6
Conclusion

First, we showed a systematic way to extract Natural Deduction systems for finitely-many-valued logics from truth-tables showing that these systems are sound and complete with respect to the intended semantics.

Next, we extended this result to non-deterministic finite-valued logics. A different framework to obtain Sequent Calculus for these logics is described in (3). One difference between our approach and those described in (3) is that we do not pass by n-sequents to achieve provability in ordinary sequent calculus. On the other hand one may say that we produce more rules than (3) when both frameworks are applied to the same logic.

Our approach is based on a combination of techniques introduced by Segerberg in (19) and the method of internal decoding of binary prints defined in (6). It's worth mentioning again that the proof-theoretical analysis is carried out in a very systematic way directly for natural deduction without any resource to auxiliary systems.

As far as we know, the first part of the thesis is the first on this subject.

In chapter 4 we achieved a bijection between normal derivations in Natural Deduction and cut-free derivations in Sequent Calculus. In order to complete the proof-theoretical isomorphism between the systems we extended, in chapter 5, the translations (f, g) and (s, t) to translate any derivation, and not just normal and cut-free ones. Finally, we showed that the extended translations preserve reductions up to equivalent derivations, which enabled us to define a translation between conversions.

Comparing our work with Herbelin's (11), the use of a substitution rule in ND is more natural as there is no need (apparently) of a formalization of explicit substitutions to achieve isomorphism. Comparing our work with Negri and von Plato's (15), we have achieved bijection between normal and cut-free derivations, although there is no gain (nor loss) when considering non-cut-free and non-normal derivations.

To give an example of a derivation whose translation, as defined in (15), does not work, let us take the implication $A \to (B \to (B \to (A \to B)))$. The derivation of this implication in Sequent Calculus has three applications of the weakening rule. All of these rules can be applied in the beginning of the derivation (in different orders) or in different levels of the derivations. For example, with the notation used in (15):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{B \Rightarrow B}{A, B \Rightarrow B}\,Wk}{A, A, B \Rightarrow B}\,Wk}{A, A, B, B \Rightarrow B}\,Wk}{A, B, B \Rightarrow A \supset B}\,R\supset}{A, B \Rightarrow B \supset (A \supset B)}\,R\supset}{A \Rightarrow B \supset (B \supset (A \supset B))}\,R\supset}{\Rightarrow A \supset (B \supset (B \supset (A \supset B)))}\,R\supset \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{B \Rightarrow B}{A, B \Rightarrow B}\,Wk}{B \Rightarrow A \supset B}\,R\supset}{\Rightarrow B \supset (A \supset B)}\,R\supset}{B \Rightarrow B \supset (A \supset B)}\,Wk}{\Rightarrow B \supset (B \supset (A \supset B))}\,R\supset}{A \Rightarrow B \supset (B \supset (A \supset B))}\,Wk}{\Rightarrow A \supset (B \supset (B \supset (A \supset B)))}\,R\supset$$

The derivation on the left side has no correspondent in Natural Deduction. The derivation on the right side corresponds to the following derivation in Natural Deduction:

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{2.}{[B]}}{A \supset B}\,I\supset, 1.}{B \to (A \supset B)}\,I\supset, 2.}{B \supset (B \to (A \supset B))}\,I\supset, 3.}{A \supset (B \supset (B \to (A \supset B)))}\,I\supset, 4.$$

where 1., 3., and 4., are "ghost" labels that correspond to vacuous discharge. In LJT and in ND there is only one possible (normal/cut-free) derivation of $A \to (B \to (B \to (A \to B)))$, respectively:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A, B; B \vdash B}\,Ax}{A, B; \vdash B}\,\mathcal{D}}{A, B; \vdash A \to B}\,\vdash\to}{A, B; \vdash B \to (A \to B)}\,\vdash\to}{A; \vdash B \to (B \to (A \to B))}\,\vdash\to}{; \vdash A \to (B \to (B \to (A \to B)))}\,\vdash\to \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A, B \vdash B}\,Ax}{A, B \vdash A \to B}\,I_{\to}}{A, B \vdash B \to (A \to B)}\,I_{\to}}{A \vdash B \to (B \to (A \to B))}\,I_{\to}}{\vdash A \to (B \to (B \to (A \to B)))}\,I_{\to}$$

A future work we would like to mention is the improvement of the complexity of the natural deduction systems presented in chapter 2. For example, in the (6, 20) we find a method to reduce both the number of rules and the number of premises in each rule (when possible). We believe these methods can be used with our approach. For example, according to the method presented in (6), the rules for elimination of the Ł$_3$ implication presented in subsection 2.3.1 (rules 1, 2 and 3) can be replaced by the following two rules:

$$\dfrac{P \to Q \quad \theta(P) \quad \begin{matrix}[Q] \\ \vdots \\ C\end{matrix}}{C} \qquad \dfrac{P \to Q \quad \theta(P) \quad \begin{matrix}[\theta(Q)] \\ \vdots \\ C\end{matrix}}{C}$$

We can also compare the systems obtained in chapter 3 for infinite many-valued logics with other more mature ones. As a first comparison we noted that, despite the use of formulas to distinguish truth-values, which are present in all frameworks, the systems produced according to our schema is more traditional in the sense that it only involves usual deductive concepts, not appealing to any semantic feature in the calculus. We have to investigate whether this is an advantage or not.

We suggest the development of the bijection shown in chapters 4 and 5 by showing a translation between the λ-calculus and a term notation for LJT (as, for example, Herbelin's syntax in (11)). It would also be interesting to define a (stoup-based) Sequent Calculus for (non-deterministic) finite-valued logics using a method similar to the ones shown in chapters 3 and 4 and then to define a translation between this Sequent Calculus and the Natural Deduction presented in those chapters.