2
Background

In this chapter, we present the basic notions related to the logics that will be considered in the thesis. We will start by introducing natural deduction that will serve as a basis for developing proof-graphs. Thus, we review the first order logic formulation in Gentzen-Prawitz' Natural Deduction.

2.1
Natural Deduction

2.1.1
A Natural Deduction System

The natural deduction (ND) is formulated in a language with logical symbols: connectives $\wedge$, $\vee$, $\rightarrow$, the quantifiers $\forall$ and $\exists$, parentheses, bound variables $x, y, z, ...$, free variables $a, b, c, ...$ and the symbol for contradiction, $\perp$(absurdity). Non-logical symbols: the propositional letters, predicates $P, Q, R, ...$, functional symbols. The set of formulas $A, B, C, ...$ can be generated with these logical symbols and non logical symbols. Negation of a formula A is expressed by $A \rightarrow \perp$.

Definition 1

- ***Minimal Implicational Logic.*** *It is a version of classical propositional calculus: Formulas are built only with propositional letters and the implication connective $\rightarrow$. There are only two simple inference rules in ND: the implication introduction ($\rightarrow$I) and the implication elimination ($\rightarrow$E) that we will see later.*
- ***Propositional Logic.*** *We will now add to propositional logic, the connectives: conjunction ($\wedge$) and disjunction ($\vee$) and their introduction and elimination rules.*
- ***First Order Logic.*** *In addition to the symbols of propositional logic (connectives, propositional letters and parentheses), It includes the quantifiers $\forall$ and $\exists$, predicates, functional symbols and constants.*

An inference rule is a scheme written as:

$$\frac{S_1, S_2, ..., S_n}{S}\rho\ ,$$

where ρ is the name of the rule, $S_1, S_2, ..., S_n$ are premise formulas and S is the conclusion formula. The basic idea of Natural Deduction is an *asymmetry*: Deductions take the form of tree-like structure, starting with one or more hypotheses as the leaves but having only one conclusion as the root.

The natural deduction system is described by means of introduction and elimination rules for each connective/quantifier plus the intuitionistic absurdity rule ($\bot_I$) and the classical absurdity rule ($\bot_C$). These inference rules are as follows (Menezes & Haeusler 2006):

Conjunction

$$\frac{\begin{array}{cc}\Pi_1 & \Pi_2 \\ A & B\end{array}}{A \wedge B}\wedge\mathrm{I} \qquad \frac{\begin{array}{c}\Pi_1 \\ A \wedge B\end{array}}{A}\wedge\mathrm{E}_R \qquad \frac{\begin{array}{c}\Pi_1 \\ A \wedge B\end{array}}{B}\wedge\mathrm{E}_L$$

Disjunction

$$\frac{\begin{array}{c}\Pi_1 \\ A\end{array}}{A \vee B}\vee\mathrm{I}_R \qquad \frac{\begin{array}{c}\Pi_1 \\ B\end{array}}{A \vee B}\vee\mathrm{I}_L \qquad \frac{\begin{array}{ccc} & [A]^u & [B]^v \\ \Pi_1 & \Pi_2 & \Pi_3 \\ A \vee B & C & C\end{array}}{C}(\vee\mathrm{E}, u, v)$$

Implication

$$\frac{\begin{array}{c}[A]^u \\ \Pi_1 \\ B\end{array}}{A \rightarrow B}(\rightarrow\mathrm{I},\ u) \qquad \frac{\begin{array}{cc}\Pi_1 & \Pi_2 \\ A & A \rightarrow B\end{array}}{B}\rightarrow\mathrm{E}$$

Existential Quantifier

$$\frac{A(t)}{\exists x A(x)}\exists\mathrm{I} \qquad \frac{\begin{array}{cc} & [A(a)] \\ & \vdots \\ \exists x A(x) & C\end{array}}{C}\exists\mathrm{E}$$

Universal Quantifier

$$\frac{A(a)}{\forall x A(x)}\forall\mathrm{I} \qquad \frac{\forall x A(x)}{A(t)}\forall\mathrm{E}$$

Intuitionistic and Classical Absurdity

$$\frac{\begin{array}{c}\Pi_1 \\ \bot\end{array}}{A}\bot_I \qquad \frac{\begin{array}{c}[A \rightarrow \bot]^u \\ \Pi_1 \\ \bot\end{array}}{\mathrm{A}}(\bot_C, u)$$

Some rules allow to discharge the hypotheses, when a formula is inferred, it becomes independent of a certain hypothesis. We denote discharged hypothesis by using square brackets and use an index to relate the hypothesis to the rule application that discharges it.

- In $\forall$I-rule the parameter a can not occur in any hypothesis on which the proof of $A(a)$ depends.
- In $\exists$E-rule the parameter a can not occur neither in $\exists x A(x)$, nor in C, nor in any hypothesis on which the upper occurrence of C depends other than $A(a)$.

Figure 2.1 shows a proof in Natural Deduction. The formulas $\forall y F(a,y)$ and $\exists x \forall y F(x,y)$ are assumed at first. Then they are discharged at the steps $\exists E_1$ and $\rightarrow I_2$ respectively.

$$\dfrac{\dfrac{[\exists x \forall y F(x,y)]^u \quad \dfrac{\dfrac{\dfrac{[\forall y F(a,y)]^v}{F(a,b)}\,\forall E}{\exists x F(x,b)}\,\exists I}{\forall y \exists x F(x,y)}\,\forall I}{\forall y \exists x F(x,y)}\,(\exists E,\, v)}{\exists x \forall y F(x,y) \rightarrow \forall y \exists x F(x,y)}\,(\rightarrow I, u)$$

Figure 2.1: A proof in Natural Deduction.

In the definition of proof-graphs for first order logic, we will only be concerned with natural deduction for intuitionistic logic, that is obtained by the set of rules for ND, excluding the classical absurdity rule ($\bot_c$).

2.1.2 Normalization

In some kinds of derivations, we have the existence of redundancies. Specifically, we have redundancies or "detours" like introduction followed by elimination of a connective that can be transformed into a proof without the two rules. Proof transformations by eliminating redundancies is called the normalization procedure, which is the main computational interest for us and was introduced by Gentzen and later on developed much further by Prawitz, who considerably extent Gentzen' techniques and results. For an extensive treatment see (van Dalen 1994).

Definition 2 *A formula occurrence γ is a* maximal formula *in a derivation when it is the conclusion of an introduction rule and the major premise of an elimination rule. γ is called a maximal formula.*

Derivations will systematically be converted into simpler ones by “elimination of maximal formulas”. The main cases are defined below:

Definition 3 (One Step Reduction)

- $\wedge$I *followed by* $\wedge \mathrm{E}_i$:

$$\dfrac{\dfrac{\begin{matrix}\Pi_1\\ A_1\end{matrix}\qquad\begin{matrix}\Pi_2\\ A_2\end{matrix}}{A_1\wedge A_2}\,\wedge\mathrm{I}}{A_i}\,\wedge\mathrm{E}_i \qquad \textit{converts to} \qquad \begin{matrix}\Pi_i\\ A_i\end{matrix}$$

- $\rightarrow$I *followed by* $\rightarrow$E:

$$\dfrac{\begin{matrix}\Pi_1\\ A\end{matrix}\qquad \dfrac{\begin{matrix}[A]^u\\ \Pi_2\\ B\end{matrix}}{A\rightarrow B}\,(\rightarrow\mathrm{I},\,u)}{B}\,\rightarrow\mathrm{E} \qquad \textit{converts to} \qquad \begin{matrix}\Pi_1\\ [A]\\ \Pi_2\\ B\end{matrix}$$

- $\vee$I *followed by* $\vee$E:

$$\dfrac{\dfrac{\begin{matrix}\Pi\\ A_i\end{matrix}}{A_1\vee A_2}\,\vee\mathrm{I}\qquad\begin{matrix}[A_1]^u\\ \Pi_1\\ B\end{matrix}\qquad\begin{matrix}[A_2]^v\\ \Pi_2\\ B\end{matrix}}{B}\,(\vee\mathrm{E},u,v) \qquad \textit{converts to} \qquad \begin{matrix}\Pi\\ [A]\\ \Pi_i\\ B\end{matrix}$$

- $\forall$I *followed by* $\forall$E:

$$\dfrac{\dfrac{\begin{matrix}\Pi\\ A\end{matrix}}{\forall x A[x/y]}\,\forall\mathrm{I}}{A[t/y]}\,\forall\mathrm{E} \qquad \textit{converts to} \qquad \begin{matrix}\Pi[t/y]\\ A[t/y]\end{matrix}$$

- $\exists$I *followed by* $\exists$E:

$$\dfrac{\dfrac{\begin{matrix}\Pi_1\\ A(t)\end{matrix}}{\exists x A(x)}\,\exists\mathrm{I}\qquad\begin{matrix}[A(y)]^u\\ \Pi_2\\ B\end{matrix}}{B}\,(\exists\mathrm{E},u) \qquad \textit{converts to} \qquad \begin{matrix}\Pi_1\\ A(t)\\ \Pi_2[t/y]\\ B\end{matrix}$$

Notation $\Pi >_1 \Pi'$ stands for “Π is converted to Π' ”. $\Pi > \Pi'$ stands for “there is a finite sequence of conversions $\Pi = \Pi_0 >_1 \Pi_1 >_1 \ldots >_1 \Pi_{n-1} = \Pi'$ ” and $\Pi \geq \Pi'$ stands for $\Pi > \Pi'$ or $\Pi = \Pi'$. (Π reduces to Π').

Definition 4 *If there is no* Π_1' *such that* $\Pi_1 >_1 \Pi_1'$ *(i.e. if* Π_1 *does not contain maximal formulas), then we call* Π_1 *a* normal derivation, *or we say that* Π_1 *is in* normal form, *and if* $\Pi \geq \Pi'$ *where* Π' *is normal, then we say that* Π normalizes *to* Π'.

We say that $>$ has the *weak normalization property* if every derivation normalizes. Moreover, when this property holds independently of the strategy that is used in applying the reduction steps, one says that the system satisfies the *strong normalization property.*

Theorem 1 (Weak Normalization) *All derivations normalise.*

Theorem 2 (Subformula Principle) *Every formula occurrence in a normal deduction of A from Γ has the shape of a subformula of A or of some formula of Γ, except for hypotheses discharged by applications of the $\perp_C$-rule and for occurrences of $\perp$ that stand immediately below such hypotheses.*