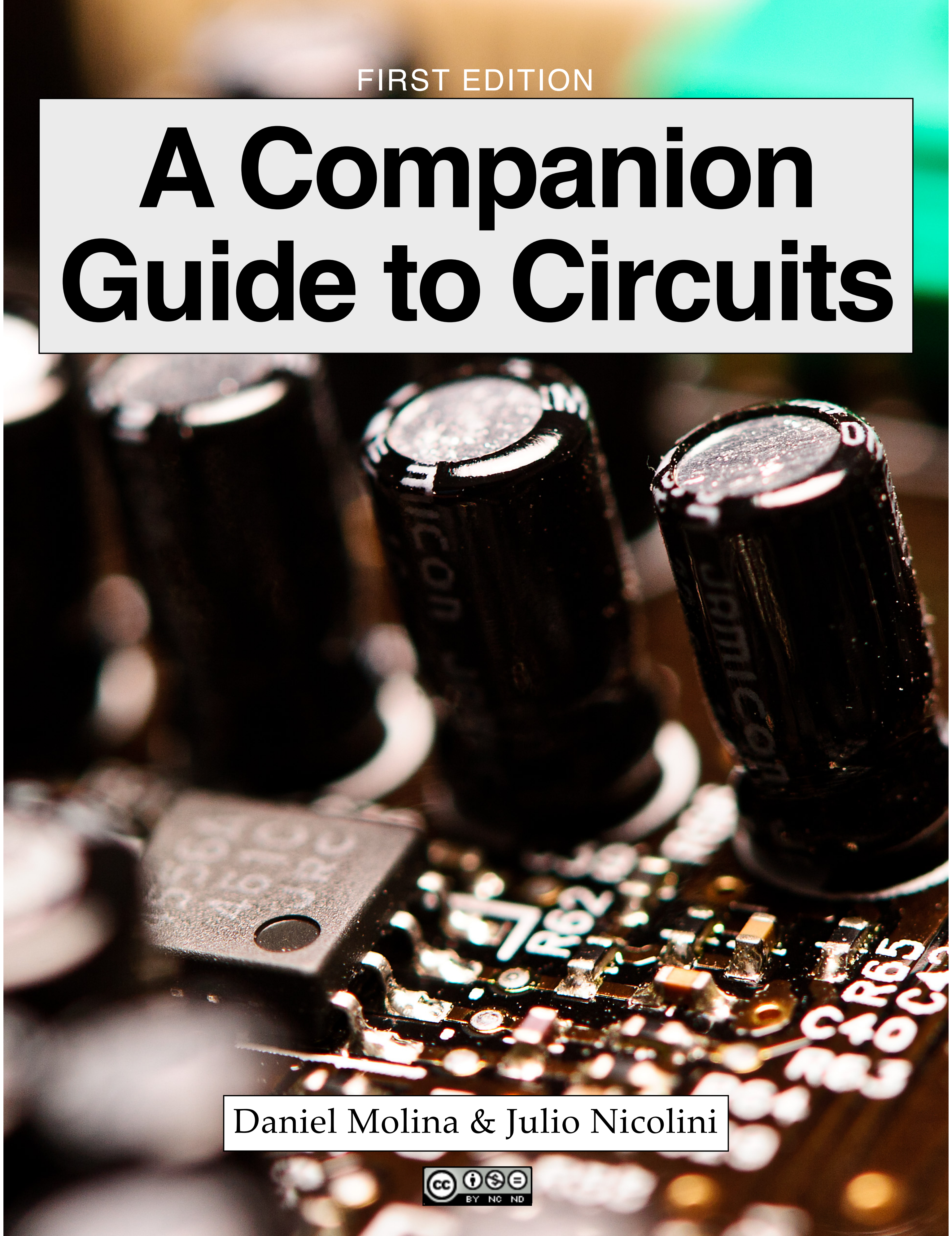
FIRST EDITION

A Companion Guide to Circuits



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About

PREFACE

Engineering is a key profession in the world. It is of paramount importance in many areas. One good example is health. When we think of engineering and health, sophisticated devices and equipment come to our minds. But health starts with the basic infrastructure of sewers and clean water that require engineers.

Engineering education is a concern and the objective of faculty, administrators, companies and funding agencies due to importance of engineers in the development of the world and society.

When two bright students chose to engage their efforts in developing an interactive book on Electrical Circuits as their Senior Project, I was delighted. This indicated that bright students are aware of the importance of Engineering Education. This is very rewarding!

In the practical side, one new object will be added to the collection *Objetos Educacionais em Engenharia Elétrica* of our university. This is wonderful since this collection makes available Learning Objects in Electrical Engineering in open access; Creative Commons licenses are used. Our objective is that they are shared and reused.

Daniel and Julio (the authors) have been my students in different courses and working with them has always been a real pleasure.

I wish that A Companion Guide to Circuits is a useful tool in the understanding of this basic "grain" of Electrical Engineering Education.

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Collaborators

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CHAPTER 1

Basic Circuit Elements

In this chapter, you will learn about basic circuit elements like voltage and current sources, resistors, capacitors and inductors. These elements are central to the design and analysis of any electrical or electronics circuit, and will accompany you throughout all of your circuit theory studies. You will also begin to familiarize yourself with some standard circuit symbology and basic properties required to analyze these elements and circuits.

Resistors

What you will learn...

1. what is a resistor.
2. how resistors are characterized.

Below, the symbols for a resistor.

Figure 1.1 Resistors with different resistances.

Resistors are one of the most common components in electrical circuits. The colored stripes on their bodies indicate both its value and its tolerance.

The concept of electrical resistance is related to that of electrical resistivity. Resistivity is a physical parameter that measures the opposition to the passage of current through an electrical conductor. It is conceptually similar to the notion of friction in classical mechanics. Resistivity is a microscopic parameter present in all materials (except superconductors, which have resistivity equal to zero).

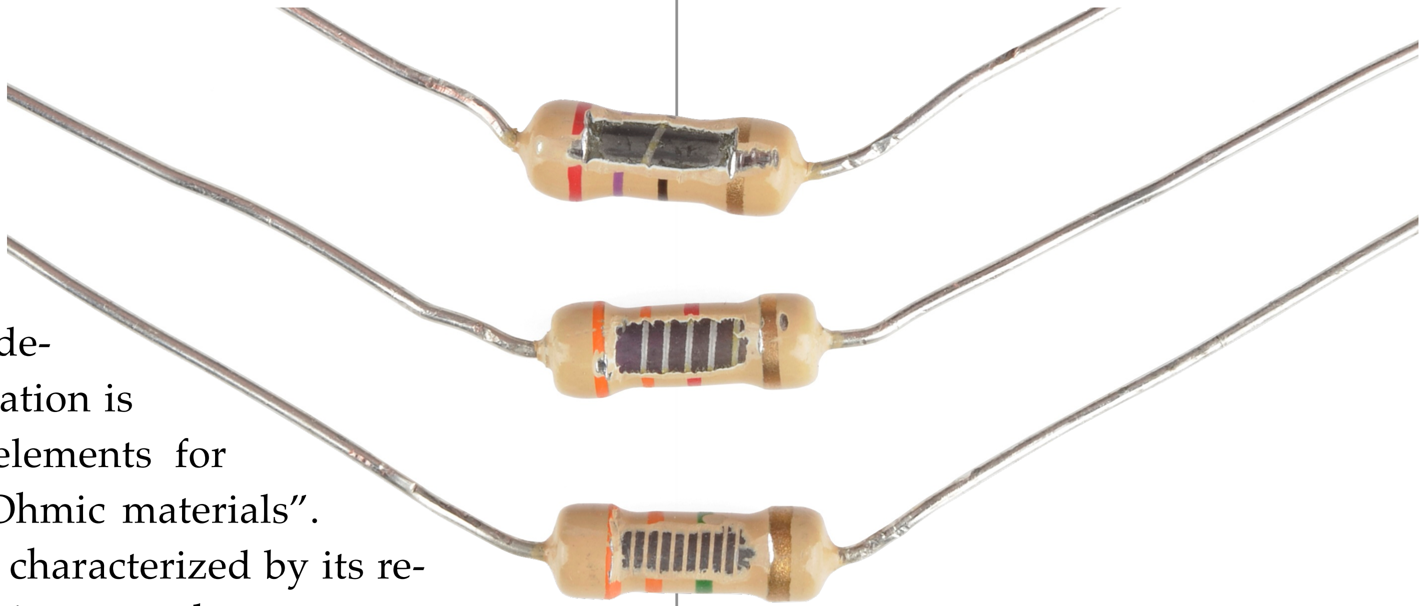
Resistance, on the other hand, is a macroscopic parameter related to the resistivity of a given material and its geometry. For a material with uniform cross section, resistance is given by

$$R = \frac{\rho l}{A}$$

where R is resistance, ρ is resistivity, l is the length of the material, and A the area of it's cross section.

The resistance of an object is defined as the ratio of voltage across its terminals to the current flowing through it, that is,

$$R = \frac{v}{i}$$



Resistance is measured in volt per ampere (V/A), a unit that is also called Ohm (Symbol: Ω) in honor of German physicist Georg Simon Ohm. When the value of R does not depend on V or I, the above equation is called Ohm's Law, and the elements for which it is valid are called "Ohmic materials". An ideal resistor is any device characterized by its resistance that functions in accordance to Ohm's Law.

Take note that an ideal resistor's resistance may still depend on exterior factors, like mechanical stress, temperature, the presence of light, or mechanical settings - it only has to be independent of the voltage and current passing through it.

If a resistor's resistance value depends on mechanical stress, it is called a piezoresistor. If it depends on temperature, it is called a thermistor. If it depends on the presence of light, it is called a photoresistor. If it depends on mechanical settings, it is called a potentiometer. All of these types of resistors are important for technological applications.

Now that we have learned about resistors, we can review our concepts of voltage and current sources and construct less idealized models for them.

Sources

What you will learn...

1. the concepts of voltage, current and power.
2. the properties of ideal sources.
3. source symbology in circuit designs

Below, the symbols for a DC voltage source, AC voltage source and current source.

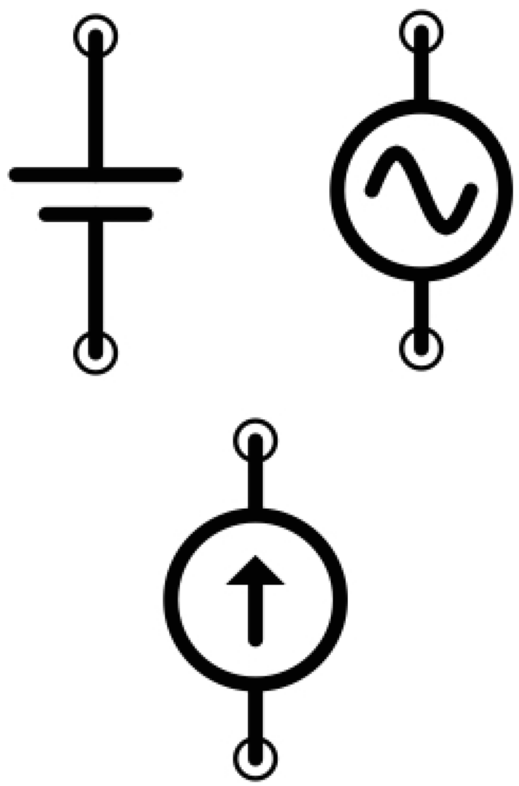


Figure 1.2 A non-ideal voltage source.

The AA batteries are a common example of a voltage source. They are not ideal, of course, as in real life nothing is ideal. Current and power sources are not as common as voltage sources.

Voltage

The concept of voltage is integral to any study in the area of electromagnetism. In the context of circuit theory, it is sufficient to know that the voltage across a generic element's terminals is proportional to amount of work necessary to move a charged particle through them. That is, the voltage across an element is proportional to the energy necessary for charge to flow through it.

It is important to remember that the voltage is defined as the difference between the electrical potential between two points. With that in mind, there is no definition of voltage for a single point, or node - instead, we have the potential at that node. We can only talk about voltages when comparing two nodes.

The voltage unit is the volt (symbol: V) in tribute to the Italian physicist Alexandre Volta.

Interactive 1.1 A simple constant voltage source.

Current

Current is defined as the flow of positive charges through a wire or, more precisely, as the rate in which charge changes with respect to time. Even though we now know that the electron, which is the particle that "carries" electricity, has negative charge, this was not known when circuit theory began its development. Therefore, we maintain the standard of the current being the flow of positive charge due to historical reasons, even though we know that it is the negative charges that are really moving. In mathematical terms, the definition of current is thus:

$$i(t) = \frac{dq(t)}{dt}$$

where $i(t)$ represent the current as a function of time and $q(t)$ represent the electric charge as a function of time.

The current unit is the ampere (symbol: A) in tribute to the French mathematician and physicist André-Marie Ampère. Note that $1\text{A} = 1^{\text{C}}/_{\text{s}}$.

Power

When an electric charges moves through a potential difference (voltage), there is the conversion of the potential electrical energy into other forms (kinetic energy, heat, etc). This conversion of energy is done through electrical work, which for circuits can be defined as

$$W(t) = q(t)v(t)$$

where W represents the work, q represents the electric charge and V represents the voltage.

In analogy with classical mechanics, we can then define electrical power as the rate of doing work, measured in watts (symbol: W) in honor of Scottish engineer James Watt. Mathematically, we define the electrical power in a generic element as the product of the voltage across its terminals with the current passing through it, or

$$p(t) = v(t)i(t)$$

where $p(t)$,$v(t)$ and $i(t)$ represent the power, voltage and current as a function of time.

Notice that this result can be either positive or negative, and both signs carry different interpretations. The interpretation for the positive sign is that energy is being absorbed by the element, while the negative sign implies that the element is providing energy to the circuit.

Ideal Voltage Source

We have defined the voltage across a generic element as the difference in the electrical potential in its terminals. It is natural that this voltage should depend on several factors such as the element's nature and properties, the circuit's design and any number of other external factors (like temperature, pressure, or even the presence of light).

We define an ideal voltage source as an element which provides a known, chosen value of voltage across its terminals regardless of external circumstances or circuit design.

Two things are worth mentioning here. First, this value need not be constant for the voltage source to be considered ideal, though constant voltage sources are a common occurrence in circuit theory. Second, there is no information given about the current that passes through an ideal voltage source, only it's voltage. Indeed, there is no way of knowing what that current is without employing some method of circuit analysis.

Ideal Current Source

Just as we have defined an ideal source for voltages, we will now define what is an ideal source for electrical currents. Again, it is natural that the current flowing through an element should depend on several factors such as the element's nature and properties, as well as external considerations such as temperature, pressure, etc, and that is indeed what happens on usual elements.

We define an ideal current source as an element which provides a known, chosen value of current flowing across it regardless of external circumstances or circuit design.

Note that, as with ideal voltage sources, this current need not be constant (though constant sources are a common topic of study). There is also no information given about the voltage across the ideal current source's terminals, only it's current. There is no way of knowing what that voltage is without employing some method of circuit analysis.

Ideal Power Source

We have defined both ideal voltage sources (that provide a known voltage value) and ideal current sources (that provide a known current value). A third type of ideal source, though less used, is the ideal power source, which provides neither a known voltage value nor a known current value, but rather a known power value. Recall that electrical power is defined by

$$p(t) = v(t)i(t)$$

Therefore, we have no actual information about current or voltage, but rather about their product. While they're generally less useful for circuit analysis than voltage or current sources, power sources are common in some technological applications, and therefore we have included them here for the sake of completeness.

Realistic Voltage Source Model

An ideal voltage source is one in which the output voltage does not change. However, realistic sources are actually con- strained by the power they are able to provide, so when using a real voltage source, if there is a large current being drawn, the output voltage should decrease so that the power output remains the same.

As such, a more realistic model would have the output voltage change according to the current that is being drawn from the source; the larger the current, the lower the voltage should be.

We can use both ideal voltage sources and resistors to model a real voltage source, by having our real voltage source be simply the series connection of an ideal voltage source of nominal value v_{int} (sometimes called electromotive force) and an internal resistance R_{int} . Therefore, we have have that the voltage source's output is

$$v = v_{int} - R_{int}i$$

where i is the current being drawn by the circuit. Alternatively, this voltage can be written as

$$v = \frac{R_{in}}{R_{in} + R_{int}} v_{int}$$

where R_{in} is the input resistance of the connected circuit.

Realistic Current Source Model

A similar analysis can be used for current sources. Rather than provide a constant current value, the output of a realistic current source should vary according to how much current the connected circuit actually draws.

Just like our realistic voltage source model, we can achieve a model for the current source by using an ideal current source and a resistor. Instead of connecting them in series, however, to make our realistic current source we connect the ideal source and the internal resistance in parallel.

By making an analysis of the resulting circuit (explained in the next chapter), we can reach the following equation for the output current:

$$i = \frac{R_{int}}{R_{int} + R_{in}} i_{int}$$

where i_{int} is the nominal current value, R_{int} is the internal resistance, and R_{in} is the input resistance of the connected circuit.

Energy Storage Elements

What you will learn...

1. what a capacitor is and which equations describe its ideal behavior.
2. what an inductor is and which equations describe its ideal behavior

Below, the symbols for a capacitor on the left and a inductor on the right.

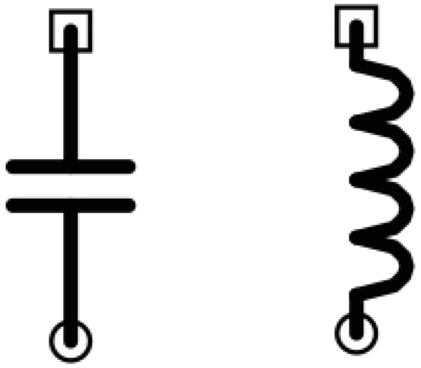
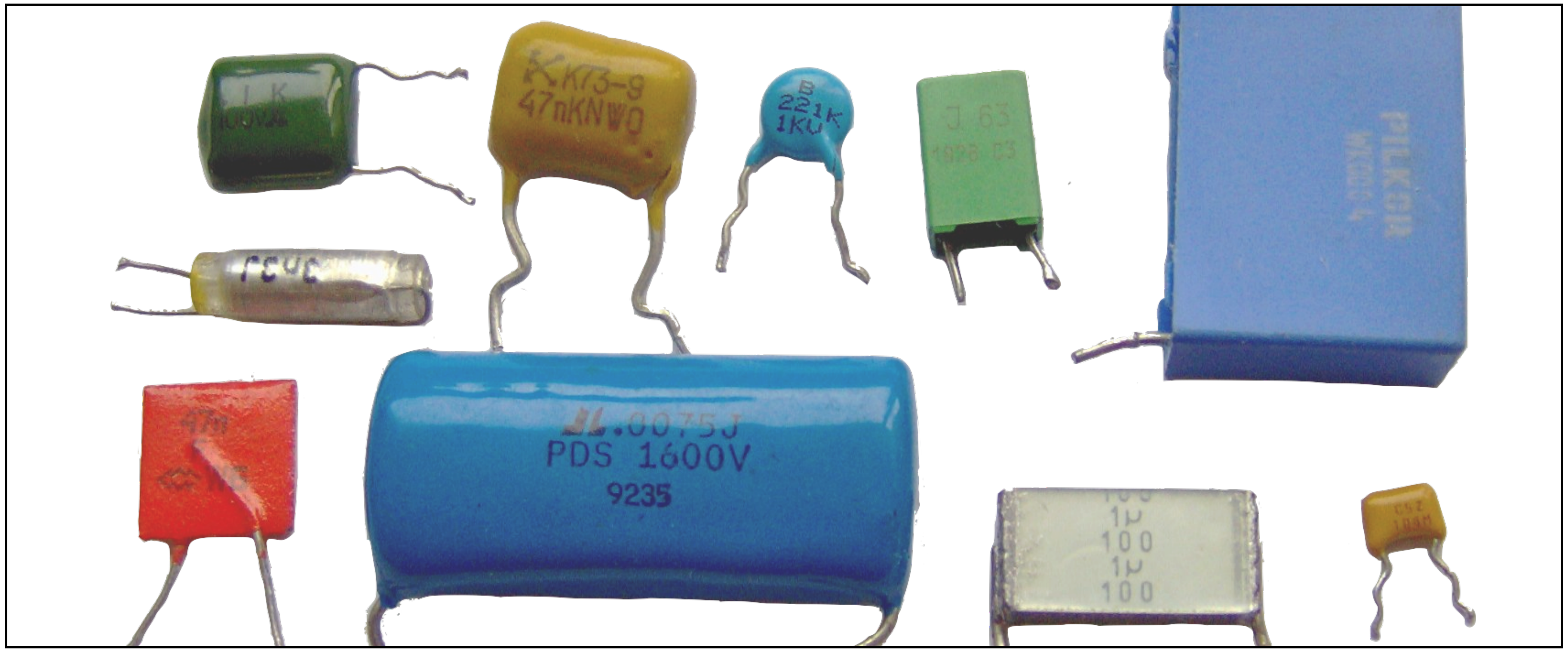


Figure 1.3 Different sized capacitors.



In this picture, you can see capacitors with different shapes, materials and capacitances.

Capacitors

The capacitor is an element widely used in electrical and electronics circuits to filter high frequency signals and to suppress ripple. It is a passive element with two terminals that may or may not be distinguishable. Each terminal is connected to a electric conductor (a plate shaped one in the simplest models) and they are separated by an insulator such as ceramic, air or glass.

You should remember from your electromagnetism studies that electric charges generate electric fields. A capacitor is based on this simple physics principle. When a voltage is applied across its terminals, a current develops. Positive and negatively charged particles accumulate in the plates due to it. This charges store energy in the form of an electric field.

We use the term "capacitance" to measure the amount of energy that a capacitor can store. It is measured in farads (symbol: F) in honor of the English physicist Michael Faraday.

There are three factors that influence the capacitance of a capacitor: the electrical permittivity. The factors that determine this attribute of this storage element are not in the scope of this text.

Lets suppose a voltage $v(t)$ is applied to a generic capacitor with capacitance C . Then some positive charge $q(t)$ will be deposited at one "side" of the element and some negative charge $-q(t)$ will be deposited at the other side. This charge can be found using the definition of capacitance, such that:

$$q(t) = Cv(t)$$

If we derive this equation we respect to time and remember the definition of current, we get:

$$i_C(t) = C\frac{dv_C(t)}{dt}$$

With this important relation in mind, we derive on of the most important facts about capacitors: the voltage cannot vary abruptly in its terminals. Mathematically, this means that

$$\lim_{t \to t_0^-} v_C(t) = \lim_{t \to t_0^+} v_C(t).$$

The reasoning behind this idea lies on the fact that is this limit relations did not hold,

$$\left.\frac{dv_C(t)}{dt}\right|_{t=t_0} \to \infty$$

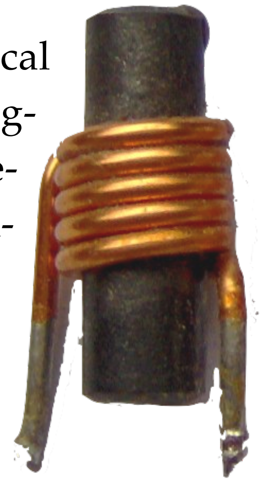
and we would violate the principle of energy conservation.

If we integrate both sides of the equation that represents the current as a function of the voltage, we can find an equation for the voltage. Do not forget to add the constant term which, in this case, is $v_C(0)$.

$$v_C(t) - v_C(0) = \frac{1}{C}\int_0^t i_C(\tau)d\tau$$

We need the constant term because the integral gives us the voltage instead of the final voltage.

Inductors



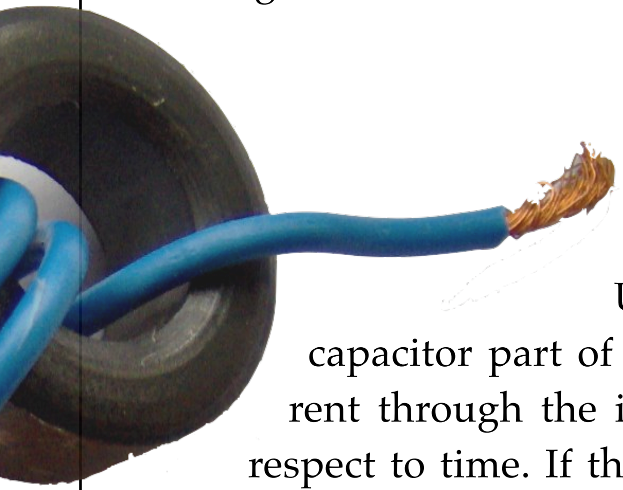
The inductor is an element widely used in electrical and electronics circuits to filter low frequency signals and to suppress ripple. It is a passive element with two terminals that are not distinguishable. The terminals are connected through a winding that may have as few as fifty turns and as many as ten thousand.

You should remember from your electromagnetism studies that electric currents generate magnetic fields. An inductor is based on this simple physics principle. When a current exists on the winding, a voltage develops across the terminals. As the current increases, a magnetic field appears in the interior of the winding where energy is stored.

We use the term "inductance" to measure the amount of energy that a inductor can store. It is measured in henries (symbol: H) in honor of the American scientist Joseph Henry.

Although this text is not going to discuss this, it can be proved that the voltage across the inductor and the current through it can be related by the following expression:

$$v_L(t) = L\frac{di_L(t)}{dt}$$



where L represent the inductance of the element.

Using the same reasoning used in the capacitor part of this section, we can infer that the current through the inductor needs to be continuous with respect to time. If that was not the case, we would violate the principle energy conservation.

Equation above represents the voltage as a function of the current, if we integrate both sides we can find an equation for the current. Do not forget to add the constant term which, in this case, is $i_C(0)$.

$$i_L(t) - i_L(0) = \frac{1}{L}\int_0^t v_L(\tau)d\tau$$

In the image, inductors with different sizes, shapes and materials.

Important Remarks

- The two terminals of a capacitor are not connected with an electric conductor. Instead, they are connected with an insulator as described earlier. Thus, if a capacitor is connected to a DC source and the system is in steady state, there is no current in the capacitor. That is, capacitors act as open-circuits when connected to DC sources as we are going to study in the following chapters.
- On the other hand, the two terminals of an inductor are connected with a conductive material. Thus, if an inductor is connected directly to a DC source and the system is in steady state, there will be an extremely large current in the system. That is, inductors act as short-circuits when connected to DC sources as we are going to study in the following chapters.
- Capacitor's terminals may or may not be distinguishable. In the case of an electrolytic capacitor, one should pay attention to the terminals as they are polarized. That is, the positive terminal cannot sustain negative values of electric potential.
- Inductors should not be connected standalone to voltage/current sources. Doing so is the same thing as short circuiting a power outlet. That is because they are simply wires that are twisted in some fashionable manner in order to generate an useful magnetic field inside them.

CHAPTER 2

Time Domain Analysis of Circuits

In this chapter we will discuss both resistive circuits and circuits with storage elements. Although they are sometimes really simple, understanding them is imperative to learn more complicated circuits that are used professionally.

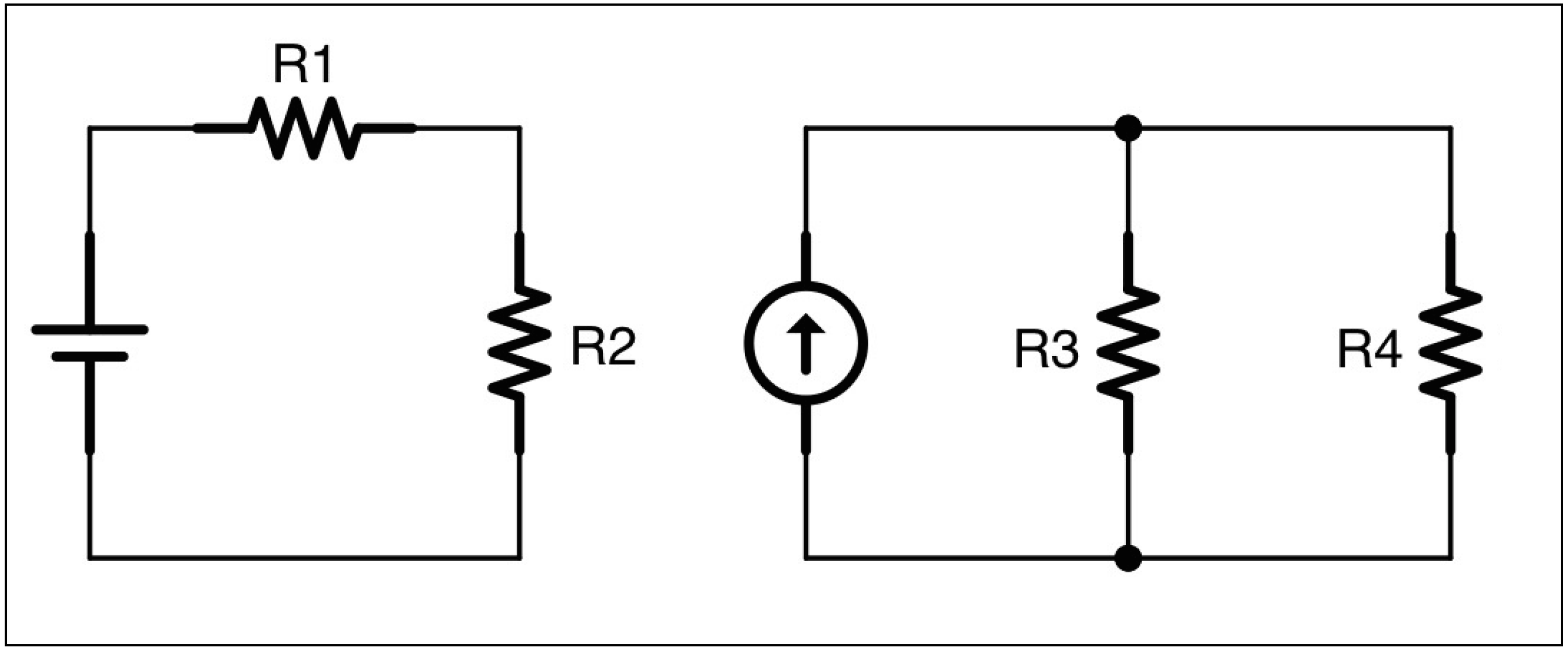
In the picture, a multimeter that is capable of measuring DC voltages and currents in real circuits.

Resistive Circuits

What you will learn...

1. how to analyze your first circuit by applying concepts learned in the previous sections.
2. two important, yet simple, circuits: the voltage and the current dividers.

Figure 2.1 A voltage divider and a current divider.



In the image, you can see a voltage divider on the left and a current divider on the right. Both circuits are extremely important and are shown in their simplest form.

A circuit is considered to be resistive if, and only if, it has only voltage/current sources and resistors. They are usually too dull for most applications but some of them are used to accomplish simple goals.

Voltage Divider

Now, we will discuss the voltage divider circuit in order to explore some characteristics of the resistive circuits. Other and more complicated examples can be found at the "Examples" section of this chapter.

The simplest voltage divider consists of a DC voltage source and two resistors. It operates under the fact described in the resistors section that $v_R(t) = Ri_R(t)$. I. e., when there is current in a resistor, a voltage appears across it and it is proportional to the resistance value (R).

Lets start the analysis of this circuit by calculating the equivalent resistance. As we already know, if the resistors are in series (same current through them) the resistances add up. This way, the equivalent resistance is $R_{eq} = R_1 + R_2$.

Knowing the equivalent resistance (R_{eq}) and the source's voltage (v_{in}) we can easily calculate the current in the circuit:

$$i = \frac{v_{in}}{R_{eq}}$$

Now lets calculate the output voltage indicated in the schematic as v_{out}. We know that the voltage drop across R_1 is $R_1 i$ according to Ohm's Law. Thus,

$$v_{out} = \frac{R_2}{R_1 + R_2} v_{in}$$

Note that the output voltage depends on a ratio between the resistor where the output is taken and the equivalent resistance. This result can be generalized for the case where we have N resistors connected in series. In this expanded case,

$$v_{out}^{(i)} = \frac{R_i}{\sum_{n=1}^{N} R_n} v_{in}$$

Current Divider

Now, we will talk about an analogous circuit. Instead of dividing voltage, this one splits current. In this case, we have a DC current source (i_{in}) instead of a voltage source. Also, the resistors are arranged in parallel instead of in series.

The equivalent resistance is calculated as shown in the section regarding resistors and it is

$$R_{eq} = \frac{R_3 R_4}{R_3 + R_4}$$

The voltage drop across the resistors can be calculated using Ohm's Law and it is simply $v = R_{eq} i_{in}$. The current in each branch is given by Ohm's Law and it is the quotient between v and the corresponding resistance yielding

$$i_{out}^{(3)} = \frac{R_4}{R_3 + R_4} i_{in}$$

for the current through R_3.

This result can also be generalized for the case in which we have N parallel resistors.

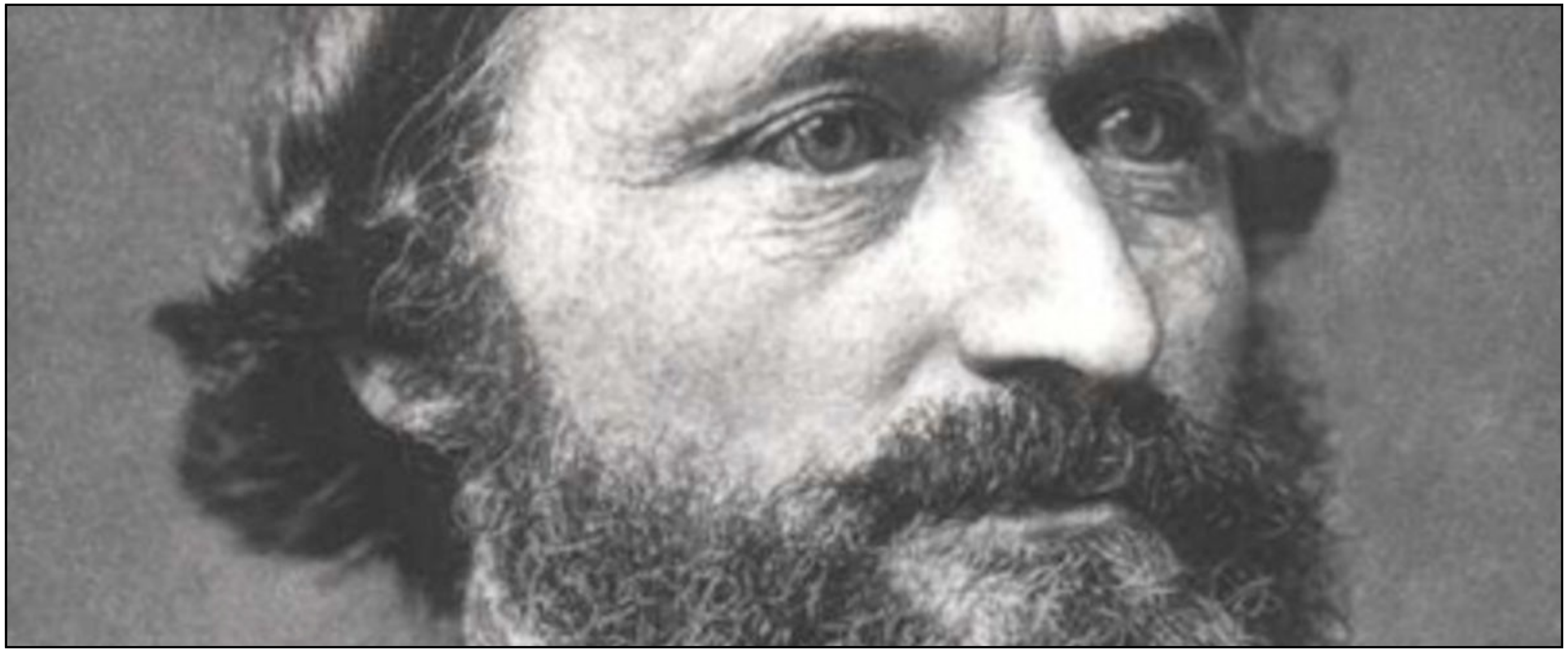
$$i_{out}^{(i)} = \frac{\sum_{n=1, n \neq i}^{N} R_n}{\sum_{n=1}^{N} R_n} i_{in}$$

Kirchhoff's Laws and the Superposition Principle

What you will learn...

1. the Kirchhoff's laws.
2. the superposition principle.

Figure 2.2 Gustav Kirchhoff



Gustav Robert Kirchhoff was a German scientist famous for his contributions in the electrical circuits and black-body radiation areas.

This section might feel out-of-place for many of the readers. However, consider it as some background necessary to the studies of electrical circuits. As soon as we dive in examples in the following sections, we will refer to the concepts presented here and everything will make more sense and will be more solidified in your minds.

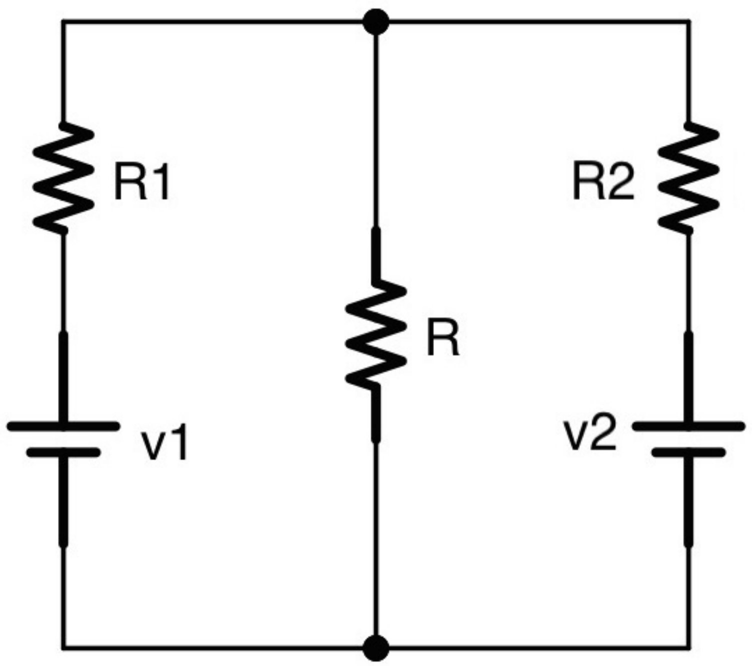
We will first talk about the voltage law and, subsequently, about the current law. At the end of this section, we will talk briefly about the Superposition Principle which is another important tool in the study of circuits.

Kirchhoff's Laws

The Kirchhoff's Circuit Laws were derived from the Maxwell's Equations by Gustav Kirchhoff, a German physicist, in 1845. They are imperative to the study of basic electric circuits and, without them, analyzing such circuits would be a harsh job.

Also known as KVL, the Kirchhoff's Voltage Law states that the algebraic sum of voltages (electric potential differences) around a closed loop in a circuit is always equal to zero.

On the other hand, the KCL, Kirchhoff's Current Law, states that there is "current conservation" in each and every node of a circuit. That is, the algebraic sum of all the currents going into a node equals the algebraic sum of all the currents going out of that same node.



Superposition Principle

You should remember from your basic math classes the concept of linear functions. They are functions such as $f(x) = ax$ and have some special properties as $f(x_1) + f(x_2) = f(x_1 + x_2)$.

Similarly to these special functions, the first circuits we are going to describe in this text as linear. Because of this property, if we have a circuit with more than one source of voltage/current we do not need to deal with all the sources at once.

Using the fact that the system is linear we can, and should, analyze the circuit with only one source turned at each time. And the result will be the combination of individual results as it was shown for the function $f(x) = ax$.

Example

To exemplify both Kirchhoff's Laws and the Superposition Principle in action, let us examine a simple case.

Imagine that you connect two voltage sources to a single resistor of value R, and you want to know the power dissipated there, P_R. Let us also assume that the voltage sources can be modeled by the realistic model presented in the previous chapter - they are therefore characterized by their voltages v_1 and v_2, and their internal resistances R_1 and R_2. Let us denote the voltage at the intersection of the three resistors by v, and set our reference at the intersection of the ideal sources and the resistor R .

First, let us solve this problem using Kirchhoff's Laws. According to the current law, the algebraic sum of currents in any node must be equal to zero. Therefore, examining the intersection node, we can write:

$$i_1 = \frac{v_1 - v}{R_1}$$

$$i_2 = \frac{v_2 - v}{R_2}$$

$$i = \frac{v - 0}{R}$$

From Kirchhoff's Law:

$$i_1 + i_2 = i$$

Therefore:

$$\frac{v_1 - v}{R_1} + \frac{v_2 - v}{R_2} = \frac{v}{R}$$

Solving for v , we get

$$v = \frac{RR_2v_1 + RR_1v_2}{R_1R_2 + RR_2 + RR_1}$$

Therefore we can find the power dissipated:

$$P_R = vi = \frac{v^2}{R} = \left(\frac{R_2v_1 + R_1v_2}{R_1R_2 + RR_2 + RR_1}\right)^2$$

Now, we can solve the same problem by applying the Superposition Principle instead. With this method, we will analyze the contribution of each source separately. We can thus write

$$v = v^{(1)} + v^{(2)}$$

where $v^{(1)}$ and $v^{(2)}$ are the contributions from source 1 and 2, respectively.

Let us examine the contribution from source 1; to do that, we must "turn off" source 2, that is, we make $v_2 = 0$. Doing that, we have R in parallel with R_2 giving us

$$R_{eq} = \frac{RR_2}{R + R_2}$$

With this, we can find $v^{(1)}$ by applying a simple voltage divider:

$$v^{(1)} = \frac{R_{eq}v_1}{R_{eq} + R_1} = \frac{RR_2v_1}{R_2R + R_1R + R_1R_2}$$

By the symmetry of the circuit, we can do the exact same calculation for the second source's contribution, giving us

$$v^{(2)} = \frac{RR_1v_2}{R_1R + R_2R + R_1R_2}$$

Therefore

$$v = v^{(1)} + v^{(2)} = \frac{RR_2v_1 + RR_1v_2}{R_1R_2 + RR_2 + RR_1}$$

And the power dissipated is, as before,

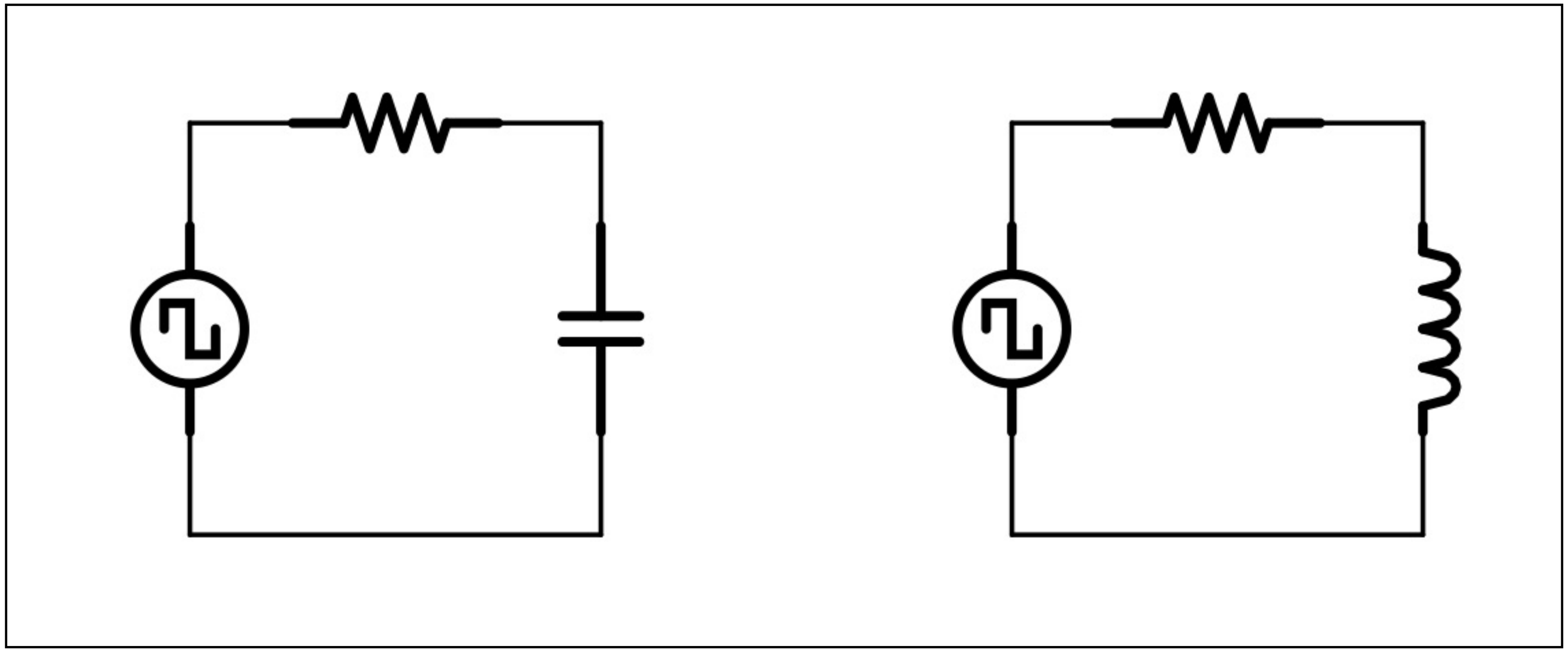
$$P_R = vi = \frac{v^2}{R} = \left(\frac{R_2v_1 + R_1v_2}{R_1R_2 + RR_2 + RR_1}\right)^2$$

First Order Circuits

What you will learn...

1. what is a first order circuit.
2. the principle of operation of a capacitive circuit.
3. the principle of operation of an inductive circuit.
4. how to use Kirchhoff's Laws to solve first order circuits.

Figure 2.3 A RC and a RL circuit.



The image depicts the schematics for a capacitive circuit on the left and an inductive circuit on the right. These circuits are going to be described in this section with a square wave as the input voltage.

In this section, we will study what is commonly called a First Order Circuit. The origin of their name lies on the fact that applying Kirchhoff's Laws on them yields a first order differential equation.

Circuits that only have one independent energy storage element are always first order circuits. Although we are not going to prove this assertion, we will show enough evidence for you to believe it in the examples.

Note that some circuits might have more than one energy storage element and still be a first order circuit. That happens because, sometimes, these elements are not independent (for example, two inductors in series).

Some Recap

Before we continue to the analysis of these particular circuits, we need to recap some math fundamentals.

First, the solution (response) of a first order differential equation is a linear combination of two parcels. The first is called homogeneous (or natural) response and the second is the particular (or forced or steady state) response.

To solve the differential equation, we find each response alone and the add them up to find the complete response.

In the study of basic electrical circuits, most homogenous responses will be exponential. Also, most forced responses will be similar to the independent term of the equation. Check the example below.

Consider the equation $\frac{dx(t)}{dt} + x(t) = C$. We know that the solution is of the form $x(t) = x_h(t) + x_p(t)$. To find the homogeneous solution, we make the constant term be zero and solve the equation by remembering the properties of the exponential function. This yields $x_h(t) = Ke^{-t}$.

To find the particular solution we assume that it is of the same kind as the independent term. That is, constant. In fact, if we make and substitute this in the equation, we see that it is a solution.

The final solution is $x(t) = Ke^{-t} + C$. Note that K can be determined using constrains particular to each situation and C is given.

Capacitive Circuits

The Idea

The first capacitive circuit we are going to describe is often referred to as the RC circuit. It it composed of a source ($v(t)$) and a resistor-capacitor serial branch. All the initial conditions (i.e., voltages and currents) are zero.

Lets assume that $v(t)$ is a square wave with period T and amplitude A. This means that

$$v(t) = \begin{cases} A & \text{for } 0 < t < \frac{T}{2} \\ 0 & \text{for } \frac{T}{2} < t < T \end{cases}$$

and it repeats periodically after $t = T$.

Without using any math, lets try to reason about what behavior will be displayed by this circuit under this conditions. During the first half-period ($0 < t < T/2$), there will be a positive voltage across the RC branch. Thus, some current should develop.

It was explored in a previous chapter that voltage must be continuous across a capacitor. Because of this, at $t = 0$ the voltage at the capacitor will still be zero and all the voltage drop will occur at the resistor. Thus, the current through the resistor and, consequently, through the capacitor (remember that they are connected in series) is $i(0) = A/R$.

We can infer that, as time goes by, this initial current $i(0)$ will start charging the capacitor. This charge will make the voltage drop across the resistor smaller and smaller reducing, this way, the current in the system. We can think of two different scenarios for the system. The capacitor can either get totally charged before $t = T/2$ or not. We will prove later that the capacitor only gets totally charged when $T \to \infty$.

For $T/2 < t < T$ the analysis is analogous. However, this time, the current will be negative through the resistor discharging the capacitor. Similarly, we will show that the capacitor only gets totally discharged when $T \to \infty$.

The Math

Now that we have already done some reasoning regarding the behavior of this simple circuit, it is time to analyze it mathematically. Lets start by applying the KVL (Kirchhoff's Voltage Law) around the only loop this circuit has. KVL states that the sum of voltages around a loop must equal zero.

Starting from the source, we have an increase of $v(t)$ volts. Then, we have a voltage drop across the resistor which, according to Ohm's Law, is $Ri_R(t)$ volts. Finally, the drop across the capacitor is $\frac{1}{C}\int_0^t i_C(t)dt$ volts. Note that $i_R(t) = i_C(t) = i(t)$ because the elements are connected in series. The equation for the KVL is, then:

$$v(t) - Ri(t) - \frac{1}{C}\int_0^t i(\tau)d\tau = 0$$

Deriving the expressing with regards to t and rearranging it, we get:

$$R\frac{di(t)}{dt} + \frac{1}{C}i(t) = \frac{dv(t)}{dt}$$

It is simpler to solve this equation in two steps. First, considering $v(t) = A$ which is is value during the first half-period. Then considering $v(t) = 0$ for the second half-period.

$$R\frac{di(t)}{dt} + \frac{1}{C}i(t) = \begin{cases} \frac{dA}{dt} = 0 & \text{for } 0 < t < \frac{T}{2} \\ \frac{d0}{dt} = 0 & \text{for } \frac{T}{2} < t < T \end{cases}$$

One must wonder why do we have to solve if separately if in both intervals the equation equals zero. It is true that we can spare some time by recycle one solution into another. However, we are dealing with equations that may be identical mathematically speaking but that do not refer to the same intervals of time. Thus, one must note that some constants will differ.

For the first half-period we have a differential equation and its solution is composed by a particular solution and a homogeneous solution. Gladly, the equation is already homogenous (because it equals zero) so the particular solution is zero. Recall from you differential equations classes that the solution to this equation will be of the form $i_1(t) = K_1 e^{-\frac{t}{RC}}$. The solution to the equation related to the second half-period is the same but with a different constant. That is, $i_2(t) = K_2 e^{-\frac{t-T/2}{RC}}$.

Please note that for $i_2(t)$, we have $t - T/2$ because this solution refers to the second half-period. This way, we need to time-shift it $T/2$ time units to the right.

To find the constants K_1 and K_2 we need to use boundary conditions. We know that at $t = 0$ the current is maximum because there is no charge (i.e., voltage) in the capacitor. Thus, all the A volt drop occurs at the resistor. Using Ohm's Law,

$A = Ri_1(0) = K_1$, we find that $K_1 = A/R$. Similarly, we know that at $t = T/2$ the source's voltage is zero and, thus, there is a $-A$ drop across the resistor. Again, using Ohm's Law we find that $K_2 = -A/R$.

$$i(t) = \begin{cases} \frac{A}{R}e^{-\frac{t}{RC}} & \text{for } 0 < t < \frac{T}{2} \\ -\frac{A}{R}e^{-\frac{t-T/2}{RC}} & \text{for } \frac{T}{2} < t < T \end{cases}$$

With both currents calculated, we can easily calculate the voltage across the resistor simply by multiplying the current by R (Ohm's Law). This yields:

$$v_R(t) = \begin{cases} Ae^{-\frac{t}{RC}} & \text{for } 0 < t < \frac{T}{2} \\ -Ae^{-\frac{t-T/2}{RC}} & \text{for } \frac{T}{2} < t < T \end{cases}$$

Also, if we want the voltage across the capacitor we can do some reasoning and see that $v_C(t) = v(t) - v_R(t)$. The voltage across the capacitor will be the input voltage minus the voltage that drops across the resistor. Note that if the capacitor is totally charged, the current in the system is zero and, thus, $v_R(t) = 0$ making $v_C(t) = v(t)$.

$$v_C(t) = \begin{cases} A - v_R(t) & \text{for } 0 < t < \frac{T}{2} \\ -v_R(t) & \text{for } \frac{T}{2} < t < T \end{cases}$$

It is a good exercise to substitute these solutions into the original equations to see if our results hold.

Remarks

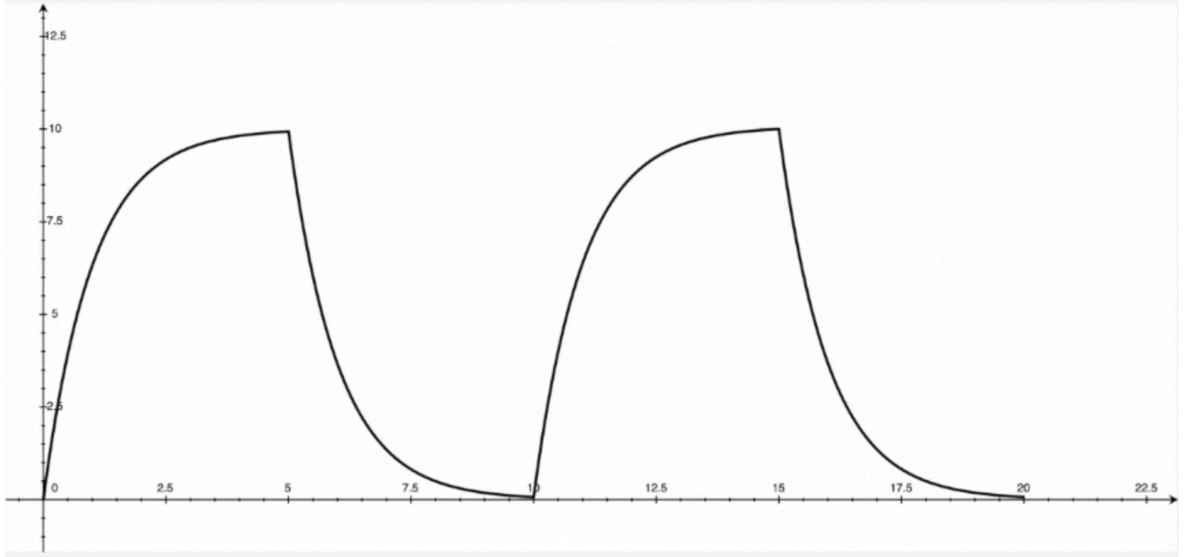
We have already given expression for all the circuit's variables, currents and voltages. However, it is wise to make some commentary regarding those solutions to achieve fluency in the topic.

First, lets calculate the voltage across the capacitor for $t = \dfrac{T}{2}$ and $t = T$.

$$v_C\left(\frac{T}{2}\right) = A - Ae^{-\frac{T}{2RC}}$$

Note that, when $T \to \infty$, $v_C(T/2) \to A$ meaning that the maximum voltage at the capacitor is the source's voltage which makes

Movie 2.1 Voltage across the capacitor as a function of time. Two periods shown. In this plot, we considered A=10 V and T=10 s. Also, R=10 kΩ and C=100 μF for starting values.



In this video, you can see the plot for the voltage across the capacitor as we increase the RC product. Note that, the higher the RC, the lower the charge acquired by the capacitor in a half period.

Also note that this graph is not showing the steady-state response (after all the transients are gone). That is the reason why the curve is not periodic.

perfect sense. However, it is not real to assume that the period goes to infinity. It is sufficient for basic applications to say that after $5RC$ seconds (often referred as the RC circuit's settling time) the capacitor is "fully" charged or discharged. Thus, if $T \geq 10RC$ the capacitor will reach "full" charge/discharge at every half-period. If $T = 10RC$ we have

$$v_C\left(\frac{T}{2}\right) = A - 0.01 \times A \approx A$$

$$v_C(T) = -0.01 \times A \approx 0$$

Inductive Circuits

The Idea

The first inductive circuit we are going to describe is often referred to as the RL circuit (the analogous for the RC circuit). It it composed of a source ($v(t)$) and a resistor-inductor serial branch. All the initial conditions (i.e., voltages and currents) are zero.

Lets assume that $v(t)$ is a square wave with period T and amplitude A. This means that

$$v(t) = \begin{cases} A & \text{for } 0 < t < \frac{T}{2} \\ 0 & \text{for } \frac{T}{2} < t < T \end{cases}$$

and it repeats itself periodically after $t = T$.

Without using any math, lets try to reason about what behavior will be displayed by this circuit under this conditions. During the first half-period ($0 < t < T/2$), there will be a positive voltage across the RL branch. Thus, some current should develop.

It was explored in a previous chapter that current must be continuous across an inductor. Because of this, at $t = 0$ the current at the inductor(i.e., the current through the circuit) will still be zero. Because there is no current in the circuit, there will be no voltage drop at the resistor meaning that all the voltage drop will occur at the inductor at $t = 0$.

We can infer that, as time goes by, this initial voltage $v(0)$ will start "charging" the inductor with current. This current will make the voltage drop across the resistor bigger and bigger. We can think of two different scenarios for the system. The inductor can either get totally charged before $t = T/2$ or not. We will prove later with math that the inductor only gets totally charged when $T \rightarrow \infty$.

For $T/2 < t < T$ the analysis is analogous. However, this time, the voltage will be negative through the inductor discharging it. Similarly, we will show that the inductor only gets totally discharged when $T \rightarrow \infty$.

The Math

Now that we have already done some reasoning regarding the behavior of this simple circuit, it is time to analyze is mathematically. Lets start by applying the KVL (Kirchhoff's Voltage Law) around the only loop this circuit has. KVL states the sum of voltages around a loop must equal zero.

Starting from the source, we have an increase $v(t)$ volts. Then, we have a voltage drop across the resistor which, according to Ohm's Law, is $Ri_R(t)$ volts. Also, across the inductor is $L\dfrac{di_L(t)}{dt}$

volts. Note that $i_R(t) = i_L(t) = i(t)$ because the elements are connected in series. The equation for the KVL is, then:

$$v(t) - Ri(t) - L\frac{di_L(t)}{dt} = 0$$

Rearranging it...

$$L\frac{di(t)}{dt} + Ri(t) = v(t)$$

For the first half-period we have a differential equation and its solution is composed by a particular solution and a homogeneous solution. Gladly, the second equation is already homogenous (because it equals zero) so the particular solution for it is zero. Recall from you differential equations classes that the solution to the first equation will be of the form

$$i_1(t) = i_{1p}(t) + i_{1h}(t) = K_1 + K_2 e^{-\frac{R}{L}t}$$

The solution to the equation related to the second half-period is similar but with a different constant and no particular solution. That is, $i_2(t) = K_3 e^{-\frac{R}{L}(t-T/2)}$.

To find the constants K_1, K_2 and K_3 we need to use boundary conditions. We know that at $t = 0$ the current is zero because there current is continuous in the inductor. Thus, $K_1 = -K_2 = K$.

To find K lets assume that $T \to \infty$. When $t \to \infty$, $i_1(t) = K_1 = K$ and we know that in that situation the inductor is totally "charged with current" and acts like a short-circuit. This way, all the voltage drop occur through the resistor. Using Ohm's Law, $A = Ri_1(t)\Big|_{t\to\infty} = RK$, we find that $K = A/R$.

Similarly, we know that the circuit's current needs to be continuos due to the inductor and, thus, $i_1(T/2) = i_2(T/2)$. This leads to $K_3 = A/R$.

$$i(t) = \begin{cases} \frac{A}{R} - \frac{A}{R}e^{-\frac{R}{L}t} & \text{for } 0 < t < \frac{T}{2} \\ \frac{A}{R}e^{-\frac{R}{L}(t-T/2)} & \text{for } \frac{T}{2} < t < T \end{cases}$$

Please note that in the second equation, we have $t - T/2$ because we are time shifting it $T/2$ time units to the right.

With both currents calculated, we can easily calculate the voltage across the resistor simply by multiplying the current by R (Ohm's Law). This yields:

$$v_R(t) = \begin{cases} A - Ae^{-\frac{R}{L}t} & \text{for } 0 < t < \frac{T}{2} \\ Ae^{-\frac{R}{L}(t-T/2)} & \text{for } \frac{T}{2} < t < T \end{cases}$$

Also, if we want the voltage across the inductor we can do some reasoning and see that $v_L(t) = v(t) - v_R(t)$. The voltage across the inductor will be the input voltage minus the voltage that drops across the resistor. Note that if the inductor is totally "discharged", the current in the system is zero and, thus, $v_R(t) = 0$ making $v_L(t) = v(t)$.

$$v_L(t) = \begin{cases} A - v_R(t) & \text{for } 0 < t < \frac{T}{2} \\ -v_R(t) & \text{for } \frac{T}{2} < t < T \end{cases}$$

It is a good exercise to substitute these solutions into the original equations to see if our results hold.

Remarks

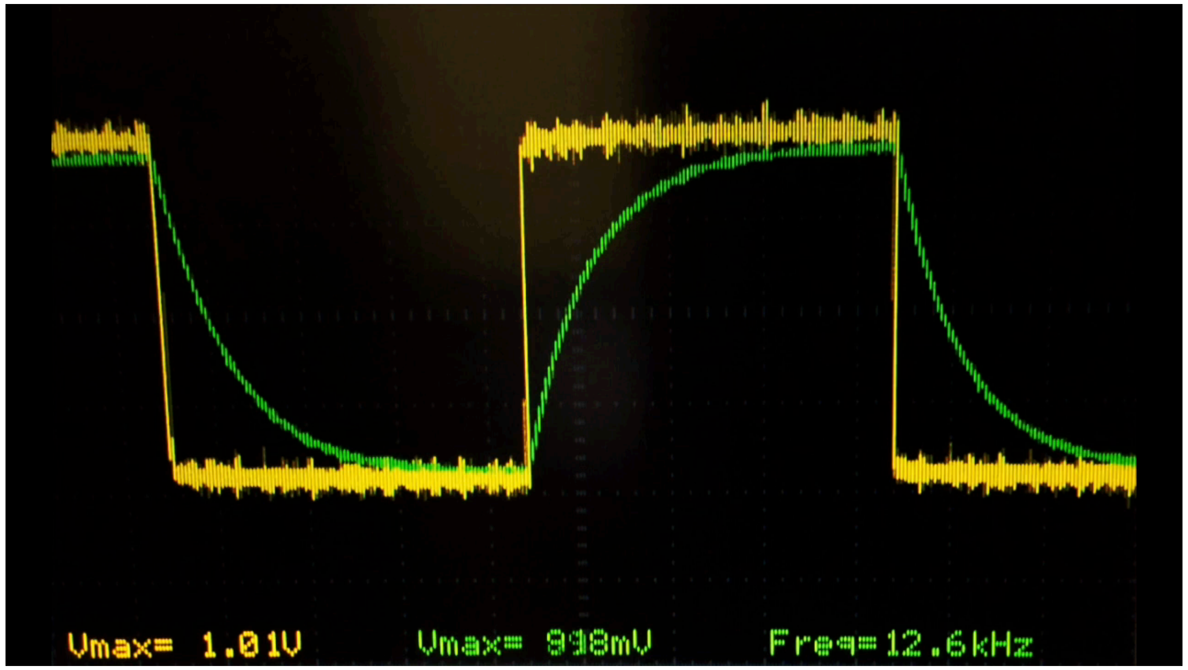
As an exercise, do the same analysis done in the RC circuit's "Remarks" part.

Note that it is sufficient for basic applications to say that after $5L/R$ seconds (often referred as the RL circuit's settling time) the inductor is "fully" charged or discharged. Thus, if $T \geq 10L/R$ the capacitor will reach "full" charge/discharge at every half-period.

Lab Experiment

In this experiment we show the practical behavior of RC circuit..

Movie 2.2 Oscilloscope measurement for capacitor charge and input.



We will be using a 1.2 kΩ resistor and a 5.6 nF capacitor.

In the low frequency range, the capacitor charge (green waveform) almost reaches the input (yellow waveform) maximum value. However, the higher the frequency, the less time the capacitor has to charge/discharge. Therefore, the capacitor is unable to reach the input's maximum value.

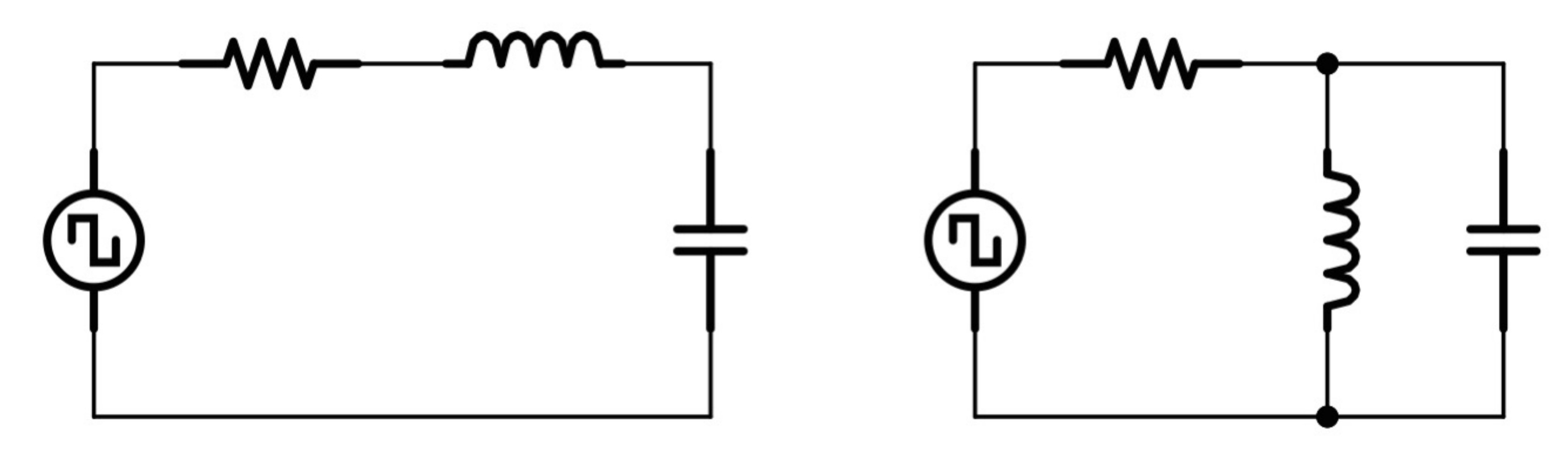
Take note that the horizontal scale has been adjusted during the recording to allow better visualization of the waveforms. The frequency, however, was increased continuously.

Second Order Circuits

What you will learn...

1. what is a second order circuit.
2. the principle of operation of a RLC series circuit.
3. the principle of operation of RLC parallel circuit.
4. how to use Kirchhoff's Laws to solve second order circuits.

Figure 2.4 A RLC series and a RLC parallel circuit.



The image depicts the schematics for a RLC series circuit on the left and a RLC parallel circuit on the right. These circuits are going to be described in this section with a square wave as the input voltage.

In this section, we will study what is commonly called a Second Order Circuit. The origin of their name lies on the fact that applying Kirchhoff's Laws on them yields a second order differential equation.

Circuits that only have two independents energy storage elements are always second order circuits (except when they are not independent). Although we are not going to prove this assertion, we will show enough evidence for you to believe it in the examples.

If you still have not read the section related to first order circuits, we really encourage you to do so.

Some Recap

Before we continue to the analysis of these particular circuits, we need to recap some math fundamentals.

Consider the following differential equation:

$$\frac{d^2x(t)}{dt^2} + B\frac{dx(t)}{dt} + Cx(t) = y(t)$$

Again, the solution to this equation is of the form $x(t) = x_p(t) + x_h(t)$. To find the homogeneous part of the solution, we need to do something more complicated then we did for a first order equation.

This time, we need to find the characteristic polynomial of the homogeneous equation by making $\frac{d}{dt} = \lambda$ yielding:

$$\lambda^2 + B\lambda + C = 0$$

This polynomial plays an important role in the form of the homogeneous solution and this will be discussed in our first example, the Series RLC Circuit.

Series RLC Circuit

This first circuit we are going to describe is fairly simple. It is composed of a voltage source ($v(t)$) and a resistor-inductor-capacitor serial branch. All the initial conditions (i.e., voltages and currents) are zero.

Lets assume that $v(t)$ is a square wave with period T and amplitude A. This means that

$$v(t) = \begin{cases} A & \text{for } 0 < t < \frac{T}{2} \\ 0 & \text{for } \frac{T}{2} < t < T \end{cases}$$

and it repeats itself periodically after $t = T$.

The Idea

Without using any math, lets try to reason about what behavior will be displayed by this circuit under these conditions. During the first half-period ($0 < t < T/2$), there will be a positive voltage across the RLC branch. Thus, some current should develop.

Recall that the voltage across the capacitor as well as the current through the inductor must be continuous as a function of time. Therefore, we can infer that at $t = 0$ there will be no current through the inductor and no voltage across the capacitor. Also there will be no voltage drop across the resistor because of the absence of current. This leads to the conclusion that all the voltage drop ($v(0)$) occurs at the inductor at $t = 0$.

This voltage drop across the inductor will start to charge it, thus, a current will develop. This current will charge the capacitor. Note that theses phenomena occur simultaneously.

If the period of the input's square wave is large enough, the capacitor will reach "full" charge and, consequently, the current in the system will reach zero at $t = T/2$.

For $T/2 < t < T$, an opposite process will occur. At first, the current will be zero and the voltage across the capacitor will be what it was just before $t = T/2$ (as long as $T/2$ is large enough for the capacitor to have fully charged). As time goes by, the energy stored in the capacitor will "feed" the circuit increasing its current and reducing the voltage across the capacitor. After some time, all the energy stored will be dissipated in the resistor and the system will reach the same state it had at $t = 0$ (again, if T is large enough).

The Math

Now that we have already done some reasoning regarding the behavior of this simple circuit, it is time to analyze it mathematically. Lets start by applying the KVL (Kirchhoff's Voltage Law) around the only loop this circuit has. KVL states that the sum of voltages around a loop must equal zero.

Starting from the source, we have an increase of $v(t)$ volts. Then, we have a voltage drop across the resistor which, according to Ohm's Law, is $Ri_R(t)$ volts. Also, the drop at the inductor is $L\dfrac{di_L(t)}{dt}$. Finally, the drop across the capacitor is $\dfrac{1}{C}\displaystyle\int_0^t i_C(t)dt$ volts. Note that $i_R(t) = i_L(t) = i_C(t) = i(t)$ because the elements are connected in series. The equation for the KVL is, then:

$$v(t) - Ri(t) - L\frac{di(t)}{dt} - \frac{1}{C}\int_0^t i(\tau)d\tau = 0$$

Deriving the expression with regards to t and rearranging it, we get:

$$\frac{d^2i(t)}{dt^2} + \frac{R}{L}\frac{di(t)}{dt} + \frac{1}{LC}i(t) = \frac{dv(t)}{dt}$$

It is simpler to solve this equation in two steps. First, considering $v(t) = A$ which is is value during the first half-period. Then considering $v(t) = 0$ for the second half-period.

$$\frac{d^2i(t)}{dt^2} + \frac{R}{L}\frac{di(t)}{dt} + \frac{1}{LC}i(t) = \begin{cases} \frac{dA}{dt} = 0 & \text{for } 0 < t < \frac{T}{2} \\ \frac{d0}{dt} = 0 & \text{for } \frac{T}{2} < t < T \end{cases}$$

Recall from what was discussed in the first order circuit's section that these equations might look identical but they have different solutions. That is because they do not describe the system in the same intervals of time and different initial conditions might apply.

For $0 < t < T/2$, we have a homogeneous equation (because it equals zero) and thus, the particular component of the solution is zero. The homogeneous solution, however, is not as simple as before because we are dealing with a second order differential equation.

The first step to find the expression for the homogeneous solution is to find the roots of the characteristic polynomial related to the equation. The polynomial, in this case, is

$$\lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC} = 0$$

and its roots are

$$\lambda_1 = -\frac{R}{2L} + \frac{1}{2}\sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}$$

$$\lambda_2 = -\frac{R}{2L} - \frac{1}{2}\sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}$$

Although we have found the general form of the roots, there are three cases which we need to explore. That is because each case will yield a different form for the solution.

The first case is when $\lambda_1 \neq \lambda_2$ giving us a solution of the form

$$i(t) = i_h(t) = k_1 e^{\lambda_1 t} + k_2 e^{\lambda_2 t}$$

where k_1 and k_2 are constants related to the initial conditions of the system (current through the inductor and voltage across the capacitor).

The second case is when $\lambda_1 = \lambda_2 = \lambda$ which will yield

$$i(t) = i_h(t) = k_1 e^{\lambda t} + k_2 t e^{\lambda t}$$

as the solution. Again, the constants are related to the initial conditions of the circuit.

Finally, if $\lambda_1 = \lambda_2$ but are complex conjugates the solution is similar to the one and differ only in the fact that the exponentials will be complex. Using Euler's Formula, it simplifies to

$$i(t) = i_h(t) = e^{\alpha t}\left[k_1 cos(\omega t) + k_2 sin(\omega t)\right]$$

where $\alpha = \Re(\lambda)$ (real part) and $\omega = \Im(\lambda)$ (imaginary part).

For $T/2 < t < T$ we have the same solutions time-shifted by a half-period ($t \to t - T/2$) and with different constants.

Now, lets assume that $\lambda_1 \neq \lambda_2$ and they are not complex conjugates and continue the analysis based on this fact. As of now we have that

$$i(t) = \begin{cases} k_1 e^{\lambda_1 t} + k_2 e^{\lambda_2 t} & \text{for } 0 < t < \frac{T}{2} \\ k_3 e^{\lambda_1 \left(t - \frac{T}{2}\right)} + k_4 e^{\lambda_2 \left(t - \frac{T}{2}\right)} & \text{for } \frac{T}{2} < t < T \end{cases}$$

Note that the constants are different in each of the equations but the λ's are not. This lies on the fact that they depend only on the topology of the circuit.

To find k_1 and k_2 lets use the conditions of the system at $t = 0$. We know that $i(0) = k_1 + k_2 = 0$ as the current through the inductor needs to be continuous. Also we know that

$$v_L(t) = L\frac{di(t)}{dt} = k_1 \lambda_1 e^{\lambda_1 t} + k_2 \lambda_2 e^{\lambda_2 t}$$

and that $v_L(0) = A$ because there is no voltage drop at the resistor ($i(0) = 0$) and also no voltage drop at the capacitor ($v_C(0) = 0$). Thus $k_1\lambda_1 + k_2\lambda_2 = A$ and we end up with the following:

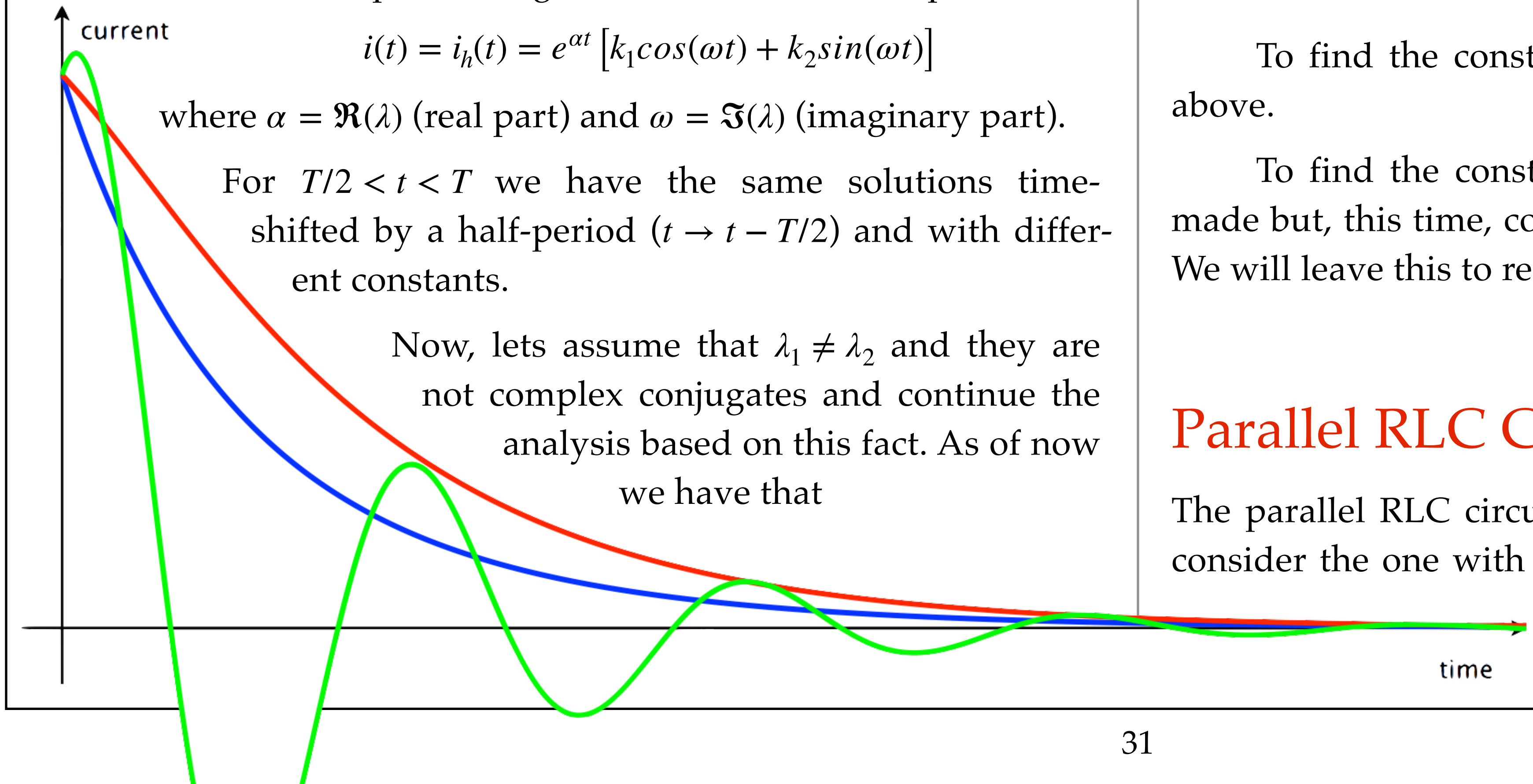
$$\begin{cases} k_1 + k_2 = 0 \\ k_1\lambda_1 + k_2\lambda_2 = A \end{cases}$$

To find the constants k_1 and k_2 one must solve the system above.

To find the constants k_3 and k_4 the same analysis must be made but, this time, considering the state of the circuit at $t = T/2$. We will leave this to reader as an exercise.

Parallel RLC Circuit

The parallel RLC circuit can have many forms. We are going to consider the one with a LC parallel branch in series with a resistor. All the initial conditions (i.e., volt-



ages and currents) are zero.

Lets assume that $v(t)$ is a square wave with period T and amplitude A. This means that

$$v(t) = \begin{cases} A & \text{for } 0 < t < \frac{T}{2} \\ 0 & \text{for } \frac{T}{2} < t < T \end{cases}$$

and it repeats itself periodically after $t = T$.

The Idea

Without using any math, lets try to reason about what behavior will be displayed by this circuit under these conditions. During the first half-period ($0 < t < T/2$), there will be a positive voltage across the circuit. Thus, some current should develop through the resistor.

Recall that the voltage across the capacitor as well as the current through the inductor must be continuous as a function of time. Therefore, we can infer that at $t = 0$ there will be no current through the inductor and no voltage across the capacitor. This leads to the conclusion that all the voltage drop ($v(0)$) occurs at the inductor at $t = 0$ and all the current goes through the resistor and and capacitor.

This current through the capacitor will start to charge it, thus, a voltage will develop. This voltage will start charge the inductor. Note that theses phenomena occur simultaneously.

If the period of the input's square wave is large enough, the capacitor and the inductor will reach "full" charge. That is, there will be no current through the capacitor and no voltage drop across the inductor.

For $T/2 < t < T$, an opposite process will occur. At first, the current through the capacitor and the voltage across the inductor will be zero (as long as $T/2$ is large enough for the capacitor to have fully charged). As time goes by, the energy stored in the capacitor/inductor will "feed" the circuit. After some time, all the energy stored will be dissipated in the resistor and the system will reach the same state it had at $t = 0$ (again, if T is large enough).

The Math

Now that we have already done some reasoning regarding the behavior of this simple circuit, it is time to analyze it mathematically. Lets start by applying the KCL (Kirchhoff's Current Law) at the node right after the resistor. Note that the voltage across the capacitor and the inductor are equal, thus, $v_C(t) = v_L(t) = v_A(t)$.

$$i_R(t) = i_C(t) + i_L(t)$$

$$\frac{v(t) - v_A(t)}{R} = C\frac{dv_A(t)}{dt} + \frac{1}{L}\int_0^t v_A(t)dt$$

If we derive with respect to time the expression above and reorganize the terms, we get

$$\frac{d^2v_A(t)}{dt^2} + \frac{1}{RC}\frac{dv_A(t)}{dt} + \frac{1}{LC}v_A(t) = \frac{dv(t)}{dt}$$

which is an differential equation that can be easily solved to find $v_A(t)$. We will leave this solution as an exercise. If in need, refer to the previous example.

With $v_A(t)$ in hand, we can find the current through the capacitor, the current through the inductor and the current through the system (the sum of the two).

$$i_C(t) = C\frac{dv_A(t)}{dt}$$
$$i_L(t) = \frac{1}{L}\int_0^t v_A(t)dt$$

CHAPTER 3

Introduction to the Frequency Domain

In this chapter we will introduce concepts related to the frequency domain. These tools are helpful in the analysis of complex systems, and understanding the frequency domain is an important step in the study of linear circuits.

To the right you can see a picture of Jean-Baptiste Joseph Fourier, a French mathematician and physicist. He is responsible for starting a branch of mathematics known as Fourier analysis, which is one of the cornerstones of frequency domain analysis.

Mathematical Tools

What you will learn...

1. what is the Laplace Transform and its properties.
2. what is the Fourier Transform.
3. what is the Fourier Series.

Figure 3.1 Pierre-Simon, marquis de Laplace.

French mathematician and astronomer. He pioneered the Laplace Transform and is often called French Newton. He is definitely one of the most important scientists of all time.

Laplace Transform

Overview

The Laplace Transform is a powerful mathematical tool for solving differential equations. We have seen in the previous chapter that linear circuits can be characterized by a differential equation for voltage or current, therefore the Laplace Transform will come in handy when performing linear circuit analysis.

Basically, the Laplace Transform is a linear operator that transforms a real-valued function in the time domain into a complex-valued function in s-domain, given by

$$\mathfrak{L}\{f(t)\} = F(s) = \int_{-\infty}^{\infty} f(t)e^{-st}dt$$

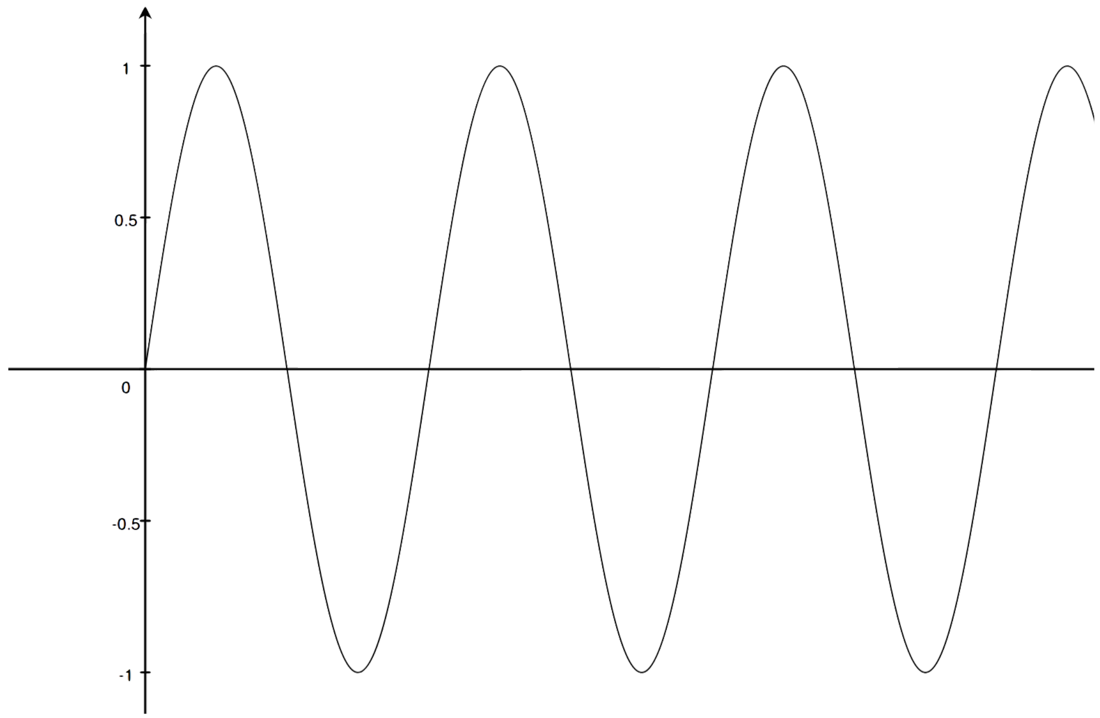
where s is a complex variable given by $s = \sigma + j\omega$. You may find the above expression called the "Bilateral Laplace Transform" in some books.

Example

Take the function $f(t) = \sin(\omega_0 t)u(t)$ where $u(t)$ is the Heaviside step function defined by

$$u(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$

Therefore, $f(t)$ is a sine function of unitary amplitude that only exists for values of t such that $t > 0$, as shown in the figure below.



The Heaviside step function is an useful tool in circuit theory, because many signals only exist after a certain point in time or until a certain point. Manipulating the Heaviside step function is a simple way of modeling those signals mathematically.

Applying the Laplace Transform equation above, we have:

$$F(s) = \int_{-\infty}^{\infty} \sin(\omega_0 t)u(t)e^{-st}dt$$

From the definition of the Heaviside step function, we can simplify that to:

$$F(s) = \int_{0}^{\infty} \sin(\omega_0 t)e^{-st}dt$$

It can be shown (do it!) that the above integral resolves into

$$\int_{0}^{\infty} \sin(\omega_0 t)e^{-st}dt = \left(\frac{se^{st}\sin(\omega_0 t)}{s^2+\omega_0^2} - \frac{\omega_0 e^{st}\cos(\omega_0 t)}{s^2+\omega_0^2} \right) \Bigg|_{0}^{\infty}$$

PROPERTY	TIME DOMAIN	S-DOMAIN
Linearity	$af(t) + bg(t)$	$aF(s) + bG(s)$
Differentiation	$\frac{d}{dt}f(t)$	$sF(s) - f(0)$
Seccond Differentiation	$\frac{d^2}{dt^2}f(t)$	$s^2F(s) - sf(0) - \frac{d}{dt}f(0)$
General Differentiation	$\frac{d^n}{dt^n}f(t)$	$s^nF(s) - \sum_{k=1}^{n} s^{k-1}\frac{d^{n-k}}{dt^{n-k}}f(0)$
Time Scaling	$f(at), a > 0$	$\frac{1}{a}F(\frac{s}{a})$
Time Shifting	$f(t - t_0)$	$e^{-st_0}F(s)$
Convolution	$\int_0^t f(\tau)g(t-\tau)d\tau$	$F(s)G(s)$

FUNCTION	TIME DOMAIN	S-DOMAIN
Impulse	$\delta(t)$	1
Unit Step	$u(t)$	$\frac{1}{s}$
Unit Ramp	$tu(t)$	$\frac{1}{s^2}$
Exponential Decay	$e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$
Exponential Approach	$(1-e^{-\alpha t})u(t)$	$\frac{\alpha}{s(s+\alpha)}$
Sine	$\sin(\omega t)u(t)$	$\frac{\omega}{s^2+\omega^2}$
Cosine	$\cos(\omega t)u(t)$	$\frac{s}{s^2+\omega^2}$
Exponentially Decaying Sine	$e^{-\alpha t}\sin(\omega t)u(t)$	$\frac{\omega}{(s+\alpha)^2+\omega^2}$
Exponentially Decaying Cosine	$e^{-\alpha t}\cos(\omega t)u(t)$	$\frac{s}{(s+\alpha)^2+\omega^2}$

Evaluating both expressions at infinity leads to zero terms, as does the sine term evaluated at $t = 0$. The only remaining term is, therefore,

$$F(s) = \left. \frac{\omega_0 e^{st}\cos(\omega_0 t)}{s^2+\omega_0^2} \right|_{t=0} = \frac{\omega_0}{s^2+\omega_0^2}$$

Now that we have wetted our feet on calculating an example of Laplace Transform, let us summarize the properties available to us. Take note that most of these properties follow from the definition of the transform and the properties of integrals, and they are summarized here for ease of access only.

Now, let us present a few useful known transforms.

Fourier Transform

The Fourier Transform is another widely used mathematical tool to solve differential equations and analyze functions by expressing them as a series of modes of vibrations (frequencies), whereas the Laplace Transform expresses them as a superposition of moments, which is more general.

Mathematically, the Fourier Transform is a restriction of the Laplace Transform to the imaginary axis, that is, whereas the Laplace Transform maps a real-valued function in the time domain to a complex-valued function in s-domain, the Fourier Transform maps a real-valued function in the time domain to a complex-valued function in the frequency domain, or ω-domain. Any result for the Laplace Transform can be carried over to the Fourier Transform by making the restriction

$$s = j\omega$$

Fourier Series

In the case of a periodic function of time, the Fourier Transform can be simplified into a series of oscillating functions (usually sinusoidal), instead of a continuous function of angular frequency. Mathematically, this means that instead of an integral over all frequencies, the time-domain function can be expressed as a summation over the discrete frequencies of the oscillating functions, that is,

$$f(t) = \sum_{-\infty}^{\infty} a_k e^{\frac{j2k\pi t}{T}}$$

where a_k are called the Fourier Series Coefficients and are defined by

$$a_k = \frac{1}{T}\int_T f(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_T f(t)e^{\frac{-j2k\pi t}{T}}dt$$

where ω_0 is the fundamental frequency and T is the fundamental period associated with that frequency.

There is an equivalent, more intuitive form of the Fourier series expansion. It is

$$f(t) = c_0 + \sum_{n=1}^{\infty}[a_n \cos(n\omega t)] + \sum_{n=1}^{\infty}[b_n \sin(n\omega t)]$$

where

$$c_0 = \frac{1}{T}\int_0^T f(t)dt$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos(n\omega t)dt$$

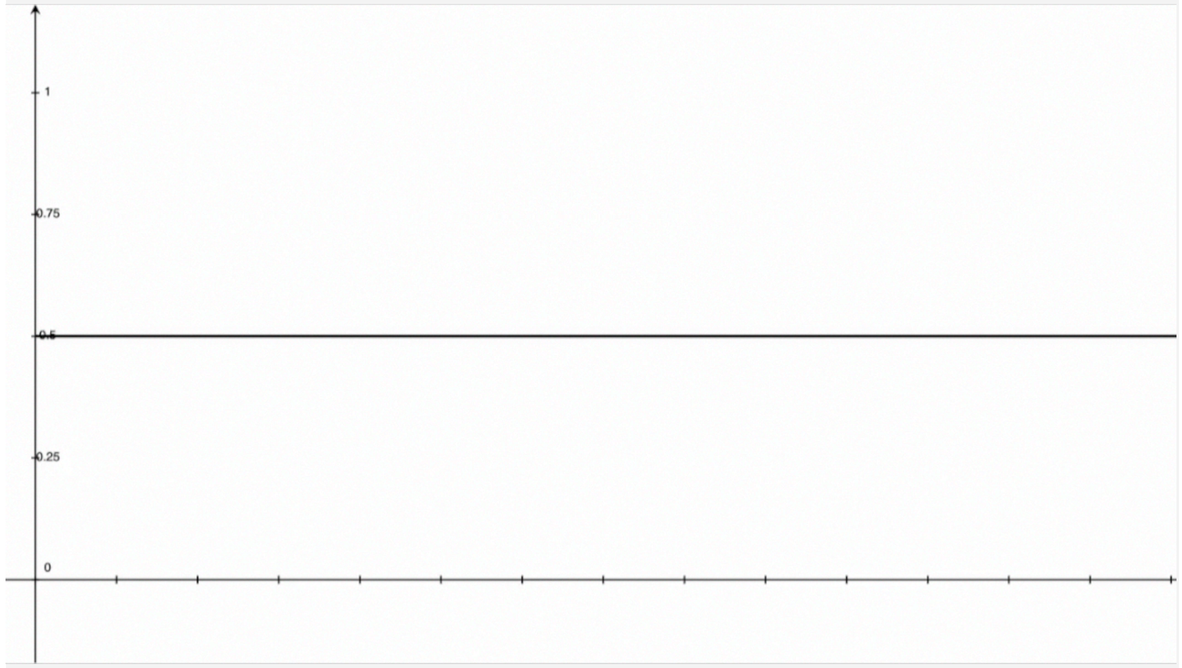
$$b_n = \frac{2}{T}\int_0^T f(t)\sin(n\omega t)dt$$

or, equivalently,

$$f(t) = c_0 + \sum_{n=1}^{\infty}\left[c_n \cos(n\omega t - \phi_n)\right]$$

where $c_n = \sqrt{a_n^2 + b_n^2}$ and $\phi_n = \arctan(b_n/a_n)$.

Movie 3.1 A square wave's Fourier Series plot.



In this animation, we have the series varying from 0 to 15 components of frequency. Note that, in this case, the summation is going up to 30 because all the even components are zero. This happens because the square wave itself is an odd function.

All representations are equivalent. One should use the one that one fells most comfortable with, either the complex exponentials or the cosines.

Example

As an example, let us calculate the Fourier Series of a square wave with period T and amplitude A. That is,

$$f(t) = \begin{cases} A & \text{for } 0 < t < \frac{T}{2} \\ 0 & \text{for } \frac{T}{2} < t < T \end{cases}$$

To specify this function Fourier Series, we need to find all its coefficients, i.e., a_n, b_n and c_0.

$$c_0 = \frac{1}{T}\int_0^{\frac{T}{2}} A\,dt = \frac{A}{2}$$

$$a_n = \frac{2}{T}\int_0^{\frac{T}{2}} A\cos(n\omega t)dt = \frac{2A}{n\omega T}\sin\left(\frac{n\omega T}{2}\right)$$

$$b_n = \frac{2}{T}\int_0^{\frac{T}{2}} A\sin(n\omega t) = \frac{2A}{n\omega T}\left(1 - \cos\left(\frac{n\omega T}{2}\right)\right)$$

It is interesting to note that $\omega T = 2\pi$, thus

$$a_n = \frac{A}{n\pi}\sin(n\pi) = 0$$

$$b_n = \frac{A}{n\pi}\left[1 - \cos(n\pi)\right]$$

Consequently,

$$f(t) = \frac{A}{2} + \frac{A}{\pi}\sum_{n=1}^{\infty}\left[\left(1 - \cos(n\pi)\right)\sin n\omega t\right]$$

which is the final form for the square wave's Fourier Series.

A Practical Problem: Gibbs Phenomenon

Discovered independently at the end of the 19th Century by English mathematician Henry Wilbraham and American scientist Josiah Williard Gibbs, the Gibbs phenomenon is a peculiar problem that arises in practical implementations of Fourier series of piecewise continuous functions - that is, functions that show discontinuities in a given number of points, like the square wave studied in the previous example.

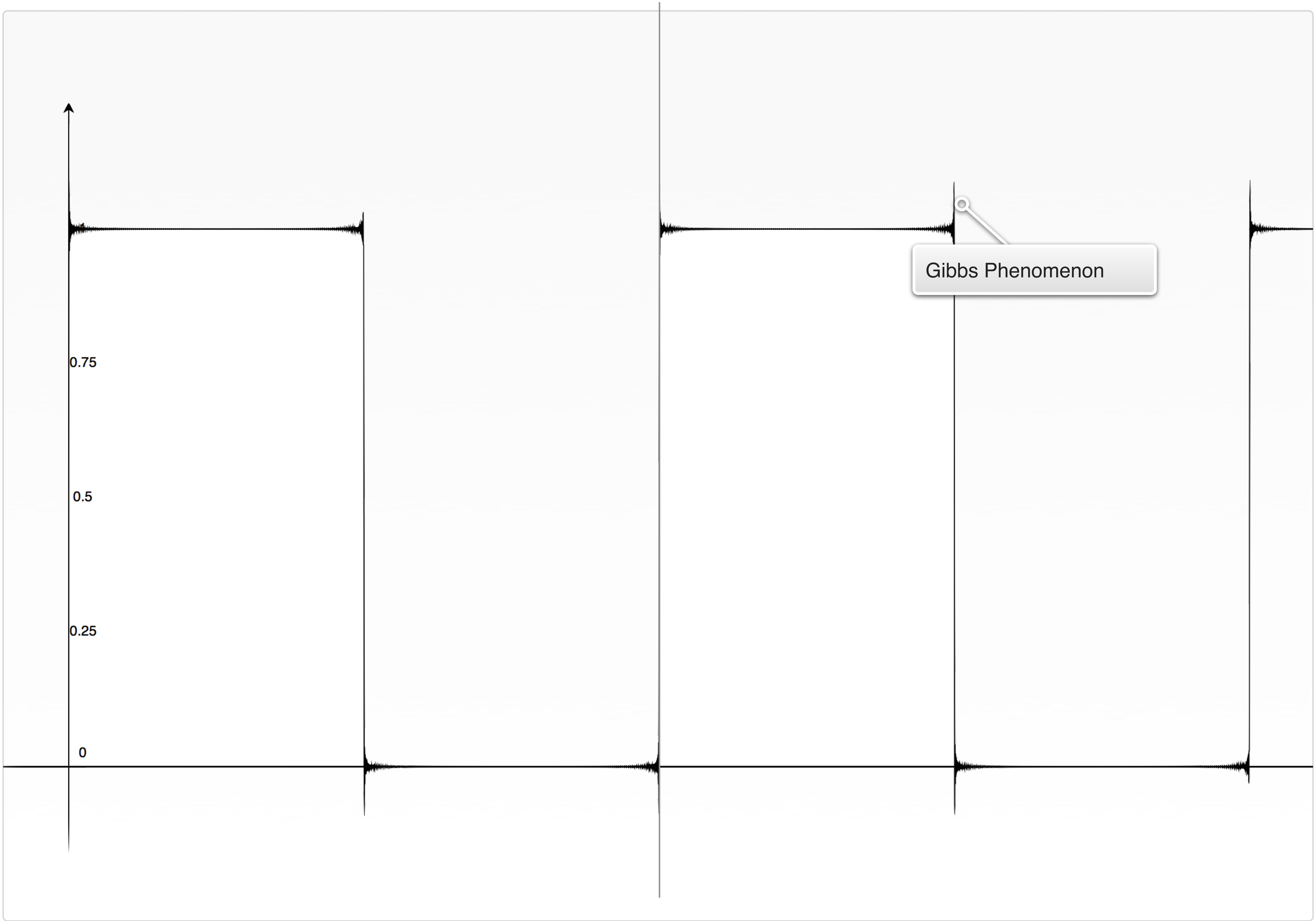
The phenomenon consists of an overshoot in the elements of the series around discontinuities, which does not die out as frequency increases, instead staying constant and approximately equal to 0.09% of the function's amplitude.

Recall, from your studies in Calculus, that a finite sum of continuous functions yields a continuous function. This does not hold with an infinite sum, so that the full Fourier series, taking infinite terms into consideration, can reproduce the square wave perfectly. When taking only a finite partial sum, however, the result must still be continuous, and so the function develops an overshoot at both sides of the point where there should be a discontinuity, while converging to the midpoint of the jump in the discontinuity itself, regardless of what the actual value of the original function is.

In signal processing theory, the Gibbs phenomenon oscillations are explained as ringing or ringing artifacts. It is important for both signal processing and practical applications of circuit theory, as there is no such thing as taking infinite terms in the real world.

The full mathematical description of the Gibbs phenomenon is not a part of the scope of this book, and thus we leave this short entry as an informative section on the subject, not a full-fledged explanation.

Interactive 3.1 In this interactive, you can see a square wave approximated using 1000 sinusoidal components.

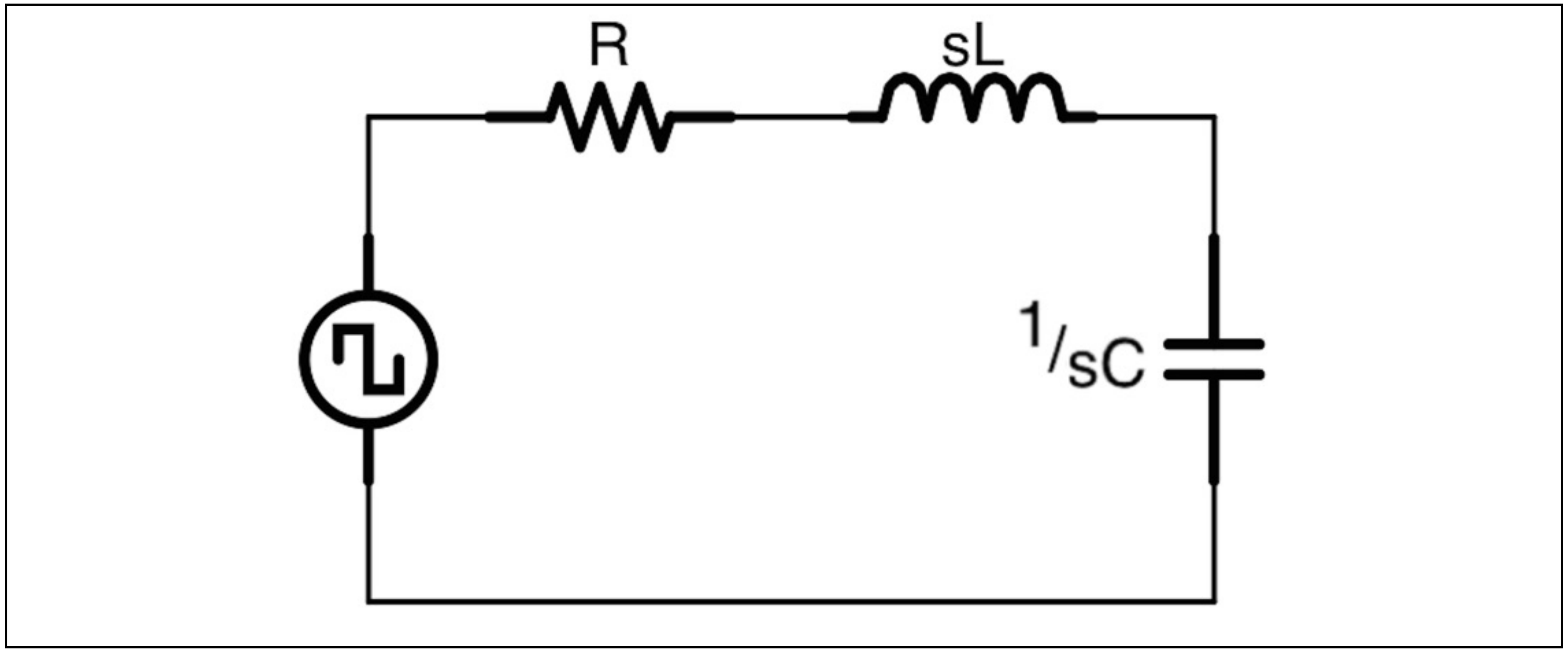


Impedance

What you will learn...

1. the mathematical definition of impedance.
2. the impedance of previously studied linear elements.

Figure 3.2 A second order circuit consisting of three impedances.



In the schematic above, you can see a RLC series circuit. The elements, however, are not being identified with their conventional metrics. Here, we are using their impedances.

We have studied linear circuits in the previous chapter. Recall that resistive circuits are easier to work with than circuits with inductors and capacitors, for they are characterized by an algebraic equation rather than a differential equation. As is usual in Mathematics, then, we ask the question: is there a way to generalize the treatment given to resistive circuits to all linear circuits?

The answer is yes, through the concept of impedance. While resistance is a real-valued property of some physical elements, impedance is the complex-valued generalization of resistance that exists for all linear elements, including capacitors and inductors. By using impedances, we transform a linear circuit's differential equations into algebraic equations, with the trade-off that now we must make use of complex analysis instead of dealing only with real numbers.

The mathematical definition of impedance (symbol: Z) is the same as that of resistance, that is, an element's impedance is the ratio of the voltage across it to the current flowing through it. However, recall that we want impedance to be a complex-valued number, and that we want it to have the ability to turn differential equations into algebraic equations. We have learned in the previous section about the Laplace and Fourier transforms - and luckily, they both have these two properties.

Therefore, we define impedance as the ratio of the Laplace transform of the voltage across an element to the Laplace transform of the current flowing through it, or

$$Z = \frac{\mathfrak{L}\{v\}}{\mathfrak{L}\{i\}}$$

Examples

First, let us calculate the impedance for the simple linear elements we are interested in: the resistor, the capacitor and the inductor.

Let us start with the resistor, as its impedance is a direct result. Since an ideal resistor's resistance is invariant with time, it follows from the definition of the Laplace Transform that

$$Z_R = \frac{V(s)}{I(s)} = R$$

For the capacitor, recall that the current through its terminals was found to follow the equation

$$i_C(t) = C\frac{dv_C(t)}{dt}$$

By applying the Laplace Transform to both sides of the equation and using the transform's properties, we find:

$$I_C(s) = sCV_C(s)$$

Therefore, we have that the impedance of the capacitor is given by:

$$Z_C = \frac{V_C(s)}{I_C(s)} = \frac{1}{sC}$$

For the inductor, recall that we have

$$v_L(t) = L\frac{di_L(t)}{dt}$$

Likewise, by applying the Laplace Transform we obtain

$$V_L(s) = sLI_L(s)$$

and thus

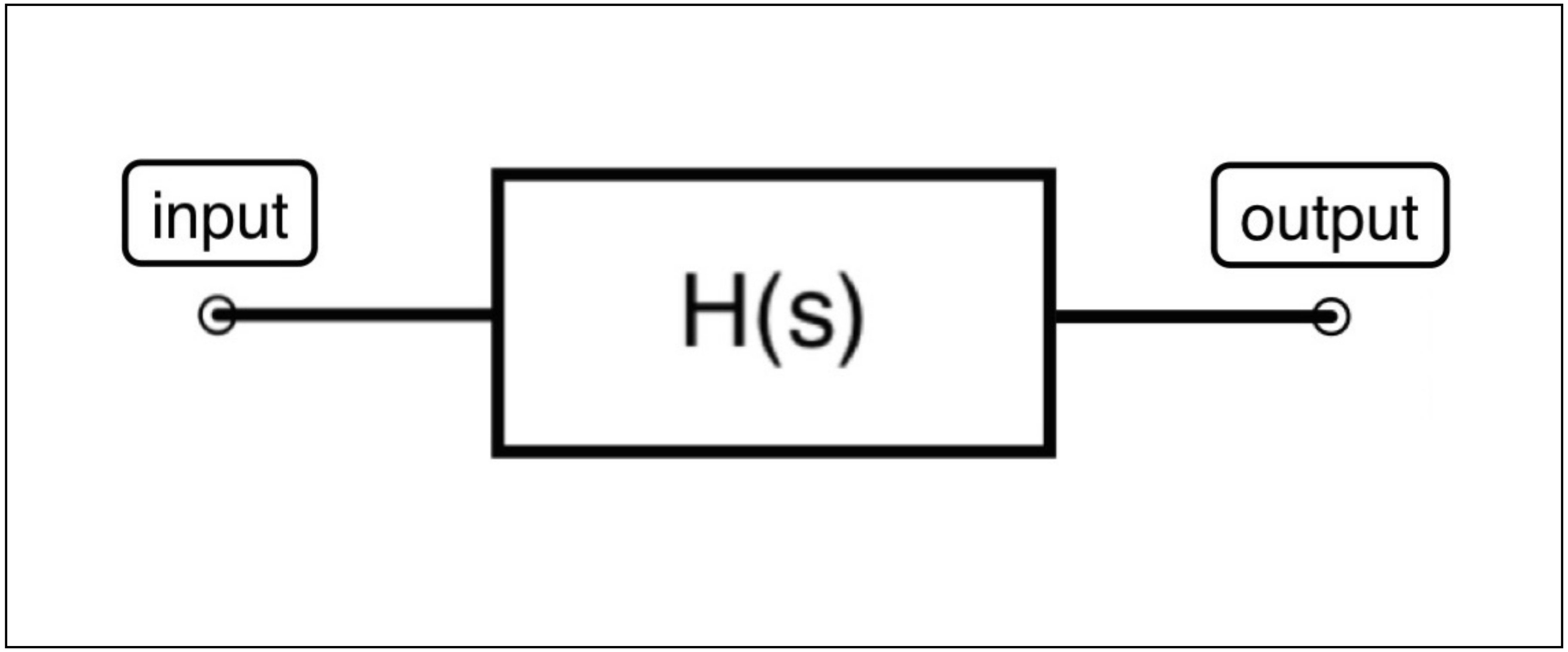
$$Z_L = \frac{V_L(s)}{I_L(s)} = sL$$

Transfer Function

What you will learn...

1. the mathematical definition of transfer function.

Figure 3.3 Transfer function block diagram.



In the image above, you can see the usual representation for a transfer function in a schematic.

Overview

Linear systems have the interesting property that their response to stimuli is given by the convolution of the stimulus itself and the system's response to an unitary impulse.

That is, if $h(t)$ is the response of the system to an input signal given by the impulse function $\delta(t)$, then the generic response $y(t)$ to a generic input $u(t)$ is given by

$$y(t) = \int_0^t u(\tau)h(t-\tau)d\tau$$

While this is an interesting property, calculating the convolution integral can be hard, so it isn't a particularly useful property.

Thankfully, however, we can also take advantage of the Laplace and Fourier Transforms that we learned about earlier in this chapter, and their property that convolutions in the time-domain become products in the transform's domain, which are simple to calculate. Using the Laplace Transform, for example, we thus have

$$Y(s) = H(s)U(s)$$

where $Y(s)$ is the transform of the system's output (that we may want to convert back to the time domain for analysis, with the help of transform tables for example), $U(s)$ is the transform of the system's input, whatever it may be, and $H(s)$ is the transform of the impulse response of the system.

The transform of the impulse response is called the transfer function of the system, because it is what "transfers" the information from the input to the output in the previous equation.

Example

As an example, let us take the simple RC circuit we studied in previous chapters. Let us say that we are interested in the voltage across the capacitor, and therefore we want to discover the transfer function that relates that particular output with the voltage input given by a generic voltage source, given by $v_0(t)$.

Therefore, for this example, our input signal $U(s)$ is given by $V_0(s)$, and our output signal $Y(s)$ is given by the voltage drop across the capacitor, $V_C(s)$.

To find the transfer function of the system, we must find a relation between $V_C(s)$ and $V_0(s)$. Remember that, by using impedances, we can treat linear circuits as if they were resistive. Notice, then, that our RC circuit can be thought of as a voltage divisor. Therefore, we can write

$$V_C(s) = \frac{Z_C}{Z_C + Z_R} V_0(s)$$

We thus have

$$H(s) = \frac{Y(s)}{U(s)} = \frac{V_C(s)}{V_0(s)} = \frac{Z_C}{Z_C + Z_R}$$

Remembering the results for the impedances of both the capacitor and the resistor, we can substitute to find

$$H(s) = \frac{\frac{1}{sC}}{\frac{1}{sC} + R} = \frac{1}{1 + sRC}$$

Equivalently, but using Fourier instead of Laplace Transform,

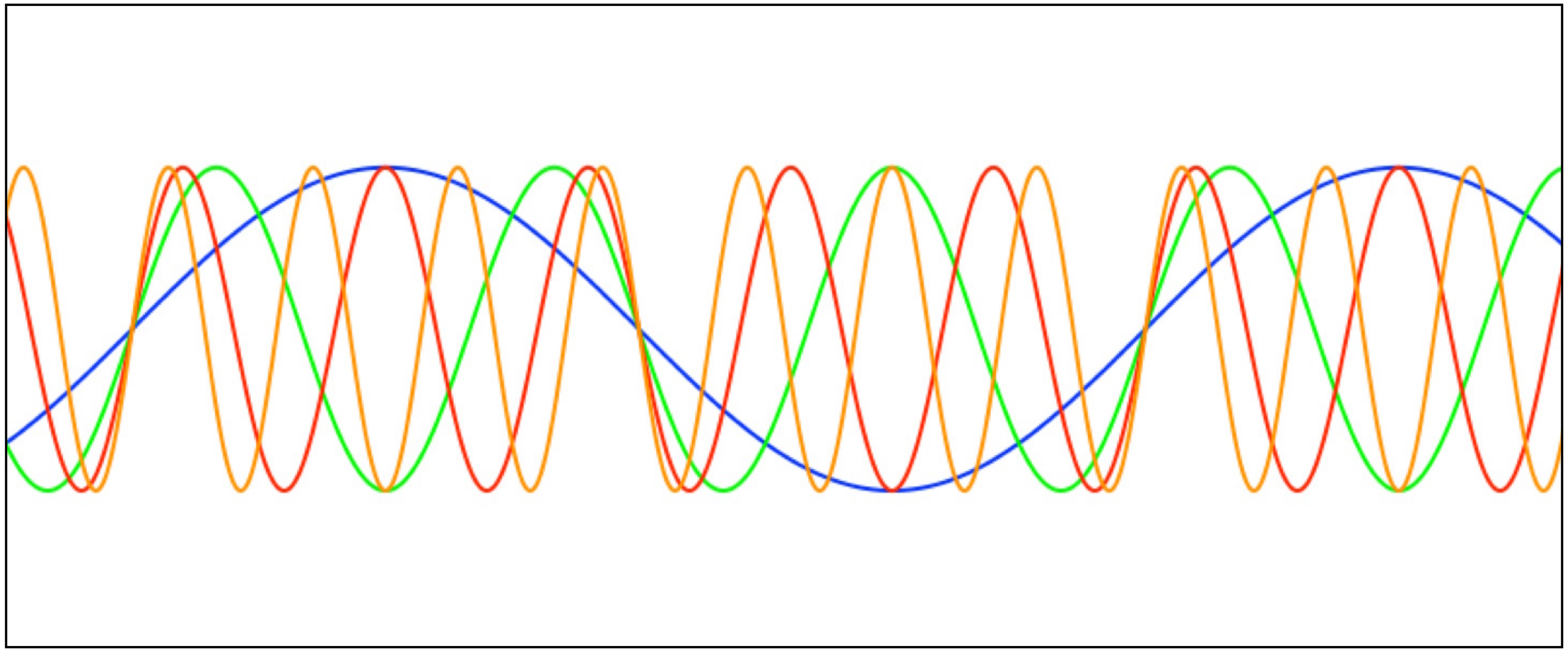
$$H(\omega) = \frac{\frac{1}{j\omega C}}{\frac{1}{j\omega C} + R} = \frac{1}{1 + j\omega RC}$$

Sinusoidal Steady State

What you will learn...

1. what is sinusoidal steady state.
2. how can it help us solve circuits with complex inputs easily.

Figure 3.4 Sinusoidal waves.



In the image above, you can see four sinusoidal waves of different frequencies. They represent the first four harmonics of the Fourier Series of the square wave.

Overview

You should recall from your Linear Algebra studies the concept of eigenvectors and eigenvalues. As a quick recap, if you have an operator A in a vector space, eigenvectors are those elements of the given vector space which suffer only a contraction or expansion when operated by A, that is, there is only a change in the amplitude of vector, and not in its direction. In mathematical terms, this is expressed as

$$A\vec{v} = \lambda\vec{v}$$

where $\vec{v}$ is the eigenvector, and λ is the eigenvalue associated to it.

This concept can be extended to linear systems as a whole, though instead of a vector space, we are now working on a function space. By making an analysis of linear systems, we discover that their eigenfunctions are complex exponentials, that is, sinusoidal functions. This means that, if the input of a linear system is a sinusoidal function, it's output will also be a sinusoidal function, properly multiplied by its related eigenvalue.

We have learned about transfer functions in the previous section. Interestingly enough, it can be shown that, if the input of a linear system is a complex exponential of angular frequency ω_0, then the eigenvalue associated with it is simply the transfer function of the system evaluated at the point $\omega = \omega_0$.

Analysis of Generic Signals

As we have seen in this section, linear systems that have sinusoidal inputs behave in a very predictable manner. Their outputs are also sinusoidal with the same frequency varying only with amplitude and phase.

We have also discussed what is called Sinusoidal Steady State that occurs when the input is sinusoidal and the transients are no longer relevant. The question is: can we also achieve a state like that in a circuit with a non-sinusoidal input?

Gladly, we can. Remember that most signals that are relevant to us can be written as an infinite sum of sines/cosines. Thus, based on the superposition principle, the output will also be a infinite sum of sines/cosines modulated in amplitude and in frequency due to the presence of an impedance.

Lets deal with this new technique by applying it to a simple example: the RL circuit with a square wave with amplitude V_0, period T and duty factor $\mathscr{D}$ as the input voltage. Please refer to the glossary if you do not know the concept of duty factor. Also, remember that we are considering steady state and, thus, all transients have already passed.

We know, from our studies of Fourier Series that

$$v(t) = c_0 + \sum_{n=1}^{\infty} \left[c_n \cos(n\omega t - \phi_n)\right]$$

where $c_n = \sqrt{a^2 + b^2}$ and $\phi_n = \arctan(b_n/a_n)$. Remember that

$$a_n = \frac{2}{T}\int_0^T f(t)\cos(n\omega t)dt$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin(n\omega t)dt$$

$$c_0 = \frac{1}{T}\int_0^T f(t)dt$$

The plan is to write in the Fourier Series form and interpret the result based on it. You shall see that the analysis is much simpler depending on the goal of it. In our case, where $v(t)$ is a square wave, we have:

$$a_n = \frac{2V_0}{T}\left(\frac{\sin(n\omega t)}{n\omega}\right)\Bigg|_0^{\mathscr{D}T} = \frac{2V_0}{n\omega T}\sin(n\omega\mathscr{D}T) = \frac{V_0}{\pi n}\sin(n\omega t_0)$$

where t_0 is the instant when the square wave goes from V_0 to zero. Similarly,

$$b_n = \frac{2V_0}{T}\left(-\frac{\cos(n\omega t)}{n\omega}\right)\Bigg|_0^{\mathscr{D}T} = \frac{V_0}{\pi n}\left(1-\cos(n\omega t_0)\right)$$

$$c_0 = \frac{1}{T}\int_0^{\mathscr{D}T} V_0 dt = \mathscr{D}V_0$$

Finally, we need to find c_n and ϕ_n in order to get an expression of the desired form.

$$c_n = \sqrt{a_n^2 + b_n^2} = \frac{2V_0}{\pi n}\left|\sin\left(\frac{n\omega t_0}{2}\right)\right|$$

$$\phi_n = \arctan\left(\frac{b_n}{a_n}\right) = \frac{1-\cos(n\omega t_0)}{\sin(n\omega t_0)}$$

Now that we have written the voltage in the Fourier Series manner, we can easily find the current through the system or any other thing.

To find the current, we use Ohm's Law that is given by $I = V/Z$ where Z is the equivalent impedance of the circuit. In this case $Z = R + j\omega L$ because the inductor and the resistor are in series. Remember that we are dealing with a RL circuit.

$$v(t) = c_0 + \sum_{n=1}^{\infty}\left[c_n \cos(n\omega t - \phi_n)\right]$$

$$v(t) = \mathscr{D}V_0 + \frac{2V_0}{\pi}\sum_{n=0}^{\infty}\left[\frac{1}{n}\left|\sin\left(\frac{n\omega t_0}{2}\right)\right|\cos\left(n\omega t - \frac{1-\cos(n\omega t_0)}{\sin(n\omega t_0)}\right)\right]$$

In order to find the current, remember that a complex number can be interpreted as a modulus and a phase. For example, if $Z = a + jb$, $|Z| = \sqrt{a^2+b^2}$ and $\phi Z = \arctan(b/a)$.

We need to divide the voltage by the impedance. And that implies that we need to divide the modulus of the voltage by the modulus of the impedance. Also, we need to phase-shift the the oscillating parts of the voltage by the phase of the impedance. This yields:

$$i(t) = \frac{\mathscr{D}V_0}{R} + \frac{2V_0}{\pi}\sum_{n=0}^{\infty}\left[\frac{\left|\sin\left(\frac{n\omega t}{2}\right)\right|}{n\sqrt{R^2+n^2\omega^2L^2}}\cos\left(n\omega t - \frac{1-\cos(n\omega t_0)}{\sin(n\omega t_0)} - \arctan\left(\frac{n\omega L}{R}\right)\right)\right]$$

If $\omega L \gg R$, then $\sqrt{R^2+n^2\omega^2L^2} \to n\omega L$ making the current

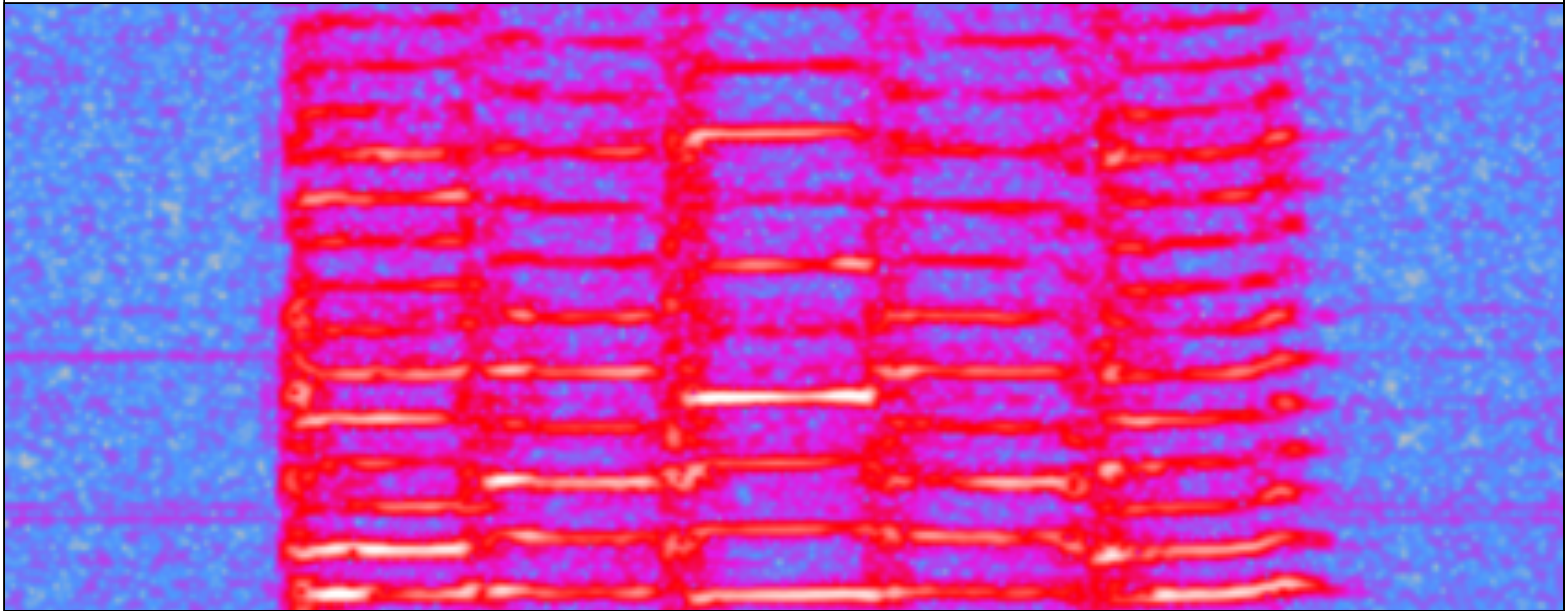
$$i(t) = \frac{\mathscr{D}V_0}{R} + \frac{2V_0}{\pi\omega L}\sum_{n=0}^{\infty}\left[\frac{\left|\sin\left(\frac{n\omega t}{2}\right)\right|}{n^2}\cos\left(n\omega t - \frac{1-\cos(n\omega t_0)}{\sin(n\omega t_0)} - \arctan\left(\frac{n\omega L}{R}\right)\right)\right]$$

This result is filled with relevant information about our answer. From it we know that the current has a DC component and infinite oscillating components. We know that the higher the component's frequency, the smaller its amplitude.

If we want a purely DC current, we can either increase the frequency of the input's square wave or we can increase the inductance of the inductor.

CHAPTER 4

Frequency Domain Analyis of Circuits



In the previous chapter, you learned fundamental concepts related to frequency such as impedance and Fourier Series. In this chapter, we will put them to use in order to efficiently analyze we are called filters. Filters are devices (in our case, circuits) that remove unwanted components from our signals. For example, imagine that we use a circuit to remove noise from a voltage signal. This circuit would be called a noise-filter.

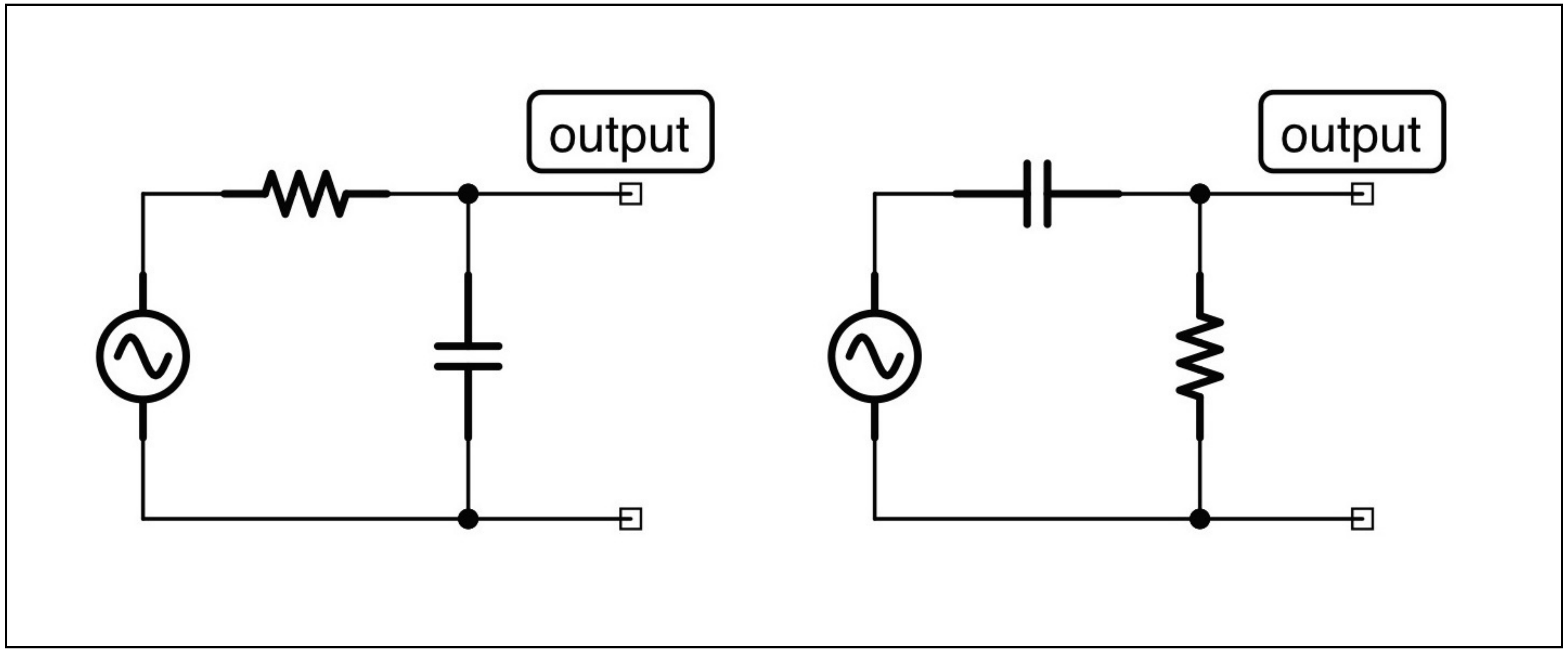
In the image, a time-frequency representation of a signal. It is called a spectrogram.

First Order Filters

What you will learn...

1. what is a first order filter.
2. a RC arrangement for a low pass filter.
3. a RC arrangement for a high pass filter.

Figure 4.1 A low-pass filter and a high-pass filter.



In the image you can see a low-pass filter on the left and a high-pass filter on the right. Note that by simply rearranging the components, we can completely change the functionality of a circuit.

Overview

In the previous chapter, you learned fundamental concepts related to frequency such as impedance and Fourier Series. In this chapter, we will put them to use in order to efficiently analyze what are called filters.

Filters are devices (in our case, circuits) that remove unwanted components from our circuits. For example, imagine that we use a circuit to remove noise from a voltage signal. This circuit would be called a noise-filter.

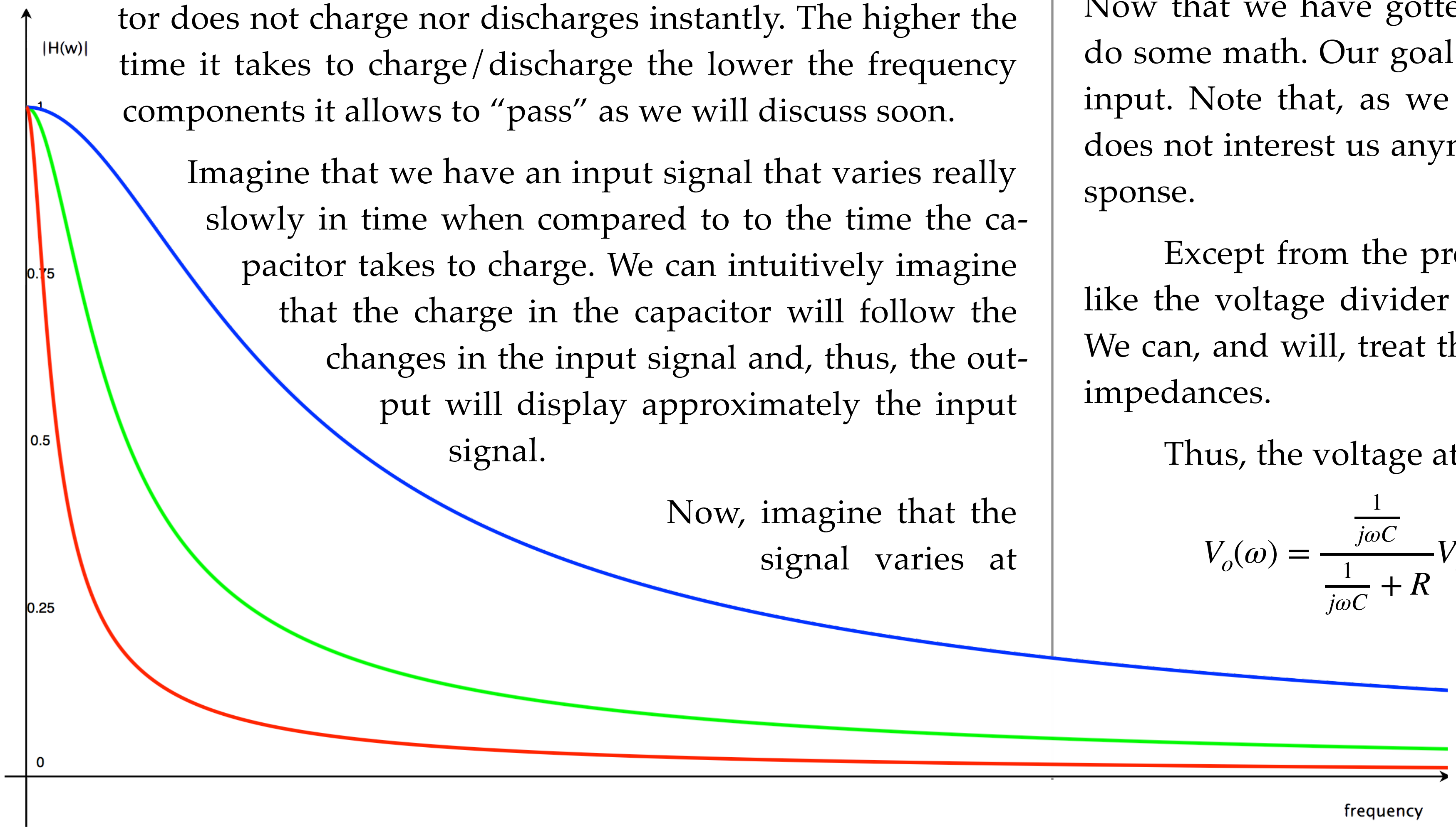
This text will be limited to linear circuits that use energy storage elements such as capacitors and inductors to perform the filtering. In this section we will also limit our study to circuits that have only one independent storage element. They are called First Order Filters.

Capacitive First Order Filter

The Idea

This first filter that we are going to describe is an old familiar: the RC circuit. The capacitor, here, will play an important role in removing unwanted high-frequency components of our signals.

Its principle of operation is based on the fact that the capacitor does not charge nor discharges instantly. The higher the time it takes to charge/discharge the lower the frequency components it allows to "pass" as we will discuss soon.



Imagine that we have an input signal that varies really slowly in time when compared to to the time the capacitor takes to charge. We can intuitively imagine that the charge in the capacitor will follow the changes in the input signal and, thus, the output will display approximately the input signal.

Now, imagine that the signal varies at high frequency. In this case, the capacitor will not be able to charge/discharge to perfectly follow the changes in the input and, thus, nothing will be reflected at the output node.

Based on these two intuitive analysis, we can argue that the RC circuit acts by allowing low frequency signals to be transmitted to the output while blocking high frequency signals. That is, the capacitor acts as an open circuit for low frequencies and as a short circuit for high frequencies. Thus it operates as a Low-Pass Filter (LPF).

The Math

Now that we have gotten the gist of the circuit's operation, lets do some math. Our goal is to find the output as a function of the input. Note that, as we are studying a filter, the time response does not interest us anymore. Now, we will find the frequency response.

Except from the presence of the capacitor, this circuit looks like the voltage divider we talked about in the second chapter. We can, and will, treat this system as a voltage divider but using impedances.

Thus, the voltage at the output is given by

$$V_o(\omega) = \frac{\frac{1}{j\omega C}}{\frac{1}{j\omega C} + R} V_i(\omega) \longrightarrow H(\omega) = \frac{V_o(\omega)}{V_i(\omega)} = \frac{1}{1 + j\omega RC}$$

Note that we can verify what was previously thought regarding the frequency of the inputs.

$$\lim_{\omega \to 0} H(\omega) = 1$$

$$\lim_{\omega \to \infty} H(\omega) = 0$$

Also, if we increase the value of the product RC we can see that $H(\omega)$ decreases faster with the increase of frequency.

Again, inputs with frequencies close to zero appear at the output. On the other hand, inputs with high frequencies are cut. This behavior can be better grasped by checking the transfer function graph. Note that the graph depicts the absolute value of the transfer function (also called gain) given by

$$|H(\omega)| = \frac{1}{\sqrt{1 + \omega^2 R^2 C^2}}$$

In some applications, one might also be interested in the phase-shift applied by the filter. It it given by:

$$\phi H(\omega) = -\arctan(\omega RC)$$

A practical characteristic of filters is the cutoff frequency. It is defined as the frequency in which the power of the output decreases to half the power of the input.

Since power is related to the square of the voltage, at the cutoff frequency the voltage of the output will have fallen by $\sqrt{2}$. Therefore, we can use the expression for the absolute value of the transfer function to find the cutoff frequency.

$$\frac{1}{\sqrt{1 + \omega_c^2 R^2 C^2}} = \frac{1}{\sqrt{2}} \longrightarrow \omega_c = \frac{1}{RC}$$

Another Arrangement

Now suppose that we switch the position of the resistor and the capacitor. If that is the case, the expression for the output will change a little and will completely change the behavior of this filter of ours.

The voltage at the output will be given by

$$V_o(\omega) = \frac{R}{\frac{1}{j\omega C} + R} V_i(\omega) \longrightarrow H(\omega) = \frac{V_o(\omega)}{V_i(\omega)} = \frac{j\omega RC}{1 + j\omega RC}$$

Note that the limits are interchanged when compared to the previous arrangement.

$$\lim_{\omega \to 0} H(\omega) = 0$$

$$\lim_{\omega \to \infty} H(\omega) = 1$$

Due to this different behavior the filter operates as a High-Pass Filter (HPF). That is, the signal's components at high frequencies are attenuated while low frequency components are not.

The expression for gain and phase-shift are given by:

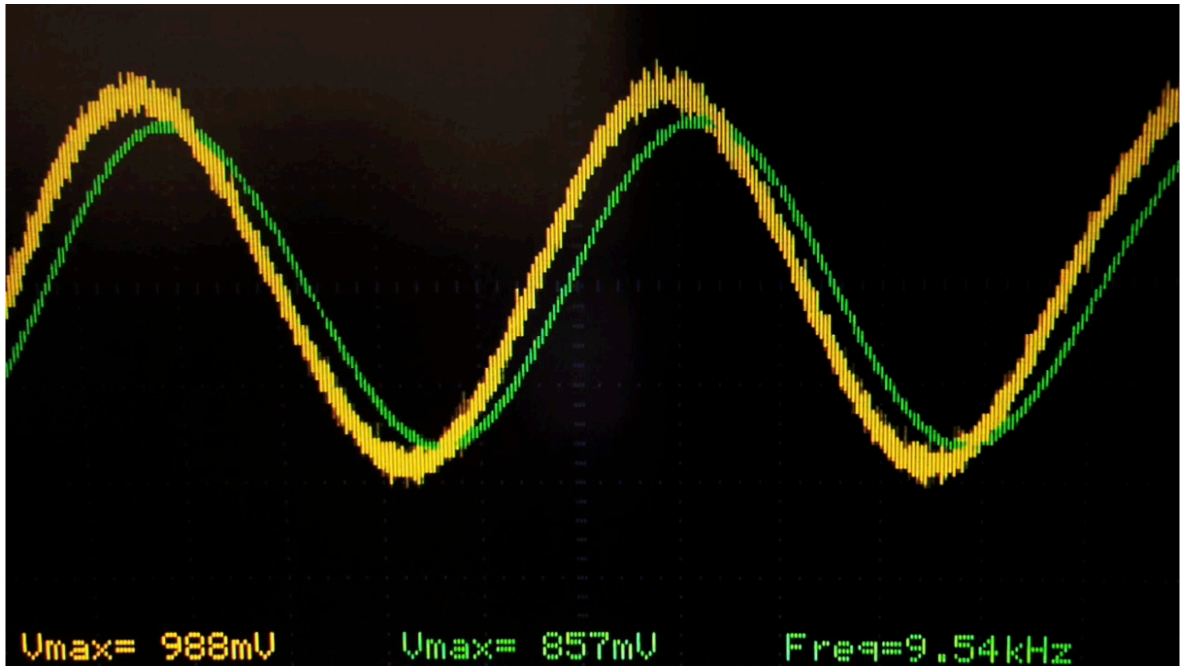
$$|H(\omega)| = \frac{\omega RC}{\sqrt{1 + \omega^2 R^2 C^2}}$$

$$\phi H(\omega) = \frac{\pi}{2} - \arctan(\omega RC)$$

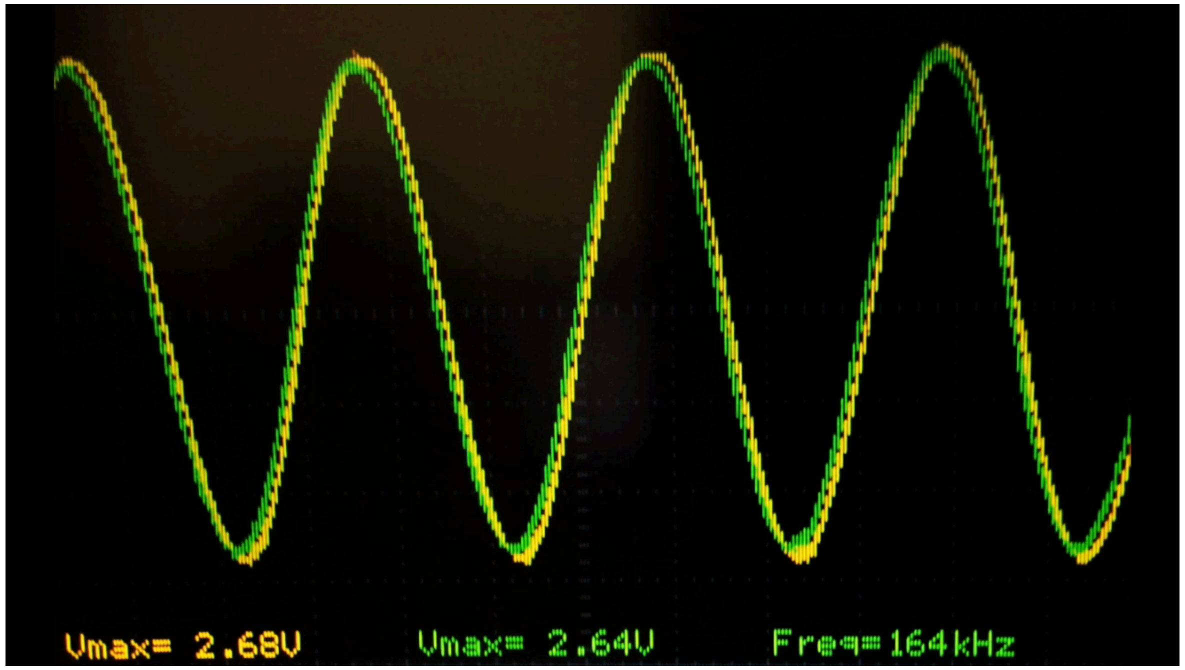
Lab Experiment

In this experiment we see the practical behavior of a low pass filter and a high pass filter. We will be using a 1.2 kΩ resistor and a 5.6 nF capacitor.

Movie 4.1 Oscilloscope measurement for LPF input and output.



Movie 4.2 Oscilloscope measurement for HPF input and output.



With this setup, we have a cutoff frequency of

$$f_c = \frac{1}{2\pi RC} \approx 23 \text{ kHz}$$

In the low frequency range for Movie 4.1, the output (green waveform) is really similar to the input (yellow waveform). However, the higher the frequency, the farther apart they are with regards to amplitude and phase.

In Movie 4.2, we have the opposite behavior. We start at a high frequency value, where both waveforms practically coincide and, and begin to decrease the frequency. The lower the frequency, the farther apart they are with regards to amplitude and phase.

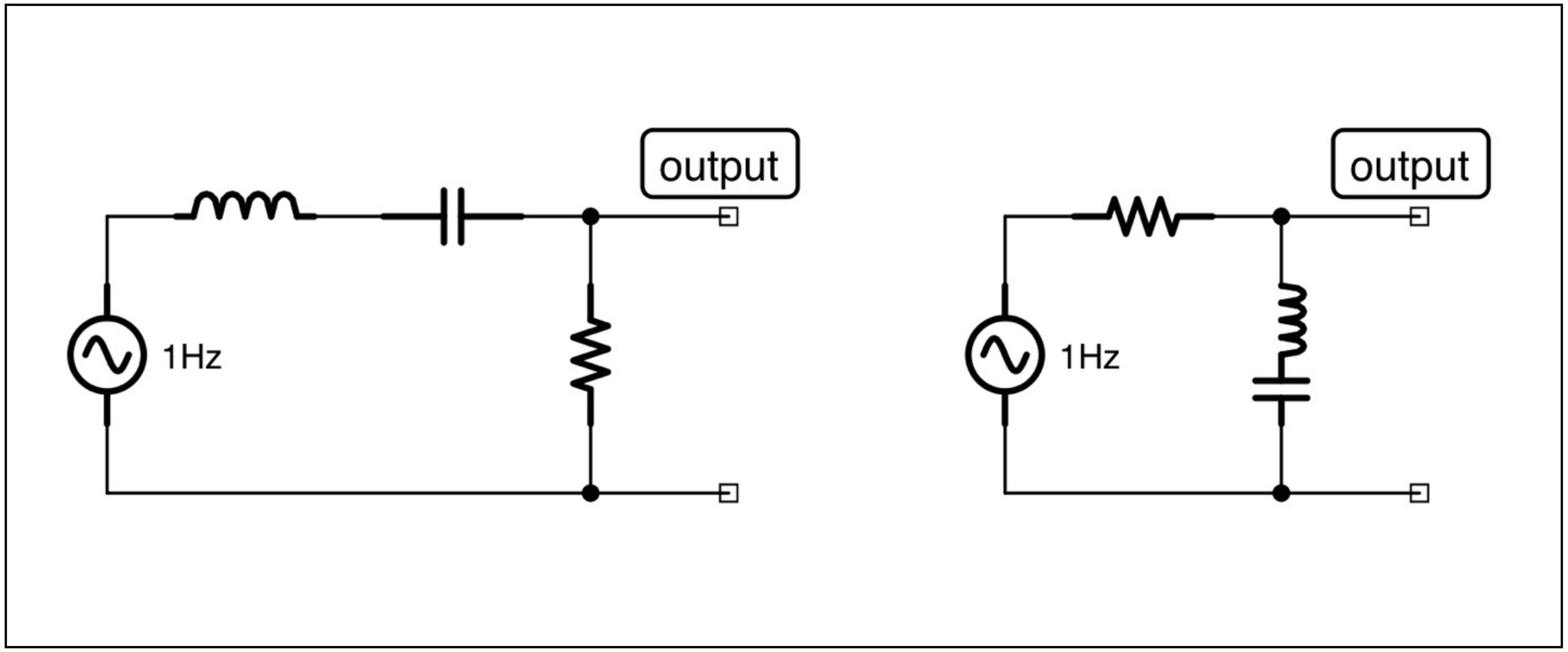
Take note that the horizontal scale has been adjusted during both recordings to allow better visualization of the waveforms. The frequency, however, was increased continuously.

Second Order Filters

What you will learn...

1. what is a second order filter.
2. a RLC arrangement for a band pass filter.
3. a RLC arrangement for a band stop filter.

Figure 4.2 A band-pass and a band-stop filter.



In the image you can see a band-pass filter on the left and a band-stop filter on the right. Note that by simply rearranging the components, we can completely change the functionality of a circuit.

Overview

In the previous section we discussed first order filters and studied some examples. As of now, the focus will be second order filters. As you might be expecting, those are filters that are implemented using two independent storage elements.

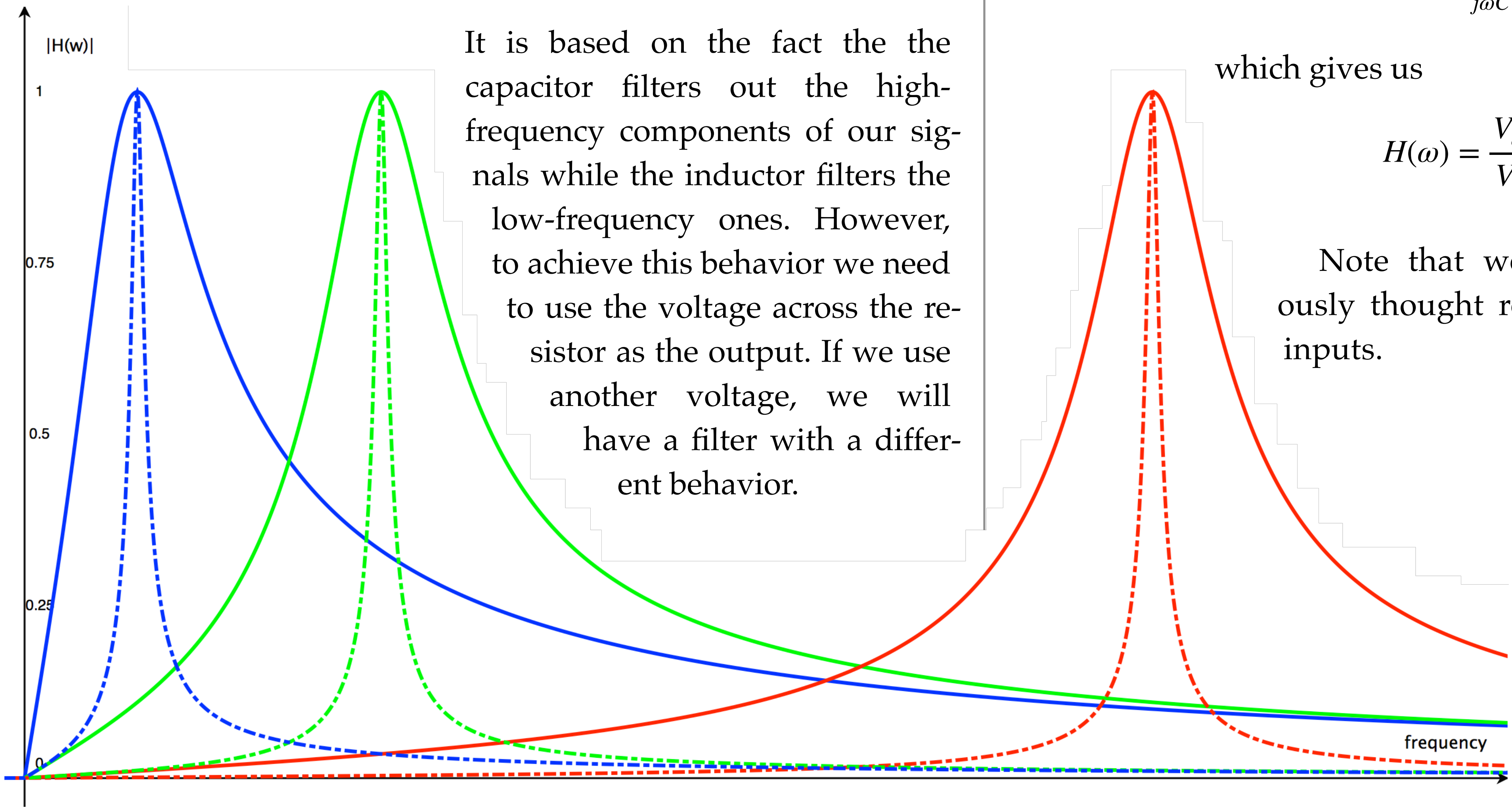
They are needed for many applications because of their versatility. Using them makes it possible to implement filters with more complex transfer functions. Such goal would not be accomplishable with the use of only one capacitor or inductor.

In this section, we are going to discuss two common types of filters. First the band-pass filter (BPF). After, the band-stop filter (BSF).

Band-Pass Filter

The Idea

This first filter that we are going to describe is an old familiar: the RLC series circuit. Both the capacitor and the inductor, here, will play an important role in removing unwanted low and high-frequency components of our signals.



It is based on the fact the the capacitor filters out the high-frequency components of our signals while the inductor filters the low-frequency ones. However, to achieve this behavior we need to use the voltage across the resistor as the output. If we use another voltage, we will have a filter with a different behavior.

The Math

Our goal is to find the output as a function of the input. Note that, as we are studying a filter, the time response does not interest us anymore. Now, we will find the frequency response.

Except from the presence of the capacitor and the inductor, this circuit looks like the voltage divider we talked about in the second chapter. We can, and will, treat this system as a voltage divider but using impedances.

Thus, the voltage at the output is given by

$$V_o(\omega) = \frac{R}{R + \frac{1}{j\omega C} + j\omega L} V_i(\omega)$$

which gives us

$$H(\omega) = \frac{V_o(\omega)}{V_i(\omega)} = \frac{j\omega}{j\omega + \frac{1}{RC} - \frac{L}{R}\omega^2}$$

Note that we can verify what was previously thought regarding the frequency of the inputs.

$$\lim_{\omega\to 0} H(\omega) = 0$$

$$\lim_{\omega\to\infty} H(\omega) = 0$$

It is harder to do an intuitive parameter analysis in this case because we are not dealing with first order expressions anymore. To give you a better perspective on what happens if we change the different components for this filter, check the transfer function graph.

Note that the graph depicts the absolute value of the transfer function (also called gain) given by

$$|H(\omega)| = \frac{\omega}{\sqrt{\omega^2 + \left(\frac{1}{RC} - \frac{L}{R}\omega^2\right)^2}}$$

In some applications, one might also be interested in the phase-shift applied by the filter. It it given by:

$$\phi H(\omega) = \frac{\pi}{2} - \arctan\left(\frac{\omega}{\frac{1}{RC} - \frac{L}{R}\omega^2}\right)$$

As we did for the first order filters, let us calculate the cutoff frequency which is the frequency in which power drops by half.

$$\frac{\omega_c}{\sqrt{\omega_c^2 + \left(\frac{1}{RC} - \frac{L}{R}\omega_c^2\right)^2}} = \frac{1}{\sqrt{2}}$$

$$\omega_c = \frac{\sqrt{C}\sqrt{R^2C + 4L} \pm RC}{2LC}$$

Note that we have two different values for the cutoff frequency as we are dealing with a second order BPF.

Band-Stop Filter

The Idea

This second filter that we are going to describe also is an old familiar: the RLC series circuit. However, the key player here is the resistor.

You can recall that in the previous filter, we took the output from the resistor and got the band-pass effect we wanted. Now, the opposite is wanted. Thus, we take the output from both the capacitor and the inductor.

The Math

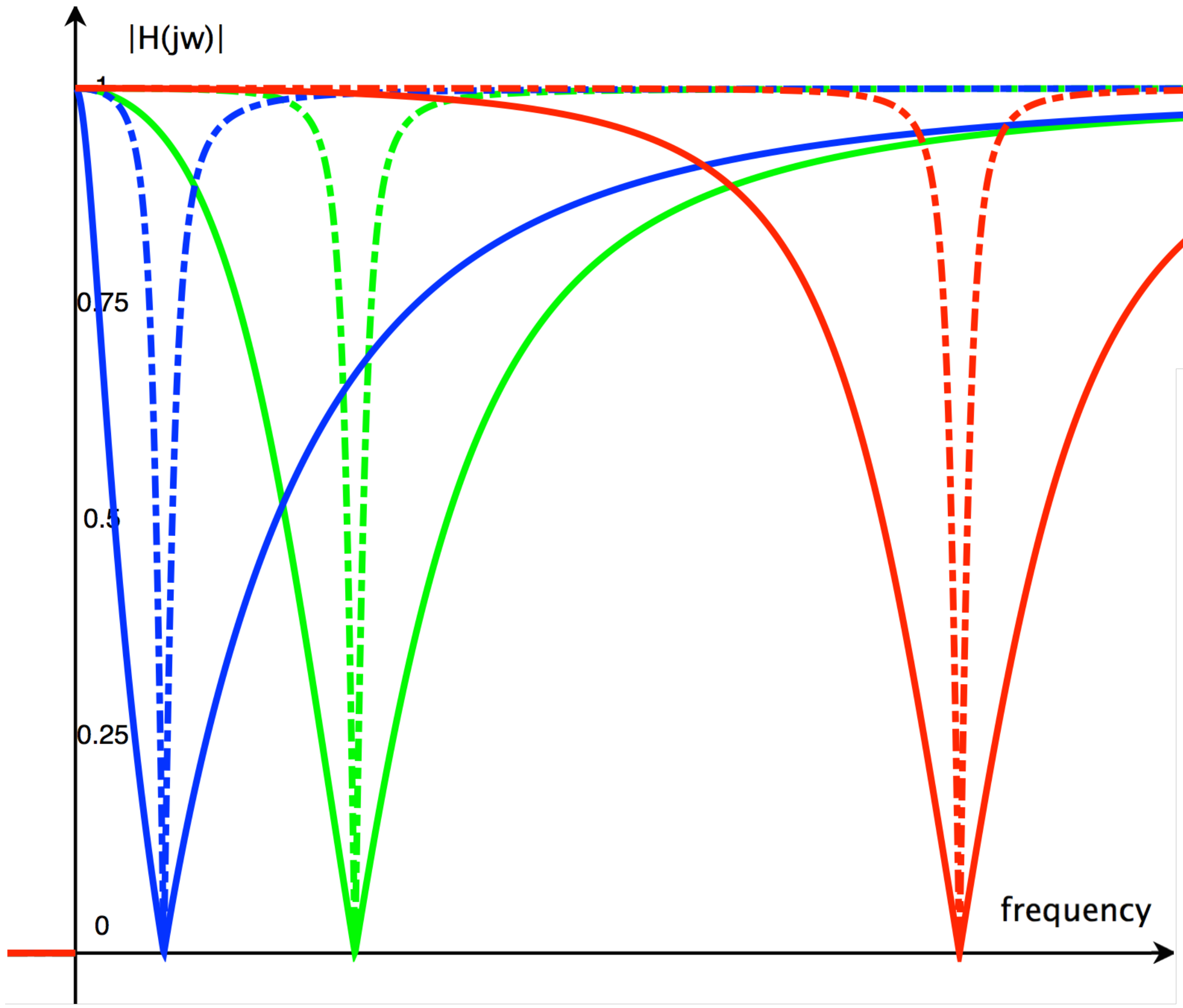
Our goal is to find the output as a function of the input. Again, we are studying a filter, thus the time response does not interest us anymore. We should find the frequency response.

Except from the presence of the capacitor and the inductor, this circuit looks like the voltage divisor we talked about in the second chapter. We can, and will, treat this system as a voltage divider but using impedances.

Thus, the voltage at the output is given by

$$V_o(\omega) = \frac{\frac{1}{j\omega C} + j\omega L}{R + \frac{1}{j\omega C} + j\omega L} V_i(\omega)$$

which gives us



It is harder to do an intuitive parameter analysis in this case because we are not dealing with first order expressions anymore. To give you a better perspective on what happens if we change the different components for this filter, check the transfer function graph.

Note that the graph depicts the absolute value of the transfer function (also called gain) given by

$$|H(\omega)| = \frac{\left|\frac{1}{LC} - \omega^2\right|}{\sqrt{\left(\frac{1}{LC} - \omega^2\right)^2 + \left(\frac{R}{L}\omega\right)^2}}$$

In some applications, one might also be interested in the phase-shift applied by the filter. It it given by:

$$\phi H(\omega) = -\arctan\left(\frac{\frac{R}{L}\omega}{\frac{1}{LC} - \omega^2}\right)$$

$$H(\omega) = \frac{V_o(\omega)}{V_i(\omega)} = \frac{\frac{1}{LC} - \omega^2}{\frac{1}{LC} - \omega^2 + j\frac{R}{L}\omega}$$

Note that we can verify what was previously thought regarding the frequency of the inputs. We surely cannot know for sure what happens in between those limits. But at least we are right regarding the limits.

$$\lim_{\omega \to 0} H(\omega) = 1$$

$$\lim_{\omega \to \infty} H(\omega) = 1$$