Referências Bibliográficas

[1] A.W.Love, *Electromagnetic Horn antennas*, IEEE Press, New York, 1976.

[2] R.E.Collin, *Introduction to Some Fundamental Experiments With Wave Guides*, Proceedings of the IEEE, v87, Issue3, Mar 1999, pp. 508-514.

[3] W.L.Barrow and L.J.Chu, *Theory of Electromagnetic Horns*, Proc. Inst. Radio Eng., 1939, v.27, pp. 51-64.

[4] A.D.Olver, P.J.Clarricoats, A.A.Kishk and L.Shafai, *Microwave Horns and Feeds*, IEE Electromagnetic Wave Series 39, IEE Press, New York, 1994.

[5] P. J. B. Clarricoats and A. D. Olver. *Corrugated Horns for Microwave Antennas*, IEE – Peter Peregrinus Ltd, London – UK, 1984.

[6] P. J. B. Clarricoats, J. R. Descardeci and A. D. Olver, "*A novel crosspolar broadband hybrid mode feed"* , USNC National Radio Science Meeting – 1988, Colorado, Proceedings of the USNC/URSI National Radio Science Meeting, 1988, v.1, pp. 13-13.

[7] P. J. Clarricoats, J. R. Descardeci and A. D. Olver, "*A Low Crosspolar Feed for Broadband Application"*, 1988-URSI-International Symp. On Electromagnetic Theory, Thessaloniki USRI, 1988, v.1. pp. 110-119.

[8] A. V. Ghiner and G. I. Surdutovich, "*Method of Integral Equations and an Extinction Theorem for Two-Dimensional Problems in Nonlinear Optics"*, Physical Review A, vol 50, N.1, July 1994, pp. 714 – 723.

[9] C. A. Balanis. *Advanced Engineering Electromagnetic*. John Wiley & Sons. 1989, USA.

[10] C. A. Balanis. *Antenna Theory.Analysis and Design*. 3rd. ed. John Wiley & Sons. 2005, USA.

[11] A. C. Ludwig. "*The Definition of Cross Polarization"*, IEEE Transactions on Antennas and Propagation, January 1973.

[12] R.F. Harrington. *Time-Harmonic Electromagnetic Fields*. John Wiley & Sons, 2001, USA.

[13] Matrix Operations, Schaum Collection, 1989.

[14] M. Abramovitz., I. A. Stegun. Handbook of mathematical functions. 10a ed., NBS, 1972.

[15] K. R. Carver, J. W. Mink, "Microstrip Antenna Technology", IEEE Transactions on Antennas and Propagation, vol.29, No. 1, January, 1981.

[16] R. C. Johnson, Antenna Engineering Handbook, 3td edition, 1993, USA, Mc Graw Hill.

[17] P. D. Potter, "A New horn antenna with suppressed sidelobes and equal beamwidth", Microwave J., pp. 71-78, June 1963.

APENDICE A
Obtenção dos Potenciais Vetores elétrico e magnético de Hertz [9].

A.1
Potencial Vetor de Hertz do Tipo Elétrico

Da equação de Maxwell, $\nabla.\bar{B}=0$, para um material não magnético, com $\mu=\mu_0$ = permeabilidade do espaço livre, tem-se $\nabla.\bar{B}=\mu_0\nabla.\bar{H}$. Tendo em vista a identidade vetorial $\nabla.(\nabla x\bar{\prod}_e)=0$, pode-se definir o potencial vetor de Hertz do tipo elétrico a partir de:

$$\bar{H}=j\omega\varepsilon_0\bar{\varepsilon}_r\nabla x\bar{\prod}_e \qquad (A.1)$$

Usando-se a equação de Maxwell:

$$\nabla x\bar{E}=-j\omega\mu_0\bar{H} \qquad (A.2)$$

Substituindo-se a equação (A-1) em (A-2), tem-se:

$$\nabla x\bar{E}=\omega^2\mu_0\varepsilon_0\bar{\varepsilon}_r\nabla x\bar{\prod}_e \qquad (A.3)$$

Tendo em vista a identidade vetorial $\nabla x(\nabla x\phi_e)=0$, pode-se escrever:

$$\bar{E}=\omega^2\mu_0\varepsilon_0\bar{\varepsilon}_r\bar{\prod}_e+\nabla\phi_e \qquad (A.4)$$

Ficando assim definido o potencial escalar de Hertz do tipo elétrico ϕ_e. Substituindo-se os campos magnético e elétrico, dados por (A.1) e (A.4), respectivamente, na equação de Maxwell:

$$\nabla x\bar{H}=j\omega\varepsilon_0\bar{\varepsilon}_r\bar{E} \qquad (A.5)$$

obtém-se:

$$(j\omega\varepsilon_0\bar{\varepsilon}_r)\nabla x\nabla x\bar{\prod}_e=(j\omega\varepsilon_0\bar{\varepsilon}_r)[\omega^2\mu_0\varepsilon_0\bar{\varepsilon}_r\bar{\prod}_e+\nabla\phi_e] \qquad (A.6)$$

Lembrando que:

$$k_0{}^2 = \omega^2 \mu_0 \varepsilon_0 \tag{A.7}$$

é o número de onda no espaço livre, pode-se utilizar a identidade vetorial

$$\nabla x \nabla x \bar{\prod}_e = -\nabla^2 \bar{\prod}_e + \nabla\nabla . \bar{\prod}_e \tag{A.8}$$

resultando em:

$$-\nabla^2 \bar{\prod}_e + \nabla\nabla . \bar{\prod}_e = k_0^2 \bar{\varepsilon}_r \bar{\prod}_e + \nabla \phi_e \tag{A.9}$$

ou

$$\nabla^2 \bar{\prod}_e + k_0^2 \bar{\varepsilon}_r \bar{\prod}_e = \nabla\nabla . \bar{\prod}_e - \nabla \phi_e \tag{A.10}$$

Para que o potencial vetor de Hertz do tipo elétrico, definido por (A.1) seja unívoco, é necessário ainda, uma condição restritiva do tipo do calibre de Lorentz. Impondo-se o calibre de Lorentz:

$$\nabla\nabla . \bar{\prod}_e - \nabla \phi_e = 0 \tag{A.11}$$

A equação (A.9) se reduz à equação homogênea de Helmholtz,

$$\nabla^2 \bar{\prod}_e + k_0^2 \bar{\varepsilon}_r \bar{\prod}_e = 0 \tag{A.12}$$

De (A.1), (A.4) e (A.9) os campos elétrico $\bar{E}$ e magnético $\bar{H}$, podem ser expressos em termos de $\bar{\prod}_e$ como seguem:

$$\bar{H} = j\omega \varepsilon_0 \bar{\varepsilon}_r \nabla x \bar{\prod}_e \tag{A.13}$$

$$\bar{E} = k_0^2 \bar{\varepsilon}_r \bar{\prod}_e + \nabla\nabla . \bar{\prod}_e \tag{A.14}$$

A.2
Potencial Vetor de Hertz do Tipo Magnético

Da equação de Maxwell, $\nabla . \bar{D} = 0$, para materiais dielétricos anisotrópicos, com $\bar{D} = \bar{\varepsilon} \bar{E}$, tem-se $\nabla . \bar{\varepsilon} \bar{E} = 0$. Se o material for também homogêneo, então $\nabla . \bar{E} = 0$.

Tendo em vista a identidade vetorial $\nabla . (\nabla x \bar{\prod}_h) = 0$, pode-se definir o potencial vetor de Hertz do tipo magnético, a partir de:

$$\bar{E} = -j\omega \mu_0 \nabla x \bar{\prod}_h \tag{A.15}$$

Com a equação de Maxwell: $\nabla x\bar{H} = j\omega\varepsilon_0\bar{\varepsilon}_r\bar{E}$, obtém-se:

$$\nabla x\bar{H} = \omega^2\mu_0\varepsilon_0\bar{\varepsilon}_r\nabla x\bar{\prod}_h \qquad \text{(A.16)}$$

Considerando a identidade vetorial $\nabla x(\nabla\phi_h) = 0$, pode-se escrever de (A.16):

$$\bar{H} = \omega^2\mu_0\varepsilon_0\bar{\varepsilon}_r\bar{\prod}_h + \nabla\phi_h \qquad \text{(A.17)}$$

Ficando assim definido o potencial escalar de Hertz do tipo magnético ϕ_h.

Substituindo $\bar{E}$ e $\bar{H}$, dados por (A.15) e (A.17), respectivamente, na equação de Maxwell: $\nabla x\bar{E} = -j\omega\mu_0\bar{H}$, obtém-se:

$$(j\omega\mu_0)\nabla x\nabla x\bar{\prod}_h = (j\omega\mu_0)[\omega^2\mu_0\varepsilon_0\bar{\varepsilon}_r\bar{\prod}_h + \nabla\phi_h] \qquad \text{(A.18)}$$

com $k_0{}^2 = \omega^2\mu_0\varepsilon_0$, então:

$$\nabla x\nabla x\bar{\prod}_h = k_0^2\bar{\varepsilon}_r\bar{\prod}_h + \nabla\phi_h \qquad \text{(A.19)}$$

Utilizando a identidade vetorial $\nabla x\nabla x\bar{\prod}_h = -\nabla^2\bar{\prod}_h + \nabla\nabla.\bar{\prod}_h$ em (A.19), resulta em:

$$\nabla^2\bar{\prod}_h + k_0^2\bar{\varepsilon}_r\bar{\prod}_h = \nabla\nabla.\bar{\prod}_h - \nabla\phi_h \qquad \text{(A.20)}$$

Para que $\bar{\prod}_h$, definido por (A.20) seja unívoco, impõe-se uma condição restritiva do tipo calibre de Lorentz:

$$\nabla\nabla.\bar{\prod}_h - \nabla\phi_h = 0 \qquad \text{(A.21)}$$

resultando:

$$\nabla^2\bar{\prod}_h + k_0^2\bar{\varepsilon}_r\bar{\prod}_h = 0 \qquad \text{(A.22)}$$

De (A.15), (A.17), (A.21) e (A.22) os campos podem ser expressos em termos de $\bar{\prod}_h$, como segue:

$$\bar{E} = -j\omega\mu_0\nabla x\bar{\prod}_h \qquad \text{(A.23)}$$

$$\bar{H} = k_0^2\bar{\varepsilon}_r\bar{\prod}_h + \nabla\nabla.\bar{\prod}_h = \nabla\nabla.\bar{\prod}_h - \nabla^2\bar{\prod}_h \qquad \text{(A.24)}$$

A.3 Solução Geral com os Potenciais Vetores de Hertz dos Tipos Elétrico e Magnético

Para a solução de um problema usando os potenciais vetores de Hertz do tipo elétrico e magnético, $\bar{\Pi}_e$ e $\bar{\Pi}_h$, respectivamente, a primeira providência é obter as soluções das equações (A.12) e (A.22). Conhecidos $\bar{\Pi}_e$ e $\bar{\Pi}_h$, os campos elétrico e magnético são obtidos como as somas de (A.14) e (A.23) para o campo elétrico; e (A.13) e (A.24) para o campo magnético:

$$\bar{E} = k_0^2 \bar{\varepsilon}_r \bar{\Pi}_e + \nabla\nabla.\bar{\Pi}_e - j\omega\mu_0 \nabla x \bar{\Pi}_h \tag{A.25}$$

$$\bar{H} = \nabla\nabla.\bar{\Pi}_h - \nabla^2 \bar{\Pi}_h + j\omega\varepsilon_0 \bar{\varepsilon}_r \nabla x \bar{\Pi}_e \tag{A.26}$$

com

$$\bar{\varepsilon}_r = \begin{pmatrix} \varepsilon_t & 0 & 0 \\ 0 & \varepsilon_t & 0 \\ 0 & 0 & \varepsilon_z \end{pmatrix}$$

onde:

ε_t é a componente tangencial do tensor da permissividade relativa do meio,

ε_z é a componente axial do tensor da permissividade relativa do meio.

APÊNDICE B
Determinação dos Campos Elétrico e Magnético a partir dos Potenciais Vetores de Hertz

Para o potencial vetor magnético, tem-se que [9]:

$$\nabla^2 \bar{\Pi}_h + k_0^2 \bar{\varepsilon}_r \bar{\Pi}_h = 0 \qquad \text{(B.1)}$$

ou

$$\nabla^2 \bar{\Pi}_h + k_0^2 \begin{pmatrix} \varepsilon_t & 0 & 0 \\ 0 & \varepsilon_t & 0 \\ 0 & 0 & \varepsilon_z \end{pmatrix} \bar{\Pi}_h = 0 \qquad \text{(B.2)}$$

Como $\bar{\Pi}_h$ só tem componente na direção z, então:

$$\nabla^2 \bar{\Pi}_h = \vec{z}\{\frac{\partial^2 \bar{\Pi}_h}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{\Pi}_h}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \bar{\Pi}_h}{\partial \phi^2} + \frac{\partial^2 \bar{\Pi}_h}{\partial z^2}\} \qquad \text{(B.3)}$$

logo

$$\frac{\partial^2 \Pi_h}{\partial r^2} + \frac{1}{r}\frac{\partial \Pi_h}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Pi_h}{\partial \phi^2} + \frac{\partial^2 \Pi_h}{\partial z^2} + k_0^2 \varepsilon_Z \Pi_h = 0 \qquad \text{(B.4)}$$

Assumindo uma propagação na forma $e^{-\gamma z}$ => $\frac{\partial^2 \Pi_h}{\partial z^2} = \gamma_Z^2 \Pi_h$,

obtém-se:

$$\frac{\partial^2 \Pi_h}{\partial r^2} + \frac{1}{r}\frac{\partial \Pi_h}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Pi_h}{\partial \phi^2} + k_{CH}^2 \Pi_h = 0 \qquad \text{(B.5)}$$

chamando

$$k_{CH}^2 = k_0^2 \varepsilon_Z + \gamma_Z^2 \qquad \text{(B.6)}$$

Então, resolvendo a equação de onda por substituição de variáveis, chega-se a:

$$\bar{\Pi}_h = \hat{z}\begin{cases} B_n J_n(k_{CH} r) e^{-\gamma z} \operatorname{sen}(n\phi), 0 \le r \le r_1 \\ \left[E_n J_n(K_1 r) + F_n Y_n(K_1 r)\right] e^{-\gamma z} \operatorname{sen}(n\phi), r_1 \le r \le r_0 \end{cases} \qquad \text{(B.7)}$$

Analisando o potencial vetor elétrico [9], tem-se:

$$\nabla^2 \bar{\Pi}_e + k_0^2 \bar{\varepsilon}_r \bar{\Pi}_e = 0 \tag{B.8}$$

Como $\bar{\Pi}_e$ só tem componente na direção z, então:

$$\bar{\Pi}_e = \Pi_e \hat{z} = \begin{bmatrix} 0\hat{r} \\ 0\hat{\phi} \\ \Pi_e \hat{z} \end{bmatrix} \tag{B.9}$$

$$\nabla^2 \bar{\Pi}_e = \hat{z}\left\{ \frac{\partial^2 \bar{\Pi}_e}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{\Pi}_e}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \bar{\Pi}_e}{\partial \phi^2} + \frac{\partial^2 \bar{\Pi}_e}{\partial z^2} \right\} \tag{B.10}$$

Assumindo uma propagação na forma $e^{-\gamma z}$, tem-se:

$$\frac{\partial^2 \Pi_e}{\partial z^2} = \gamma_z^2 \Pi_e \tag{B.11}$$

então:

$$\frac{\partial^2 \Pi_e}{\partial r^2} + \frac{1}{r}\frac{\partial \Pi_e}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Pi_e}{\partial \phi^2} + k_{CE}^2 \Pi_e = 0 \tag{B.12}$$

com

$$k_{CE}^2 = k_0^2 \varepsilon_Z + \gamma_Z^2 \tag{B.13}$$

Portanto:

$$\bar{\Pi}_e = \hat{z}\{A_n J_n(k_{CE} r) e^{-\gamma z} \cos(n\phi)\}, 0 \le r \le r_1 \tag{B.14}$$

$$\bar{\Pi}_e = \hat{z}\left\{ \left(C_n J_n(K_1 r) + D_n Y_n(K_1 r) \right) e^{-\gamma z} \cos(n\phi) \right\}, r_1 \le r \le r_0 \tag{B.15}$$

O campo elétrico, obtido através dos potenciais vetores é dado por [9]:

$$\vec{E} = k_0^2 \varepsilon_z \bar{\Pi}_e + \nabla(\nabla.\bar{\Pi}_e) - j\omega\mu_0 \nabla x \bar{\Pi}_h \tag{B.16}$$

$$\nabla.\bar{\Pi}_e = \nabla.(\Pi_e \hat{z}) = \frac{\partial \Pi_e}{\partial z} \tag{B.17}$$

$$\nabla(\nabla.\bar{\Pi}_e) = \hat{r}\frac{\partial^2 \Pi_e}{\partial r \partial z} + \hat{\phi}\frac{1}{r}\frac{\partial^2 \Pi_e}{\partial \phi \partial z} + \hat{z}\frac{\partial^2 \Pi_e}{\partial z^2} \tag{B.18}$$

$$\nabla x \bar{\Pi}_h = \hat{r}\frac{1}{r}\frac{\partial \Pi_h}{\partial \phi} - \hat{\phi}\frac{\partial \Pi_h}{\partial r} \tag{B.19}$$

Desta forma, substituindo (B.18) e (B.19) em (B.16) pode-se reescrever o campo elétrico como:

$$\vec{E} = k_0^2 \varepsilon_z \overline{\prod}_e \hat{z} + \{\hat{r} \frac{\partial^2 \prod_e}{\partial r \partial z} + \hat{\phi} \frac{1}{r} \frac{\partial^2 \prod_e}{\partial \phi \partial z} + \hat{z} \frac{\partial^2 \prod_e}{\partial z^2}\} - j\omega\mu_0 (\hat{r} \frac{1}{r} \frac{\partial \prod_h}{\partial \phi} - \hat{\phi} \frac{\partial \prod_h}{\partial r}) \quad \text{(B.20)}$$

$$\vec{E} = E_r \hat{r} + E_\phi \hat{\phi} + E_z \hat{z} \quad \text{(B.21)}$$

com

$$E_r = \frac{\partial^2 \prod_e}{\partial r \partial z} - j\omega\mu \frac{1}{r} \frac{\partial \prod_h}{\partial \phi} \quad \text{(B.22)}$$

$$E_\phi = \frac{1}{r} \frac{\partial^2 \prod_e}{\partial \phi \partial z} + j\omega\mu \frac{\partial \prod_h}{\partial r} \quad \text{(B.23)}$$

$$E_z = \frac{\partial^2 \prod_e}{\partial z^2} + k_0^2 \varepsilon_z \prod_e \quad \text{(B.24)}$$

Analisando a equação (B.24) junto com a equação (B.13), verifica-se que:

$$\begin{cases} E_z^i = \left[k_0^2 \varepsilon_z + {\gamma_z}^2\right] \prod_e = k_{CE}^2 \prod_e, 0 \le r \le r_1 \\ E_z^o = \left[k_0^2 + {\gamma_z}^2\right] \prod_e = K_1^2 \prod_e, r_1 \le r \le r_0 \end{cases} \quad \text{(B.25)}$$

onde ${\gamma_z}^2 = \frac{\partial^2}{\partial z^2}$

então

$$\begin{cases} \prod_e = \frac{1}{k_{CE}^2} E_z^i, 0 \le r \le r_1 \\ \prod_e = \frac{1}{K_1^2} E_z^o, r_1 \le r \le r_2 \end{cases} \quad \text{(B.26)}$$

O campo magnético, obtido pelos potenciais vetores é dado por:

$$\vec{H} = k_0^2 \bar{\varepsilon}_r \overline{\prod}_h + \nabla(\nabla . \overline{\prod}_n) + j\omega\varepsilon_0 \bar{\varepsilon}_r \nabla x \overline{\prod}_e \quad \text{(B.27)}$$

$$\nabla . \overline{\prod}_h = \nabla . (\overline{\prod}_h \hat{z}) = \frac{\partial \prod_h}{\partial z} \quad \text{(B.28)}$$

$$\nabla(\nabla . \overline{\prod}_h) = \hat{r} \frac{\partial^2 \prod_h}{\partial r \partial z} + \hat{\phi} \frac{1}{r} \frac{\partial^2 \prod_h}{\partial \phi \partial z} + \hat{z} \frac{\partial^2 \prod_h}{\partial z^2} \quad \text{(B.29)}$$

$$\nabla x \overline{\prod}_e = \hat{r} \frac{1}{r} \frac{\partial \prod_e}{\partial \phi} - \hat{\phi} \frac{\partial \prod_e}{\partial r} \quad \text{(B.30)}$$

e

$$\bar{\varepsilon}_r = \begin{pmatrix} \varepsilon_t & 0 & 0 \\ 0 & \varepsilon_t & 0 \\ 0 & 0 & \varepsilon_z \end{pmatrix} \quad \text{(B.31)}$$

resultando em:

$$\vec{H} = k_0^2 \varepsilon_z \prod_h \hat{z} + \{\hat{r}\frac{\partial^2 \prod_h}{\partial r \partial z} + \hat{\phi}\frac{1}{r}\frac{\partial^2 \prod_h}{\partial \phi \partial z} + \hat{z}\frac{\partial^2 \prod_h}{\partial z^2}\} + j\omega\varepsilon_0\varepsilon_t(\hat{r}\frac{1}{r}\frac{\partial \prod_e}{\partial \phi} - \hat{\phi}\frac{\partial \prod_e}{\partial r}) \quad \text{(B.32)}$$

$$\vec{H} = H_r\hat{r} + H_\phi\hat{\phi} + H_z\hat{z} \quad \text{(B.33)}$$

com

$$H_r = \frac{\partial^2 \prod_h}{\partial r \partial z} + j\omega\varepsilon_0\varepsilon_t \frac{1}{r}\frac{\partial \prod_e}{\partial \phi} \quad \text{(B.34)}$$

$$H_\phi = \frac{1}{r}\frac{\partial^2 \prod_h}{\partial \phi \partial z} - j\omega\varepsilon_0\varepsilon_t \frac{\partial \prod_e}{\partial r} \quad \text{(B.35)}$$

$$H_z = \frac{\partial^2 \prod_h}{\partial z^2} + k_0^2 \varepsilon_z \prod_h \quad \text{(B.36)}$$

Analisando as equações (B.36) e (B.6), verifica-se que:

$$\begin{cases} H_z = [\varepsilon_z k_0^2 + \gamma_z{}^2]\prod_h = k_{CH}^2 \prod_h, 0 \le r \le r_1 \\ H_z = [k_0^2 + \gamma_z{}^2]\prod_h = K_1^2 \prod_h, r_1 \le r \le r_0 \end{cases} \quad \text{(B.37)}$$

então:

$$\begin{cases} \prod_h = \frac{1}{k_{CH}^2} H_z^i, 0 \le r \le r_1 \\ \prod_h = \frac{1}{K_1^2} H_z^o, r_1 \le r \le r_0 \end{cases} \quad \text{(B.38)}$$

com as equações (B.26) e (B.38) pode-se reescrever as equações (B.22), (B.23), (B.34) e (B.35), para as duas regiões de estudo:

1) região i ($0 \le r \le r_1$):

$$E_r^i = \frac{1}{k_{CE}^2}\frac{\partial^2 E_z^i}{\partial r \partial z} - \frac{j\omega\mu_0}{r k_{CE}^2}\frac{\partial H_z^i}{\partial \phi} = \frac{1}{k_{CE}^2}\left\{\frac{\partial^2 E_z^i}{\partial r \partial z} - \frac{j\omega\mu_0}{r}\frac{\partial H_z^i}{\partial \phi}\right\} \quad \text{(B.39)}$$

$$E_\phi^i = \frac{1}{k_{CE}^2}\frac{1}{r}\frac{\partial^2 E_z^i}{\partial \phi \partial z} + \frac{j\omega\mu_0}{k_{CE}^2}\frac{\partial H_z^i}{\partial r} = \frac{1}{k_{CE}^2}\left\{\frac{1}{r}\frac{\partial^2 E_z^i}{\partial \phi \partial z} + j\omega\mu_0\frac{\partial H_z^i}{\partial r}\right\} \quad \text{(B.40)}$$

$$H_r^i = \frac{1}{k_{CH}^2}\frac{\partial^2 H_z^i}{\partial r \partial z} + \frac{j\omega\varepsilon_0}{k_{CH}^2}\frac{\varepsilon_{ti}}{r}\frac{\partial E_z^i}{\partial \phi} = \frac{1}{k_{CH}^2}\left\{\frac{\partial^2 H_z^i}{\partial r \partial z} + \frac{j\omega\varepsilon_0\varepsilon_{ti}}{r}\frac{\partial E_z^i}{\partial \phi}\right\} \quad \text{(B.41)}$$

$$H_\phi^i = \frac{1}{k_{CH}^2 r}\frac{\partial^2 H_z^i}{\partial \phi \partial z} - \frac{j\omega\varepsilon_0\varepsilon_{ti}}{k_{CH}^2}\frac{\partial E_z^i}{\partial r} = \frac{1}{k_{CH}^2}\left\{\frac{1}{r}\frac{\partial^2 H_z^i}{\partial \phi \partial z} - j\omega\varepsilon_0\varepsilon_{ti}\frac{\partial E_z^i}{\partial r}\right\} \quad \text{(B.42)}$$

onde, considerando $\gamma = j\beta$ (desconsiderando a atenuação, $\alpha = 0$):

$$k_{CE}^2 = k_{CH}^2 = K^2 = k_0^2\varepsilon_{zi} + \gamma_z^{\ 2} = k_0^2\varepsilon_{zi} - \beta_z^{\ 2} \quad \text{(B.43)}$$

2) região o ($r_1 \le r \le r_0$): $\varepsilon_{to} = \varepsilon_{ro} = 1$

$$E_r^o = \frac{1}{K_1^2}\left\{\frac{\partial^2 E_z^o}{\partial r \partial z} - \frac{j\omega\mu_0}{r}\frac{\partial H_z^o}{\partial \phi}\right\} \quad \text{(B.44)}$$

$$E_\phi^o = \frac{1}{K_1^2}\left\{\frac{1}{r}\frac{\partial^2 E_z^o}{\partial \phi \partial z} + j\omega\mu_0\frac{\partial H_z^o}{\partial r}\right\} \quad \text{(B.45)}$$

$$H_r^o = \frac{1}{K_1^2}\left\{\frac{\partial^2 H_z^o}{\partial r \partial z} + \frac{j\omega\varepsilon_0\varepsilon_{to}}{r}\frac{\partial E_z^o}{\partial \phi}\right\} \quad \text{(B.46)}$$

$$H_\phi^o = \frac{1}{k_1^2}\left\{\frac{1}{r}\frac{\partial^2 H_z^o}{\partial \phi \partial z} - j\omega\varepsilon_0\varepsilon_{to}\frac{\partial E_z^o}{\partial r}\right\} \quad \text{(B.47)}$$

e

$$K_1^2 = k_0^2 + \gamma_z^{\ 2} = k_0^2 - \beta_z^{\ 2} \quad \text{(B.48)}$$

É interessante comparar estes resultados com os conhecidos do caso isotrópico, onde:

$$\varepsilon_z = \varepsilon_t = \varepsilon_r$$

$$E_r = \frac{1}{K^2}\left\{\frac{\partial^2 E_z}{\partial r \partial z} - \frac{j\omega\mu_0}{r}\frac{\partial H_z}{\partial \phi}\right\}$$

$$E_\phi = \frac{1}{K^2}\left\{\frac{1}{r}\frac{\partial^2 E_z}{\partial \phi \partial z} + j\omega\mu_0\frac{\partial H_z}{\partial r}\right\}$$

$$H_r = \frac{1}{K^2}\left\{\frac{\partial^2 H_z}{\partial r \partial z} + \frac{j\omega\varepsilon_0\varepsilon_r}{r}\frac{\partial E_z}{\partial \phi}\right\}$$

$$H_\phi = \frac{1}{K^2}\left\{\frac{1}{r}\frac{\partial^2 H_z}{\partial \phi \partial z} - j\omega\varepsilon_0\varepsilon_r\frac{\partial E_z}{\partial r}\right\}$$

e

$$K^2 = k_0^2\varepsilon_r + \gamma^2 = k_0^2\varepsilon_r - \beta^2$$, para meios sem perdas ($\alpha = 0$).

Pode-se verificar que, quando o meio é isotrópico, as equações ficam as mesmas conhecidas na literatura.

APENDICE C
Obtenção da Admitância de Superfície em r = r_0 no Lado da Corrugação.

Os modos transverso-magnéticos a “z” são obtidos a partir dos potenciais vetores [9]. Fazendo-se:

$$\begin{cases} \bar{\Pi}_e = \Pi_{ez}\,\hat{z} \\ \bar{\Pi}_h = 0 \end{cases} \qquad \text{(C.1)}$$

O potencial vetor $\bar{\Pi}_e$ deve satisfazer a equação de onda:

$$\nabla^2 \Pi_{ez}(r,\phi,z) + K_1^2 \Pi_{ez}(r,\phi,z) = 0 \qquad \text{(C.2)}$$

Na região de corrugação, a constante de propagação é zero, pois na corrugação não existe propagação. Então: $r_0 \le r \le r_2$ => $e^{-\gamma_z Z} = e^{-0.Z} = 1$.

A solução da equação (C.2) é dada por:

$$\Pi_{ez} = [C_n J_n(K_1 r) + D_n Y_n(K_1 r)]\cos(n\phi) \qquad \text{(C.3)}$$

Sabe-se que: $\varepsilon_z -> 1$,

$$E_z = K_1^2 \Pi_e,$$

$$K_1^2 = k_0^2 + \gamma_Z$$

então:

$$E_z(r,\phi,z) = K_1^2[C_n J_n(K_1 r) + D_n Y_n(K_1 r)]\cos(n\phi) \qquad \text{(C.4)}$$

como em r = r2,

$$E_z(r,\phi,z) = 0 \qquad \text{(C.5)}$$

logo:

$$C_n J_n(K_1 r_2) + D_n Y_n(K_1 r_2) = 0 \qquad \text{(C.6)}$$

resultando em:

$$D_n = -\frac{J_n(K_1 r_2)}{Y_n(K_1 r_2)} C_n \qquad \text{(C.7)}$$

Substituindo (C.7) na equação (C.4) tem-se:

$$E_z = K_1^2 C_n [J_n(K_1 r) - \frac{J_n(K_1 r_2)}{Y_n(K_1 r_2)} Y_n(K_1 r)] \cos(n\phi) \quad \text{(C.8)}$$

considerando que não existe propagação na corrugação, ou seja, $\gamma_Z = 0 \Rightarrow K_1 = k_0$, obtém-se a equação:

$$E_z = \frac{k_0^2 C_n}{Y_n(k_0 r_2)} [J_n(k_0 r) Y_n(k_0 r_2) - J_n(k_0 r_2) Y_n(k_0 r)] \cos(n\phi) \quad \text{(C.9)}$$

O campo magnético na direção $\hat{\phi}$ é dado por:

$$H_\phi = \frac{1}{r} \frac{\partial^2 \prod_h}{\partial \phi \partial z} - j\omega \varepsilon_0 \varepsilon_t \frac{\partial \prod_e}{\partial r} \quad \text{(C.10)}$$

Mas, $\prod_h = 0$, então:

$$H_\phi = -j\omega \varepsilon_0 \varepsilon_t \frac{\partial \prod_e}{\partial r} \quad \text{(C.11)}$$

Na região de corrugação, a permissividade elétrica é igual a 1 e só existe componente na direção z, ou seja:

$$H_\phi(r,\phi,z) = \frac{-j\omega\varepsilon_0}{K_1^2} \left\{ K_1^2 \frac{C_n}{Y_n(K_1 r_2)} K_1 [J'_n(K_1 r) Y_n(K_1 r_2) - J_n(K_1 r_2) Y'_n(K_1 r)] \right\} \cos(n\phi) \quad \text{(C.12)}$$

ou

$$H_\phi(r,\phi,z) = -j\omega\varepsilon_0 K_1 \frac{C_n}{Y_n(K_1 r_2)} [J'_n(K_1 r) Y_n(K_1 r_2) - J_n(K_1 r_2) Y'_n(K_1 r)] \cos(n\phi) \quad \text{(C.13)}$$

com $K_1 = k_0$ na corrugação:

$$H_\phi(r,\phi,z) = -j\omega\varepsilon_0 k_0 \frac{C_n}{Y_n(k_0 r_2)} [J'_n(k_0 r) Y_n(k_0 r_2) - J_n(k_0 r_2) Y'_n(k_0 r)] \cos(n\phi) \quad \text{(C.14)}$$

A admitância de superfície é dada por:

$$Y_S(r) = \frac{H_\phi}{E_z} = \frac{-j\omega\varepsilon_0 k_0}{k_0^2} \left[\frac{\left\{ \frac{C_n}{Y_n(k_0 r_2)} [J'_n(k_0 r) Y_n(k_0 r_2) - J_n(k_0 r_2) Y'_n(k_0 r)] \right\} \cos(n\phi)}{\frac{C_n}{Y_n(k_0 r_2)} [J_n(k_0 r) Y_n(k_0 r_2) - J_n(k_0 r_2) Y_n(k_0 r)] \cos(n\phi)} \right] \quad \text{(C.15)}$$

ou

$$Y_S(r) = \frac{-j\omega\varepsilon_0}{k_0}\left[\frac{J'_n(k_0 r)Y_n(k_0 r_2) - J_n(k_0 r_2)Y'_n(k_0 r)}{J_n(k_0 r)Y_n(k_0 r_2) - J_n(k_0 r_2)Y_n(k_0 r)}\right] \quad \text{(C.16)}$$

sabendo que $y_0 = \dfrac{\omega\varepsilon_0}{k_0}$, dividindo e multiplicando $Y_S(r)$ por rk_0, tem-se:

$$Y_S(r) = -j\frac{y_o}{rk_0}S_n(k_0, r, r_2) \quad \text{(C.17)}$$

com

$$S_n(k, x, y) = (kx)\left\{\frac{J'_n(kx)Y_n(ky) - J_n(ky)Y'_n(kx)}{J_n(kx)Y_n(ky) - J_n(ky)Y_n(kx)}\right\} \quad \text{(C.18)}$$

Portanto, a admitância de superfície em $r = r_0$ é dada por:

$$Y_S(r_0) = \frac{-jy_0}{k_0 r_0}.S_n(k_0, r_0, r_2) \quad \text{(C.19)}$$

onde

$y_0 = \sqrt{\dfrac{\varepsilon_0}{\mu_0}}$ - admitância do espaço livre

e

$$S_n(k_0, r_0, r_2) = (k_0 r_0)\left\{\frac{J'_n(k_0 r_0)Y_n(k_0 r_2) - J_n(k_0 r_2)Y'_n(k_0 r_0)}{J_n(k_0 r_0)Y_n(k_0 r_2) - J_n(k_0 r_2)Y_n(k_0 r_0)}\right\} \quad \text{(C.20)}$$

APENDICE D
Obtenção dos Coeficientes A_n, B_n, C_n, D_n, E_n, F_n

O sistema de equações dado pelas Equações 2.27, 2.29, 2.32, 2.34, 2.36 e 2.38 do Capítulo 2, pode ser escrito na forma matricial: [M].[Cref] = [0], onde M é uma matriz 6 x 6 e Cref é uma matriz coluna. Tem-se:

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} & \alpha_{15} & \alpha_{16} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} & \alpha_{25} & \alpha_{26} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} & \alpha_{35} & \alpha_{36} \\ \alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44} & \alpha_{45} & \alpha_{46} \\ \alpha_{51} & \alpha_{52} & \alpha_{53} & \alpha_{54} & \alpha_{55} & \alpha_{56} \\ \alpha_{61} & \alpha_{62} & \alpha_{63} & \alpha_{64} & \alpha_{65} & \alpha_{66} \end{pmatrix} \begin{pmatrix} A_n \\ B_n \\ C_n \\ D_n \\ E_n \\ F_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \text{(D.1)}$$

onde:

$$\alpha_{22} = \alpha_{25} = \alpha_{26} = \alpha_{31} = \alpha_{32} = \alpha_{41} = \alpha_{42} = \alpha_{61} = \alpha_{63} = \alpha_{64} = 0 \qquad \text{(D.2)}$$

$$\alpha_{11} = \frac{n\gamma}{K^2 r_1} J_n(Kr_1) \qquad \text{(D.3)}$$

$$\alpha_{12} = \frac{j\omega\mu_0}{K} J_n^{'}(Kr_1) \qquad \text{(D.4)}$$

$$\alpha_{13} = -\frac{n\gamma}{K_1^2 r_1} J_n(K_1 r_1) \qquad \text{(D.5)}$$

$$\alpha_{14} = -\frac{n\gamma}{K_1^2 r_1} Y_n(K_1 r_1) \qquad \text{(D.6)}$$

$$\alpha_{15} = -\frac{j\omega\mu_0}{K_1} J_n^{'}(K_1 r_1) \qquad \text{(D.7)}$$

$$\alpha_{16} = -\frac{j\omega\mu_0}{K_1} Y_n^{'}(K_1 r_1) \qquad \text{(D.8)}$$

$$\alpha_{21} = J_n(Kr_1) \qquad \text{(D.9)}$$

$$\alpha_{23} = -J_n(K_1 r_1) \qquad \text{(D.10)}$$

$$\alpha_{24} = -Y_n(K_1 r_1) \qquad \text{(D.11)}$$

$$\alpha_{33} = K_1^2 J_n(K_1 r_0) Ys(r_0) + j\omega\varepsilon_0 K_1 J_n^{'}(K_1 r_0) \qquad \text{(D.12)}$$

$$\alpha_{34} = K_1^2 Y_n(K_1 r_0) Ys(r_0) + j\omega\varepsilon_0 K_1 Y_n^{'}(K_1 r_0) \qquad \text{(D.13)}$$

$$\alpha_{35} = \frac{n\gamma}{r_0} J_n(K_1 r_0) \tag{D.14}$$

$$\alpha_{36} = \frac{n\gamma}{r_0} Y_n(K_1 r_0) \tag{D.15}$$

$$\alpha_{43} = \frac{n\gamma}{r_0} J_n(K_1 r_0) \tag{D.16}$$

$$\alpha_{44} = \frac{n\gamma}{r_0} Y_n(K_1 r_0) \tag{D.17}$$

$$\alpha_{45} = j\omega\mu_0 K_1 J_n^{'}(K_1 r_0) \tag{D.18}$$

$$\alpha_{46} = j\omega\mu_0 K_1 Y_n^{'}(K_1 r_0) \tag{D.19}$$

$$\alpha_{51} = -\frac{j\omega\varepsilon_0\varepsilon_{ti}}{K} J_n^{'}(K r_1) \tag{D.20}$$

$$\alpha_{52} = -\frac{n\gamma\varepsilon_{ti}}{K^2 r_1} J_n(K r_1) \tag{D.21}$$

$$\alpha_{53} = \frac{j\omega\varepsilon_0}{K_1} J_n^{'}(K_1 r_1) \tag{D.22}$$

$$\alpha_{54} = \frac{j\omega\varepsilon_0}{K_1} Y_n^{'}(K_1 r_1) \tag{D.23}$$

$$\alpha_{55} = \frac{n\gamma}{r_1 K_1^2} J_n(K_1 r_1) \tag{D.24}$$

$$\alpha_{56} = \frac{n\gamma}{r_1 K_1^2} Y_n(K_1 r_1) \tag{D.25}$$

$$\alpha_{62} = J_n(K r_1) \tag{D.26}$$

$$\alpha_{65} = -J_n(K_1 r_1) \tag{D.27}$$

$$\alpha_{66} = -Y_n(K_1 r_1) \tag{D.28}$$

onde, $\gamma = \alpha + j\beta \cong j\beta = \gamma_Z$ = constante de propagação na direção z para meios sem perdas, e $Y_S(r_0)$ é a admitância de superfície em r = r_0, dada pela equação C.17 (apêndice C).

Sabendo que:

$$k_0^2 = \omega^2 \mu_0 \varepsilon_0 \tag{D.29}$$

$$\omega\varepsilon_0 = k_0 y_o \tag{D.30}$$

$$\omega\mu_0 = \frac{k_0}{y_o} \tag{D.31}$$

E, substituindo D.29, D.30 e D.31 nas Equações D.3 a D.28, pode-se reescrever a matriz como:

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} & \alpha_{15} & \alpha_{16} \\ \alpha_{21} & 0 & \alpha_{23} & \alpha_{24} & 0 & 0 \\ 0 & 0 & \alpha_{33} & \alpha_{34} & \alpha_{35} & \alpha_{36} \\ 0 & 0 & \alpha_{43} & \alpha_{44} & \alpha_{45} & \alpha_{46} \\ \alpha_{51} & \alpha_{52} & \alpha_{53} & \alpha_{54} & \alpha_{55} & \alpha_{56} \\ 0 & \alpha_{62} & 0 & 0 & \alpha_{65} & \alpha_{66} \end{pmatrix} \begin{pmatrix} A_n \\ B_n \\ C_n \\ D_n \\ E_n \\ F_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{(D.32)}$$

com

$$\alpha_{22} = \alpha_{25} = \alpha_{26} = \alpha_{31} = \alpha_{32} = \alpha_{41} = \alpha_{42} = \alpha_{61} = \alpha_{63} = \alpha_{64} = 0 \quad \text{(D.33)}$$

$$\alpha_{11} = \frac{j\beta n}{K^2 r_1} J_n(Kr_1) \quad \text{(D.34)}$$

$$\alpha_{12} = \frac{jk_0}{Ky_0} J_n^{'}(Kr_1) \quad \text{(D.35)}$$

$$\alpha_{13} = -\frac{j\beta n}{K_1^2 r_1} J_n(K_1 r_1) \quad \text{(D.36)}$$

$$\alpha_{14} = -\frac{j\beta n}{K_1^2 r_1} Y_n(K_1 r_1) \quad \text{(D.37)}$$

$$\alpha_{15} = -\frac{jk_0}{K_1 y_0} J_n^{'}(K_1 r_1) \quad \text{(D.38)}$$

$$\alpha_{16} = -\frac{jk_0}{K_1 y_0} Y_n^{'}(K_1 r_1) \quad \text{(D.39)}$$

$$\alpha_{21} = J_n(Kr_1) \quad \text{(D.40)}$$

$$\alpha_{23} = -J_n(K_1 r_1) \quad \text{(D.41)}$$

$$\alpha_{24} = -Y_n(K_1 r_1) \quad \text{(D.42)}$$

$$\alpha_{33} = K_1^2 J_n(K_1 r_0) Ys(r_0) + jk_0 y_0 K_1 J_n^{'}(K_1 r_0) \quad \text{(D.43)}$$

$$\alpha_{34} = K_1^2 Y_n(K_1 r_0) Ys(r_0) + jk_0 y_0 K_1 Y_n^{'}(K_1 r_0) \quad \text{(D.44)}$$

$$\alpha_{35} = \frac{j\beta n}{r_0} J_n(K_1 r_0) \quad \text{(D.45)}$$

$$\alpha_{36} = \frac{j\beta n}{r_0} Y_n(K_1 r_0) \quad \text{(D.46)}$$

$$\alpha_{43} = \frac{j\beta n}{r_0} J_n(K_1 r_0) \quad \text{(D.47)}$$

$$\alpha_{44} = \frac{j\beta n}{r_0} Y_n(K_1 r_0) \tag{D.48}$$

$$\alpha_{45} = j\frac{k_0}{y_0} K_1 J_n^{'}(K_1 r_0) \tag{D.49}$$

$$\alpha_{46} = j\frac{k_0}{y_0} K_1 Y_n^{'}(K_1 r_0) \tag{D.50}$$

$$\alpha_{51} = -\frac{jk_0 y_0 \varepsilon_{ti}}{K} J_n^{'}(K r_1) \tag{D.51}$$

$$\alpha_{52} = -\frac{j\beta n}{K^2 r_1} \varepsilon_{ti} J_n(K r_1) \tag{D.52}$$

$$\alpha_{53} = \frac{jk_0 y_0}{K_1} J_n^{'}(K_1 r_1) \tag{D.53}$$

$$\alpha_{54} = \frac{jk_0 y_0}{K_1} Y_n^{'}(K_1 r_1) \tag{D.54}$$

$$\alpha_{55} = \frac{j\beta n}{r_1 K_1^2} J_n(K_1 r_1) \tag{D.55}$$

$$\alpha_{56} = \frac{j\beta n}{r_1 K_1^2} Y_n(K_1 r_1) \tag{D.56}$$

$$\alpha_{62} = J_n(K r_1) \tag{D.57}$$

$$\alpha_{65} = -J_n(K_1 r_1) \tag{D.58}$$

$$\alpha_{66} = -Y_n(K_1 r_1) \tag{D.59}$$

Gerando a matriz aumentada e triangularizando a matriz, ou seja, zerando os valores abaixo da diagonal principal [13] obtém-se o determinante da matriz. Para isso, usando a matriz aumentada:

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} & \alpha_{15} & \alpha_{16} & 0 \\ \alpha_{21} & 0 & \alpha_{23} & \alpha_{24} & 0 & 0 & 0 \\ 0 & 0 & \alpha_{33} & \alpha_{34} & \alpha_{35} & \alpha_{36} & 0 \\ 0 & 0 & \alpha_{43} & \alpha_{44} & \alpha_{45} & \alpha_{46} & 0 \\ \alpha_{51} & \alpha_{52} & \alpha_{53} & \alpha_{54} & \alpha_{55} & \alpha_{56} & 0 \\ 0 & \alpha_{62} & 0 & 0 & \alpha_{65} & \alpha_{66} & 0 \end{pmatrix} \tag{D.60}$$

Trocando algumas linhas para facilitar a triangularização, escreve-se a nova matriz:

$$\begin{pmatrix} \alpha_{51} & \alpha_{52} & \alpha_{53} & \alpha_{54} & \alpha_{55} & \alpha_{56} & 0 \\ 0 & \alpha_{62} & 0 & 0 & \alpha_{65} & \alpha_{66} & 0 \\ 0 & 0 & \alpha_{43} & \alpha_{44} & \alpha_{45} & \alpha_{46} & 0 \\ \alpha_{21} & 0 & \alpha_{23} & \alpha_{24} & 0 & 0 & 0 \\ \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} & \alpha_{15} & \alpha_{16} & 0 \\ 0 & 0 & \alpha_{33} & \alpha_{34} & \alpha_{35} & \alpha_{36} & 0 \end{pmatrix} \qquad \text{(D.61)}$$

Zerando a coluna 1 das linhas 4 e 5 com a linha 1 (α_{5n}):

$$\begin{pmatrix} \alpha_{51} & \alpha_{52} & \alpha_{53} & \alpha_{54} & \alpha_{55} & \alpha_{56} & 0 \\ 0 & \alpha_{62} & 0 & 0 & \alpha_{65} & \alpha_{66} & 0 \\ 0 & 0 & \alpha_{43} & \alpha_{44} & \alpha_{45} & \alpha_{46} & 0 \\ 0 & -\alpha_{21}\frac{\alpha_{52}}{\alpha_{51}} & \alpha_{23}-\alpha_{21}\frac{\alpha_{53}}{\alpha_{51}} & \alpha_{24}-\alpha_{21}\frac{\alpha_{54}}{\alpha_{51}} & -\alpha_{21}\frac{\alpha_{55}}{\alpha_{51}} & -\alpha_{21}\frac{\alpha_{56}}{\alpha_{51}} & 0 \\ 0 & \alpha_{12}-\alpha_{11}\frac{\alpha_{52}}{\alpha_{51}} & \alpha_{13}-\alpha_{11}\frac{\alpha_{53}}{\alpha_{51}} & \alpha_{14}-\alpha_{11}\frac{\alpha_{54}}{\alpha_{51}} & \alpha_{15}-\alpha_{11}\frac{\alpha_{55}}{\alpha_{51}} & \alpha_{16}-\alpha_{11}\frac{\alpha_{56}}{\alpha_{51}} & 0 \\ 0 & 0 & \alpha_{33} & \alpha_{34} & \alpha_{35} & \alpha_{36} & 0 \end{pmatrix} \qquad \text{(D.62)}$$

Zerando a coluna 2 das linhas 4 e 5 a partir da linha 2 (α_{6n}):

$$\begin{pmatrix} \alpha_{51} & \alpha_{52} & \alpha_{53} & \alpha_{54} & \alpha_{55} & \alpha_{56} & 0 \\ 0 & \alpha_{62} & 0 & 0 & \alpha_{65} & \alpha_{66} & 0 \\ 0 & 0 & \alpha_{43} & \alpha_{44} & \alpha_{45} & \alpha_{46} & 0 \\ 0 & 0 & \left(\alpha_{23}-\alpha_{21}\frac{\alpha_{53}}{\alpha_{51}}\right) & \left(\alpha_{24}-\alpha_{21}\frac{\alpha_{54}}{\alpha_{51}}\right) & \begin{pmatrix} -\alpha_{21}\frac{\alpha_{55}}{\alpha_{51}}+ \\ \left(\alpha_{21}\frac{\alpha_{52}}{\alpha_{51}}\right)\frac{\alpha_{65}}{\alpha_{62}} \end{pmatrix} & \begin{pmatrix} -\alpha_{21}\frac{\alpha_{56}}{\alpha_{51}}+ \\ \left(\alpha_{21}\frac{\alpha_{52}}{\alpha_{51}}\right)\frac{\alpha_{66}}{\alpha_{62}} \end{pmatrix} & 0 \\ 0 & 0 & \left(\alpha_{13}-\alpha_{11}\frac{\alpha_{53}}{\alpha_{51}}\right) & \left(\alpha_{14}-\alpha_{11}\frac{\alpha_{53}}{\alpha_{51}}\right) & \begin{pmatrix} \alpha_{15}-\alpha_{11}\frac{\alpha_{53}}{\alpha_{51}}- \\ \left(\alpha_{12}-\alpha_{11}\frac{\alpha_{52}}{\alpha_{51}}\right)\frac{\alpha_{65}}{\alpha_{62}} \end{pmatrix} & \begin{pmatrix} \alpha_{16}-\alpha_{11}\frac{\alpha_{53}}{\alpha_{51}}- \\ \left(\alpha_{12}-\alpha_{11}\frac{\alpha_{52}}{\alpha_{51}}\right)\frac{\alpha_{66}}{\alpha_{62}} \end{pmatrix} & 0 \\ 0 & 0 & \alpha_{33} & \alpha_{34} & \alpha_{35} & \alpha_{36} & 0 \end{pmatrix} \qquad \text{(D.63)}$$

Zerando a coluna 3 das linhas 4, 5 e 6 a partir da linha 3 (α_{4n}):

$$\begin{pmatrix} \alpha_{51} & \alpha_{52} & \alpha_{53} & \alpha_{54} & \alpha_{55} & \alpha_{56} & 0 \\ 0 & \alpha_{62} & 0 & 0 & \alpha_{65} & \alpha_{66} & 0 \\ 0 & 0 & \alpha_{43} & \alpha_{44} & \alpha_{45} & \alpha_{46} & 0 \\ 0 & 0 & 0 & L44n & L45n & L46n & 0 \\ 0 & 0 & 0 & L54n & L55n & L56n & 0 \\ 0 & 0 & 0 & \alpha_{34}\frac{\alpha_{44}}{\alpha_{43}} & \alpha_{35}\frac{\alpha_{45}}{\alpha_{43}} & \alpha_{36}\frac{\alpha_{46}}{\alpha_{43}} & 0 \end{pmatrix} \qquad \text{(D.64)}$$

com

$$L44n = \left(\alpha_{24} - \alpha_{21}\frac{\alpha_{54}}{\alpha_{51}}\right) - \left(\alpha_{23} - \alpha_{21}\frac{\alpha_{53}}{\alpha_{51}}\right)\frac{\alpha_{44}}{\alpha_{43}} \quad \text{(D.65)}$$

$$L45n = -\alpha_{21}\frac{\alpha_{55}}{\alpha_{51}} + \left(\alpha_{21}\frac{\alpha_{52}}{\alpha_{51}}\right)\frac{\alpha_{65}}{\alpha_{62}} - \left(\alpha_{23} - \alpha_{21}\frac{\alpha_{53}}{\alpha_{51}}\right)\frac{\alpha_{45}}{\alpha_{43}} \quad \text{(D.66)}$$

$$L46n = -\alpha_{21}\frac{\alpha_{56}}{\alpha_{51}} + \left(\alpha_{21}\frac{\alpha_{52}}{\alpha_{51}}\right)\frac{\alpha_{66}}{\alpha_{62}} - \left(\alpha_{23} - \alpha_{21}\frac{\alpha_{53}}{\alpha_{51}}\right)\frac{\alpha_{46}}{\alpha_{43}} \quad \text{(D.67)}$$

$$L54n = \alpha_{14} - \alpha_{11}\frac{\alpha_{53}}{\alpha_{51}} - \left(\alpha_{13} - \alpha_{11}\frac{\alpha_{53}}{\alpha_{51}}\right)\frac{\alpha_{44}}{\alpha_{43}} \quad \text{(D.68)}$$

$$L55n = \alpha_{15} - \alpha_{11}\frac{\alpha_{53}}{\alpha_{51}} - \left(\alpha_{12} - \alpha_{11}\frac{\alpha_{52}}{\alpha_{51}}\right)\frac{\alpha_{65}}{\alpha_{62}} - \left(\alpha_{13} - \alpha_{11}\frac{\alpha_{53}}{\alpha_{51}}\right)\frac{\alpha_{45}}{\alpha_{43}} \quad \text{(D.69)}$$

$$L56n = \alpha_{16} - \alpha_{11}\frac{\alpha_{53}}{\alpha_{51}} - \left(\alpha_{12} - \alpha_{11}\frac{\alpha_{52}}{\alpha_{51}}\right)\frac{\alpha_{66}}{\alpha_{62}} - \left(\alpha_{13} - \alpha_{11}\frac{\alpha_{53}}{\alpha_{51}}\right)\frac{\alpha_{46}}{\alpha_{43}} \quad \text{(D.70)}$$

Zerando a coluna 4 das linhas 5 e 6 com linha 4:

$$\begin{pmatrix} \alpha_{51} & \alpha_{52} & \alpha_{53} & \alpha_{54} & \alpha_{55} & \alpha_{56} & 0 \\ 0 & \alpha_{62} & 0 & 0 & \alpha_{65} & \alpha_{66} & 0 \\ 0 & 0 & \alpha_{43} & \alpha_{44} & \alpha_{45} & \alpha_{46} & 0 \\ 0 & 0 & 0 & L44n & L45n & L46n & 0 \\ 0 & 0 & 0 & 0 & L55n & L56n & 0 \\ 0 & 0 & 0 & 0 & L65n & L66n & 0 \end{pmatrix} \quad \text{(D.71)}$$

$$L44n = \alpha_{24} - \alpha_{21}\frac{\alpha_{54}}{\alpha_{51}} + \alpha_{21}\frac{\alpha_{52}}{\alpha_{51}}\frac{\alpha_{64}}{\alpha_{62}} - \left(\alpha_{23} - \alpha_{21}\frac{\alpha_{53}}{\alpha_{51}}\right)\frac{\alpha_{44}}{\alpha_{43}} \quad \text{(D.72)}$$

$$L45n = -\alpha_{21}\frac{\alpha_{55}}{\alpha_{51}} + \left(\alpha_{21}\frac{\alpha_{52}}{\alpha_{51}}\right)\frac{\alpha_{65}}{\alpha_{62}} - \left(\alpha_{23} - \alpha_{21}\frac{\alpha_{53}}{\alpha_{51}}\right)\frac{\alpha_{45}}{\alpha_{43}} \quad \text{(D.73)}$$

$$L46n = -\alpha_{21}\frac{\alpha_{56}}{\alpha_{51}} + \left(\alpha_{21}\frac{\alpha_{52}}{\alpha_{51}}\right)\frac{\alpha_{66}}{\alpha_{62}} - \left(\alpha_{23} - \alpha_{21}\frac{\alpha_{53}}{\alpha_{51}}\right)\frac{\alpha_{46}}{\alpha_{43}} \quad \text{(D.74)}$$

$$L555n = \alpha_{15} - \alpha_{11}\frac{\alpha_{53}}{\alpha_{51}} - \left(\alpha_{12} - \alpha_{11}\frac{\alpha_{52}}{\alpha_{51}}\right)\frac{\alpha_{65}}{\alpha_{62}} - \left(\alpha_{13} - \alpha_{11}\frac{\alpha_{53}}{\alpha_{51}}\right)\frac{\alpha_{45}}{\alpha_{43}} - \left(\alpha_{14} - \alpha_{11}\frac{\alpha_{53}}{\alpha_{51}} - \left(\alpha_{13} - \alpha_{11}\frac{\alpha_{53}}{\alpha_{51}}\right)\frac{\alpha_{44}}{\alpha_{43}}\right)\frac{\left(-\alpha_{21}\frac{\alpha_{55}}{\alpha_{51}} + \left(\alpha_{21}\frac{\alpha_{52}}{\alpha_{51}}\right)\frac{\alpha_{65}}{\alpha_{62}} - \left(\alpha_{23} - \alpha_{21}\frac{\alpha_{53}}{\alpha_{51}}\right)\frac{\alpha_{45}}{\alpha_{43}}\right)}{\left(\alpha_{24} - \alpha_{21}\frac{\alpha_{54}}{\alpha_{51}}\right) - \left(\alpha_{23} - \alpha_{21}\frac{\alpha_{53}}{\alpha_{51}}\right)\frac{\alpha_{44}}{\alpha_{43}}} \quad \text{(D.75)}$$

$$L565n = \alpha_{16} - \alpha_{11}\frac{\alpha_{53}}{\alpha_{51}} - \left(\alpha_{12} - \alpha_{11}\frac{\alpha_{52}}{\alpha_{51}}\right)\frac{\alpha_{66}}{\alpha_{62}} - \left(\alpha_{13} - \alpha_{11}\frac{\alpha_{53}}{\alpha_{51}}\right)\frac{\alpha_{46}}{\alpha_{43}} - \left(\alpha_{14} - \alpha_{11}\frac{\alpha_{53}}{\alpha_{51}} - \left(\alpha_{13} - \alpha_{11}\frac{\alpha_{53}}{\alpha_{51}}\right)\frac{\alpha_{44}}{\alpha_{43}}\right)\frac{\left(-\alpha_{21}\frac{\alpha_{56}}{\alpha_{51}} + \left(\alpha_{21}\frac{\alpha_{52}}{\alpha_{51}}\right)\frac{\alpha_{66}}{\alpha_{62}} - \left(\alpha_{23} - \alpha_{21}\frac{\alpha_{53}}{\alpha_{51}}\right)\frac{\alpha_{46}}{\alpha_{43}}\right)}{\left(\left(\alpha_{24} - \alpha_{21}\frac{\alpha_{54}}{\alpha_{51}}\right) - \left(\alpha_{23} - \alpha_{21}\frac{\alpha_{53}}{\alpha_{51}}\right)\frac{\alpha_{44}}{\alpha_{43}}\right)} \quad \text{(D.76)}$$

$$L65n = \alpha_{35}\frac{\alpha_{45}}{\alpha_{43}} - \left(\alpha_{34}\frac{\alpha_{44}}{\alpha_{43}}\right)\frac{\left(-\alpha_{21}\frac{\alpha_{55}}{\alpha_{51}} + \left(\alpha_{21}\frac{\alpha_{52}}{\alpha_{51}}\right)\frac{\alpha_{65}}{\alpha_{62}} - \left(\alpha_{23} - \alpha_{21}\frac{\alpha_{53}}{\alpha_{51}}\right)\frac{\alpha_{45}}{\alpha_{43}}\right)}{\left(\left(\alpha_{24} - \alpha_{21}\frac{\alpha_{54}}{\alpha_{51}}\right) - \left(\alpha_{23} - \alpha_{21}\frac{\alpha_{53}}{\alpha_{51}}\right)\frac{\alpha_{44}}{\alpha_{43}}\right)} \quad \text{(D.77)}$$

$$L66n = \alpha_{36}\frac{\alpha_{46}}{\alpha_{43}}\left(\alpha_{34}\frac{\alpha_{44}}{\alpha_{43}}\right)\frac{\left(-\alpha_{21}\frac{\alpha_{56}}{\alpha_{51}} + \left(\alpha_{21}\frac{\alpha_{52}}{\alpha_{51}}\right)\frac{\alpha_{66}}{\alpha_{62}} - \left(\alpha_{23} - \alpha_{21}\frac{\alpha_{53}}{\alpha_{51}}\right)\frac{\alpha_{46}}{\alpha_{43}}\right)}{\left(\left(\alpha_{24} - \alpha_{21}\frac{\alpha_{54}}{\alpha_{51}}\right) - \left(\alpha_{23} - \alpha_{21}\frac{\alpha_{53}}{\alpha_{51}}\right)\frac{\alpha_{44}}{\alpha_{43}}\right)} \quad \text{(D.78)}$$

Usando a matriz resultante da Equação (D.71) para calcular o determinante, tem-se os coeficientes em função de An (normalizando em An=1).

Da linha 5:

$$L555n.An + L556n.Cn = 0 => \qquad Cn = -\frac{L555n}{L556n}An \quad \text{(D.79)}$$

ou

$$Cn = NAn \quad \text{(D.80)}$$

com

$$N = -\frac{L555n}{L556n} \quad \text{(D.81)}$$

Da linha 4:

$$L44nBn + L45nAn + L46nCn = 0 => \qquad L44nBn + L45nAn - L46n\frac{L555}{L556}An = 0 \qquad Bn = -\frac{\left(L45n + \frac{L46nL555}{L556}\right)}{L44n}An \quad \text{(D.82)}$$

ou

$$Bn = MAn \quad \text{(D.83)}$$

com

$$M = -\frac{\left(L45n + \frac{L46nL555}{L556}\right)}{L44n} \quad \text{(D.84)}$$

Da linha 3:

$$\alpha_{43}Dn + \alpha_{44}Bn + \alpha_{45}An + \alpha_{46}Cn = 0 =>$$

$$\alpha_{43}Dn - \alpha_{44}\left(\frac{\left(L45n + \frac{L46nL555}{L556}\right)}{L44n}\right)An + \alpha_{45}An - \alpha_{46}\left(\frac{L555n}{L556n}An\right) = 0 \quad \text{(D.85)}$$

$$Dn = \frac{\alpha_{44}\left(\frac{\left(L45n + \frac{L46nL555}{L556}\right)}{L44n}\right) - \alpha_{45} + \alpha_{46}\left(\frac{L555n}{L556n}\right)}{\alpha_{43}}An$$

ou

$$Dn = PAn \quad \text{(D.86)}$$

com

$$P = \frac{\alpha_{44}\left(\frac{\left(L45n + \frac{L46nL555}{L556}\right)}{L44n}\right) - \alpha_{45} + \alpha_{46}\left(\frac{L555n}{L556n}\right)}{\alpha_{43}} \quad \text{(D.87)}$$

Da linha 2:

$$\alpha_{62}Fn + \alpha_{65}An + \alpha_{66}Cn = 0 => \alpha_{62}Fn + \alpha_{65}An - \alpha_{66}\left(\frac{L555n}{L556n}An\right) = 0$$

$$=> Fn = \frac{\alpha_{66}\left(\frac{L555n}{L556n}\right) - \alpha_{65}}{\alpha_{62}}An \quad \text{(D.88)}$$

ou

$$Fn = LAn \quad \text{(D.89)}$$

com

$$L = \frac{\alpha_{66}\left(\frac{L555n}{L556n}\right) - \alpha_{65}}{\alpha_{62}} \tag{D.90}$$

Da linha 1:

$$\alpha_{51}En + \alpha_{52}Fn + \alpha_{53}Dn + \alpha_{54}Bn + \alpha_{55}An + \alpha_{56}Cn = 0 \Rightarrow$$

$$\alpha_{51}En + \alpha_{52}\left(\frac{\alpha_{66}\left(\frac{L555n}{L556n}\right) - \alpha_{65}}{\alpha_{62}}\right)An + \alpha_{53}\left(\frac{\alpha_{44}\left(\frac{\left(L45n + \frac{L46nL555}{L556}\right)}{L44n}\right) - \alpha_{45} + \alpha_{46}\left(\frac{L555n}{L556n}\right)}{\alpha_{43}}\right)An - \tag{D.91}$$

$$\alpha_{54}\frac{\left(L45n + \frac{L46nL555}{L556}\right)}{L44n}An + \alpha_{55}An - \alpha_{56}\frac{L555n}{L556n}An = 0 \Rightarrow$$

logo:

$$En = \frac{An}{\alpha_{51}}\left\{\begin{array}{l} -\alpha_{52}\left(\frac{\alpha_{66}\left(\frac{L555n}{L556n}\right) - \alpha_{65}}{\alpha_{62}}\right) - \alpha_{53}\left(\frac{\alpha_{44}\left(\frac{\left(L45n + \frac{L46nL555}{L556}\right)}{L44n}\right) - \alpha_{45} + \alpha_{46}\left(\frac{L555n}{L556n}\right)}{\alpha_{43}}\right) + \\ \alpha_{54}\frac{\left(L45n + \frac{L46nL555}{L556}\right)}{L44n} - \alpha_{55} + \alpha_{56}\frac{L555n}{L556n} \end{array}\right\} \tag{D.92}$$

ou

$$En = QAn \tag{D.93}$$

com

$$Q = \left\{\begin{array}{l} -\alpha_{52}\left(\frac{\alpha_{66}\left(\frac{L555n}{L556n}\right) - \alpha_{65}}{\alpha_{62}}\right) - \alpha_{53}\left(\frac{\alpha_{44}\left(\frac{\left(L45n + \frac{L46nL555}{L556}\right)}{L44n}\right) - \alpha_{45} + \alpha_{46}\left(\frac{L555n}{L556n}\right)}{\alpha_{43}}\right) + \\ \alpha_{54}\frac{\left(L45n + \frac{L46nL555}{L556}\right)}{L44n} - \alpha_{55} + \alpha_{56}\frac{L555n}{L556n} \end{array}\right\}\frac{1}{\alpha_{51}} \tag{D.94}$$

APÊNDICE E
Desenvolvimento de Integrais de Campos usadas [10,14]

E.1
Cos(nϕ)sen(ϕ-ϕ')

$$\int_0^{2\pi} \cos(n\phi')\,\mathrm{sen}(\phi-\phi')e^{jkr'\mathrm{sen}\,\theta\cos(\phi-\phi')}d\phi' =$$

$$\alpha = \phi-\phi' => \phi' = \phi-\alpha$$

$$d\alpha = -d\phi'$$

$$= -\int_{\phi}^{\phi-2\pi} \cos n(\phi-\alpha)\,\mathrm{sen}\,\alpha e^{jkr'\mathrm{sen}\,\theta\cos\alpha}d\alpha =$$

$$-\int_{\phi}^{\phi-2\pi} \cos n\phi\cos n\alpha\,\mathrm{sen}\,\alpha e^{jkr'\mathrm{sen}\,\theta\cos\alpha}d\alpha -$$

$$\int_{\phi}^{\phi-2\pi} \sin n\phi\sin(n\alpha)\,\mathrm{sen}\,\alpha e^{jkr'\mathrm{sen}\,\theta\cos\alpha}d\alpha =$$

$$= -\cos n\phi\int_0^{2\pi} \frac{(e^{jn\alpha}+e^{-jn\alpha})}{2}\frac{(e^{j\alpha}-e^{-j\alpha})}{2j}e^{jkr'\mathrm{sen}\,\theta\cos\alpha}d\alpha -$$

$$\mathrm{sen}\,n\phi\int_0^{2\pi} \frac{(e^{jn\alpha}-e^{-jn\alpha})}{2j}\frac{(e^{j\alpha}-e^{-j\alpha})}{2j}e^{jkr'\mathrm{sen}\,\theta\cos\alpha}d\alpha =$$

$$= -\frac{\cos n\phi}{4j}\int_0^{2\pi} \begin{matrix}(e^{j\alpha(n+1)}+e^{-j\alpha(n-1)}\\ -e^{j\alpha(n-1)}-e^{-j\alpha(n+1)})\end{matrix} e^{jkr'\mathrm{sen}\,\theta\cos\alpha}d\alpha -$$

$$\frac{\mathrm{sen}\,n\phi}{4}\int_0^{2\pi} \begin{matrix}(e^{j\alpha(n+1)}-e^{-j\alpha(n-1)}\\ -e^{j\alpha(n-1)}+e^{-j\alpha(n+1)})\end{matrix} e^{jkr'\mathrm{sen}\,\theta\cos\alpha}d\alpha =$$

$$= -\frac{\cos n\phi}{4j}\left\{\begin{array}{l}\frac{2\pi}{j^{-(n+1)}}J_{(n+1)}(kr'\mathrm{sen}\,\theta)+\frac{2\pi}{j^{(n-1)}}J_{-(n-1)}(kr'\mathrm{sen}\,\theta)\\ -\frac{2\pi}{j^{-(n-1)}}J_{(n-1)}(kr'\mathrm{sen}\,\theta)-\frac{2\pi}{j^{(n+1)}}J_{-(n+1)}(kr'\mathrm{sen}\,\theta)\end{array}\right\}-$$

$$\frac{\mathrm{sen}\,n\phi}{4}\left\{\begin{array}{l}\frac{2\pi}{j^{-(n+1)}}J_{(n+1)}(kr'\mathrm{sen}\,\theta)-\frac{2\pi}{j^{(n-1)}}J_{-(n-1)}(kr'\mathrm{sen}\,\theta)\\ -\frac{2\pi}{j^{-(n-1)}}J_{(n-1)}(kr'\mathrm{sen}\,\theta)+\frac{2\pi}{j^{(n+1)}}J_{-(n+1)}(kr'\mathrm{sen}\,\theta)\end{array}\right\}=$$

$$= -\frac{2\pi\cos n\phi}{4j}\left\{\begin{array}{l}\frac{1}{j^{-(n+1)}}J_{(n+1)}(kr'\mathrm{sen}\,\theta)+\frac{(-1)^{(n-1)}}{j^{(n-1)}}J_{(n-1)}(kr'\mathrm{sen}\,\theta)\\ -\frac{1}{j^{-(n-1)}}J_{(n-1)}(kr'\mathrm{sen}\,\theta)-\frac{(-1)^{(n+1)}}{j^{(n+1)}}J_{(n+1)}(kr'\mathrm{sen}\,\theta)\end{array}\right\}-$$

$$\frac{2\pi\,\mathrm{sen}\,n\phi}{4}\left\{\begin{array}{l}\frac{1}{j^{-(n+1)}}J_{(n+1)}(kr'\mathrm{sen}\,\theta)-\frac{(-1)^{(n-1)}}{j^{(n-1)}}J_{(n-1)}(kr'\mathrm{sen}\,\theta)\\ -\frac{1}{j^{-(n-1)}}J_{(n-1)}(kr'\mathrm{sen}\,\theta)+\frac{(-1)^{(n+1)}}{j^{(n+1)}}J_{(n+1)}(kr'\mathrm{sen}\,\theta)\end{array}\right\}=$$

$$= -\frac{\pi\cos n\phi}{2j}\left\{\left(\frac{1}{j^{-(n+1)}}-\frac{(-1)^{(n+1)}}{j^{(n+1)}}\right)J_{(n+1)}(kr'\mathrm{sen}\,\theta)+\left(\frac{(-1)^{(n-1)}}{j^{(n-1)}}-\frac{1}{j^{-(n-1)}}\right)J_{(n-1)}(kr'\mathrm{sen}\,\theta)\right\}$$

$$-\frac{\pi\,\mathrm{sen}\,n\phi}{2}\left\{\left(\frac{1}{j^{-(n+1)}}+\frac{(-1)^{(n+1)}}{j^{(n+1)}}\right)J_{(n+1)}(kr'\mathrm{sen}\,\theta)-\left(\frac{(-1)^{(n-1)}}{j^{(n-1)}}+\frac{1}{j^{-(n-1)}}\right)J_{(n-1)}(kr'\mathrm{sen}\,\theta)\right\}$$

ou

$$\int_0^{2\pi}\cos(n\phi')\sin(\phi-\phi')e^{jkr'\operatorname{sen}\theta\cos(\phi-\phi')}d\phi' = -\frac{\pi\cos n\phi}{2j}\left\{\left(\frac{1}{j^{-(n+1)}}-\frac{(-1)^{(n+1)}}{j^{(n+1)}}\right)J_{(n+1)}(kr'\operatorname{sen}\theta)+\left(\frac{(-1)^{(n-1)}}{j^{(n-1)}}-\frac{1}{j^{-(n-1)}}\right)J_{(n-1)}(kr'\operatorname{sen}\theta)\right\}+$$

$$+\frac{\pi\sin n\phi}{2}\left\{\left(\frac{1}{j^{-(n+1)}}+\frac{(-1)^{(n+1)}}{j^{(n+1)}}\right)J_{(n+1)}(kr'\operatorname{sen}\theta)-\left(\frac{(-1)^{(n-1)}}{j^{(n-1)}}+\frac{1}{j^{-(n-1)}}\right)J_{(n-1)}(kr'\operatorname{sen}\theta)\right\}=$$

$$=\frac{\pi}{2}\left\{\begin{array}{l}-\cos n\phi\left\{\left(j^{(n)}+\frac{(j)^{(2n)}}{j^{(n+2)}}\right)J_{(n+1)}(kr'\operatorname{sen}\theta)+\left(-\frac{(j)^{(2n)}}{j^{(n)}}-j^{(n-2)}\right)J_{(n-1)}(kr'\operatorname{sen}\theta)\right\}+\\ \sin n\phi\left\{\left(j^{(n+1)}-\frac{(j)^{(2n)}}{j^{(n+1)}}\right)\left\{J_{(n+1)}(kr'\operatorname{sen}\theta)\right\}-\left(-\frac{(j)^{(2n)}}{j^{(n-1)}}+j^{(n-1)}\right)J_{(n-1)}(kr'\operatorname{sen}\theta)\right\}\end{array}\right\}=$$

$$\frac{\pi}{2}\left\{-\cos n\phi\left(j^{(n)}+j^{(n-2)}\right)\left\{J_{(n+1)}(kr'\operatorname{sen}\theta)-J_{(n-1)}(kr'\operatorname{sen}\theta)\right\}+\operatorname{sen} n\phi\left(j^{(n+1)}-j^{(n-1)}\right)\left\{J_{(n+1)}(kr'\operatorname{sen}\theta)+J_{(n-1)}(kr'\operatorname{sen}\theta)\right\}\right\}=$$

$$=\frac{\pi}{2}\left\{-\cos n\phi\left(j^{(n)}-j^{(n)}\right)\left\{J_{(n+1)}(kr'\operatorname{sen}\theta)+-J_{(n-1)}(kr'\operatorname{sen}\theta)\right\}+\operatorname{sen} n\phi\left(j^{(n+1)}-j^{(n-1)}\right)\left\{J_{(n+1)}(kr'\operatorname{sen}\theta)+J_{(n-1)}(kr'\operatorname{sen}\theta)\right\}\right\}=$$

$$\frac{\pi}{2}\sin n\phi\left(j^{(n+1)}-j^{(n-1)}\right)\left\{J_{(n+1)}(kr'\operatorname{sen}\theta)+J_{(n-1)}(kr'\operatorname{sen}\theta)\right\}==\frac{\pi}{2}\operatorname{sen} n\phi\, j^{(n)}\left(\frac{-1-1}{j}\right)\left\{J_{(n+1)}(kr'\operatorname{sen}\theta)+J_{(n-1)}(kr'\operatorname{sen}\theta)\right\}=$$

$$-\pi\sin n\phi\, j^{(n-1)}\left\{J_{(n+1)}(kr'\operatorname{sen}\theta)+J_{(n-1)}(kr'\operatorname{sen}\theta)\right\}=-\frac{2\pi\operatorname{sen} n\phi\, j^{(n-1)}}{k\sin\theta}J_{(n)}(kr'\operatorname{sen}\theta)$$

Então

$$\int_0^{2\pi}\cos(n\phi')\operatorname{sen}(\phi-\phi')e^{jkr'\operatorname{sen}\theta\cos(\phi-\phi')}d\phi' = -\frac{2\pi\operatorname{sen} n\phi\, j^{(n-1)}}{k\operatorname{sen}\theta}J_{(n)}(kr'\operatorname{sen}\theta) \quad \text{(E.1)}$$

E.2

Cos(nϕ)cos(ϕ-ϕ’)

$$\int_0^{2\pi}\cos(n\phi')\cos(\phi-\phi')e^{jkr'\operatorname{sen}\theta\cos(\phi-\phi')}d\phi' =$$

$$\alpha=\phi-\phi' => \phi'=\phi-\alpha$$

$$d\alpha=-d\phi'$$

$$=-\int_\phi^{\phi-2\pi}\cos n(\phi-\alpha)\cos\alpha e^{jkr'\operatorname{sen}\theta\cos\alpha}d\alpha=-\int_\phi^{\phi-2\pi}\cos n\phi\cos n\alpha\cos\alpha e^{jkr'\operatorname{sen}\theta\cos\alpha}d\alpha-\int_\phi^{\phi-2\pi}\operatorname{sen} n\phi\operatorname{sen}(n\alpha)\cos\alpha e^{jkr'\operatorname{sen}\theta\cos\alpha}d\alpha=$$

$$=\cos n\phi\int_0^{2\pi}\frac{(e^{jn\alpha}+e^{-jn\alpha})}{2}\frac{(e^{j\alpha}+e^{-j\alpha})}{2}e^{jkr'\operatorname{sen}\theta\cos\alpha}d\alpha-\operatorname{sen} n\phi\int_0^{2\pi}\frac{(e^{jn\alpha}-e^{-jn\alpha})}{2j}\frac{(e^{j\alpha}+e^{-j\alpha})}{2}e^{jkr'\operatorname{sen}\theta\cos\alpha}d\alpha=$$

$$=\frac{\cos n\phi}{4}\int_0^{2\pi}(e^{jn\alpha}+e^{-jn\alpha})(e^{j\alpha}+e^{-j\alpha})e^{jkr'\operatorname{sen}\theta\cos\alpha}d\alpha-\frac{\operatorname{sen} n\phi}{4j}\int_0^{2\pi}(e^{jn\alpha}-e^{-jn\alpha})(e^{j\alpha}+e^{-j\alpha})e^{jkr'\operatorname{sen}\theta\cos\alpha}d\alpha=$$

$$=\frac{\cos n\phi}{4}\int_0^{2\pi}(e^{j\alpha(n+1)}+e^{-j\alpha(n-1)}+e^{j\alpha(n-1)}+e^{-j\alpha(n+1)})e^{jkr'\sin\theta\cos\alpha}d\alpha-\frac{\operatorname{sen} n\phi}{4j}\int_0^{2\pi}(e^{j\alpha(n+1)}-e^{-j\alpha(n-1)}+e^{j\alpha(n-1)}-e^{-j\alpha(n+1)})e^{jkr'\operatorname{sen}\theta\cos\alpha}d\alpha=$$

$$=\frac{\cos n\phi}{4}\left\{\frac{2\pi}{j^{-(n+1)}}J_{(n+1)}(kr'\operatorname{sen}\theta)+\frac{2\pi}{j^{(n-1)}}J_{-(n-1)}(kr'\operatorname{sen}\theta)+\frac{2\pi}{j^{-(n-1)}}J_{(n-1)}(kr'\operatorname{sen}\theta)+\frac{2\pi}{j^{(n+1)}}J_{-(n+1)}(kr'\operatorname{sen}\theta)\right\}$$

$$-\frac{\operatorname{sen} n\phi}{4j}\left\{\frac{2\pi}{j^{-(n+1)}}J_{(n+1)}(kr'\operatorname{sen}\theta)-\frac{2\pi}{j^{(n-1)}}J_{-(n-1)}(kr'\operatorname{sen}\theta)+\frac{2\pi}{j^{-(n-1)}}J_{(n-1)}(kr'\operatorname{sen}\theta)-\frac{2\pi}{j^{(n+1)}}J_{-(n+1)}(kr'\operatorname{sen}\theta)\right\}=$$

$$=\frac{2\pi\cos n\phi}{4}\left\{\frac{1}{j^{-(n+1)}}J_{(n+1)}(kr'\operatorname{sen}\theta)+\frac{(-1)^{(n-1)}}{j^{(n-1)}}J_{(n-1)}(kr'\operatorname{sen}\theta)+\frac{1}{j^{-(n-1)}}J_{(n-1)}(kr'\operatorname{sen}\theta)+\frac{(-1)^{(n+1)}}{j^{(n+1)}}J_{(n+1)}(kr'\sin\theta)\right\}$$

$$-\frac{2\pi\sin n\phi}{4j}\left\{\frac{1}{j^{-(n+1)}}J_{(n+1)}(kr'\operatorname{sen}\theta)-\frac{(-1)^{(n-1)}}{j^{(n-1)}}J_{(n-1)}(kr'\operatorname{sen}\theta)+\frac{1}{j^{-(n-1)}}J_{(n-1)}(kr'\operatorname{sen}\theta)-\frac{(-1)^{(n+1)}}{j^{(n+1)}}J_{(n+1)}(kr'\operatorname{sen}\theta)\right\}=$$

ou

$$= \frac{\pi\cos n\phi}{2}\left\{\left(\frac{1}{j^{-(n+1)}}+\frac{(-1)^{(n+1)}}{j^{(n+1)}}\right)J_{(n+1)}(kr'\mathrm{sen}\,\theta)+\left(\frac{1}{j^{-(n-1)}}+\frac{(-1)^{(n-1)}}{j^{(n-1)}}\right)J_{(n-1)}(kr'\mathrm{sen}\,\theta)\right\}$$

$$-\frac{\pi\sin n\phi}{2j}\left\{\left(\frac{1}{j^{-(n+1)}}-\frac{(-1)^{(n+1)}}{j^{(n+1)}}\right)J_{(n+1)}(kr'\mathrm{sen}\,\theta)+\left(\frac{1}{j^{-(n-1)}}-\frac{(-1)^{(n-1)}}{j^{(n-1)}}\right)J_{(n-1)}(kr'\mathrm{sen}\,\theta)\right\}=$$

$$= \frac{\pi\cos n\phi}{2}\left\{\left(j^{(n+1)}-\frac{(-1)^{(n)}}{j^{(n+1)}}\right)J_{(n+1)}(kr'\mathrm{sen}\,\theta)+\left(j^{(n-1)}-\frac{(-1)^{(n)}}{j^{(n-1)}}\right)J_{(n-1)}(kr'\mathrm{sen}\,\theta)\right\}+$$

$$-\frac{\pi\,\mathrm{sen}\,n\phi}{2}\left\{\left(j^{n}+\frac{(-1)^{(n)}}{j^{(n+2)}}\right)J_{(n+1)}(kr'\mathrm{sen}\,\theta)+\left(j^{(n-2)}+\frac{(-1)^{(n)}}{j^{n}}\right)J_{(n-1)}(kr'\mathrm{sen}\,\theta)\right\}=$$

$$= \frac{\pi\cos n\phi}{2}\left\{\left(j^{(n+1)}-\frac{(j)^{(2n)}}{j^{(n+1)}}\right)J_{(n+1)}(kr'\mathrm{sen}\,\theta)+\left(j^{(n-1)}-\frac{(j)^{(2n)}}{j^{(n-1)}}\right)J_{(n-1)}(kr'\mathrm{sen}\,\theta)\right\}+$$

$$-\frac{\pi\,\mathrm{sen}\,n\phi}{2}\left\{\left(j^{n}+\frac{(j)^{(2n)}}{j^{(n+2)}}\right)J_{(n+1)}(kr'\mathrm{sen}\,\theta)+\left(j^{(n-2)}+\frac{(j)^{(2n)}}{j^{n}}\right)J_{(n-1)}(kr'\mathrm{sen}\,\theta)\right\}=$$

$$= \frac{\pi\cos n\phi}{2}\left\{\left(j^{(n+1)}-j^{(n-1)}\right)J_{(n+1)}(kr'\mathrm{sen}\,\theta)+\left(j^{(n-1)}-j^{(n+1)}\right)J_{(n-1)}(kr'\mathrm{sen}\,\theta)\right\}+$$

$$-\frac{\pi\sin n\phi}{2}\left\{\left(j^{n}-j^{(n)}\right)J_{(n+1)}(kr'\mathrm{sen}\,\theta)+\left(-j^{(n)}+j^{(n)}\right)J_{(n-1)}(kr'\mathrm{sen}\,\theta)\right\}=$$

$$= \frac{\pi\cos n\phi}{2}\left(j^{(n+1)}-j^{(n-1)}\right)\left\{J_{(n+1)}(kr'\mathrm{sen}\,\theta)-J_{(n-1)}(kr'\mathrm{sen}\,\theta)\right\}=-\pi\cos n\phi\left(j^{(n+1)}-j^{(n-1)}\right)J'_{(n)}(kr'\mathrm{sen}\,\theta)=$$

$$= 2\pi\cos n\phi\, j^{(n-1)}J'_{(n)}(kr'\mathrm{sen}\,\theta)$$

logo:

$$\int_0^{2\pi}\cos(n\phi')\cos(\phi-\phi')e^{jkr'\mathrm{sen}\,\theta\cos(\phi-\phi')}d\phi' = 2\pi\cos n\phi\, j^{(n-1)}J_n'(kr'\mathrm{sen}\,\theta) \qquad \text{(E.2)}$$

E.3
Sen(nϕ’)sen(ϕ-ϕ’)

$$\int_0^{2\pi}\mathrm{sen}\,n\phi'\,\mathrm{sen}(\phi-\phi')e^{jkr'\mathrm{sen}\,\theta\cos(\phi-\phi')}d\phi' = ?$$

$$\alpha=\phi-\phi' \Rightarrow \phi'=\phi-\alpha$$

$$d\alpha=-d\phi'$$

$$x=kr'\mathrm{sen}\,\theta$$

$$=-\int_{\phi}^{\phi-2\pi}\mathrm{sen}(n(\phi-\alpha))\,\mathrm{sen}\,\alpha e^{jkr'\sin\theta\cos\alpha}d\alpha =$$

$$\mathrm{sen}(a-b)=\mathrm{sen}\,a\cos b-\cos a\,\mathrm{sen}\,b \Rightarrow$$

$$=-\int_{\phi}^{\phi-2\pi}\mathrm{sen}(n\phi)\cos(n\alpha)\,\mathrm{sen}\,\alpha e^{jx\cos\alpha}d\alpha+\int_{\phi}^{\phi-2\pi}\cos n\phi\,\mathrm{sen}(n\alpha)\,\mathrm{sen}\,\alpha e^{jx\cos\alpha}d\alpha =$$

$$=-\mathrm{sen}(n\phi)\int_{\phi}^{\phi-2\pi}\cos(n\alpha)\,\mathrm{sen}\,\alpha e^{jx\cos\alpha}d\alpha+\cos(n\phi)\int_{\phi}^{\phi-2\pi}\mathrm{sen}(n\alpha)\,\mathrm{sen}\,\alpha e^{jx\cos\alpha}d\alpha =$$

mas

$$\cos(n\alpha)\,\mathrm{sen}\,\alpha = \frac{1}{2}\left\{\mathrm{sen}(n\alpha+\alpha)-\mathrm{sen}(n\alpha-\alpha)\right\} = \frac{1}{2}\left\{\frac{(e^{j\alpha(n+1)}-e^{-j\alpha(n+1)})}{2j}-\frac{(e^{j\alpha(n-1)}-e^{-j\alpha(n-1)})}{2j}\right\}$$

$$\mathrm{sen}(n\alpha)\,\mathrm{sen}\,\alpha = \frac{1}{2}\left\{-\cos(n\alpha+\alpha)+\cos(n\alpha-\alpha)\right\} = \frac{1}{2}\left\{-\frac{(e^{j\alpha(n+1)}+e^{-j\alpha(n+1)})}{2}+\frac{(e^{j\alpha(n-1)}+e^{-j\alpha(n-1)})}{2}\right\}$$

então

$$\int_0^{2\pi}\mathrm{sen}\,n\phi'\,\mathrm{sen}(\phi-\phi')e^{jkr'\mathrm{sen}\theta\cos(\phi-\phi')}d\phi' = -\frac{\pi}{2j}\mathrm{sen}(n\phi)\begin{pmatrix}\frac{1}{j^{-(n+1)}}J_{(n+1)}(u)-\frac{1}{j^{(n+1)}}J_{-(n+1)}(u)- \\ \frac{1}{j^{-(n-1)}}J_{(n-1)}(u)+\frac{1}{j^{(n-1)}}J_{-(n-1)}(u)\end{pmatrix}+$$

$$+\frac{\pi}{2}\cos(n\phi)\begin{pmatrix}-\frac{1}{j^{-(n+1)}}J_{(n+1)}(u)-\frac{1}{j^{(n+1)}}J_{-(n+1)}(u)+ \\ \frac{1}{j^{-(n-1)}}J_{(n-1)}(u)+\frac{1}{j^{(n-1)}}J_{-(n-1)}(u)\end{pmatrix}$$

mas

$$J_{-(n+1)}(u) = (-1)^{(n+1)}J_{(n+1)}(u)$$

logo

$$\int_0^{2\pi}\mathrm{sen}\,n\phi'\,\mathrm{sen}(\phi-\phi')e^{jkr'\mathrm{sen}\theta\cos(\phi-\phi')}d\phi' = \frac{\pi}{2}\left\{\begin{matrix}\frac{1}{j}\mathrm{sen}(n\phi)\begin{bmatrix}\left(-\frac{1}{j^{-(n+1)}}+\frac{(-1)^{(n+1)}}{j^{(n+1)}}\right)J_{(n+1)}(u)+ \\ \left(\frac{1}{j^{-(n-1)}}-\frac{(-1)^{(n-1)}}{j^{(n-1)}}\right)J_{(n-1)}(u)\end{bmatrix}+ \\ +\cos(n\phi)\begin{bmatrix}\left(-\frac{1}{j^{-(n+1)}}-\frac{(-1)^{(n+1)}}{j^{(n+1)}}\right)J_{(n+1)}(u)+ \\ \left(\frac{1}{j^{-(n-1)}}+\frac{(-1)^{(n-1)}}{j^{(n-1)}}\right)J_{(n-1)}(u)\end{bmatrix}\end{matrix}\right\}=$$

$$=\frac{\pi}{2}\left\{\begin{matrix}\frac{1}{j}\mathrm{sen}(n\phi)\left[\left(-\frac{1}{j^{-(n+1)}}+\frac{j^{2(n+1)}}{j^{(n+1)}}\right)J_{(n+1)}(u)+\left(\frac{1}{j^{-(n-1)}}-\frac{j^{2(n-1)}}{j^{(n-1)}}\right)J_{(n-1)}(u)\right]+ \\ +\cos(n\phi)\left[\left(-\frac{1}{j^{-(n+1)}}-\frac{j^{2(n+1)}}{j^{(n+1)}}\right)J_{(n+1)}(u)+\left(\frac{1}{j^{-(n-1)}}+\frac{j^{2(n-1)}}{j^{(n-1)}}\right)J_{(n-1)}(u)\right]\end{matrix}\right\}=$$

$$=\frac{\pi}{2}\left\{\begin{matrix}\frac{1}{j}\mathrm{sen}(n\phi)\left[\left(-j^{(n+1)}+j^{(n+1)}\right)J_{(n+1)}(u)+\left(j^{(n-1)}-j^{(n-1)}\right)J_{(n-1)}(u)\right]+ \\ +\cos(n\phi)\left[\left(-j^{(n+1)}-j^{(n+1)}\right)J_{(n+1)}(u)+\left(j^{(n-1)}+j^{(n-1)}\right)J_{(n-1)}(u)\right]\end{matrix}\right\}=$$

$$=\frac{\pi}{2}\left\{\cos(n\phi)\left[\left(-j^{(n+1)}-j^{(n+1)}\right)J_{(n+1)}(u)+\left(j^{(n-1)}+j^{(n-1)}\right)J_{(n-1)}(u)\right]\right\}=$$

$$=\frac{\pi}{2}\left\{\cos(n\phi)\left[-2j^{(n+1)}J_{(n+1)}(u)+2j^{(n-1)}J_{(n-1)}(u)\right]\right\}=$$

$$=\pi j^{n}\cos(n\phi)\left[-jJ_{(n+1)}(u)+j^{-1}J_{(n-1)}(u)\right]=$$

$$=\pi j^{n-1}\cos(n\phi)\left[J_{(n+1)}(u)+J_{(n-1)}(u)\right]=$$

$$=2\pi j^{n-1}\cos(n\phi)\frac{J_n(u)}{k\,\mathrm{sen}\,\theta}$$

Portanto:

$$\int_0^{2\pi} \operatorname{sen}(n\phi')\operatorname{sen}(\phi-\phi')e^{jkr'\operatorname{sen}\theta\cos(\phi-\phi')}d\phi' = 2\pi j^{n-1}\cos(n\phi)\frac{J_n(kr'\operatorname{sen}\theta)}{k\operatorname{sen}\theta} \quad \text{(E.3)}$$

E.4
Sen(nϕ)cos(ϕ-ϕ')

$$\int_0^{2\pi} \operatorname{sen}(n\phi')\cos(\phi-\phi')e^{jkr'\operatorname{sen}\theta\cos(\phi-\phi')}d\phi' =$$

$$\alpha = \phi-\phi' => \phi' = \phi-\alpha$$

$$d\alpha = -d\phi'$$

$$= -\int_\phi^{\phi-2\pi} \operatorname{sen} n(\phi-\alpha)\cos\alpha e^{jkr'\operatorname{sen}\theta\cos\alpha}d\alpha = -\int_\phi^{\phi-2\pi} \operatorname{sen} n\phi\cos n\alpha\cos\alpha e^{jkr'\operatorname{sen}\theta\cos\alpha}d\alpha + \int_\phi^{\phi-2\pi} \cos n\phi\operatorname{sen}(n\alpha)\cos\alpha e^{jkr'\sin\theta\cos\alpha}d\alpha =$$

$$= -\operatorname{sen} n\phi\int_0^{2\pi} \frac{(e^{jn\alpha}+e^{-jn\alpha})}{2}\frac{(e^{j\alpha}+e^{-j\alpha})}{2}e^{jkr'\operatorname{sen}\theta\cos\alpha}d\alpha + \cos n\phi\int_0^{2\pi} \frac{(e^{jn\alpha}-e^{-jn\alpha})}{2j}\frac{(e^{j\alpha}+e^{-j\alpha})}{2}e^{jkr'\operatorname{sen}\theta\cos\alpha}d\alpha =$$

$$= -\frac{\operatorname{sen} n\phi}{4}\int_0^{2\pi}(e^{jn\alpha}+e^{-jn\alpha})(e^{j\alpha}+e^{-j\alpha})e^{jkr'\operatorname{sen}\theta\cos\alpha}d\alpha + \frac{\cos n\phi}{4j}\int_0^{2\pi}(e^{jn\alpha}-e^{-jn\alpha})(e^{j\alpha}+e^{-j\alpha})e^{jkr'\operatorname{sen}\theta\cos\alpha}d\alpha =$$

$$= -\frac{\operatorname{sen} n\phi}{4}\int_0^{2\pi}(e^{j\alpha(n+1)}+e^{-j\alpha(n-1)}+e^{j\alpha(n-1)}+e^{-j\alpha(n+1)})e^{jkr'\operatorname{sen}\theta\cos\alpha}d\alpha + \frac{\cos n\phi}{4j}\int_0^{2\pi}(e^{j\alpha(n+1)}-e^{-j\alpha(n-1)}+e^{j\alpha(n-1)}-e^{-j\alpha(n+1)})e^{jkr'\operatorname{sen}\theta\cos\alpha}d\alpha =$$

$$= -\frac{\operatorname{sen} n\phi}{4}\left\{\frac{2\pi}{j^{-(n+1)}}J_{(n+1)}(kr'\operatorname{sen}\theta) + \frac{2\pi}{j^{(n-1)}}J_{-(n-1)}(kr'\operatorname{sen}\theta) + \frac{2\pi}{j^{-(n-1)}}J_{(n-1)}(kr'\operatorname{sen}\theta) + \frac{2\pi}{j^{(n+1)}}J_{-(n+1)}(kr'\operatorname{sen}\theta)\right\} +$$

$$+\frac{\cos n\phi}{4j}\left\{\frac{2\pi}{j^{-(n+1)}}J_{(n+1)}(kr'\operatorname{sen}\theta) - \frac{2\pi}{j^{(n-1)}}J_{-(n-1)}(kr'\operatorname{sen}\theta) + \frac{2\pi}{j^{-(n-1)}}J_{(n-1)}(kr'\operatorname{sen}\theta) - \frac{2\pi}{j^{(n+1)}}J_{-(n+1)}(kr'\operatorname{sen}\theta)\right\} =$$

$$= -\frac{2\pi\operatorname{sen} n\phi}{4}\left\{\frac{1}{j^{-(n+1)}}J_{(n+1)}(kr'\operatorname{sen}\theta) + \frac{(-1)^{(n-1)}}{j^{(n-1)}}J_{(n-1)}(kr'\operatorname{sen}\theta) + \frac{1}{j^{-(n-1)}}J_{(n-1)}(kr'\operatorname{sen}\theta) + \frac{(-1)^{(n+1)}}{j^{(n+1)}}J_{(n+1)}(kr'\operatorname{sen}\theta)\right\} +$$

$$+\frac{2\pi\cos n\phi}{4}\left\{\frac{1}{j^{-(n+1)}}J_{(n+1)}(kr'\operatorname{sen}\theta) - \frac{(-1)^{(n-1)}}{j^{(n-1)}}J_{(n-1)}(kr'\operatorname{sen}\theta) + \frac{1}{j^{-(n-1)}}J_{(n-1)}(kr'\operatorname{sen}\theta) - \frac{(-1)^{(n+1)}}{j^{(n+1)}}J_{(n+1)}(kr'\operatorname{sen}\theta)\right\} =$$

$$= -\frac{\pi\operatorname{sen} n\phi}{2}\left\{\left(\frac{1}{j^{-(n+1)}}+\frac{(-1)^{(n+1)}}{j^{(n+1)}}\right)J_{(n+1)}(kr'\operatorname{sen}\theta) + \left(\frac{1}{j^{-(n-1)}}+\frac{(-1)^{(n-1)}}{j^{(n-1)}}\right)J_{(n-1)}(kr'\operatorname{sen}\theta)\right\} +$$

$$+\frac{\pi\cos n\phi}{2j}\left\{\left(\frac{1}{j^{-(n+1)}}-\frac{(-1)^{(n+1)}}{j^{(n+1)}}\right)J_{(n+1)}(kr'\operatorname{sen}\theta) + \left(\frac{1}{j^{-(n-1)}}-\frac{(-1)^{(n-1)}}{j^{(n-1)}}\right)J_{(n-1)}(kr'\operatorname{sen}\theta)\right\} =$$

ou

$$\int_0^{2\pi} \operatorname{sen}(n\phi')\cos(\phi-\phi')e^{jkr'\operatorname{sen}\theta\cos(\phi-\phi')}d\phi' = -\frac{\pi\operatorname{sen} n\phi}{2}\left\{\begin{matrix}\left(j^{(n+1)}-\frac{(-1)^{(n)}}{j^{(n+1)}}\right)J_{(n+1)}(kr'\operatorname{sen}\theta)+\\ \left(j^{(n-1)}-\frac{(-1)^{(n)}}{j^{(n-1)}}\right)J_{(n-1)}(kr'\operatorname{sen}\theta)\end{matrix}\right\} + \frac{\pi\cos n\phi}{2j}\left\{\begin{matrix}\left(j^{(n+1)}+\frac{(-1)^{(n)}}{j^{(n+1)}}\right)J_{(n+1)}(kr'\operatorname{sen}\theta)+\\ \left(j^{(n-1)}+\frac{(-1)^{(n)}}{j^{(n-1)}}\right)J_{(n-1)}(kr'\operatorname{sen}\theta)\end{matrix}\right\} =$$

$$= -\frac{\pi\operatorname{sen} n\phi}{2}\left\{\begin{matrix}\left(j^{(n+1)}-\frac{(j)^{(2n)}}{j^{(n+1)}}\right)J_{(n+1)}(kr'\operatorname{sen}\theta)+\\ \left(j^{(n-1)}-\frac{(j)^{(2n)}}{j^{(n-1)}}\right)J_{(n-1)}(kr'\operatorname{sen}\theta)\end{matrix}\right\} + \frac{\pi\cos n\phi}{2j}\left\{\begin{matrix}\left(j^{(n+1)}+\frac{(j)^{(2n)}}{j^{(n+1)}}\right)J_{(n+1)}(kr'\operatorname{sen}\theta)+\\ \left(j^{(n-1)}+\frac{(j)^{(2n)}}{j^{(n-1)}}\right)J_{(n-1)}(kr'\operatorname{sen}\theta)\end{matrix}\right\} =$$

$$= -\frac{\pi\operatorname{sen} n\phi}{2}\left\{\left(j^{(n+1)}-j^{(n-1)}\right)J_{(n+1)}(kr'\operatorname{sen}\theta) + \left(j^{(n-1)}-j^{(n+1)}\right)J_{(n-1)}(kr'\operatorname{sen}\theta)\right\} +$$

$$+\frac{\pi\cos n\phi}{2j}\left\{\left(j^{(n+1)}+j^{(n-1)}\right)J_{(n+1)}(kr'\operatorname{sen}\theta) + \left(j^{(n-1)}+j^{(n+1)}\right)J_{(n-1)}(kr'\operatorname{sen}\theta)\right\} =$$

ou

$$\int_0^{2\pi} \operatorname{sen}(n\phi')\cos(\phi-\phi')e^{jkr'\operatorname{sen}\theta\cos(\phi-\phi')}d\phi' =$$

$$= -\frac{\pi\operatorname{sen} n\phi}{2}\left(j^{(n+1)}-j^{(n-1)}\right)\left\{J_{(n+1)}(kr'\operatorname{sen}\theta) - J_{(n-1)}(kr'\operatorname{sen}\theta)\right\} + \frac{\pi\cos n\phi}{2j}\left(j^{(n+1)}+j^{(n-1)}\right)\left\{J_{(n+1)}(kr'\operatorname{sen}\theta) + J_{(n-1)}(kr'\operatorname{sen}\theta)\right\} =$$

$$= -\frac{\pi\operatorname{sen} n\phi}{2}\left(j^{(n+1)}-j^{(n-1)}\right)\left\{J_{(n+1)}(kr'\operatorname{sen}\theta) - J_{(n-1)}(kr'\operatorname{sen}\theta)\right\} = -\frac{\pi\operatorname{sen} n\phi}{2}j^{(n)}\left(j-j^{-1}\right)\left\{J_{(n+1)}(kr'\operatorname{sen}\theta) - J_{(n-1)}(kr'\operatorname{sen}\theta)\right\} =$$

$$= -\frac{\pi\operatorname{sen} n\phi}{2}j^{(n)}\left(-2j^{-1}\right)\left\{J_{(n+1)}(kr'\operatorname{sen}\theta) - J_{(n-1)}(kr'\operatorname{sen}\theta)\right\} = \pi\sin n\phi j^{(n-1)}\left\{J_{(n+1)}(kr'\sin\theta) - J_{(n-1)}(kr'\sin\theta)\right\} =$$

$$= -2\pi\sin n\phi j^{(n-1)}J'_{(n)}(kr'\sin\theta)$$

logo:

$$\int_0^{2\pi} \text{sen}(n\phi')\cos(\phi-\phi')e^{jkr'\text{sen}\,\theta\cos(\phi-\phi')}d\phi' = -2\pi\,\text{sen}\,n\phi\, j^{(n-1)} J_n^{'}(kr'\text{sen}\,\theta) \quad \text{(E.4)}$$

E.5 Obtenção das Integrais para o Caso Especial em que n = 1

Com n=1, a Equação (E.2) pode ser reescrita como:

$$\int_0^{2\pi} \cos(\phi')\cos(\phi-\phi')e^{jkr'\text{sen}\,\theta\cos(\phi-\phi')}d\phi' = 2\pi\cos\phi\, j^{(-1)} J_1^{'}(kr'\text{sen}\,\theta)$$

mas,

$$\int_0^{2\pi} \cos(\phi-\phi')e^{jkr'\text{sen}\,\theta\cos(\phi-\phi')}d\phi' = -\int_\phi^{\phi-2\pi}\cos(\alpha)e^{jkr'\text{sen}\,\theta\cos\alpha}d\alpha = -\frac{1}{2}\int_\phi^{\phi-2\pi}(e^{j\alpha}+e^{-j\alpha})e^{jkr'\text{sen}\,\theta\cos\alpha}d\alpha =$$

$$\phi-\phi' = \alpha => \phi' = \phi-\alpha$$

$$d\phi' = -d\alpha$$

$$= -\frac{1}{2}\left(\int_\phi^{\phi-2\pi} e^{j\alpha}e^{jkr'\text{sen}\,\theta\cos\alpha}d\alpha + \int_\phi^{\phi-2\pi} e^{-j\alpha}e^{jkr'\text{sen}\,\theta\cos\alpha}d\alpha\right) =$$

$$= -2\pi\cos\phi J_2(kr'\text{sen}\,\theta)$$

então:

$$\int_0^{2\pi}\cos 2\phi' e^{jkr'\text{sen}\,\theta\cos(\phi-\phi')}d\phi' = \frac{1}{2}\int_0^{2\pi}(e^{j2\phi'}+e^{-j2\phi'})e^{jkr'\text{sen}\,\theta\cos(\phi-\phi')}d\phi' =$$

$$= \frac{1}{2}\int_0^{2\pi} e^{j2\phi'}e^{jkr'\text{sen}\,\theta\cos(\phi-\phi')}d\phi' + \frac{1}{2}\int_0^{2\pi} e^{-j2\phi'}e^{jkr'\text{sen}\,\theta\cos(\phi-\phi')}d\phi' = \pi J_2(kr'\text{sen}\,\theta)(e^{j2\phi}+e^{-j2\phi}) =$$

$$= 2\pi\cos 2\phi J_2(kr'\text{sen}\,\theta)$$

$$\frac{1}{2}\int_0^{2\pi} e^{j2\phi'}e^{jkr'\text{sen}\,\theta\cos(\phi-\phi')}d\phi' = \frac{1}{2}\int_{-\phi}^{2\pi-\phi} e^{j2\phi}e^{j(kr'\text{sen}\,\theta\cos\beta+2\beta)}d\beta = -\frac{e^{j2\phi}}{2}\int_0^{2\pi} e^{j(kr'\text{sen}\,\theta\cos\beta+2\beta)}d\beta =$$

$$= -\frac{e^{j2\phi}}{2}\int_0^{2\pi} e^{j2\beta}e^{j(kr'\text{sen}\,\theta\cos\beta)}d\beta = -\frac{e^{j2\phi}}{2}\frac{2\pi}{j^{-2}}J_2(kr'\text{sen}\,\theta)e^{-j2\phi} = \pi e^{j2\phi}J_2(kr'\text{sen}\,\theta)$$

$$\beta = \phi'-\phi => \phi' = \beta+\phi$$

$$d\beta = d\phi'$$

$$\phi' = 0 => \beta = -\phi$$

$$\phi'' = 2\pi => \beta = 2\pi-\phi$$

$$\frac{1}{2}\int_0^{2\pi} e^{-j2\phi'}e^{jkr'\text{sen}\,\theta\cos(\phi-\phi')}d\phi' = \frac{1}{2}\int_{-\phi}^{2\pi-\phi} e^{-j2\phi}e^{j(kr'\text{sen}\,\theta\cos\beta-2\beta)}d\beta = -\frac{e^{-j2\phi}}{2}\int_0^{2\pi} e^{j(kr'\text{sen}\,\theta\cos\beta-2\beta)}d\beta =$$

$$= -\frac{e^{-j2\phi}}{2}\int_0^{2\pi} e^{-j2\beta}e^{j(kr'\text{sen}\,\theta\cos\beta)}d\beta = -\frac{e^{-j2\phi}}{2}\frac{2\pi}{j^2}J_{-2}(kr'\text{sen}\,\theta)e^{-j2\phi} = \pi e^{-j2\phi}J_2(kr'\text{sen}\,\theta)$$

$$\frac{j^{-n}}{2\pi}\int_0^{2\pi} e^{jn\beta}e^{jx\cos\beta}d\beta = J_n(x)$$

logo:

$$\int_0^{2\pi} \cos\phi' \cos(\phi-\phi') e^{jkr'\operatorname{sen}\theta\cos(\phi-\phi')} d\phi' = \pi\cos\phi[J_0(kr'\operatorname{sen}\theta) - J_2(kr'\operatorname{sen}\theta)]$$

Do mesmo modo, obtém-se:

$$\int_0^{2\pi} \cos\phi' \operatorname{sen}(\phi-\phi') e^{jkr'\operatorname{sen}\theta\cos(\phi-\phi')} d\phi' = \pi\operatorname{sen}\phi\left(J_0(kr'\operatorname{sen}\theta) + J_2(kr'\operatorname{sen}\theta)\right)$$

$$\int_0^{2\pi} \operatorname{sen}\phi' \cos(\phi-\phi') e^{jkr'\operatorname{sen}\theta\cos(\phi-\phi')} d\phi' = \pi\operatorname{sen}\phi[J_0(kr'\operatorname{sen}\theta) - J_2(kr'\operatorname{sen}\theta)]$$

$$\int_0^{2\pi} \operatorname{sen}\phi' \operatorname{sen}(\phi-\phi') e^{jkr'\operatorname{sen}\theta\cos(\phi-\phi')} d\phi' = -\pi\cos\phi[J_0(kr'\operatorname{sen}\theta) + J_2(kr'\operatorname{sen}\theta)]$$

$$\int_0^{2\pi} \cos 2\alpha e^{ju\cos\alpha} d\alpha = \int_0^{2\pi} \left(\frac{e^{i2\alpha} + e^{-i2\alpha}}{2}\right) e^{ju\cos\alpha} d\alpha = -2\pi J_2(u)$$

$$\int_0^{2\pi} \operatorname{sen} 2\alpha e^{ju\cos\alpha} d\alpha = \int_0^{2\pi} \left(\frac{e^{i2\alpha} - e^{-i2\alpha}}{2j}\right) e^{ju\cos\alpha} d\alpha = 0$$

$$\int_0^{2\pi} e^{i2\alpha} e^{ju\cos\alpha} d\alpha = -2\pi J_2(u)$$

$$\int_0^{2\pi} e^{ju\cos\alpha} d\alpha = -2\pi J_0(u)$$

De [14, (IV-35)]:

$$\int_0^{2\pi} e^{jn\alpha} e^{ju\cos\alpha} d\alpha = \frac{2\pi}{j^{-n}} J_n(u)$$

então:

$$\int_0^{2\pi} e^{jn\alpha} e^{ju\cos\alpha} d\alpha = \frac{2\pi}{j^{-n}} J_n(u)$$

$$\int_0^{2\pi} \cos n\alpha e^{ju\cos\alpha} d\alpha = \int_0^{2\pi} \left(\frac{e^{in\alpha} + e^{-in\alpha}}{2} \right) e^{ju\cos\alpha} d\alpha =$$

$$= \frac{1}{2}\left(\frac{2\pi}{j^{-n}} J_n(u) + \frac{2\pi}{j^{n}} J_{-n}(u) \right) = \frac{1}{2}\left(\frac{2\pi}{j^{-n}} J_n(u) + \frac{2\pi}{j^{n}} (-1)^n J_n(u) \right)$$

$$J_{-n}(X) = (-1)^n J_n(X)$$

$$n = 0 \Rightarrow 2\pi J_0(u)$$

$$n = 1 \Rightarrow \pi J_1(u)\left(\frac{1}{j^{-1}} - \frac{1}{j^{1}} \right) =$$

$$N = 2 \Rightarrow \frac{1}{2}\left(\frac{2\pi}{j^{-2}} J_2(u) + \frac{2\pi}{j^{2}} (-1)^2 J_2(u) \right) = -2\pi J_2(u)$$

ou

$$\int_0^{2\pi} \cos 2\phi' e^{jkr'\operatorname{sen}\theta\cos(\phi-\phi')} d\phi' = 2\pi \cos 2\phi J_2(kr'\operatorname{sen}\theta)$$

E.6
Desenvolvimento para os Casos Apresentados

a)

$$\int_0^{2\pi} \operatorname{sen}^2\phi' e^{jkr'\operatorname{sen}\theta\cos(\phi-\phi')} d\phi' = -\int_\phi^{\phi-2\pi} \operatorname{sen}^2(\phi-\alpha) e^{jkr'\operatorname{sen}\theta\cos\alpha} d\alpha = -\frac{1}{2}\int_\phi^{\phi-2\pi} [1-\cos(2\phi-2\alpha)] e^{jkr'\operatorname{sen}\theta\cos\alpha} d\alpha =$$

$$\alpha = \phi - \phi'$$

$$d\alpha = -d\phi'$$

$$= \frac{1}{2}\int_0^{2\pi} e^{jkr'\operatorname{sen}\theta\cos\alpha} d\alpha + \frac{1}{2}\int_\phi^{\phi-2\pi} \cos 2\phi \cos 2\alpha e^{jkr'\operatorname{sen}\theta\cos\alpha} d\alpha + \frac{1}{2}\int_\phi^{\phi-2\pi} \operatorname{sen} 2\phi \operatorname{sen} 2\alpha e^{jkr'\operatorname{sen}\theta\cos\alpha} d\alpha =$$

$$= -\pi J_0(kr'\operatorname{sen}\theta) + \frac{1}{2}\cos 2\phi \int_\phi^{\phi-2\pi} \frac{(e^{j2\phi'} + e^{-j2\phi'})}{2} e^{jkr'\operatorname{sen}\theta\cos\alpha} d\alpha + \frac{1}{2}\operatorname{sen} 2\phi \int_\phi^{\phi-2\pi} \frac{(e^{j2\phi'} - e^{-j2\phi'})}{2j} e^{jkr'\operatorname{sen}\theta\cos\alpha} d\alpha =$$

$$= -\pi J_0(kr'\operatorname{sen}\theta) - \frac{1}{4}\cos 2\phi \left\{ \int_0^{2\pi} e^{j2\phi'} e^{jkr'\operatorname{sen}\theta\cos\alpha} d\alpha + \int_0^{2\pi} e^{-j2\phi'} e^{jkr'\operatorname{sen}\theta\cos\alpha} d\alpha \right\} +$$

$$\frac{1}{4j}\operatorname{sen} 2\phi \left\{ \int_0^{2\pi} e^{j2\phi'} e^{jkr'\operatorname{sen}\theta\cos\alpha} d\alpha - \int_0^{2\pi} e^{-j2\phi'} e^{jkr'\operatorname{sen}\theta\cos\alpha} d\alpha \right\} =$$

$$= -\pi J_0(kr'\operatorname{sen}\theta) - \frac{1}{4}\cos 2\phi \left\{ -2\pi J_2(kr'\operatorname{sen}\theta) - 2\pi J_2(kr'\operatorname{sen}\theta) \right\} =$$

$$= -\pi J_0(kr'\operatorname{sen}\theta) + \pi \cos 2\phi J_2(kr'\operatorname{sen}\theta)$$

b)

$$\int_0^{2\pi} \cos\phi' \operatorname{sen}(\phi-\phi') e^{jkr'\operatorname{sen}\theta\cos(\phi-\phi')} d\phi' =$$
$$\alpha = \phi - \phi' => \phi' = -(\alpha-\phi)$$
$$d\alpha = -d\phi'$$
$$= -\int_\phi^{\phi-2\pi} \cos(\alpha-\phi)\operatorname{sen}\alpha e^{jkr'\operatorname{sen}\theta\cos\alpha} d\alpha = -\int_\phi^{\phi-2\pi} \cos\phi\cos\alpha\operatorname{sen}\alpha e^{jkr'\operatorname{sen}\theta\cos\alpha} d\alpha - \int_\phi^{\phi-2\pi} \operatorname{sen}\phi\operatorname{sen}^2\alpha e^{jkr'\operatorname{sen}\theta\cos\alpha} d\alpha =$$
$$= -\frac{\cos\phi}{2}\int_\phi^{\phi-2\pi} \operatorname{sen}2\alpha e^{jkr'\operatorname{sen}\theta\cos\alpha} d\alpha - \frac{\operatorname{sen}\phi}{2}\int_\phi^{\phi-2\pi} e^{jkr'\operatorname{sen}\theta\cos\alpha} d\alpha + \frac{\operatorname{sen}\phi}{2}\int_\phi^{\phi-2\pi} \cos 2\alpha e^{jkr'\operatorname{sen}\theta\cos\alpha} d\alpha =$$
$$= -\frac{\cos\phi}{2}\int_0^{2\pi} \frac{(e^{j2\phi'}-e^{-j2\phi'})}{2j} e^{jkr'\operatorname{sen}\theta\cos\alpha} d\alpha - \frac{\operatorname{sen}\phi}{2}(-2\pi J_0(kr'\operatorname{sen}\theta)) - \frac{\operatorname{sen}\phi}{2}\int_0^{2\pi} \frac{(e^{j2\phi'}+e^{-j2\phi'})}{2} e^{jkr'\operatorname{sen}\theta\cos\alpha} d\alpha$$
$$= -\frac{\cos\phi}{4j}\left\{-2\pi J_2(kr'\operatorname{sen}\theta) + 2\pi J_2(kr'\operatorname{sen}\theta)\right\} + \operatorname{sen}\phi\pi J_0(kr'\operatorname{sen}\theta) - \frac{\operatorname{sen}\phi}{4}\left\{-2\pi J_2(kr'\operatorname{sen}\theta) - 2\pi J_2(kr'\operatorname{sen}\theta)\right\} =$$
$$= \pi\operatorname{sen}\phi[J_0(kr'\operatorname{sen}\theta) + J_2(kr'\operatorname{sen}\theta)]$$

c)

$$\int_0^{2\pi} \operatorname{sen}\phi' \operatorname{sen}(\phi-\phi') e^{jkr'\operatorname{sen}\theta\cos(\phi-\phi')} d\phi' = ?$$
$$\alpha = \phi - \phi' => \phi' = -(\alpha-\phi)$$
$$d\alpha = -d\phi'$$
$$= -\int_\phi^{\phi-2\pi} \operatorname{sen}(\alpha-\phi)\operatorname{sen}\alpha e^{jkr'\operatorname{sen}\theta\cos\alpha} d\alpha = -\int_\phi^{\phi-2\pi} \cos\phi\operatorname{sen}^2\alpha e^{jkr'\operatorname{sen}\theta\cos\alpha} d\alpha + \int_\phi^{\phi-2\pi} \operatorname{sen}\phi\cos\alpha\operatorname{sen}\alpha e^{jkr'\operatorname{sen}\theta\cos\alpha} d\alpha =$$
$$= -\frac{\cos\phi}{2}\int_\phi^{\phi-2\pi} e^{jkr'\operatorname{sen}\theta\cos\alpha} d\alpha + \frac{\cos\phi}{2}\int_\phi^{\phi-2\pi} \cos 2\alpha e^{jkr'\operatorname{sen}\theta\cos\alpha} d\alpha + \frac{\operatorname{sen}\phi}{2}\int_\phi^{\phi-2\pi} \operatorname{sen}2\alpha e^{jkr'\operatorname{sen}\theta\cos\alpha} d\alpha =$$
$$= -\frac{\cos\phi}{2}(2\pi J_0(kr'\operatorname{sen}\theta)) + \frac{\cos\phi}{2}\int_0^{2\pi} \frac{(e^{j2\phi'}+e^{-j2\phi'})}{2} e^{jkr'\operatorname{sen}\theta\cos\alpha} d\alpha + \frac{\operatorname{sen}\phi}{2}\int_0^{2\pi} \frac{(e^{j2\phi'}-e^{-j2\phi'})}{2j} e^{jkr'\operatorname{sen}\theta\cos\alpha} d\alpha =$$
$$= -\cos\phi\pi J_0(kr'\operatorname{sen}\theta)) + \frac{\cos\phi}{4}\left\{-2\pi J_2(kr'\operatorname{sen}\theta) - 2\pi J_2(kr'\operatorname{sen}\theta)\right\} + \frac{\operatorname{sen}\phi}{4}\left\{-2\pi J_2(kr'\operatorname{sen}\theta) + 2\pi J_2(kr'\operatorname{sen}\theta)\right\} =$$
$$-\cos\phi\pi J_0(kr'\operatorname{sen}\theta)) - \frac{4\pi\cos\phi}{4}\left\{J_2(kr'\operatorname{sen}\theta)\right\} = -\pi\cos\phi J_0(kr'\operatorname{sen}\theta) - \pi\cos\phi J_2(kr'\operatorname{sen}\theta) =$$
$$= -\pi\cos\phi[J_0(kr'\operatorname{sen}\theta) + J_2(kr'\operatorname{sen}\theta)]$$