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Marcos Azevedo da Silveira

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Marcos Azevedo da Silveira

Electrical Engineering Department, PUC-Rio, Rua Marquês de São Vicente, 225, 22453-900, Rio de Janeiro, RJ, BRAZIL; marcos@ele.puc-rio.br

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Abstract

In this paper the extension of Wiener-Hopf design to irrational transfer functions is investigated. Infinite-dimensional servomechanisms are considered. A spectral factorization theorem and an optimization result are showed conditioned by the necessity to have a bounded Youla's parameter, i.e., an optimal Youla's parameter in the Hardy class H_+^∞. Some ideas are presented to solve this last problem by the use of an auxiliary constrained optimal control problem.

Keywords: Irrational transfer functions, Infinite-dimensional systems, Servomechanisms, Wiener-Hopf optimal design, Weighted Hardy spaces.

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1. Introduction

Wiener-Hopf design, following Youla's terminology, do not fit easily in H^2 optimal control problems on infinite-dimensional linear systems with irrational transfer functions because the optimal parameter should belong to H_+^∞ (Smith, 1989), (Oostveen, 1999) and quadratic functionals are naturally defined on weighted H_+^2 spaces (Da Silveira & Ades, 2000). Finite-dimensional linear systems, with rational transfer functions, do not have this problem because rational functions in $H_+^{2,-1}$ necessarily belong to H_+^∞. This paper investigates this problem introducing some conditions on the optimal control problem such that the Youla parameter belongs to H_+^∞. The weighted Hardy spaces are used as a tool for this investigation.

Youla and his co-workers (Youla, Jabr & Bongiorno Jr, 1976) introduced Wiener-Hopf optimal design on finite-dimensional linear systems. They differ from the usual LQG problems because the optimization is on the stabilizing controllers set, parameterized by real-rational proper stable matrices. From these initial works several results and methods were developed, working also with controllers solving the servomechanism problem and with special criteria fitting the transitory behavior of the controlled system: see (Youla & Bongiorno, 1985), (Park & Bongiorno, 1989), (Park & Bongiorno, 1990), (Da Silveira & Corrêa, 1992), (Corrêa & Da Silveira, 1995), (Zhuo, Doyle & Glover, 1996), (Corrêa, Sales & Soares, 1997), (Da Silveira & Ades, 1998), (Da Silveira & Ades, 2000), (Xie, Xue & Tso, 2000), and the references therein. The extension to infinite-dimensional systems begins with some works relating coprime factorizations on H_{+}^{∞} and the description of the stabilizing controller set: see (Smith, 1989), (Curtain & Zwart, 1995) and its references. In spite of many results on the LQ optimal control problems by the use of suitable Riccati equations – (Lions, 1971), (Curtain & Pritchard, 1978), (Curtain & Zwart, 1995), (Oostveen, 1999), for instance – early Wiener-Hopf methods were not extended to irrational transfer functions.

In the second section a typical infinite-dimensional Wiener-Hopf (or a H^2) optimal control problem will be presented, extending the ideas of (Corrêa & da Silveira, 1995) and (Xie et al., 2000) to the infinite-dimensional setting. The problem of asymptotic tracking with known dynamics for finite-dimensional inputs and stochastic disturbance rejection will be considered. The criterion will select a controller achieving good transitory properties. In section three the weighted Hardy spaces, theirs properties and some of their relations with the H_{+}^{∞} setting will be presented. In section four a spectral factorization theorem will be presented, which will be needed to solve the optimal control problem proposed in section 2. In the fifth section Wiener-Hopf methods (in the sense of Youla) will be investigated for infinite-dimensional linear systems in the light of the tools presented in previous sections. Another approach to solve the same problem via a constrained optimal control problem will be discussed in section six. The last section will present some conclusions and open problems.

Notations: Let $\mathbb{R}$ and $\mathbb{C}$ denote the real numbers and the complex number sets, respectively, $\mathbb{C}_+=\{s\in\mathbb{C} \therefore Re\{s\}>0\}$, $\mathbb{C}_-=\{s\in\mathbb{C} \therefore Re\{s\}<0\}$. The usual euclidean norm on $\mathbb{R}^n$ will be denoted by $\|x\|$. L^2 denotes the class of bounded quadratic norm functions on the imaginary axis with, L^∞ the class of essential bounded functions on the imaginary axis, H_+^2 the usual Hardy class of analytic functions in $\mathbb{C}_+$ with bounded quadratic norm in any vertical line in $\mathbb{C}_+$, H_-^2 the corresponding space obtained by changing $\mathbb{C}_+$ by $\mathbb{C}_-$.

The norm and the inner product in L^2 will be denoted by $\|f\|_2$ and $<f,g>$, respectively. The norm on L^∞ will be denoted by $\|f\|_\infty$. Remember that $L^2 = H_+^2 + H_-^2$, these two sub-spaces being orthogonal (Hoffmann). H_+^∞ is the usual space of functions analytic on $\mathbb{C}_+$ which are bounded on each vertical line in $\mathbb{C}_+$ (Hoffmann, 1962). This space can be viewed as a subspace of L^∞. L^1 will denote the space of integrable functions on the imaginary axis, with norm given by $\|f\|_1$.

$M[X]$ will denote the set of matrices with entries in X. If A is a matrix, A^T denotes its transpose, det{A} its determinant, Tr{A} its trace. $A\geq 0$ means that the matrix A is hermitian non-negative, $A>0$ denotes that A is also definite positive. If F(s) is a matrix function, $F^\sim(s) = F^T(-s)$. A para-hermitian matrix is a functional matrix F(s) such that $F^\sim(s)=F(s)$.

The symbol $\hat{f}(s)$ denotes the Laplace transform of a function f(t). If f(s) is a rational function, $\partial_r(f)$ denotes the relative degree of f(.), i.e. the difference between the degree of its denominator from the degree of its numerator. If K, A, B are matrices, column{K} denotes the stacking of K-rows, A⊗B the Kronecker product of A and B. With these notations, Tr{AB} = Tr{BA}, column{AKB} = A⊗B(column{K}). E{.} will denote the mathematical expectance of its argument and [p] the integer part of a real number p.

2. H^2 optimal control problems for infinite-dimensional linear systems

Let a time-invariant linear system be described by:

$$\dot{x}(t) = \mathcal{A}x(t)+\mathcal{B}u(t)+\mathcal{E}v(t),\ x(0)=x_o,$$

$$y(t) = \mathcal{C}x(t)+\mathcal{D}u(t)+\mathcal{F}v(t),$$

where $x(t)\in\mathcal{X}$, $u(t)\in\mathbb{R}^m$, $y(t)\in\mathbb{R}^n$, $v(t)\in\mathbb{R}^m$, when $\mathcal{X}$ is a separable Hilbert space. The operator $\mathcal{A}$ on $\mathcal{X}$ generates a C_0-semigroup of bounded operators T(t), $\mathcal{B}$ is a bounded operator from $\mathbb{R}^m$ to $\mathcal{X}$, $\mathcal{E}$ is a bounded operator from $\mathbb{R}^m$ to $\mathcal{X}$, $\mathcal{C}$ is a bounded operator from $\mathcal{X}$ to $\mathbb{R}^n$, $\mathcal{D}$ and $\mathcal{F}$ are nxm real matrices. Here x(t) denotes the state variable, u(t) the control variable, v(t) a disturbance signal and y(t) the measured output.

Assume that the transfer-functions

$$G(s) = \mathcal{C}(sI-\mathcal{A})^{-1}\mathcal{B}+\mathcal{D},\quad G'(s) = \mathcal{C}(sI-\mathcal{A})^{-1}\mathcal{E}+\mathcal{F}$$

are well-posed (i.e., analytic in some right half-plane) (Curtain & Zwart, 1995), that G(s) has a right-coprime and a left-coprime factorizations on H_+^∞:

$$G(s) = N(s)D^{-1}(s) = \tilde{D}^{-1}(s)\tilde{N}(s),$$

N(s), D(s), $\tilde{N}(s)$ and $\tilde{D}(s)$ being nxm, mxm, nxm, nxn H_+^∞ matrices, D(s) and $\tilde{D}(s)$ invertible (also at s=∞), and that G'(s) has the same property:

$$G'(s) = M(s)D^{-1}(s) = \tilde{D}^{-1}(s)\tilde{M}(s),$$

M(s) and $\tilde{M}(s)$ being pxm H_+^∞ matrices.

By convenience, it will be developed here only the robust asymptotic tracking problem for one-parameter controllers with a quadratic criterion on transient performance and disturbance attenuation. The theory of the complete servomechanism problem for four-block linear systems and two-degree-of-freedom controllers, considering asymptotic tracking and disturbance and sensor noise asymptotic rejection, can be developed extending the results shown here, following (Da Silveira & Corrêa, 1992).

For the asymptotic tracking problem, let the reference signals r(t) be in the class of signals with Laplace transform given by $\hat{r}(s) = \Psi_r^{-1}(s)\mu(s)$, $\Psi_r(s)$ a stable, biproper and invertible rational matrix with all its zeroes out of the stability region, μ(s) any strictly proper and stable rational matrix. The choice of μ(s) will fix the specific reference signal to be tracked. For the disturbance attenuation problem, assume v(t) a wide-sense stationary stochastic process with zero-mean and power density spectra $\phi_v(i\omega)$, a real-rational strictly proper matrix such that $\phi_v(i\omega)=[\phi_v(-i\omega)]^T$, i.e., para-hermitian. One-parameter controllers solving the robust asymptotic tracking are described by:

$$\hat{u}(s) = H(s)[\hat{r}(s) - \hat{y}(s)],$$

$$H(s) = [\tilde{D}_c(s)\varphi_r(s)]^{-1}\tilde{N}_c(s),$$

when H(s) internally stabilizes the closed control loop. Here, $\tilde{N}_c(s)$ and $\tilde{D}_c(s)$ are mxm and mxn H_+^∞ matrices with $\tilde{D}_c(s)$ invertible (also at s=∞), $\varphi_r(s)$ is the maximum invariant factor of $\Psi_r(s)$.

Notation: By convenience, the argument "s" will be dropped in the sequel when there is no chance of ambiguity.

The existence of such controllers is equivalent to the existence of mxm and mxp H_+^∞ matrices, Y and X, solving the diophantine equation:

$$Y(\varphi_r D)+XN=I$$

(see the Appendix). Note that

$$N(\varphi_r D)^{-1} = (\varphi_r \tilde{D})^{-1}\tilde{N}.$$

The right-coprimeness of ($\varphi_r D$,N) means exactly the existence of such X, Y. Actually, the existence of such controller needs to be verified for each particular system. Meanwhile, for finite-dimensional systems, exponentially stabilizable systems (Curtain & Zwart, 1995) and collocated infinite-dimensional systems (where $C = B^*$) (Oostveen, 1999) it is possible to show some general conditions for the existence of such controllers.

With these assumptions and applying some stabilizing results from (Curtain, Weiss and Weiss, 2001), all one-parameter controllers stabilizing the linear system and solving the asymptotic tracking problem considered here are given by:

$$H(s) = [\tilde{D}_c\varphi_r]^{-1}\tilde{N}_c = [(Y + L\tilde{N})\varphi_r]^{-1}(X - L\varphi_r\tilde{D}),$$

for any mxn H_+^∞ matrix L(s) such that the controller is well posed. In (Da Silveira & Corrêa, 1992), (Corrêa & Da Silveira, 1995) and (Xie et al., 2000) there are sets of conditions to solve this problem when the linear system is finite-dimensional. All the H_+^∞ matrices are being real-rational proper and stable matrices.

Closing the loop and defining the tracking error (when $v(t) \equiv 0$) by $e(t) = r(t)-y(t)$, it is easy to verify that the proposed controller solves the robust asymptotic tracking problem (see the Appendix), when:

$$\hat{e}(s) = D(s)\tilde{D}_c(s)\varphi_r(s)\hat{r}(s) = D(s)\tilde{D}_c(s)R(s)\mu(s),$$

$$\hat{u}(s) = D(s)\tilde{N}_c(s)\hat{r}(s),$$

where $R(s) = \varphi_r(s)\Psi_r^{-1}(s)$, a stable biproper real-rational matrix.

Extending a result from (Corrêa, da Silveira), define the filtered error by:

$$\tilde{e}(s) = W(s)\hat{e}(s),$$

where $W(s)$ is a real-rational stable and strictly proper matrix without zeroes on the imaginary axis. From a known result on Fourier transform (Rudin, 1966),

$$\begin{aligned}
\sup_{t\geq 0} \| \tilde{e}_k(t) \| &\leq \| W_k(s)[D(s)\tilde{D}_c(s)]R(s)\mu(s) \|_1 \\
&\leq \sum_j \| W_k(s)[D(s)\tilde{D}_c(s)]_j [R(s)\mu(s)]_j \|_1 \\
&\leq \sum_j \{ \| W_k(s)[D(s)\tilde{D}_c(s)]_j \|_2 \| [R(s)\mu(s)]_j \|_2 \} \\
&\leq \| [W(s)D(s)\tilde{D}_c(s)]_k \|_2 \| R(s)\mu(s) \|_2,
\end{aligned}$$

where $[\Theta]_k$ denote the k^{th} row of the matrix Θ and the Cauchy-Schwarz inequality was used in the two last lines. Therefore,

$$\begin{aligned}
\sup_{t\geq 0} \| \tilde{e}_k(t) \|_2^2 = \sup_{t\geq 0} \sum_k \| \tilde{e}_k(t) \|^2 &\leq \sum_k \sup_{t\geq 0} \| \tilde{e}_k(t) \|^2 \\
&\leq \{ \sum_k \| [W(s)D(s)\tilde{D}_c(s)]_k \|_2^2 \} \| R(s)\mu(s) \|_2^2 \\
&= \| W(s)D(s)\tilde{D}_c(s) \|_F^2 \| R(s)\mu(s) \|_2^2,
\end{aligned}$$

where

$$\| Z(s) \|_F^2 = \int_{-\infty}^{\infty} \mathrm{Tr}\{H^{\sim}(i\omega)H(i\omega)\}d\omega$$

(the Frobenius norm) and $H^{\sim}(s) = H^T(-s)$. Thus, it can be seen that the term $\| W(s)D(s)\tilde{D}_c(s) \|_F^2$ has a bearing on the magnitude of the error signal in the time domain, as it was verified for finite-dimensional systems (Corrêa & Da Silveira, 1995).

To reflect the magnitude of the control signals u(t) it will be used the suggestion of (Xie et al., 2000), i.e., the Frobenius norm of the transfer function from r(t) to u(t), also filtered by a real-rational stable and strictly proper filter denoted by V(s): $\| V(s)D(s)\tilde{N}_c(s) \|_F^2$.

To minimize the stochastic disturbance influence note that when $v \neq 0$:

$$\hat{y} = N\tilde{N}_c\hat{r} - N\tilde{N}_c MD^{-1}\hat{v}.$$

Whence, the mean quadratic error originated from the disturbance can be measured by:

$$E\{\| N\tilde{N}_c MD^{-1}v \|_F^2\} = \int_{-\infty}^{\infty} \mathrm{Tr}\{\phi_v [N\tilde{N}_c MD^{-1}]^{\sim}[N\tilde{N}_c MD^{-1}]\}_{s=i\omega} d\omega.$$

By simplicity, assume M=D, i.e. v(t) affects all the state components.

Now, recalling that $\tilde{D}_c = Y + L\tilde{N}$ and $\tilde{N}_c = X - L\varphi_r\tilde{D}$, a criterion for a quadratic optimal control problem can be defined summing all the three presented terms, which gives:

$$\begin{aligned}
J[L] = & \int_{-\infty}^{\infty} \mathrm{Tr}\{[WD(Y + L\tilde{N})]^{\sim}[WD(Y + L\tilde{N})]\}_{s=i\omega} d\omega \\
& + \rho \int_{-\infty}^{\infty} \mathrm{Tr}\{[VD(X - L\varphi_r\tilde{D})]^{\sim}[VD(X - L\varphi_r\tilde{D})]\}_{s=i\omega} d\omega \\
& + \mu \int_{-\infty}^{\infty} \mathrm{Tr}\{\phi_v[N(X - L\varphi_r\tilde{D})]^{\sim}[N(X - L\varphi_r\tilde{D})]\}_{s=i\omega} d\omega,
\end{aligned}$$

for ρ and μ non-negative real numbers (weighting coefficients).

By the use of the trace and Kronecker product properties cited in the first section, the criterion J[L] can be written as:

$$J[K] = \int_{-\infty}^{\infty} \{K^{\sim}(s)\Gamma(s)K(s) - 2K^{\sim}(s)\gamma(s)\}_{s=i\omega} d\omega + J[0],$$

where K = column{L} and:

$$\Gamma = [(D^{\sim}W^{\sim}WD)\otimes(\tilde{N}\,\tilde{N}^{\sim})]+\rho[(D^{\sim}V^{\sim}VD)\otimes(\tilde{D}\,\varphi_r(\varphi_r)^{\sim}\,\tilde{D}^{\sim})]+$$

$$+\mu[(N^{\sim}N)\otimes(\tilde{D}\,\varphi_r\phi_v(\varphi_r)^{\sim}\,\tilde{D}^{\sim})$$

$$\gamma(s) = \text{column}\{\rho(D^{\sim}V^{\sim}VDX(\varphi_r)^{\sim}\,\tilde{D}^{\sim})\text{-}(D^{\sim}W^{\sim}WDY\,\tilde{N}^{\sim})+\mu(N^{\sim}NX(\varphi_r)^{\sim}\phi_v\,\tilde{D}^{\sim})\}.$$

From this definition we can see, by inspection, that $\Gamma^{\sim}=\Gamma$. Moreover, $\Gamma = \Lambda\Theta$, where Λ is a real-rational para-hermitian ($\Lambda^{\sim}=\Lambda$) and strictly proper matrix with relative degree greater than or equal to 2, i.e, $(1+s^2)^p\Lambda(s)$ is biproper for some $p\geq 1$. Indeed, this is a consequence of the presence of $W^{\sim}W$, $V^{\sim}V$ or ϕ_v in the terms defining Γ. All the other matrices are in $\mathcal{M}[L^{\infty}]$, as H_+^{∞} matrices or its para-conjugates, which implies that $\Gamma(i\omega)\in\mathcal{M}[L^{\infty}]$ and $(1+\omega^2)^p\Gamma(i\omega)\in\mathcal{M}[L^{\infty}]$ for some $p\geq 1$.

Reasoning as in the proof of Proposition 1 in (Corrêa & da Silveira, 1995), which is presented in (Corrêa & Da Silveira, 1993), proves that $\Gamma(i\omega)$ has zeroes on the imaginary axis only and exactly at $D^{\sim}(i\omega)D(i\omega)$ zeroes. The argument uses the non-negativeness of the three terms in Γ definition and the coprimeness of the pair $(\tilde{N}, \tilde{D}\,\varphi_r)$. Therefore, if D(s) has no zeroes on the imaginary axis, $\Gamma(i\omega)$ will have no zeroes in this set. If $\Gamma(i\omega)$ is continuous this implies that $\det\{(1+\omega^2)^p\Gamma(i\omega)\}\geq\varepsilon>0$. In general, if we know only that $\Gamma(i\omega)\in\mathcal{M}[L^{\infty}]$, this last property is not warranted and it needs a particular verification.

New terms can be added to the criterion to allow plant poles on the imaginary axis, as discussed in (Da Silveira & Corrêa, 1992) or such that p=0. In the last case, $L\in \mathcal{M}[H_+^2]$, as it will be show in section 5.

Analogously to $\Gamma(i\omega)$, it can be shown that $\gamma(i\omega)\in \mathcal{M}[L^\infty]$ and $(1+\omega^2)^p\gamma(i\omega)\in \mathcal{M}[L^\infty]$, for which the presence of $W^\sim W$, $V^\sim V$ or ϕ_V in the terms defining $\gamma(i\omega)$ is essential.

The criterion J[K] defined above is only an example of functional appearing on optimal quadratic control problems. Also, the control problem considered in this paper is only an example between another interesting control problems. Our choice tries to grasp the principal features of these problems in relation to the solvability of quadratic optimal control problems for infinite-dimensional systems.

3. Weighted Hardy spaces

The purpose of this section is to collect some facts about the weighted L^2 and Hardy spaces to be used later in this paper. All results are slightly adaptations of results that can be found in (Hoffman, 1962), (Aubin, 1977) and (Da Silveira & Ades, 2000).

Definition 1 – For $p\in\mathbb{R}$, the weighted L^2 spaces on $i\mathbb{R}$ are defined by:

$$L^{2,p} = \{f(.):i\mathbb{R}\mapsto\mathbb{C} \therefore f(i\omega)(\omega^2+1)^{p/2}\in L^2\} = \{f(.):i\mathbb{R}\mapsto\mathbb{C} \therefore f(i\omega)(i\omega+1)^p\in L^2\}.$$

These spaces are Hilbert spaces with inner product and norm given by:

$$\langle f,g\rangle_p = \int_{-\infty}^{\infty} f(i\omega)(\omega^2+1)^p g(i\omega)d\omega,\ \|f\|_{2,p} = \sqrt{\langle f,f\rangle_p}\,,$$

respectively. When p=0 we recognize the usual L^2 space on $i\mathbb{R}$. The corresponding weighted (quadratic) Hardy spaces are defined by:

$$H_+^{2,p} = \{f(.):\mathbb{C}_+\mapsto\mathbb{C} \text{ analytic in } \mathbb{C}_+ \therefore f(\sigma+i\omega)\in L^{2,p} \text{ for any } \sigma>0\},$$

$H_-^{2,p} = \{f(.):\mathbb{C}_- \mapsto \mathbb{C}$ analytic in $\mathbb{C}_+ \therefore f(\sigma+i\omega) \in L^{2,p}$ for any $\sigma<0\}$.

When p=0 we recognize the usual quadratic Hardy spaces H_+^2 and H_-^2.

Fact 1 - $H_+^{2,p}$ and $H_-^{2,p}$ can be identified to closed subspaces of $L^{2,p}$ with the same inner product and norm. Moreover, $L^{2,p} = H_+^{2,p} + H_-^{2,p}$, the sum being orthogonal if $p \geq 0$.

Notations - $[f]_{p,+}$ and $[f]_{p,-}$ will denote the projections of a function $f(.) \in L^{2,p}$ in $H_+^{2,p}$ and in $H_-^{2,p}$, respectively. By the use of the identification above, we will say that $f(.) \in L^{2,p}$ if and only if $(s+1)^p f(.) \in L^2$ and $f(.) \in H_+^{2,p}$ if and only if $(s+1)^p f(.) \in H_+^2$, with $\| f \|_{2,p} = \| (s+1)^p f(s) \|_2$. Also, a analytic function (or matrix) in $\mathbb{C}_+$ with no poles on the imaginary axis will be called stable.

A rational function f(s) belongs to $L^{2,p}$ if and only if it has no poles on the imaginary axis and $\partial_r(f) \geq [p]+1 \geq p+1$, where [p] denotes the integer part of the real p and $\partial_r(f)$ denotes the relative degree of f(.).

Fact 2 – The set of real-rational functions with no poles on the imaginary axis and with $\partial_r(f) \geq [p]+1 \geq p+1$ is dense in $L^{2,p}$. The set of stable (i.e. analytic in $\mathbb{C}_+$ with no poles on the imaginary axis) real-rational functions with $\partial_r(f) \geq [p]+1 \geq p+1$ is dense in $H_+^{2,p}$.

Fact 3 – $L^{2,p}$ and $H_+^{2,p}$ are isometric to $L^{2,q}$ and $H_+^{2,q}$, respectively, for any p, q. Such isometry is given by $f(s) \mapsto (s+1)^{q-p} f(s)$. Moreover, $f(s) \in H_+^{2,p}$ if and only if

$(s+1)^{q-p} f(s) \in H_+^{2,q}$.

This last fact shows that $L^{2,p}$ and $H_+^{2,p}$ constitute chains of spaces: if p>q>0,

$$L^{2,p} \subset L^{2,q} \subset L^2 \subset L^{2,-q} \subset L^{2,-p}$$

and

$$H_+^{2,p} \subset H_+^{2,q} \subset H_+^2 \subset H_+^{2,-q} \subset H_+^{2,-p}.$$

Note that the topology on $L^{2,p}$ is stronger than that on $L^{2,q}$ if p>q. Also, each space in these chains is dense in the subsequent, and $H_+^{2,-p}$ is identifiable (as a Hilbert space) to the dual space of $H_+^{2,p}$.

Some relations between L^∞, H_+^∞ and $L^{2,p}$, $H_+^{2,p}$ will be used.

Fact 4 - $H_+^\infty \subset H_+^{2,-1}$, the topology in H_+^∞ being stronger than that on $H_+^{2,-1}$. Also, bounded and closed subsets in H_+^∞ are also bounded and closed as $H_+^{2,-1}$ subsets. See (Da Silveira & Ades, 2000) for a proof.

Fact 5 – $H_+^2 \cap L^\infty \subset H_+^\infty$.

4. A spectral factorization theorem

Now, a spectral factorization result to be used in the sequel will be presented.

Theorem 1 – Let $\Gamma(.)$ be a mnxmn functional matrix on the imaginary axis such that $\Gamma(i\omega)=\Gamma^{\sim}(i\omega)$ and $\Gamma(i\omega)\geq 0$. Assume also that, for some $p\geq 0$, there are real numbers ε and δ such that $0<\varepsilon\leq\det\{\Gamma(i\omega)(1+\omega^2)^p\}\leq\delta$ almost everywhere (which means $\Gamma(i\omega)(1+\omega^2)^p\in M[L^\infty]$ and $\Gamma^{-1}(i\omega)(1+\omega^2)^{-p}\in M[L^\infty]$). Then there is a mnxmn functional matrix $\Phi(s)$ with the following properties:

(a) $\Gamma(i\omega)=\Phi^{\sim}(i\omega)\Phi(i\omega)$ almost everywhere on the imaginary axis;

(b) $\Phi(s)(1+s)^p\in M[H_+^\infty]$ and $\Phi^{-1}(s)(1+s)^{-p}\in M[H_+^\infty]$:

(c) $\Phi(s)\in M[H_+^{2,p-1}]$ and $\Phi^{-1}(s)\in M[H_+^{2,-p-1}]$.

Any two such matrices are related by right multiplication by an orthogonal constant matrix.

Proof: Theorem 1.3, page 204, in (Caines, 1988), says that, if $\Theta(e^{i\theta})$, $\theta\in[0,2\pi]$, is a nxn matrix verifying $\Theta(e^{i\theta})^{\sim}=\Theta(e^{-i\theta})$ with $\Theta(e^{i\theta})\geq 0$ and $\Theta(e^{i\theta})$, $\Theta^{-1}(e^{i\theta})\in M[L^\infty]$ (on the unity circle), there exists a nxn matrix $Z(z) \in \mathcal{M}[\,H^{\infty}\,]$ (on the unity disk) which satisfies $Z^{-1}(z) \in \mathcal{M}[\,H^{\infty}_{+}\,]$ (on the unity disk) and $Z(e^{i\theta})Z^{T}(e^{-i\theta}) = \Theta(e^{i\theta})$ almost everywhere for $\theta \in [0,2\pi]$.

Now, define:

$$F(i\omega) = \Theta\left(\frac{1-i\omega}{1+i\omega}\right),\; G(s) = Z\left(\frac{1-s}{1+s}\right).$$

This transformation being a Banach isomorphism from the space H^{∞} on the unitary disk to H^{∞}_{+} (on $\mathbb{C}_{+}$), the transposed version of this theorem applies with F(iω) and $F^{-1}(i\omega)$ in $\mathcal{M}[\,L^{\infty}\,]$, $G(s) \in \mathcal{M}[\,H^{\infty}_{+}\,]$, $G^{-1}(s) \in \mathcal{M}[\,H^{\infty}_{+}\,]$, $G^{\sim}(i\omega)G(i\omega)=F(i\omega)$.

Define $F(i\omega) = (1+\omega^2)^p\Gamma(i\omega)$ and $\Phi(s) = (1+s)^{-p}G(s)$. The assumptions on $\Gamma(i\omega)$ implies the assumptions just above on F(iω), and then:

$$\Gamma(i\omega) = F(i\omega)(1+\omega^2)^{-p} = (1-i\omega)^p\, G^{\sim}(i\omega)G(i\omega)(1+i\omega)^p = \Phi^{\sim}(i\omega)\Phi(i\omega),$$

with $\Phi(s)(1+s)^p$, $\Phi(s)^{-1}(1+s)^{-p}$ in $\mathcal{M}[\,H^{\infty}_{+}\,]$ (on $\mathbb{C}_{+}$), proving statements (a) and (b).

Statement (c) is a consequence of the two implication chains below:

(i) $\Phi(s)(1+s)^p \in \mathcal{M}[\,H^{\infty}_{+}\,] \Rightarrow \Phi(s)(1+s)^p(1+s)^{-1} \in \mathcal{M}[\,L^{2}\,] \Rightarrow \Phi(s) \in \mathcal{M}[\,H^{2,p-1}_{+}\,]$;

(ii) $\Phi^{-1}(s)(1+s)^{-p} \in \mathcal{M}[\,H^{\infty}_{+}\,] \Rightarrow \Phi^{-1}(s)(1+s)^{-p}(1+s)^{-1} \in \mathcal{M}[\,L^{2}\,] \Rightarrow \Phi^{-1}(s) \in \mathcal{M}[\,H^{2,-p-1}_{+}\,]$.

5. H^2 problem for irrational matrices

In this section the quadratic functional arising in the H^2 problem described in the second section will be minimized with calculations similar to those in (Da Silveira & Corrêa, 1992). The factorization in Theorem 1 allows us to write the functional as:

$$J[K] = \int_{-\infty}^{\infty} \mathrm{Tr}\{K^{\sim}(i\omega)\Gamma(i\omega)K(i\omega) - 2K^{\sim}(i\omega)\gamma(i\omega)\}d\omega$$

$$= \|\,\Phi K\,\|_2^2 - 2\langle \Phi K, (\Phi^{\sim})^{-1}\gamma\rangle,$$

Then, J[K] is finite only if $\Phi K \in \mathcal{M}[\mathrm{L}^2]$ and $(\Phi^\sim)^{-1}\gamma \in \mathcal{M}[\mathrm{L}^2]$.

If $K \in \mathcal{M}[\mathrm{H}_+^{2,-p}]$, $\Phi K = [\Phi(1+s)^p][(1+s)^{-p}K] \in \mathcal{M}[\mathrm{L}^2]$ because the first parcel belongs to $\mathcal{M}[\mathrm{H}_+^\infty]$ (by Theorem 1) and the second to $\mathcal{M}[\mathrm{L}^2]$ (by the $\mathrm{H}_+^{2,-p}$ definition). If $\gamma \in \mathcal{M}[\mathrm{L}^{2,p}]$, the same arguments show that $(\Phi^\sim)^{-1}\gamma = [(\Phi^\sim)^{-1}(1-s)^{-p}][(1-s)^p\gamma] \in \mathcal{M}[\mathrm{L}^2]$. Note that, if $\gamma \in \mathcal{M}[\mathrm{L}^{2,2p}]$, this argument shows that $(\Phi^\sim)^{-1}\gamma \in \mathcal{M}[\mathrm{L}^{2,p}]$. These considerations prove the next result.

Lemma 1 – Under Theorem 1 assumptions, $\gamma \in \mathcal{M}[\mathrm{L}^{2,p}]$ implies $J[K]<\infty$ on $\mathcal{M}[\mathrm{H}_+^{2,-p}]$.

As $(\Phi^\sim)^{-1}\gamma \in \mathcal{M}[\mathrm{L}^2]$, then $(\Phi^\sim)^{-1}\gamma = [(\Phi^\sim)^{-1}\gamma]_{0,+} + [(\Phi^\sim)^{-1}\gamma]_{0,-}$, where the first term is in $\mathcal{M}[\mathrm{H}_+^2]$ and the second in $\mathcal{M}[\mathrm{H}_-^2]$. Whence,

$$J[K] = \|\Phi K\|_2^2 - 2\langle \Phi K, [(\Phi^\sim)^{-1}\gamma]_{0,+} + [(\Phi^\sim)^{-1}\gamma]_{0,-}\rangle$$

$$= \|\Phi K\|_2^2 - 2\langle \Phi K, [(\Phi^\sim)^{-1}\gamma]_{0,+}\rangle,$$

because the unstable term in $\mathcal{M}[\mathrm{H}_-^2]$ is orthogonal to $\Phi K \in \mathcal{M}[\mathrm{H}_+^2]$. Completing squares,

$$J[K] = \|\Phi K\|_2^2 - 2\langle \Phi K, [(\Phi^\sim)^{-1}\gamma]_{0,+}\rangle + \|[(\Phi^\sim)^{-1}\gamma]_{0,+}\|_2^2 - \|[(\Phi^\sim)^{-1}\gamma]_{0,+}\|_2^2$$

$$= \|\Phi\{K - \Phi^{-1}[(\Phi^\sim)^{-1}\gamma]_{0,+}\}\|_2^2 - \|[(\Phi^\sim)^{-1}\gamma]_{0,+}\|_2^2.$$

Now, $\hat{K} = \Phi^{-1}[(\Phi^\sim)^{-1}\gamma]_{0,+} = \{\Phi^{-1}(1+s)^{-p}\}\{(1+s)^p[(\Phi^\sim)^{-1}\gamma]_{0,+}\} \in \mathcal{M}[\mathrm{H}_+^{2,-p}]$ because the first term in brackets belongs to $\mathcal{M}[\mathrm{H}_+^\infty]$ and the second to $\mathcal{M}[\mathrm{H}_+^{2,-p}]$. This implies $J[\hat{K}]<\infty$.

If $\gamma \in \mathcal{M}[\mathrm{L}^{2,2p}]$ and $p \geq 0$, then $(\Phi^\sim)^{-1}\gamma$ belongs to $\mathrm{L}^{2,p} \subset \mathrm{L}^2$ and can be projected in $\mathrm{H}_+^{2,p}$. Whence $\hat{K} = \Phi^{-1}[(\Phi^\sim)^{-1}\gamma]_{p,+} = \{\Phi^{-1}(1+s)^{-p}\}\{(1+s)^p[(\Phi^\sim)^{-1}\gamma]_{p,+}\} \in \mathcal{M}[\mathrm{H}_+^2]$. Analogously, if $\gamma \in \mathcal{M}[\mathrm{L}^{2,2p-1}]$, then $\hat{K} \in \mathcal{M}[\mathrm{H}_+^{2,-1}]$.

But $\hat{K}$ should belong to $\mathcal{M}[H_+^\infty] \subset \mathcal{M}[H_+^{2,-1}]$, a stronger statement. For that, from the above calculations, it is sufficient that $(1+s)^p[(\Phi^\sim)^{-1}\gamma]_{0,+}\} \in \mathcal{M}[H_+^\infty]$. This condition will be taken as an assumption in the next theorem.

Theorem 2 – Let $p \geq 0$ and the assumptions of Theorem 1 verified. Assume also $\gamma \in \mathcal{M}[L^{2,p}]$ with $(1+s)^p[(\Phi^\sim)^{-1}\gamma]_{0,+} \in \mathcal{M}[H_+^\infty]$. Therefore $\hat{K} = \Phi^{-1}[(\Phi^\sim)^{-1}\gamma]_{0,+}$ minimizes J[K] on $\mathcal{M}[H_+^{2,-p}]$ with $\hat{K} \in \mathcal{M}[H_+^\infty]$.

The last assumption is problematic. Actually, by the definition of $H_+^{2,p}$ and Fact 5, we need only that $[(\Phi^\sim)^{-1}\gamma]_{0,+} \in \mathcal{M}[H_+^{2,p} \cap L^\infty]$. But $(\Phi^\sim)^{-1}\gamma \in \mathcal{M}[L^\infty]$ does not imply that the functional matrix $[(\Phi^\sim)^{-1}\gamma]_{0,+}$ is bounded on the imaginary axis. An open problem is to find particular classes of problems where this last property is verified.

Theorem 2 assumptions can be simplified in some situations.

Corollary 1 – If $\gamma \in \mathcal{M}[L^{2,2p}]$ with $(1+s)^p[(\Phi^\sim)^{-1}\gamma]_{p,+} \in \mathcal{M}[L^\infty]$, then
$\hat{K} = \Phi^{-1}[(\Phi^\sim)^{-1}\gamma]_{p,+} \in \mathcal{M}[H_+^\infty]$.

Proof - If $\gamma \in \mathcal{M}[L^{2,2p}]$, $(\Phi^\sim)^{-1}\gamma \in \mathcal{M}[L^{2,p}]$, then $[(\Phi^\sim)^{-1}\gamma]_{p,+} \in \mathcal{M}[H_+^{2,p}]$. Now, $(1+s)^p[(\Phi^\sim)^{-1}\gamma]_{0,+} \in \mathcal{M}[H_+^2 \cap L^\infty] \subset \mathcal{M}[H_+^\infty]$ by the $H_+^{2,p}$ definition and Fact 5. Therefore, $\hat{K} \in \mathcal{M}[H_+^\infty]$.

Corollary 2 – If $p=0$, $\gamma \in \mathcal{M}[L^2]$ and $[(\Phi^\sim)^{-1}\gamma]_{0,+} \in \mathcal{M}[L^\infty]$, then $\hat{K} \in \mathcal{M}[H_+^\infty]$.

Proof – For $p=0$, $\hat{K} \in \mathcal{M}[H_+^2]$. By the assumptions and from Fact 5, $\Phi^{-1} \in \mathcal{M}[H_+^\infty]$ and $[(\Phi^\sim)^{-1}\gamma]_{0,+} \in \mathcal{M}[H_+^\infty]$, which implies $\hat{K} \in \mathcal{M}[H_+^\infty]$.

Actually, if we change, in Theorem 1 assumptions, $\mathcal{M}[H_+^\infty]$ by $\mathcal{M}[H_+^{2,-1}]$ and $\mathcal{M}[L^\infty]$ by $\mathcal{M}[L^{2,-1}]$, then $\hat{K} \in \mathcal{M}[H_+^{2,-1}]$ (Da Silveira, 2001). But this conclusion is not sufficient to have the Youla parameters in H_+^∞.

6. Another approach: constrained optimization problems.

Another approach to this question is to minimize J[K] under a constraint on the H_+^∞ norm of the Youla parameter K. Explicitly, solve a new problem: find $\breve{K} \in \mathcal{M}[H_+^\infty]$ such that

$$J[\breve{K}] = \inf J[K] \text{ for } K \in \Omega(\lambda),$$

where $\Omega(\lambda)=\{K(.)\in \mathcal{M}[H_+^\infty] \therefore \|K\|_\infty \leq \lambda\}$ is a $\mathcal{M}[H_+^\infty]$ closed ball centered in the origin, for a sufficiently big real number λ. Here, $\|K\|_\infty = \max\{[\|[K]_k\|_\infty, k=1,...,mn\}$, $[K]_k$ being the k[th] entrie of the functional vector K. If the original problem has a solution $\hat{K}$ with $\lambda \geq \|\hat{K}\|_\infty$, then $\hat{K}$ equals $\breve{K}$. Therefore, $\breve{K}$ solves the original problem.

This approach is based on the next theorem.

Theorem 3 – Under the assumptions of Theorem 1 and if p≥1, there is one and only one $\breve{K} \in \mathcal{M}[H_+^\infty]$ minimizing the criterion J[K] above on $\Omega(\lambda)=\{K\in \mathcal{M}[H_+^\infty] \therefore \|K\|_\infty \leq \lambda\}$, for each λ≥0.

Proof – Assumptions on Theorem 1 imply that the functional J[K] is continuous in $\mathcal{M}[H_+^{2,-p}]$, as it is shown in the proof of Theorem 2 (da Silveira, Ades). Also, J[K] is a strictly convex functional in this space. If p≥1, $H_+^\infty \subset H_+^{2,-p}$ and Ω(λ), a bounded closed convex subset of $\mathcal{M}[H_+^\infty]$, is also a bounded closed convex subset of $\mathcal{M}[H_+^{2,-p}]$ by Facts 3 and 4. Then, Theorem 1.4.1 from (Balakrishnan, 1976), page 9, implies the statement.

Presently we only know how to obtain approximations for $\breve{K}$ by Gallerkin methods (Da Silveira & Ades, 2000) and by a dual method (Corrêa et. al., 1997). An attentive reading of the former paper shows that the proposed method do not need rational matrices in the definition of the criterion, but only the possibility to project the infinite-dimensional equations on the finite-dimensional spaces used therein. The knowledge of $\Phi(s)$ and $\gamma(i\omega)$ is sufficient for these calculations. The convergence of this algorithm was proved in the same paper, stronlgy on $H_+^{2,-p}$ and weakly on H_+^∞. The algorithm can be applied here without changes if the constraint is assumed active ($\|K\|_\infty=\lambda$), or, with some easy generalizations, for inequality constraints ($\|K\|_\infty\leq\lambda$). The dual method from (Corrêa et al., 1997) can be also applied to the problem, furnishing an estimation to the approximation error in terms of the criterion values. This dual method necessarily assumes equality constraints ($\|K\|_\infty=\lambda$).

The use of both methods allows to find a real interval [a,b] such that $J[\breve{K}]\in[a,b]$, i.e., an estimation for the optimal criterion value. The lowest value “a” is determined from the dual method and the greater value “b” is determined by the Gallerkin method.

It will be useful to prove that the solution for the constrained problem in $\Omega(\lambda)$ belongs to its H_+^∞-boundary if the unconstrained solution $\hat{K}$ is in the exterior of $\Omega(\lambda)$, i.e., $\|\hat{K}\|_\infty>\lambda$. For that we will used a Lemma proved in (Corrêa et. al., 1997). The proof presented here will clarify the geometry of this problem.

Lemma 2 - If $\|\hat{K}\|_\infty>\lambda$ the solution $\breve{K}$ for the constrained problem on $\Omega(\lambda)$ verifies $\|\breve{K}\|_\infty=\lambda$.

Proof – Consider the line segment defined by $K=(1-\tau)\breve{K}+\tau\hat{K}$, $\tau\in[0,1]$ a real number. By assumption, $\breve{K}$ (τ=0) belongs to $\Omega(\lambda)$ and $\hat{K}$ (τ=1) is in its exterior. As $\Omega(\lambda)$ is closed and convex, its trace on this one-dimensional segment is a closed real interval, say $\tau\in[0,e]\subset\mathbb{R}$. The quadratic criterion defines a real quadratic function on the segment, $J[\tau] = J[(1-\tau)\breve{K}+\tau\hat{K}]$. Now, this real function has its minimum in the whole segment at τ=1, by assumption. Therefore, the criterion minimum on the constrained segment [0,e] is at the point corresponding to τ=e, say $\overline{K}=(1\text{-}e)\breve{K}+e\hat{K}$. Then, $\overline{K}=\breve{K}$. But $\|\overline{K}\|_\infty=\lambda$, which proves the Lemma.

Now, we will use this Lemma to verify if the approximation obtained by the Gallerkin method approaches the unconstrained problem solution $\hat{K}$. For that, solve the constrained problem for $\Omega(\eta)$, $\eta>\lambda$. If the criterion value interval associated to the problem in $\Omega(\eta)$ is [c,d] with $[c,d]\subset[a,b]$, c>d, then the unconstrained problem solution $\hat{K}$ has a H_+^∞ norm lower than η. Indeed, if $\hat{K}$ is exterior to $\Omega(\eta)$, the solution of the constrained problem has a norm exactly equal to η by Lemma 2. But we have here the opposite. This implies that this optimal solution is the unconstrained solution. In other words, $\|\breve{K}\|_\infty<\eta$ implies $\breve{K}=\hat{K}$ with $\hat{K}\in\mathcal{M}[H_+^\infty]$.

If η is not sufficiently big, the procedure follows by the choice of a bigger value for this parameter. Mathematically speaking, if there is a solution, the procedure will find it after a finite number of iterations. As we do not know at this moment conditions to the existence of a H_+^∞ minimum for J[K], we can not warrant this finite convergence in the general situation.

7. Conclusions

In this paper the quadratic optimal control problem for servomechanisms on infinite-dimensional methods was investigated. In spite of the results showed here, some problems remain open. First, to find conditions on problem data to $0<\varepsilon\leq\det\{\Gamma(i\omega)(1+\omega^2)^p\}$ for some real number ε when $\Gamma(i\omega)$ is not continuous. Second, to find conditions on problem data such that the limitation of $\|(\Phi^*)^{-1}\gamma\|_\infty$ on the imaginary axis implies the limitation of $\|[(\Phi^*)^{-1}\gamma]_{0,+}\|_\infty$. Third, the alternative approach in section 6 needs an independent proof for the existence of a H_+^∞ solution for the unconstrained problem. In any case, the proposed algorithm obtains finite-dimensional approximations for the optimal solution.

The implementation of the Gallerkin method in (Da Silveira & Ades, 2000) – see also (Ades, 1999) – use the MATLAB toolboxes performing state variable calculations. It will be interesting to follow this way to investigate the approximation of the optimal control problem solution by finite-dimensional parameters beginning from finite-dimensional approximations for the semigroup S(t) considered in section 2. There is a strong interest on H_+^∞ convergence, but they are not easy to obtain, as showed in (Xiao & Basar, 1999).

We are developing, at this moment, some applications of these ideas in the optimal vibration control of mechanical structures comparing several different approximation procedures, which will be presented in another paper due to the amount of preparation work needed.

Appendix

From the system equations,

$$\hat{u} = \overline{D}_c^{-1}\tilde{N}_c(\hat{r} - ND^{-1}\hat{u}),$$

where $\overline{D}_c = \varphi_r \tilde{D}_c$. Then,

$$\hat{u} = D(\varphi_r \tilde{D}_c D+N\tilde{N}_c)^{-1}\tilde{N}_c\,\hat{r} = D\tilde{N}_c\,\hat{r},$$

where the diofantine equation $\varphi_r \tilde{D}_c D\text{-}N\tilde{N}_c = I$ was used. Also,

$$\hat{e} = \hat{r} - \hat{y} = \hat{r} - ND^{-1}D\tilde{N}_c\,\hat{r} = (I\text{-}N\tilde{N}_c)\hat{r} = \varphi_r \tilde{D}_c D\hat{r},$$

where the same identitie was used. As φ_r is a polynomial,

$$\hat{e} = \tilde{D}_c D\varphi_r \hat{r} = \tilde{D}_c D\varphi_r \Psi_r^{-1}\mu = \tilde{D}_c DR\mu.$$

The converse is also true, but its proof needs an incursion on the robust servomechanism problem with a lot of definitions. The theory in (Corrêa & Da Silveira, 1995b) can be applied without changes because the reference signals are assumed rational functions.

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