mathcal{W}, \$, \mathcal{V}, \chi_\Psi \rangle \models \gamma^\Theta$. So, by the definition of accessibility of worlds, for every $\chi_\Psi \in \mathcal{A}(\chi_{\Gamma'})$: $\langle \mathcal{W}, \$, \mathcal{V}, \chi_\Psi \rangle \models \gamma^\Theta$ and, by definition, $\mathcal{M} \models \beta^\Omega \rightarrow \gamma^\Theta$. From $\beta^\Omega \in \Gamma'$ we know that by i.h. $\mathcal{M} \models \beta^\Omega$. So, from $\chi_{\Gamma'} \in \mathcal{A}(\chi_{\Gamma'})$ and $\mathcal{M} \models \beta^\Omega \rightarrow \gamma^\Theta$, we get $\mathcal{M} \models \gamma^\Theta$ and by i.h. $\gamma^\Theta \in \Gamma'$. But, in this case, $\beta^\Omega \rightarrow \gamma^\Theta \in \Gamma'$ by a single $\rightarrow$ -introduction and the fact that a prime theory is closed by derivability;

$\alpha^\Sigma = \neg\beta^\Omega$. This case is treated by the previous case, because $\neg\beta^\Omega \equiv \beta^\Omega \rightarrow \perp_n$;

$\alpha^\Sigma = \beta^{\Omega, \otimes}$. This wff do not require the existence of a prime w -theory to represent a neighbourhood in which β^Ω holds, on the contrary, it only requires that there is no prime w -theory, accepted by Υ , in which γ^Θ does not hold.

$\alpha^\Sigma = \beta^{\Omega, \odot}$. We take any enumeration $\rho_1, \rho_2, \dots$ in $\mathbf{F}_w$, such that $\rho_i^\odot \in \Gamma'$ and $\rho_0 = \beta^\Omega$. We construct a w-theory by following:

- $\Upsilon_0 = \{\beta^\Omega\}$;
- $\Upsilon_i = \Upsilon_{i-1} \cup \{\rho_i\}$, if $(\varphi_0 \wedge \dots \wedge \varphi_m \wedge \rho_i)^\odot \in \Gamma'$, where $\Upsilon_{i-1} = \{\varphi_0, \dots, \varphi_m\}$.
 $\Upsilon_i = \Upsilon_{i-1}$ otherwise;
- $\Upsilon = \bigcup_{n \in \mathbb{N}} \Upsilon_n$.

Now we must prove the following:

1. Υ is a w-theory:

- $\Upsilon \not\vdash \perp_w$. A derivation in iPUC build only with wff from $\mathbf{F}_w$ must have all rules with odd sized context. To do so, we choose a neighbourhood variable, that do not occur during the derivation, to be the leftmost label at every context that appears during the derivation. This new variable may be understood as the representation of the neighbourhood in which the inference is made. So, if Υ is not w-consistent, then, given some derivation $\mathcal{D}$ that concludes $\perp_w$ from wff of Υ , there is an index $n \in \mathbb{N}$, such that Υ_n contains all wff that appears in $\mathcal{D}$. So, by definition, that there is a wff $(\varphi_0 \wedge \dots \wedge \varphi_m)^\odot \in \Gamma'$, such that $\varphi_0, \dots, \varphi_m$ represent all wff of Υ_n and $\perp_n \in \Gamma'$ by the derivation below, which is a contradiction by the definition of Γ' .

$$\frac{\frac{\frac{(\varphi_0 \wedge \dots \wedge \varphi_m)^\odot}{(\varphi_0 \wedge \dots \wedge \varphi_m)^\odot} \odot}{\varphi_0 \wedge \dots \wedge \varphi_m} \odot}{\perp_n} \quad \frac{\frac{[\varphi_0 \wedge \dots \wedge \varphi_m]}{\mathcal{D}' N} N}{\perp_w N} \quad \perp_n$$

The derivation $\mathcal{D}'$ is obtained from $\mathcal{D}$ by 1) adding $\varphi_0 \wedge \dots \wedge \varphi_m$ in the place of the hypothesis φ_i and the following $\wedge$ -eliminations to recover φ_i ; 2) binding the variable N added to produce $\mathcal{D}$.

- If $\varpi \vee \vartheta \in \Upsilon$, then $\exists i \in \mathbb{N} : \rho_i = \varpi \vee \vartheta \in \Upsilon_i$ and $(\varphi_0 \wedge \dots \wedge \varphi_m \wedge (\varpi \vee \vartheta))^\odot \in \Gamma'$. By the distribution of $\vee$ over $\wedge$, we know that $((\varphi_0 \wedge \dots \wedge \varphi_m \wedge \varpi) \vee (\varphi_0 \wedge \dots \wedge \varphi_m \wedge \vartheta))^\odot \in \Gamma'$. By the following derivation (Π_1), we know that $(\varphi_0 \wedge \dots \wedge \varphi_m \wedge \varpi)^\odot \vee (\varphi_0 \wedge \dots \wedge \varphi_m \wedge \vartheta)^\odot \in \Gamma'$ and, by the definition of prime theory, $(\varphi_0 \wedge \dots \wedge \varphi_m \wedge \varpi)^\odot \in \Gamma'$ or $(\varphi_0 \wedge \dots \wedge \varphi_m \wedge \vartheta)^\odot \in \Gamma'$. We proceed now considering the possibilities of the order provided by the enumeration ρ . Without loss of generality, we suppose that $\rho_k = \varpi$, $\rho_l = \vartheta$ and $k < l$. In the case $l < i$, we are done, because, from $(\varphi_0 \wedge \dots \wedge \varphi_m \wedge \varpi)^\odot \in \Gamma'$

or $(\varphi_0 \wedge \dots \wedge \varphi_m \wedge \vartheta)^\circ \in \Gamma'$, we can eliminate the extra formulas to recover Υ_k or Υ_l by the rule of construction of the set Υ . In the case $i < m$, at the moment of verification of ρ_m , we know that for all formulas of $\Upsilon_{m-1} = \{\varphi_0, \dots, (\varpi \vee \vartheta), \dots, \varphi_t\}$ we have a conjunction $(\varphi_0 \wedge \dots \wedge (\varpi \vee \vartheta) \wedge \dots \wedge \varphi_t)^\circ \in \Gamma'$. By a derivation similar to Π_1 , we know that $(\varphi_0 \wedge \dots \wedge \varpi \wedge \dots \wedge \varphi_t)^\circ \in \Gamma'$ or $(\varphi_0 \wedge \dots \wedge \vartheta \wedge \dots \wedge \varphi_t)^\circ \in \Gamma'$ and by means of a $\vee$ -introduction in another derivation we know that $(\varphi_0 \wedge \dots \wedge (\varpi \vee \vartheta) \wedge \dots \wedge \varphi_t \wedge \varpi)^\circ \in \Gamma'$ or $(\varphi_0 \wedge \dots \wedge (\varpi \vee \vartheta) \wedge \dots \wedge \varphi_t \wedge \vartheta)^\circ \in \Gamma'$. If we have $(\varphi_0 \wedge \dots \wedge (\varpi \vee \vartheta) \wedge \dots \wedge \varphi_t \wedge \varpi)^\circ \in \Gamma'$, then Υ_m and Υ contain ϖ . If we have $(\varphi_0 \wedge \dots \wedge (\varpi \vee \vartheta) \wedge \dots \wedge \varphi_t \wedge \vartheta)^\circ \in \Gamma'$, at the moment of verification of ρ_l , we make the distribution of the disjunction $(\varpi \vee \vartheta)$ again. Given $\Upsilon_{l-1} = \{\varphi_0, \dots, (\varpi \vee \vartheta), \dots, \varphi_s\}$ we have a conjunction $(\varphi_0 \wedge \dots \wedge (\varpi \vee \vartheta) \wedge \dots \wedge \varphi_s)^\circ \in \Gamma'$. By a derivation similar to Π_1 , we know that $(\varphi_0 \wedge \dots \wedge \varpi \wedge \dots \wedge \varphi_s)^\circ \in \Gamma'$ or $(\varphi_0 \wedge \dots \wedge \vartheta \wedge \dots \wedge \varphi_s)^\circ \in \Gamma'$ and by means of a $\vee$ -introduction in another derivation we know that $(\varphi_0 \wedge \dots \wedge (\varpi \vee \vartheta) \wedge \dots \wedge \varphi_s \wedge \varpi)^\circ \in \Gamma'$ or $(\varphi_0 \wedge \dots \wedge (\varpi \vee \vartheta) \wedge \dots \wedge \varphi_s \wedge \vartheta)^\circ \in \Gamma'$. If $(\varphi_0 \wedge \dots \wedge (\varpi \vee \vartheta) \wedge \dots \wedge \varphi_s \wedge \vartheta)^\circ \in \Gamma'$, then we are done. If $(\varphi_0 \wedge \dots \wedge (\varpi \vee \vartheta) \wedge \dots \wedge \varphi_s \wedge \varpi)^\circ \in \Gamma'$, then we have a contradiction, because we supposed that $(\varphi_0 \wedge \dots \wedge (\varpi \vee \vartheta) \wedge \dots \wedge \varphi_t \wedge \varpi)^\circ \in \Gamma'$ were not the case and we can recover it by eliminating the extra formulas. The the case $m < i < l$ is analogous.

$$\Pi_1 \frac{\frac{((\varphi_0 \wedge \dots \wedge \varphi_m \wedge \varpi) \vee (\varphi_0 \wedge \dots \wedge \varphi_m \wedge \vartheta))^{\odot}}{((\varphi_0 \wedge \dots \wedge \varphi_m \wedge \varpi) \vee (\varphi_0 \wedge \dots \wedge \varphi_m \wedge \vartheta))^{\odot}} \odot \quad \frac{\frac{(\varphi_0 \wedge \dots \wedge \varphi_m \wedge \varpi) \vee (\varphi_0 \wedge \dots \wedge \varphi_m \wedge \vartheta)}{\Pi_2} N}{(\varphi_0 \wedge \dots \wedge \varphi_m \wedge \varpi)^{\odot} \vee (\varphi_0 \wedge \dots \wedge \varphi_m \wedge \vartheta)^{\odot}}$$

$$\Pi_2 \frac{\frac{(\varphi_0 \wedge \dots \wedge \varphi_m \wedge \varpi) \vee (\varphi_0 \wedge \dots \wedge \varphi_m \wedge \vartheta)}{(\varphi_0 \wedge \dots \wedge \varphi_m \wedge \varpi) \vee (\varphi_0 \wedge \dots \wedge \varphi_m \wedge \vartheta)} N \quad \frac{\frac{\frac{[\varphi_0 \wedge \dots \wedge \varphi_m \wedge \varpi]}{\varphi_0 \wedge \dots \wedge \varphi_m \wedge \varpi} N \quad \frac{[\varphi_0 \wedge \dots \wedge \varphi_m \wedge \vartheta]}{\varphi_0 \wedge \dots \wedge \varphi_m \wedge \vartheta} N}{\Pi_3} \quad \frac{\Pi_4}{(\varphi_0 \wedge \dots \wedge \varphi_m \wedge \varpi)^{\odot} \vee (\varphi_0 \wedge \dots \wedge \varphi_m \wedge \vartheta)^{\odot}}}{(\varphi_0 \wedge \dots \wedge \varphi_m \wedge \varpi)^{\odot} \vee (\varphi_0 \wedge \dots \wedge \varphi_m \wedge \vartheta)^{\odot}}$$

$$\Pi_3 \frac{\frac{\frac{((\varphi_0 \wedge \dots \wedge \varphi_m \wedge \varpi) \vee (\varphi_0 \wedge \dots \wedge \varphi_m \wedge \vartheta))^{\odot}}{((\varphi_0 \wedge \dots \wedge \varphi_m \wedge \varpi) \vee (\varphi_0 \wedge \dots \wedge \varphi_m \wedge \vartheta))^{\odot}} \odot \quad \frac{\varphi_0 \wedge \dots \wedge \varphi_m \wedge \varpi}{\varphi_0 \wedge \dots \wedge \varphi_m \wedge \varpi} N}{\frac{\varphi_0 \wedge \dots \wedge \varphi_m \wedge \varpi}{(\varphi_0 \wedge \dots \wedge \varphi_m \wedge \varpi)^{\odot}} \odot} \quad \frac{(\varphi_0 \wedge \dots \wedge \varphi_m \wedge \varpi)^{\odot} \vee (\varphi_0 \wedge \dots \wedge \varphi_m \wedge \vartheta)^{\odot}}{(\varphi_0 \wedge \dots \wedge \varphi_m \wedge \varpi)^{\odot} \vee (\varphi_0 \wedge \dots \wedge \varphi_m \wedge \vartheta)^{\odot}}$$

Π_4 is analogous.

- Υ is closed by derivation. Given a derivation $\mathcal{D}$ of wff in $\mathbf{F}_w$ that concludes ϖ and that has a new neighbourhood variable N like described above. Then there is $i \in \mathbb{N}$, such that $\Upsilon_i = \{\varphi_0, \dots, \varphi_m\}$ contains all hypothesis in $\mathcal{D}$. It means that $(\varphi_0 \wedge \dots \wedge \varphi_m)^\odot \in \Gamma'$ and, by the following derivation, we know that $(\varphi_0 \wedge \dots \wedge \varphi_m \wedge \varpi)^\odot \in \Gamma'$ because Γ' is a prime theory. If $\rho_k = \varpi$ and $k \leq i$, then $\varpi \in \Upsilon$, because we can obtain $(\varphi_0 \wedge \dots \wedge \varphi_j \wedge \varpi)^\odot \in \Gamma'$, where $\Upsilon_{k-1} = \{\varphi_0, \dots, \varphi_j\}$, by eliminating the extra formulas. If $k > i$, then, from $(\varphi_0 \wedge \dots \wedge \varphi_j)^\odot$, we can obtain, by a similar derivation, $(\varphi_0 \wedge \dots \wedge \varphi_j \wedge \varpi)^\odot$ because all the wff of Υ_i are in Υ_k .

$$\begin{array}{c}
\frac{(\varphi_0 \wedge \dots \wedge \varphi_m)^\odot}{(\varphi_0 \wedge \dots \wedge \varphi_m)^\odot} \odot \quad \frac{\frac{(\varphi_0 \wedge \dots \wedge \varphi_m)^\odot}{(\varphi_0 \wedge \dots \wedge \varphi_m)^\odot} \odot \quad \frac{\frac{[\varphi_0 \wedge \dots \wedge \varphi_m]}{\varphi_0 \wedge \dots \wedge \varphi_m} N \quad \frac{[\varphi_0 \wedge \dots \wedge \varphi_m]}{\varpi} \frac{\mathcal{D}'}{N} N}{\varphi_0 \wedge \dots \wedge \varphi_m \wedge \varpi} N}{\varphi_0 \wedge \dots \wedge \varphi_m \wedge \varpi} \odot \\
\frac{\varphi_0 \wedge \dots \wedge \varphi_m \wedge \varpi}{(\varphi_0 \wedge \dots \wedge \varphi_m \wedge \varpi)^\odot} \odot
\end{array}$$

The derivation $\mathcal{D}'$ is obtained from $\mathcal{D}$ by 1) adding $\varphi_0 \wedge \dots \wedge \varphi_m$ in the place of the hypothesis φ_i and the following $\wedge$ -eliminations to recover φ_i ; 2) binding the variable N added to $\mathcal{D}$.

2. Υ is accepted by Γ' : for every wff $\varpi \in \Upsilon$, there is $i \in \mathbb{N}$, such that $\varpi \in \Upsilon_i$, what means that there is a conjunction such that $(\varphi_0 \wedge \dots \wedge \varphi_m \wedge \varpi)^\odot \in \Gamma'$. We obtain $\varpi^\odot \in \Gamma'$ by a derivation with some $\wedge$ -eliminations.
3. The wff of the form $\gamma^{\Theta, \bullet} \in \Upsilon$ demand a prime n-theory to represent a world in which γ^Θ holds. The argument is, like in the classical situation, analogous to the case of the wff $\beta^{\Omega, \odot} \in \Gamma'$.
4. The wff of the form $\gamma^{\Theta, *}$ do not require the existence of a prime n-theory to represent a world in which γ^Θ holds, on the contrary, it only requires that there is no prime n-theory, accepted by Υ , in which γ^Θ does not hold.

■

Corollary 75 $\Gamma \not\vdash \alpha^\Sigma$ iff there is a model $\mathcal{M}$, such that $\mathcal{M} \models \phi^\Theta$, for every $\phi^\Theta \in \Gamma$, and $\mathcal{M} \not\models \alpha^\Sigma$.

Proof: $\Gamma \not\vdash \alpha^\Sigma$ iff $\Gamma \cup \{\neg \alpha^\Sigma\}$ is n-consistent by lemma 69 and the definition of n-consistent set. By lemmas 68 and 74, $\Gamma \cup \{\neg \alpha^\Sigma\}$ is n-consistent iff there is a model $\mathcal{M}$, such that $\mathcal{M} \models \phi^\Theta$, for every $\phi^\Theta \in \Gamma \cup \{\neg \alpha^\Sigma\}$. It means that $\mathcal{M}$ satisfies every formula of Γ and $\mathcal{M} \not\models \alpha^\Sigma$. ■

Theorem 76 $\Gamma \models \alpha^\Sigma$ implies $\Gamma \vdash \alpha^\Sigma$ (Completeness).

Proof: $\Gamma \not\models \alpha^\Sigma$ implies $\Gamma \not\vdash \alpha^\Sigma$, by the corollary 75 and the definition of logical consequence. ■

Theorem 77 Every derivation of *iPUC-Logic* is normalizable.

Proof: To the PUC normalization arguments we add some permutation reductions following the approach of van Dalen [19] for the operators not in $\mathcal{L}_-$. ■

The arguments about decidability and complexity of iPUC are analogous to PUC and produce analogous results.

Theorem 78 The problem of satisfiability is **PS**-complete for *iPUC-Logic*.

Proof: We repeat the arguments for theorem 57, but with reference to the article of Statman[24]. ■

III.2 Constructive Counterfactuals

In the proof of the relative completeness of PUC-Logic for the Lewis V-Logic, the proofs of CONNEX and CPR demanded the classical absurd rule. In order to do provide a constructive approach over the counterfactual logic, we need to find another way to prove them. Lewis [1] gave us (page 124) an alternative formulation for the CPR rule with a simpler rule and an axiom schema:

$$\frac{\phi \rightarrow \psi}{\psi \preceq \phi} \text{ and } (\phi \preceq (\phi \vee \psi)) \vee (\psi \preceq (\phi \vee \psi))$$

That rule is a derived rule in iPUC as we can see by the derivation below, using lemma 59 (proof II):

$$\frac{\frac{\frac{1[\phi^\bullet]}{\phi^\bullet} N}{\phi} N, \bullet}{\frac{\frac{2[\phi]}{\phi} N, u \quad \frac{\frac{\Pi}{\phi \rightarrow \psi} N, u}{\phi \rightarrow \psi} N, u}{\frac{\psi}{\psi} N, u}{\frac{\psi}{\psi} N, \bullet}{\frac{\psi}{\psi^\bullet} N}{N}} \frac{1 \frac{\psi^\bullet}{\phi^\bullet \rightarrow \psi^\bullet} N}{\frac{\phi^\bullet \rightarrow \psi^\bullet}{\phi^\bullet \rightarrow \psi^\bullet} \textcircled{*}} \textcircled{*}$$

Inside iPUC-Logic, the axioms $(\phi \preceq (\phi \vee \psi)) \vee (\psi \preceq (\phi \vee \psi))$ are unnecessary to recover the completeness of the V-Logic, because they are derived formulas in the presence of CONNEX:

$$\begin{array}{c}
\frac{\frac{\frac{^3[(\phi^\bullet \rightarrow \psi^\bullet)^\circ]}{(\phi^\bullet \rightarrow \psi^\bullet)^\circ} \circ}{\phi^\bullet \rightarrow \psi^\bullet} \circ}{\frac{\frac{\frac{\psi^\bullet}{\psi} \circ, \bullet}{\phi \vee \psi} \circ, \bullet}{(\phi \vee \psi)^\bullet} \circ} \frac{^1[\phi^\bullet]}{\phi^\bullet} \circ}{^1 \frac{\phi^\bullet \rightarrow (\phi \vee \psi)^\bullet}{((\phi^\bullet \rightarrow (\phi \vee \psi)^\bullet)^\circ)} \circ} \\
\frac{(\phi^\bullet \rightarrow \psi^\bullet)^\circ \vee (\psi^\bullet \rightarrow \phi^\bullet)^\circ}{(\phi^\bullet \rightarrow \psi^\bullet)^\circ \vee (\psi^\bullet \rightarrow \phi^\bullet)^\circ} \quad \frac{\frac{\frac{\frac{^3[(\psi^\bullet \rightarrow \phi^\bullet)^\circ]}{(\psi^\bullet \rightarrow \phi^\bullet)^\circ} \circ}{\psi^\bullet \rightarrow \phi^\bullet} \circ}{\frac{\frac{\frac{\phi^\bullet}{\phi} \circ, \bullet}{\phi \vee \psi} \circ, \bullet}{(\phi \vee \psi)^\bullet} \circ} \frac{^2[\psi^\bullet]}{\psi^\bullet} \circ}{^2 \frac{\psi^\bullet \rightarrow (\phi \vee \psi)^\bullet}{((\psi^\bullet \rightarrow (\phi \vee \psi)^\bullet)^\circ)} \circ} \\
\frac{(\phi^\bullet \rightarrow \psi^\bullet)^\circ \vee (\psi^\bullet \rightarrow \phi^\bullet)^\circ}{((\phi^\bullet \rightarrow (\phi \vee \psi)^\bullet)^\circ \vee ((\psi^\bullet \rightarrow (\phi \vee \psi)^\bullet)^\circ)} \quad \frac{((\phi^\bullet \rightarrow (\phi \vee \psi)^\bullet)^\circ \vee ((\psi^\bullet \rightarrow (\phi \vee \psi)^\bullet)^\circ)}{((\phi^\bullet \rightarrow (\phi \vee \psi)^\bullet)^\circ \vee ((\psi^\bullet \rightarrow (\phi \vee \psi)^\bullet)^\circ)}
\end{array}$$

So, in order to recover the full V-Logic expressivity, we need to introduce some rules into the iPUC-Logic. The first approach is a single rule, motivated by the article of López-Escobar [15].

$$31: \frac{\frac{\frac{[\alpha^{\Sigma, \bullet} \rightarrow \beta^{\Omega, \bullet}]}{\Pi_1} \Theta}{\gamma^{\Lambda}} \Delta, \otimes \quad \frac{\frac{[\beta^{\Omega, \bullet} \rightarrow \alpha^{\Sigma, \bullet}]}{\Pi_2} \Theta}{\gamma^{\Lambda}} \Delta, \otimes}{\gamma^{\Lambda}} \Theta$$

(a) the formulas fit into their contexts; (b) Δ has no universal quantifier.

Given the rule 31, the CONNEX axioms are theorems:

$$\frac{\frac{\frac{[\alpha^{\Sigma, \bullet} \rightarrow \beta^{\Omega, \bullet}]}{\alpha^{\Sigma, \bullet} \rightarrow \beta^{\Omega, \bullet}} \otimes}{(\alpha^{\Sigma, \bullet} \rightarrow \beta^{\Omega, \bullet})^{\otimes}} \quad \frac{\frac{[\beta^{\Omega, \bullet} \rightarrow \alpha^{\Sigma, \bullet}]}{\beta^{\Omega, \bullet} \rightarrow \alpha^{\Sigma, \bullet}} \otimes}{(\beta^{\Omega, \bullet} \rightarrow \alpha^{\Sigma, \bullet})^{\otimes}}}{(\alpha^{\Sigma, \bullet} \rightarrow \beta^{\Omega, \bullet})^{\otimes} \vee (\beta^{\Omega, \bullet} \rightarrow \alpha^{\Sigma, \bullet})^{\otimes}} \quad \frac{(\alpha^{\Sigma, \bullet} \rightarrow \beta^{\Omega, \bullet})^{\otimes} \vee (\beta^{\Omega, \bullet} \rightarrow \alpha^{\Sigma, \bullet})^{\otimes}}{(\alpha^{\Sigma, \bullet} \rightarrow \beta^{\Omega, \bullet})^{\otimes} \vee (\beta^{\Omega, \bullet} \rightarrow \alpha^{\Sigma, \bullet})^{\otimes}}$$

Definition 79 Given any model $\mathcal{M} = \langle \mathcal{W}, \$, \mathcal{V}, \chi \rangle$, the set of testimonials of α^{Σ} is the set of neighbourhoods $\mathcal{T}(\alpha^{\Sigma}) = \{N \in \$(\chi) \mid \langle \mathcal{W}, \$, \mathcal{V}, \chi, N \rangle \models \alpha^{\Sigma, \bullet}\}$ and the set of believers of α^{Σ} is the set of neighbourhoods $\mathcal{B}(\alpha^{\Sigma}) = \{N \in \$(\chi) \mid \langle \mathcal{W}, \$, \mathcal{V}, \chi, N \rangle \models \alpha^{\Sigma, *}\}$.

$\mathcal{T}(\alpha^{\Sigma})$ is an hereditary set [6] in $\$(\chi)$, because given any neighbourhood $M \in \mathcal{T}$, for every $L \in \$(\chi)$, such that $M \subset L$, then $\langle \mathcal{W}, \$, \mathcal{V}, \chi, L \rangle \models \uparrow M$ and, by world existential propagation, $\langle \mathcal{W}, \$, \mathcal{V}, \chi, L \rangle \models \alpha^{\Sigma, \bullet}$ and $L \in \mathcal{T}$. But the hereditary sets are in total order in $\$(\chi)$, so, given any set $\mathcal{T}(\beta^{\Omega})$, either $\mathcal{T}(\alpha^{\Sigma}) \subset \mathcal{T}(\beta^{\Omega})$ or $\mathcal{T}(\beta^{\Omega}) \subset \mathcal{T}(\alpha^{\Sigma})$. In the case where $\mathcal{T}(\alpha^{\Sigma}) \subset \mathcal{T}(\beta^{\Omega})$, we know that every neighbourhood, that has a α^{Σ} -world, must have a β^{Ω} -world and we conclude that $\mathcal{M} \models (\alpha^{\Sigma, \bullet} \rightarrow \beta^{\Omega, \bullet})^{\otimes}$. By analogy, in the other case $\mathcal{M} \models (\beta^{\Omega, \bullet} \rightarrow \alpha^{\Sigma, \bullet})^{\otimes}$. So, by definition, $\mathcal{M} \models (\alpha^{\Sigma, \bullet} \rightarrow \beta^{\Omega, \bullet})^{\otimes} \vee (\beta^{\Omega, \bullet} \rightarrow \alpha^{\Sigma, \bullet})^{\otimes}$ and the rule 31 is sound.

We can see that the soundness of rule 31 relies upon the hereditary sets definition. But the rule depends on the index of two formulas, the operator $\rightarrow$ and the scope of the rule and it makes this rule too much complex. So, instead of adding rule 31 into iPUC-Logic, we prefer to add the representation of the hereditary sets into the system. We can also treat the hereditary sets of believers to recover some symmetry between the world quantifiers, because we have, by a similar argument, $\mathcal{M} \models (\alpha^{\Sigma, *} \rightarrow \beta^{\Omega, *})^{\otimes} \vee (\beta^{\Omega, *} \rightarrow \alpha^{\Sigma, *})^{\otimes}$. This extension of iPUC will be called iPUC^V-Logic.

$$\begin{array}{c}
\begin{array}{ccc}
32: \frac{}{\alpha^{\Sigma, \bullet}} \Delta, \mathcal{T}(\alpha^{\Sigma}) & 33: \frac{}{\beta^{\Omega, \bullet} \rightarrow \alpha^{\Sigma}} \Delta, \mathcal{T}(\beta^{\Omega}) & 34: \frac{}{\beta^{\Omega, \bullet} \rightarrow \alpha^{\Sigma}} \Delta, \mathcal{T}(\beta^{\Omega}) \\
\frac{[\alpha^{\Sigma, \bullet}]}{\Pi_1} \Delta, \mathcal{T}(\beta^{\Omega}) & \frac{[\beta^{\Omega, \bullet}]}{\Pi_2} \Delta, \mathcal{T}(\alpha^{\Sigma}) & \\
\frac{\gamma^{\Psi}}{\Theta} & \frac{\gamma^{\Psi}}{\Theta} & 36: \frac{}{\alpha^{\Sigma, *}} \Delta, \mathcal{B}(\alpha^{\Sigma}) \\
35: \frac{}{\gamma^{\Psi}} \Theta & & 37: \frac{}{\beta^{\Omega, *} \rightarrow \alpha^{\Sigma}} \Delta, \mathcal{B}(\beta^{\Omega}) \\
\frac{[\alpha^{\Sigma, *}]}{\Pi_1} \Delta, \mathcal{B}(\beta^{\Omega}) & \frac{[\beta^{\Omega, *}]}{\Pi_2} \Delta, \mathcal{B}(\alpha^{\Sigma}) & \\
\frac{\gamma^{\Psi}}{\Theta} & \frac{\gamma^{\Psi}}{\Theta} & \\
38: \frac{}{\alpha^{\Sigma}} \Delta, \mathcal{B}(\beta^{\Omega}) & 39: \frac{}{\gamma^{\Psi}} \Theta &
\end{array}
\end{array}$$

Figure III.1: The additional rules of $iPUC^V$ -ND

Restriction for rules from 32 to 39: All formulas must fit into their contexts.
 Restriction for the rules 32 and 39: Δ has no universal quantifier.

Lemma 80 $iPUC^V$ is sound.

Proof: First of all, we need to add $\mathcal{T}$ and $\mathcal{B}$ as labels. Those sets may be empty and may be defined by an universal quantification. For example, $\mathcal{T}(\alpha^{\Sigma})$ may be defined by the sentence: for all $N \in \$(\chi)$, such that α^{Σ} holds in some of its worlds. So, we consider $\mathcal{T}$ and $\mathcal{B}$ as restrictions of the label $\otimes$, thus as universal quantifiers over neighbourhoods. For the semantics:

19. $\langle \mathcal{W}, \$, \mathcal{V}, \chi \rangle \models \beta^{\Omega, \mathcal{T}(\alpha^{\Sigma})}$ iff: $\alpha^{\Sigma} \in \mathbf{F}_n$, $\beta^{\Omega} \in \mathbf{F}_w$ and $\forall \lambda \in \mathcal{A}(\chi)$, $\forall N \in \$(\lambda)$: if $\langle \mathcal{W}, \$, \mathcal{V}, \lambda, N \rangle \models \alpha^{\Sigma, \bullet}$, then $\langle \mathcal{W}, \$, \mathcal{V}, \lambda, N \rangle \models \beta^{\Omega}$;
20. $\langle \mathcal{W}, \$, \mathcal{V}, \chi \rangle \models \beta^{\Omega, \mathcal{B}(\alpha^{\Sigma})}$ iff: $\alpha^{\Sigma} \in \mathbf{F}_n$, $\beta^{\Omega} \in \mathbf{F}_w$ and $\forall \lambda \in \mathcal{A}(\chi)$, $\forall N \in \$(\lambda)$: if $\langle \mathcal{W}, \$, \mathcal{V}, \lambda, N \rangle \models \alpha^{\Sigma, *}$, then $\langle \mathcal{W}, \$, \mathcal{V}, \lambda, N \rangle \models \beta^{\Omega}$.

For the definition of the previous rules, we only need to change the first restriction of the rule 9: (a) Δ must have no occurrence of universal quantifiers over neighbourhoods. For the proof of the lemmas and theorems for soundness and completeness, the arguments for $\mathcal{T}$ and $\mathcal{B}$ should follow the arguments for $\otimes$. To prove that $iPUC^V$ is sound, we follow the strategy for PUC and iPUC. So, we need to prove that the additional rules preserve resolution.

32. From the fitting relation and lemma 19, $s(\{\Delta, \mathcal{T}(\alpha^{\Sigma})\})$ must be odd, because $\mathcal{C}(\alpha^{\Sigma, \bullet}) = \ddagger$. So, $s(\Delta)$ is even. If we take some model $\mathcal{H} = \langle \mathcal{W}, \$, \mathcal{V}, z \rangle$, such that $\mathcal{M} \multimap_{s(\Delta)} \mathcal{H}$, then we know, by definition, that $\forall \lambda \in \mathcal{A}(\chi)$, $\forall N \in \$(\lambda)$: if $\langle \mathcal{W}, \$, \mathcal{V}, \lambda, N \rangle \models \alpha^{\Sigma, \bullet}$, then $\langle \mathcal{W}, \$, \mathcal{V}, \lambda, N \rangle \models \alpha^{\Sigma}$, which means that $\mathcal{M} \models^{\Delta} \alpha^{\Sigma, \bullet, \mathcal{T}(\alpha^{\Sigma})}$ and, by rule 13, $\mathcal{M} \models^{\Delta, \mathcal{T}(\alpha^{\Sigma})} \alpha^{\Sigma, \bullet}$;

33. If $\mathcal{M} \models^{\Delta, \mathcal{T}(\beta^\Omega)} \alpha^\Sigma$, then, by the rule 14, $\mathcal{M} \models^\Delta \alpha^{\Sigma, \mathcal{T}(\beta^\Omega)}$. From $\mathcal{C}(\alpha^{\Sigma, \mathcal{T}(\beta^\Omega)}) = \dagger$, the fitting relation and lemma 19, we know that $s(\Delta)$ is even. If we take some model $\mathcal{H} = \langle \mathcal{W}, \$, \mathcal{V}, z \rangle$, such that $\mathcal{M} \multimap_{s(\Delta)} \mathcal{H}$ and $\mathcal{H} \models \alpha^{\Sigma, \mathcal{T}(\beta^\Omega)}$, then $\forall \lambda \in \mathcal{A}(z)$, $\forall N \in \$(\lambda) : \text{if } \langle \mathcal{W}, \$, \mathcal{V}, \lambda, N \rangle \models \beta^{\Omega, \bullet}$, then $\langle \mathcal{W}, \$, \mathcal{V}, \lambda, N \rangle \models \alpha^\Sigma$. By definition $\forall \lambda \in \mathcal{A}(z) : \$(\lambda) = \$(\lambda)$, then we know that $\forall N \in \$(\lambda) : \forall \lambda \in \mathcal{A}(z)$, if $\langle \mathcal{W}, \$, \mathcal{V}, \lambda, N \rangle \models \beta^{\Omega, \bullet}$, then $\langle \mathcal{W}, \$, \mathcal{V}, \lambda, N \rangle \models \alpha^\Sigma$, what means that $\forall N \in \$(\lambda) : \beta^{\Omega, \bullet} \rightarrow \alpha^\Sigma$. We conclude that $\mathcal{H} \models (\beta^{\Omega, \bullet} \rightarrow \alpha^\Sigma)^*$ and, by definition, $\alpha^{\Sigma, \mathcal{T}(\beta^\Omega)} \models_{\mathcal{M}:s(\Delta)} (\beta^{\Omega, \bullet} \rightarrow \alpha^\Sigma)^*$, which means, by lemma 21, that $\mathcal{M} \models^\Delta (\beta^{\Omega, \bullet} \rightarrow \alpha^\Sigma)^*$ and, by rule 13, $\mathcal{M} \models^{\Delta, *} \beta^{\Omega, \bullet} \rightarrow \alpha^\Sigma$;
34. Analogous to the rule 33, but in the reverse order of the conclusions;
35. From rule 14, the fitting relation, and lemma 19, we know that $s(\Delta)$ is even. If we take some model $\mathcal{H} = \langle \mathcal{W}, \$, \mathcal{V}, z \rangle$, such that $\mathcal{M} \multimap_{s(\Delta)} \mathcal{H}$, we know that $\mathcal{T}(\alpha^\Sigma) \subset \mathcal{T}(\beta^\Omega)$ or $\mathcal{T}(\beta^\Omega) \subset \mathcal{T}(\alpha^\Sigma)$. This means that $\forall N \in \$(\lambda) : \text{if } \langle \mathcal{W}, \$, \mathcal{V}, z, N \rangle \models \alpha^{\Sigma, \bullet}$ then $\langle \mathcal{W}, \$, \mathcal{V}, z, N \rangle \models \beta^{\Omega, \bullet}$ or $\forall N \in \$(\lambda) : \text{if } \langle \mathcal{W}, \$, \mathcal{V}, z, N \rangle \models \beta^{\Omega, \bullet}$ then $\langle \mathcal{W}, \$, \mathcal{V}, z, N \rangle \models \alpha^{\Sigma, \bullet}$. We can expressed it by $\mathcal{H} \models \alpha^{\Sigma, \bullet, \mathcal{T}(\beta^\Omega)} \vee \beta^{\Omega, \bullet, \mathcal{T}(\alpha^\Sigma)}$, because $\forall \lambda \in \mathcal{A}(z) : \$(\lambda) = \$(\lambda)$. By definition, $\mathcal{H} \models \alpha^{\Sigma, \bullet, \mathcal{T}(\beta^\Omega)}$ or $\mathcal{H} \models \beta^{\Omega, \bullet, \mathcal{T}(\alpha^\Sigma)}$, then, by definition, $\mathcal{M} \models^\Delta \alpha^{\Sigma, \bullet, \mathcal{T}(\beta^\Omega)}$ or $\mathcal{M} \models^\Delta \beta^{\Omega, \bullet, \mathcal{T}(\alpha^\Sigma)}$ and, using rule 13, $\mathcal{M} \models^{\Delta, \mathcal{T}(\beta^\Omega)} \alpha^{\Sigma, \bullet}$ or $\mathcal{M} \models^{\Delta, \mathcal{T}(\alpha^\Sigma)} \beta^{\Omega, \bullet}$. Using the same argument for the subderivations Π_1 and Π_2 as in lemma 28, then $\mathcal{M} \models^\Theta \gamma^\Psi$ and the hypothesis may be discharged;
36. Analogous to the case of rule 32;
37. Analogous to the case of rule 33;
38. Analogous to the rule 37, but in the reverse order of the conclusions;
39. Analogous to the case of rule 35.

■

Lemma 81 *iPUC^V is normalizing.*

Proof: We consider only the additional rules in this proof. The rules 32 and 36 introduce no maximum formula and no detour, because they take no hypothesis. The rules 35 and 39 introduce no maximum formula and no detour, because they do not change the formulas and because they left no open hypothesis to produce a detour. The rules 33 and 34 can only produce a maximum formula if the conclusion of the rule 33 is taken as hypothesis of the rule 34. This situation can also be seen as a detour and is easily removed by the following reduction rule:

$$\begin{array}{c} \frac{}{\alpha^{\Sigma, \bullet}} \Delta, \mathcal{T}(\beta^{\Omega}) \\ 33: \frac{}{\beta^{\Omega, \bullet} \rightarrow \alpha^{\Sigma, \bullet}} \Delta, \otimes \\ 34: \frac{}{\alpha^{\Sigma, \bullet}} \Delta, \mathcal{T}(\beta^{\Omega}) \quad \triangleright \quad \frac{}{\Pi} \Delta, \mathcal{T}(\beta^{\Omega}) \\ \Pi \end{array}$$

The case of rules 37 and 38 is similar to the case of rules 33 and 34. ■

Lemma 82 *iPUC^V is complete for V-Logic.*

Proof:

$$\begin{array}{c} \frac{\frac{\frac{{}^1[\alpha^{\bullet}]}{\alpha^{\bullet}} \mathcal{T}(\beta)}{\beta^{\bullet} \rightarrow \alpha^{\bullet}} \otimes}{(\beta^{\bullet} \rightarrow \alpha^{\bullet})^{\otimes}} \quad \frac{\frac{\frac{{}^1[\beta^{\bullet}]}{\beta^{\bullet}} \mathcal{T}(\alpha)}{\alpha^{\bullet} \rightarrow \beta^{\bullet}} \otimes}{(\alpha^{\bullet} \rightarrow \beta^{\bullet})^{\otimes}} \\ 1 \frac{(\alpha^{\bullet} \rightarrow \beta^{\bullet})^{\otimes} \vee (\beta^{\bullet} \rightarrow \alpha^{\bullet})^{\otimes} \quad (\alpha^{\bullet} \rightarrow \beta^{\bullet})^{\otimes} \vee (\beta^{\bullet} \rightarrow \alpha^{\bullet})^{\otimes}}{(\alpha^{\bullet} \rightarrow \beta^{\bullet})^{\otimes} \vee (\beta^{\bullet} \rightarrow \alpha^{\bullet})^{\otimes}} \end{array}$$

Considering the arguments of the beginning of this section, we only need to prove that the CONNEX axioms are theorems. ■