II
Vehicle Routing Problems

This chapter intends to contextualize the ideas discussed in this dissertation. In view of this, we provide a brief introduction and some definitions to the VRP and some of its main variants.

II.1 The Capacitated Vehicle Routing Problem

The roots of Vehicle Routing Problems rely in one of the most known problems in Computer Science, the Traveling Salesman Problem (TSP). The TSP aims to find the shortest route that passes through n given points. Assuming that each pair of points is joint by a link, the total number of such possible routes is $\frac{1}{2}n!$, which is a great amount of routes, even for small values of n.

The Vehicle Routing Problem is closely related to the TSP. In fact the VRP generalizes the TSP by imposing new conditions. For instance it may be required that a quantity q_i must be delivered to each point i. If the capacity C of the carrier is such that:

$$C \geq \sum_i q_i \tag{II.1}$$

then the problem would be identical to the TSP. In the VRP, the relation between the capacity C of the carrier and the quantities q_i is such that a single carrier will not be able to attend all deliveries. More precisely, in the VRP this relationship commonly is:

$$C \ll \sum_i q_i \tag{II.2}$$

Therefore the VRP can be seen as the problem of finding loops such that all loops have a point in common (depot) and the total loop length is minimum. This problem was first introduced by (DANTZIG; RAMSER, 1959) in the context of a real world application concerning the delivery of gasoline to service stations. Dantzig also suggested a name for it, the “Clover Leaf Problem”.

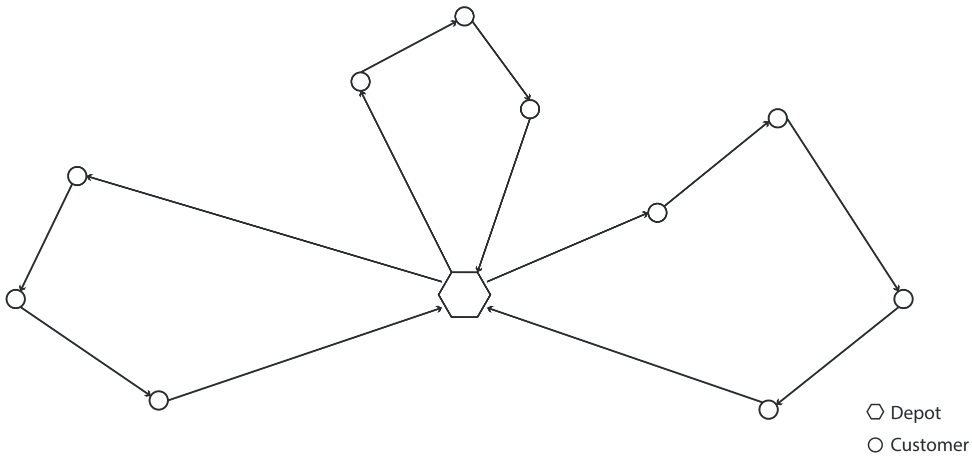

Figure II.1: Example Solution of a Vehicle Routing Problem.

Generally speaking, the Vehicle Routing Problem may be described as the problem of routing a given number of vehicles to serve a set of customers with known demands. The idea is to find the best set of routes, starting and ending at the depot, that covers all customer demands and complies with operating restrictions of the vehicles.

The VRP has been intensively studied in the literature (MAGNANTI, 1981), (BODIN *et al.*, 1983), (TOTH; VIGO, 2002) both for its rich combinatorial characteristics and for the large number of applications that it has. Up to this day, the best exact solution approaches for the VRP are those from (FUKASAWA *et al.*, 2006) and (BALDACCI *et al.*, 2011), which are based on cuts and column generation methods over a set partitioning formulation. It is not in the scope of this chapter to give much detail about these solution methods. For that purpose we refer the reader to (BALDACCI *et al.*, 2012), which is a recent review of the state of the art exact methods for solving the VRP under capacity and time constraints.

Vehicle Routing Problems naturally arise in the fields of transportation, distribution and logistics. Common applications of this type are, school bus routing, dial-a-ride systems, newspaper and mail delivery, to name just a few. Some applications from real world problems impose new conditions that lead to all kind of variants of the VRP. Some of the main variants for the VRP are shown in the figure II.2. These are:

- The Capacity Constrained VRP (CVRP) where vehicles are able to deliver or collect goods from customer up to a given maximum capacity.
- The Distance-Constrained VRP (DCVRP) where the capacity of the vehicles is expressed as maximum length (or time) constraint.

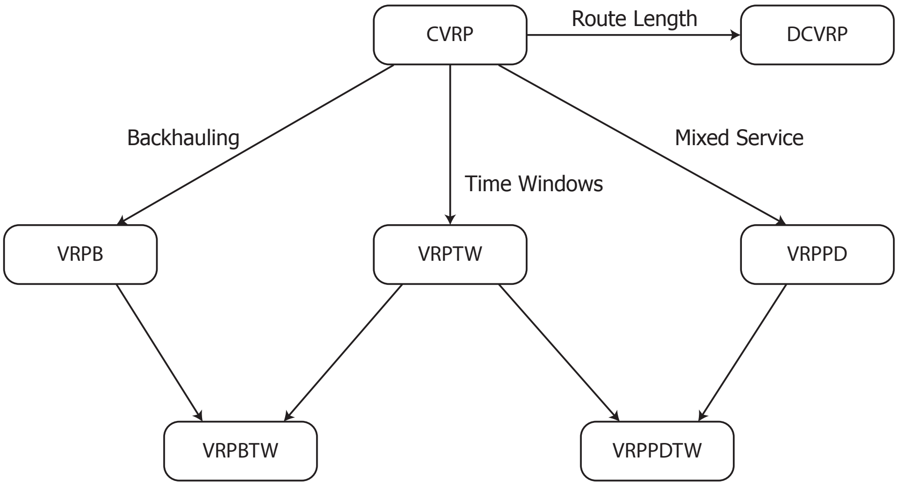

Figure II.2: The basic problems of the VRP class. (TOTH; VIGO, 2002)

– The VRP with Backhauls, where the customer set is partitioned in two subsets depending on their needs: the delivery and the *backhaul* customers. Vehicles can perform pick-up and delivery of goods, but all deliveries must be performed before the collections.

– The VRP with Time Windows, where capacity constraints are imposed and each customer is associated with a time interval in which he can be served. Further details about this variant will be given in section II.2.

– The Pick-up and Delivery VRP (VRPPD) where vehicles can perform both collection and delivery of goods, and the goods collected from the pick-up customers must be carried to the delivery customers by the same vehicle, which imposes visits precedence.

The other two variants shown in the figure correspond to mixed forms of the above cases. As one can see, the Capacitated Vehicle Routing Problem (CVRP) is the basic member of the VRP family and from it a whole group of other VRP variants arise. A formal definition of the CVRP follows.

(a) CVRP Problem Definition

The Capacitated Vehicle Routing Problem (CVRP) is defined over a graph $G = (V, A)$ where V is the vertex set and A is the arc set. Vertices $i = \{1, \ldots, n\}$ correspond to the customers and vertex 0 corresponds to the depot. The edge set A defines a strongly connected graph, generally assumed to be complete.

A non-negative cost function $c_{i,j} : A \to \mathbb{Z}^+$ is associated with the arcs of G while a demand function $d : V \to \mathbb{Z}^+$ is associated with its vertices; the depot is given a demand $d_0 = 0$. The use of loop arcs (i, i) is forbidden usually by setting $c_{i,i} = +\infty$.

An homogeneous fleet of $\mathcal{K}$ vehicles with capacity $\mathcal{Q}$ are initially stationed at the depot. The objective is to find a set of routes with minimum cost such that:

1. All routes start and end at the depot.

2. All customer demands are covered exactly once by a single vehicle.

3. All of the $|\mathcal{K}|$ vehicles are used, each one associated to only one route.

4. The capacity of each vehicle is not exceeded.

The CVRP is $\mathcal{NP}$-hard in the strong sense, as it generalizes the TSP, which is well known to be strongly $\mathcal{NP}$-hard as well (GAREY; JOHNSON, 1979).

II.2 The VRP with Time Windows

The Vehicle Routing Problem with Time Windows is the extension of the CVRP where each customer i have an associated service time s_i and a time interval $[\alpha_i, \beta_i]$ that indicates the time periods when customer i can be visite. Values α_i and β_i are known as the release date and due date for customer i, respectively. This values define the so called time windows. When time windows are treated as hard constraints, the vehicles are not allowed to arrive late at a customer. However, if a vehicle arrives to early, it is allowed to wait until the customer is ready for the beginning of service.

The goal in the VRPTW is to find a set of minimum cost routes with the same characteristics described for the CVRP (II.1(a)) and where for each customer i the service starts within time interval $[\alpha_i, \beta_i]$ and the vehicle remains s_i time instants.

The temporal aspect of routing problems became of great importance to the industry as manufacturing, service and transportation companies became more interested in not only cutting logistic costs, but also in achieving service differentiation (DESROSIERS *et al.*, 1995). Indeed, by adding the time dimension to routing problems it is possible to better handle realistic complications arising from real-world applications. Examples to this include bank deliveries, where tight schedules must be met, among others, for security reasons, dial-a-ride services where a pick-up point and a time window is given to customers for collection, and in school bus routing where the kids must be collected and delivered to school within specific times. Specific cases where

Vehicle Routing Problem with Time Windows have been successfully applied are the routing and scheduling of New York City sanity workers (BODIN, 1990) and the transporting of mentally handicapped adults (SUTCLIFFE; BOARDMAN, 1990).

The VRPTW have been intensively studied in the literature. This research has been reviewed in surveys written by (MAGNANTI, 1981) and (DESROSIERS *et al.*, 1995), and more recently by (BRÄYSY; GENDREAU, 2005a) and (BRÄYSY; GENDREAU, 2005b). The set of VRPTW instances that have been the reference benchmark in the literature was introduced by (SOLOMON, 1987). The most successful exact methods that constitute the state of the art in solving the VRPTW are based on branch-cut-and-price algorithms, except for the most efficient one which uses a mixed integer programming solver and variable elimination based on reduced cost to solve all but one of the Solomon's benchmark instances (DESAULNIERS *et al.*, 2010). In the next section we give a formal definition to the VRPTW.

(a) VRPTW Problem Definition

Starting from the definition given for the CVRP (II.1(a)) the Vehicle Routing Problem with Time Windows (VRPTW) as stated by (CORDEAU *et al.*, 2002) is defined over the graph $G = (V, A)$ where vertices $i = \{1, \ldots, n\}$ represent the customers, and the vertices 0 and $n + 1$ represent the depot.

Each vertex i has an associated service time s_i and an associated time window $[\alpha_i, \beta_i]$ where α_i and β_i correspond to the earliest and latest times, respectively, in which the customer i can start to be serviced.

The time window associated to depot nodes, 0 and $n + 1$, is $[\alpha_0, \beta_0] = [\alpha_{n+1}, \beta_{n+1}] = [S, T]$ where S and T represent the start and end of the planning time horizon, respectively. Furthermore, the depot nodes demands and service times are defined as $d_0 = d_{n+1} = s_0 = s_{n+1} = 0$.

For each pair of vertices (i, j) there is an associated non-negative travel time t_{ij}. Note that an arc $(i, j) \in A$ can be eliminated if $\alpha_i + s_i + t_{ij} > \beta_j$.

Similarly to the CVRP, the objective in the VRPTW is to find a set of routes with minimum cost such that:

1. All routes start at end at the depot.

2. All customer demands are covered exactly once by a single vehicle.

3. The capacity of each vehicle is not exceeded.

4. For each customer i the service starts within time interval $[\alpha_i, \beta_i]$ and the vehicle remains s_i time instants.

It is important to notice that the constraint, imposing that $|\mathcal{K}|$ vehicles must be used, was omitted. This is due to the fact that in some cases the objective of the VRPTW involves minimizing the total number of vehicles used. In that case the arc $(0, n+1)$ with cost $c_{0,n+1} = 0$ is added to the network.

Finally let us note that the VRPTW is strongly $\mathcal{NP}$-hard, as it is a generalization of the CVRP. It is easy to see that when time windows $[0, +\infty]$ are defined for each customer in the VRPTW, it in fact becomes equivalent to a CVRP.