1
Introduction

Segerberg presented a general completeness proof for (2-valued) propositional logics. For this purpose, a Natural Deduction system was defined in a way that its rules were rules for an arbitrary boolean operator in a given propositional logic. Each of these rules corresponds to a row in the operator's truth-table. In the first part of this thesis we extend Segerberg's idea to finite-valued propositional logic and to non-deterministic logic. We maintain the idea of defining a deductive system whose rules correspond to rows of truth-tables, but instead of having n types of rules (one for each truth-value), we use a bivalent representation that makes use of the technique of separating formulas as defined by Carlos Caleiro and João Marcos. We go further and extend again our framework to include non-deterministic semantics.

Apart from its philosophical and mathematical importance, many-valued logics have provided a vast field of study in model theory and proof-theory. The definition of a complete and sound deductive system for a class of many-valued logics can certainly be seen as a contribution for this vast field. As possible applications of the results presented here, it's worth mentioning the use of many-valued logics in computer science to deal with problems of epistemic gaps, paradoxical knowledge and degrees of believe.

The systems defined have, in general, so many rules it might be laborious to work with it. We believe that a sequent calculus system defined in a similar way would be more intuitive. Motivated by this observation, in the second part of this thesis we work out translations between Sequent Calculus and Natural Deduction, searching for a better bijective relationship than those already existing, such as the translations defined in (8, 21, 15).

Logic has a strong syntactical and deductive tradition, semantics is relatively new in logic. From the model-theoretic point of view there might be many approaches to provide semantics. Algebras, categories and Tarski-based semantics are some examples. There is also proof-theoretical semantics. The Curry-Howard isomorphism can be seen as one of the most well-known representatives of this kind of semantics. Categorical models can be also considered as representatives of this proof-theoretical approach. However, even for the most well-known propositional logics, proof-theoretical semantics faces some problems. Natural Deduction and Sequent Calculus are mostly taken into account when discussing such problems. One of the points that deserve special attention is the (potential?) isomorphism between both systems. When considering normal and cut-free proofs, the literature has reported some problems as discussed in chapter 4.

Equivalences between natural deduction and sequent calculus have been discussed since their definition by Gentzen (8). By equivalence between the systems we mean that every derivation in one system can be transformed into a derivation in the other. Such equivalence being established, the search for a stronger equivalence starts. Some examples are Zucker (21), who shows a correspondence between normalization and cut-elimination for the fragment $\{\wedge, \rightarrow, \forall, \perp\}$, followed by Pottinger (17), who improved Zucker's method by simplifying it and extending it to the full intuitionistic propositional logic. Danos, Joinet and Schellinx (7) have an isomorphism between Sequent Calculus and Natural Deduction passing through Linear Logic. Nigam and Miller (16) showed that different proof systems, including Natural Deduction and Sequent Calculus, have the same provable sets of formulas by encoding the systems into a Focused Linear Logic. In (10), Henriksen showed that Linear Logic is not needed and showed a similar result from that of (16) by encoding the systems into a focused intuitionistic system. Negri and von Plato (15) showed the relation between structural rules in sequent calculus and discharge of formulas in natural deduction. Due to the structural rules, the correspondence shown in (15) is not one-to-one (see chapter 4).

In chapter 2, we extend Segerberg's (19) showing a schema that allow us to define the rules for any connective in any finite-valued propositional logic and we show soundness and completeness of such a system. In chapter 3 we extend the result obtained in chapter 2 to non-deterministic propositional logic. In chapter 4 we show a bijection between a Sequent Calculus system and a Natural Deduction system but only when dealing with cut-free and normal derivations. Non-cut-free and non-normal derivations are the subject of chapter 5 and in chapter 6 we discuss how our proposed bijection can be considered as a better solution to our relationship between Natural Deduction and Sequent Calculus as stated by (15): "cut-free proofs in sequent calculus and normal proofs in natural deduction became mere notational variants of one and the same proof".