II
A Review of the Problems

This chapter details and reviews the problems tackled in this work, particularly the Capacitated Arc Routing Problem, the Capacitated Vehicle Routing Problem and the Generalized Vehicle Routing Problem. We also explain how other routing problems can be solved using the GVRP. Furthermore, some detailed and comprehensive surveys on routing problems can be found in [36], [75], [20], [39].

II.1 The Capacitated Arc Routing Problem

The Capacitated Arc Routing Problem (CARP) can be defined as follows. Consider a connected undirected graph $G = (V, E)$, with vertex set V and edge set E, costs $c : E \rightarrow \mathbb{Z}_0^+$, demands $d : E \rightarrow \mathbb{Z}_0^+$, a set of identical vehicles $\mathcal{K}$ with capacity Q and a distinguished depot vertex labeled 0. Define $E_R = \{e \in E | d_e > 0\}$ as the set of required edges. Let $\mathscr{R}$ be a set of closed routes which start and end at the depot, where the edges in a route can be either *serviced* or *deadheaded*. An edge is serviced when a vehicle traverses it collecting all its demand and is deadheaded when a vehicle traverses it without collecting any demand. The set $\mathscr{R}$ is a feasible CARP solution if: (i) each required edge is serviced by exactly one route in $\mathscr{R}$; (ii) The sum of demands of the serviced edges in each route in $\mathscr{R}$ does not exceed the vehicle capacity Q and (iii) $|\mathscr{R}| = |\mathcal{K}|$. We want to find a solution minimizing the sum of the costs of the routes. Notice that it corresponds to minimize the sum of the deadheaded edges' cost in the routes.

The CARP has several applications in real life. One can think of this problem as a garbage collection problem. Suppose a company is responsible for collecting the garbage of a neighborhood. This company has a set of identical vehicles available in its base and knows *a priori* the amount of garbage generated on each street. As a company, its objective is to minimize the operational costs. So, the company needs to define routes for the vehicles which minimize the total distance traveled. These routes must begin and end at the company's base, none of the vehicles can have its capacity violated, the garbage of a given street may be collected only by a single vehicle and all the garbage must be collected. Other possible applications for this problem are street sweeping, winter gritting, electric meter reading and airline scheduling, as described in the work of Wøhlk [77].

This problem is strongly $\mathcal{NP}$-hard as shown by Golden and Wong in 1981 [38], where it was first proposed. Since then, several solution algorithms were suggested for it, following different approaches. We can cite some specific primal heuristics like Augment-Merge [38, 37], Path-Scanning [37], Parallel-Insert [18], Construct-Strike [64] and Augment-Insert [65]. There are several heuristics for obtaining lower bounds, like Matching [38], Node Scanning [3], Matching-Node Scanning [66], Node Duplication [43], Hierarchical Relaxations [2] and Multiple Cuts Node Duplication [78].

Recently, most of the non-exact algorithms suggested for the CARP are based on metaheuristics. All kinds of metaheuristics have already been proposed for the CARP, some of them are Simulated Annealing [26], the CARPET Tabu Search [42], Genetic Algorithm [48], Memetic Algorithm [49], Ant Colony [23], Guided Local Search [13], Deterministic Tabu Search [17], Improved Tabu Search [61], Improved Memetic Algorithm [62], Improved Ant Colony [73] and Improved Local Search through a transformation for the Capacitated Vehicle Routing Problem [60].

Being a hard problem, it is very difficult to solve the CARP to optimality. The works which tried to achieve this usually use integer programming. The first integer programming formulation was proposed by Golden and Wong [38] and since then some other formulations were proposed by Belenguer and Benavent [9] and Letchford [52], who also proposed valid inequalities for the problem. These integer formulations were solved using techniques such as Branch-and-Bound [44], Cutting Planes [10, 1], Column Generation [40, 53, 58], Branch-and-Cut [50], Branch-Cut-and-Price [59], Cut-First Branch-and-Price-Second [16] and Cut-and-Column Generation based technique combined with a Set Partitioning Approach [7].

II.2 The Capacitated Vehicle Routing Problem

The Capacitated Vehicle Routing Problem (CVRP) is defined over a graph $G = (V, E)$, where V is the vertex set and $E = \{(i, j) | i, j \in V, i < j\}$ is the edge set. From the definition of the edge set, the graph G is a complete graph. The vertex 0 is the depot and the others are the customers. The depot is the location of a fleet of vehicles containing a set $\mathcal{K}$ of identical vehicles of capacity Q. A cost function $c : E \to \mathbb{Z}_0^+$ is associated with the edges of G while a demand function $d : V \to \mathbb{Z}_0^+$ is associated with its vertices (the depot has demand $d_0 = 0$). The objective is to find a set of closed routes with minimum total cost such that: (i) each customer must be visited exactly once and by a single vehicle which must service all of its demand; (ii) all of the $|\mathcal{K}|$ vehicles must be used and each vehicle must be associated with only one route; (iii) Vehicle routes must start and end at the depot, servicing at least one vertex; (iv) The total serviced demand by each vehicle must not exceed its capacity Q.

This problem was proposed on the work of Dantzig and Ramser [21] and is strongly $\mathcal{NP}$-hard, since it is a generalization of the TSP, which, as mentioned before, is known to be strongly $\mathcal{NP}$-hard [32].

Since it was proposed, several works have been published for the CVRP. Since it is not our intention to do a complete bibliographic review of the problem, we refer the reader the following recent CVRP surveys: the annotated bibliography on metaheuristics done by Gendreau et al. in 2008 [33], the survey on exact and heuristic algorithms done by Laporte in 2009 [51] and the survey on the latest exact approaches done by Baldacci et al. in 2010 [6].

Nevertheless, these surveys do not cite the most recent exact approaches for the CVRP. Recently, Baldacci et al. [5] introduced the state-of-the-art exact solution algorithm for vehicle routing problems, which is further improved in the current work. The approach from Baldacci et al. [5] is used to improve the solutions obtained by the best algorithms for the CVRP which before were the branch-and-cut from Lysgaard et al. [57], the robust branch-cut-and-price of Fukasawa et al. [31], the set partitioning with additional cuts based algorithm of Baldacci et al. [4] and the extended formulation with capacity-indexed variables of Pessoa et al. [69].

II.3 The Generalized Vehicle Routing Problem

The Generalized Vehicle Routing Problem (GVRP) can be defined as follows. Let $G = (V, E)$ be a graph with vertex set V and edge set E. There is a special vertex 0 called the depot. The vertices are partitioned into disjoint sets, called clusters, $C = \{C_0, C_1, \ldots, C_t\}$, where $C_0 = \{0\}$ contains only the depot. Given the cluster index set $M = \{0, 1, \ldots, t\}$, let $\mu(i) \in M$ be defined, for each vertex $i \in V$, as the index of the cluster which contains i. There exists a demand function $d : M \to \mathbb{Z}^+$ associated with all clusters, in which the depot has demand $d_0 = 0$. These demands are to be serviced by a set $\mathcal{K}$ of identical vehicles with capacity Q, located at the depot. The edge set $E = \{\{i, j\} | i, j \in V, \mu(i) \neq \mu(j)\}$ contains the edges between all pairs of vertices from different clusters. Associated with these edges, there exists a traversal cost function $c : E \rightarrow \mathbb{Z}_0^+$. Let $\mathscr{R}$ be the set of all possible closed routes starting and ending at the depot. The objective of the GVRP is to select a subset of k routes from $\mathscr{R}$ which: (i) minimizes the total traversal cost; (ii) the demand from every cluster is serviced by a single vehicle on exactly one vertex from each cluster; (iii) the total demand serviced by each route does not exceed the vehicle capacity Q.

The GVRP is a generalization of the Capacitated Vehicle Routing Problem (CVRP) and the Generalized Traveling Salesman Problem (GTSP). When all the clusters contain only one vertex, it is simply the CVRP. Similarly, when there is only one vehicle, it is simply the GTSP. It is clear that, when both conditions are true, it is simply the Traveling Salesman Problem (TSP). In the view of this, it is easy to see that any solution algorithm given to the GVRP can be directly used to solve these problems. In the chapters that follow, any algorithm designed for the GVRP will also be applied to the CVRP in this way.

This problem is strongly $\mathcal{NP}$-hard and has gained attention in the literature in recent years. As far as we know, the first published work to deal with this problem is Ghiani and Improta [34], where a transformation to the Capacitated Arc Routing Problem (CARP) is presented in order to use the existing algorithms for the CARP. One instance was proposed and solved using this approach.

Since then, few works have been published on the GVRP. Recently, the work of Bektaş et al. [8] has proposed four formulations for the GVRP. After extensive experiments with these formulations, a branch-and-cut algorithm was devised using one of them, an undirected formulation with an exponential number of constraints. The reader may consult this paper for further details on these formulations.

Furthermore, the work of Bektaş et al. [8] also points out some applications for the GVRP. We can cite some of them: routing vessels in maritime transportation, health-care logistics, urban waste collection and survivable telecommunication design.