THE SOLUTION OF THE H^2/H^∞ PROBLEM BY DIRECT METHODS

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Publicação Interna DEE 01-01

Junho de 2001

Draft Paper
(submitted to SIAM J. Control and Optimization)

THE SOLUTION OF THE H^2/H^∞ PROBLEM BY DIRECT METHODS

M. A. DA SILVEIRA† AND R. ADES‡

Abstract. The H^2/H^∞ problem is formulated in a Hilbertian context. It has a unique solution, which is the strong limit of sequences generated by Gallerkin methods based on convenient and not necessarily orthogonal generator sets. Using these results, a methodology to solve the problem by the Gallerkin method is proposed and an example is solved and compared to other approachs.

Key Words. optimal control, robust control, H^2/H^∞ problem, linear control systems, weighted Hardy spaces.

AMS subject classifications. 49J02, 34H05, 41A20, 65D02.

1. Introduction. The simplest H^2/H^∞ problem is to find a function $K(.)$ in the Hardy class H_+^∞ minimizing the quadratic criterion

$$J[K(.)] = \int_{-\infty}^{\infty} \{K(-i\omega)\Gamma(i\omega)K(i\omega) + K(-i\omega)\gamma(i\omega)\} d\omega ,$$

under the H^∞ constraint

$$ess.sup. \, | \, A(i\omega)K(i\omega) + B(i\omega) \, | \leq \lambda,$$

where $\Gamma(.)$, $\gamma(.)$, $A(.)$ and $B(.)$ are given rational functions, λ is a given positive real number and the essential supremum is taken on the set of real numbers ω. This problem arises in quadratic optimal control theory for linear systems when robustness conditions or filtering constraints are imposed on the controller. This paper shows that, in spite of the H^∞ constraint, the above optimal control problem is well-posed in a larger space, $H_+^{2,-1}$, a Hilbert space to be defined here. It means that the optimal control problem has a unique solution in this space with desired regularity properties, under suitable conditions on the functional $J[.]$. Moreover, this functional setting leads to the definition of generator sets such that Gallerkin methods converge to the optimal control problem solution. A significant remark is that it is possible to measure the approached solution quality when the proposed method is coupled with the dual method presented in [1]. The design of a pitch optimal control of a fight airplane will be showed as an example to show the numerical viability and to allow the comparison with others design methods.

The crucial point in this paper is the construction of a Hilbert space containing the usual Hardy spaces H_+^2, H_+^∞ and such that bounded closed sets in both spaces are also bounded closed in this new space. The embedding of the original problem in this new setting allows the use of the Hilbert spaces convex optimization tools to solve the problem. The H_+^∞ constraint carries the optimal solution into this last space with no further considerations about the H_+^∞ non-Hilbertian topology. Besides this construction

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it will be necessary to build a chain of Hilbert spaces to well represent the optimal solution regularity, an essential information for the Gallerkin method convergence properties.

In the remaining of this section it will be presented a survey of the H^2/H^∞ problem and the notations to be used. The geometry of the Hilbert spaces $H_+^{2,-1}$ and $H_+^{2,-k}$ is presented in section 2. The unconstrained H^2 optimal control problem is rewritten in section 3 as a minimum norm problem in a suitable space $H_+^{2,-k}$ according to the data. This clarifies its existence and regularity properties. The constrained H^2/H^∞ optimal control problem is solved in section 4, Theorem 7 containing the existence and uniqueness results cited above. The convergence of Gallerkin methods is the subject of section 5 and a numerical example is presented and discussed in section 6. Some extensions of those results are shown in the last section, in particular, to multivariable problems. All the proofs not presented in the main text can be found in Appendix 1.

After the introduction of the Youla-Kučera parameterization [2], [3], quadratic criteria for Wiener-Hopf linear-quadratic optimal control problems have been considered, which allow the manipulation of well-defined technical or physical optimal solution characteristics, as rms transient error, plant saturation and closed loop sensitivity [2], transient specifications against shape-deterministic exogenous inputs [4], performance measures [5], [6] and [7], servomechanism specifications [8] and transient specifications [9]. The work in [9] presents a heuristic procedure to choose the criteria weighting filters in such a way that a trade-off between overshoot and time constant can be obtained. All these papers consider the controller as optimization variable, the set of controllers being parameterized by real-rational proper stable rational matrices. They arrived to explicit expressions for the optimal solutions.

These linear-quadratic criteria were enriched by quadratic or H^∞ constraints to consider performance or robustness conditions in [6], [7], [10] and [11]. In special, H^∞ constraints have been used to impose a pre-specified robustness degree to the optimal solution - see [12], but they can be used also to impose other specifications, as filter constraints - see [13].

A set of different methods was proposed to solve the H^2/H^∞ problem. Some of them modify the original optimization criteria, loosing the original physical interpretations in view to obtain new soluble mathematical problems. Examples are the methods exposed in [5], [10] and [11]. Direct methods, using expansions in series, do not modify the original criteria. They were proposed in [14], without major developments, and in [15], [16] and [17]. These papers show examples using Laguerre functions as generator set, do not proving the optimal solution existence or uniqueness. Also, they do not discuss the numerical viability of the algorithms. In [18] is proposed the use of linear matrix inequalities (LMI) to solve the H^2/H^∞ problem, but under assumptions too restrictive and non natural.

A last methodology was presented in [1] where a sequence of H^2 constraints approaching the original H^∞ constraint was built. In this method, each H^2 constraint defines a pure H^2 problem solved by a dual problem whose solution is explicitly given. The present paper shows that this solution defines a lower bound to the original optimal cost, the sequence of these solutions approaching monotonically the optimal solution, when it exists. Such an algorithm will be used here as a part of a methodology to give lower bounds to the optimal criterion value.

Actually, if finite dimensional controllers are imposed, it is possible to obtain only approximate solutions. Indeed, [19] proved that the optimal solution is infinite dimensional when the H^∞ constraint is active. If this constraint is inactive and the criterion is quadratic, then an explicit formula for the optimal solution can be obtained as is in [8].

A first proof for existence and uniqueness results for the H^2/H^∞ problem was gived by the authors in [20], searching the solution in the space generated by completing the set of real-rational proper stable functions under the norm defined by the quadratic criterion term. This result was further developed in [21], allowing a complete methodology to solve the H^2/H^∞ problem without changes in the criteria and in the constraints. This methodology will be, in part, showed here. The present paper develops a more complete mathematical theory for the problem, proving the existence, uniqueness and regularity to the solutions under natural assumptions, and proves the convergence of the Gallerkin approximating sequence to the optimal solution.

NOTATIONS: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ denote the natural numbers (i.e. the positive integers), the integers, the real and the complex numbers, respectively. $|s|$, $\bar{s}$ and $Re\{s\}$ denote the module, a conjugate and real part of a complex number s, respectively.

If $i=\sqrt{-1}$, $i\mathbb{R}$ = $\{i\omega,\ \omega\in\mathbb{R}\}$, C_+^0 = $\{s\in\mathbb{C} \therefore Re\{s\}>0\}$, C_-^0 = $\{s\in\mathbb{C} \therefore Re\{s\}<0\}$. Functions f:A→B are denoted as f, $f(.)$ or $f(s)$, $f(s)$ also denoting its value at the number $s\in$ A. A function $f(.)$ is real if it transforms real numbers in real numbers.

If $f(.) = n(.)/d(.)$ is rational, $n(.)$ and $d(.)$ being polynomials, $\partial_r(f)$ denotes its relative degree, defined as the integer "degree of $d(.)$ - degree of $n(.)$". Also $f^*(s)=f(-s)$, $|f(i\omega)|^2=\bar{f}(i\omega)f(i\omega)$. The last expression equals $f^*(i\omega)f(i\omega)$ if $f(.)$ is real.

The usual inner product and the usual quadratic norm are defined by:

$$\langle f,g\rangle_2=\int_{-\infty}^{\infty}\bar{f}(i\omega)f(i\omega)d\omega \text{ and } \|f\|_2=[\langle f,f\rangle_2]^{1/2}=[\int_{-\infty}^{\infty}|f(i\omega)|^2\ d\omega]^{1/2}.$$

The H^∞ norm is defined by

$$\|f\|_\infty = ess.sup\,|f(i\omega)|,$$

the supremum taken on $\omega\in\mathbb{R}$. The symbols R_m, R_m^+ and R_m^- denote the classes of rational functions with relative degree greater or equal to m, respectively without poles in $i\mathbb{R}$, in $i\mathbb{R}\cup C_+^0$ (stable functions) and in $i\mathbb{R}\cup C_-^0$ (completely unstable functions).

The symbols H_+^2, H_-^2, H_+^∞ and H_-^∞ represent the usual Hardy classes studied in [22], [23]. The principal features of these spaces to be used here are given in the sequence. The two spaces of stable functions are defined by

$$H_+^2 = \{f\colon \mathbb{C}\to\mathbb{C} \textit{ analytic in } C_+^0 \therefore \|f(a+i\omega)\|_2<\infty,\ \forall\ a>0\},$$

$$H_+^\infty = \{f\colon \mathbb{C}\to\mathbb{C} \textit{ analytic in } C_+^0 \therefore \|f(a+i\omega)\|_\infty<\infty,\ \forall\ a>0\},$$

H_-^2 and H_-^∞ defined analogously changing the symbol "+" by "-" and assuming $a<0$.

Also, if

$$L^2(i\mathbb{R}) = \{f\colon i\mathbb{R}\to\mathbb{C} \therefore \|f\|_2<\infty\},$$

it can be proved that $\langle f,g\rangle_2$ is an inner product in $L^2(i\mathbb{R})$, H_+^2 and H_-^2, these spaces being Hilbert spaces under this inner product. The functional spaces H_+^2 and H_-^2 can be identified to orthogonal subspaces of $L^2(i\mathbb{R})$ so that $L^2(i\mathbb{R}) = H_+^2 \oplus H_-^2$ (an orthogonal sum of subspaces). The symbols $[f]_+$ and $[f]_-$ denote the orthogonal projection of $f\in L^2(i\mathbb{R})$ in H_+^2 and H_-^2, respectively. H_+^∞ and H_-^∞ are Banach spaces under the norm $\|\cdot\|_\infty$.

If H represents a locally convex topological vector space [24], H' denotes its topological dual endowed with its strong topology. If H and V are such spaces, $H+V$ and $H\oplus V$ denote their sums and their direct sums, respectively. The latter means $H\cap V = \{0\}$, the trivial subspace. Further information about these concepts can be found in [24], [25] or [26].

2. A functional setting for the optimal control problem. This section presents the functional setting to formulate the H^2/H^∞ problem as a well-posed problem. The subjacent idea is to define spaces containing H_+^2 and H_+^∞ such that the quadratic functional to be minimized be continuous and the constraints convex, closed and bounded. Actually, it will be defined a chain of spaces like H_+^2 to grasp the problem regularity.

DEFINITION 1. Let $\Phi_{-k} = (s+1)^{-k}$, $k\in\mathbb{Z}$,

$$\langle f,g\rangle_{2,-k} = \int_{-\infty}^{\infty} f^*(i\omega)\Phi_{-k}^*(i\omega)\Phi_{-k}(i\omega)f(i\omega)d\omega \text{ and}$$

$$\|f\|_{2,-k} = [\langle f,f\rangle_{2,-k}]^{1/2} = [\int_{-\infty}^{\infty}|\Phi_{-k}(i\omega)f(i\omega)|^2\, d\omega]^{1/2}.$$

Set

$L_{-k}^2(i\mathbb{R}) = \{f\colon i\mathbb{R}\to\mathbb{C} \therefore \|f\|_{2,-k} < \infty\}$,

$H_+^{2,-k} = \{f\colon\mathbb{C}\to\mathbb{C}$ *is analytic in* $C_+^0 \therefore \|f(a+i\omega)\|_{2,-k} < \infty$ *for all* $a > 0\}$,

$H_-^{2,-k}$ the analogous space using $a < 0$ and C_-^0 in its definition, □

As $\langle f,g\rangle_{2,-k} = \langle\Phi_{-k}f,\Phi_{-k}g\rangle_2$ and $\|f\|_{2,-k} = \|\Phi_{-k}f\|_2$, it is easy to proof that $\langle f,g\rangle_{2,-k}$ defines an inner product and $\|f\|_{2,-k}$ the associated norm in the spaces defined above (see Appendix 1). Moreover, the usual $L^2(i\mathbb{R})$, H_+^2 and H_-^2 spaces are the special cases where k=0. The next theorem presents the geometrical properties of the spaces defined here.

THEOREM 1. Let the spaces $L_{-k}^2(i\mathbb{R})$, $H_+^{2,-k}$, $H_-^{2,-k}$ be as in Definition 1.

a) $L_{-k}^2(i\mathbb{R})$, $H_+^{2,-k}$ and $H_-^{2,-k}$ are the completion of the sets R_{1-k}, R_{1-k}^+ and R_{1-k}^- in the norm $\|\cdot\|_{2,-k}$, respectively. Moreover, they are Hilbert spaces with respect to the corresponding inner product.

b) $H_+^{2,-k}$ and $H_-^{2,-k}$ are closed subspaces of $L_{-k}^2(i\mathbb{R})$.

c) $L_{-k}^2(i\mathbb{R}) = H_+^{2,-k} + H_-^{2,-k}$. If $k \le 0$, $H_+^{2,-k}\cap H_-^{2,-k}$ is empty. If $k \ge 1$, $H_+^{2,-k}\cap H_-^{2,-k}$ contains the polynomials in s with degree less than or equal to k-1 and the functions defined by $\sum_{m=1}^{\infty}\alpha_m e^{st_m}$, where $\sum_{m=1}^{\infty}|\alpha_m| < \infty$. □

REMARK 1. It is worth noticing that a rational function $f(s)$ without poles in $i\mathbb{R}$ belongs to $L^2_{-k}(i\mathbb{R})$ if and only if $\partial_r(f) \geq 1\text{-}k$. Alternatively, if $\partial_r(f) = m$ and $f(s)$ has no poles in $i\mathbb{R}$, $f(s)\in L^2_{-k}(i\mathbb{R})$ for each $k \geq 1\text{-}m$. □

The next theorem collect some results relating the topologies of H^∞_+, $L^2_{-k}(i\mathbb{R})$ and $H^{2,-k}_+$ for different indexes k.

THEOREM 2. Let the spaces presented in Definition 1 and $k < m$.

(a) $L^2_{-k}(i\mathbb{R}) \subset L^2_{-m}(i\mathbb{R})$. The linear spaces $L^2_{-k}(i\mathbb{R})$ and $L^2_{-m}(i\mathbb{R})$ are isometricaly isomorphic, the isometry from $L^2_{-k}(i\mathbb{R})$ to $L^2_{-m}(i\mathbb{R})$ being injective and the inverse isometry being surjective. Therefore, the $L^2_{-k}(i\mathbb{R})$ topology is strictly finer than the $L^2_{-m}(i\mathbb{R})$ topology.

(b) $H^{2,-k}_+ \subset H^{2,-m}_+$. The $H^{2,-k}_+$ topology is strictly finer than the $H^{2,-m}_+$ topology.

(c) $L^2_{-k}(i\mathbb{R})$ is dense in $L^2_{-m}(i\mathbb{R})$, $H^{2,-k}_+$ is dense in $H^{2,-m}_+$. In particular, if $k \geq 1$, the sets R_0, R^+_0 and R^-_0 are dense in $L^2_{-k}(i\mathbb{R})$, $H^{2,-k}_+$ and $H^{2,-k}_-$, respectively.

(d) $H^\infty_+ \subset H^{2,-1}_+$, the H^∞_+ topology being strictly finer than the one of $H^{2,-1}_+$. □

REMARK 2. Property (c) says that biproper rational functions can be approached in $H^{2,-1}_+$ by strictly proper rational functions, diminishing the relative degree at the limit. As an example, $f_n(s) = n(s+n)^{-1}$ converges to the constant function $f(s) \equiv 1$ in the $H^{2,-1}_+$ topology. This explains why it is possible to find complete sets for $H^{2,-k}_+$, $k\geq 1$, formed by strictly proper real-rational stable functions (R^+_0 functions). □

REMARK 3. Let $S(i\mathbb{R})$ denotes the space of functions going quickly to zero at infinity, $(S(i\mathbb{R}))'$ its topological dual (the space of temperate distributions) [25], [26]. Define $S_+(i\mathbb{R})$ as $S(i\mathbb{R}) \cap H^2_+$, $(S_+(i\mathbb{R}))'$ as its closure in the $(S(i\mathbb{R}))'$ topology. With these notations it is possible to prove that, for any $k > 1$,

$$
\begin{array}{lllllll}
S(i\mathbb{R}) \subset & L^2_k(i\mathbb{R}) \subset & L^2_1(i\mathbb{R}) \subset & L^2(i\mathbb{R}) \subset & L^2_{-1}(i\mathbb{R}) \subset & L^2_{-k}(i\mathbb{R}) \subset & (S(i\mathbb{R}))' \\
\cup & \cup & \cup & \cup & \cup & \cup & \cup \\
S_+(i\mathbb{R}) \subset & H^{2,k}_+ \quad \subset & H^{2,1}_+ \quad \subset & H^2_+ \quad \subset & H^{2,-1}_+ \quad \subset & H^{2,-k}_+ \quad \subset & (S_+(i\mathbb{R}))' \\
 & & & & \cup & & \\
 & & & & H^\infty_+ & &
\end{array}
$$

Each space is dense in the next bigger one in the chain. An analogous sequence can be build for unstable function spaces. The original H^2/H^∞ problem will be embeded in these chains of Hilbert spaces, as it will be showed in the next section.

REMARK 4. Let $\mathbb{H}_k$ denote the order k Sobolev space [25], [26]. As $\mathbb{H}_k$ is the Fourier transform image of $L^2_k(i\mathbb{R})$ (by an adaptation of a construction found in [25]), it is possible to define stable Sobolev spaces $[\mathbb{H}_k]^+$ as the inverse Fourier transform image of H^k_+. Then it is possible to build a corresponding sequence of stable Sobolev spaces also beginning in $S_+(i\mathbb{R})$ and ending in $(S_+(i\mathbb{R}))'$. Also, from the Structure Theorem

([25], page 255), it is possible to show that the temperate distributions in $(S_+(i\mathbb{R}))'$ are derivatives of some finite order of functions in $[\mathbb{H}_2]^+$. □

Now, the crucial point for embeding the H^2/H^∞ problem in $H_+^{2,-k}$ will be considered.

THEOREM 3. Consider the spaces presented in Definition 1.

(a) If $k \le m$, the bounded subsets of $L^2_{-k}(i\mathbb{R})$ are bounded in $L^2_{-m}(i\mathbb{R})$, and the bounded closed subsets of $L^2_{-k}(i\mathbb{R})$ are bounded and closed in $L^2_{-m}(i\mathbb{R})$, the same relations arriving between sets in $H_+^{2,-k}$ and $H_+^{2,-m}$.

(b) The bounded subsets of H_+^∞ are bounded in $H_+^{2,-1}$, and the bounded closed subsets of H_+^∞ are bounded and closed in $H_+^{2,-1}$. □

REMARK 5. Here it is essential that the subset be bounded. The spaces H_+^∞ and H_+^2 are closed and unbounded in its own topologies, but they are dense in $H_+^{2,-1}$ in the coarser topology. Also, closed balls in H_+^∞ have empty interior in relation to $H_+^{2,-1}$ topology. □

The next step is to collect the properties of linear and quadratic functional in $H_+^{2,-k}$, preparing more tools for minimizing the quadratic criteria showed in section 1.

THEOREM 4. Let $\gamma(s)$ be a real-rational function without poles in $i\mathbb{R}$.

(a) The linear functional

$$F(f) = \int_{-\infty}^{\infty} f^*(i\omega)\gamma(i\omega)d\omega$$

is continuous on $H_+^{2,-k}$ if and only if $\partial_r(\gamma) \ge k+1$.

(b) The space of continuous linear functional on $H_+^{2,-k}$ can be identified to $H_+^{2,k}$, for any k. □

THEOREM 5. Let Γ(s) a real rational para-hermitian function in R_{2k} without poles or zeroes in $i\mathbb{R}$, i.e., $\Gamma(s) = \Gamma^*(s)$ and $|\Gamma(i\omega)|>0$ for each finite ω.

(a) $\Gamma(s) = \Phi^*(s)\Phi(s)$, $\Phi(s)$ being a real rational stable function in R_k^+ with all its zeroes in C_+^0 (i.e., minimum-phase).

(b) The quadratic functional

$$f \mapsto \int_{-\infty}^{\infty} f^*(i\omega)\Gamma(i\omega)f(i\omega)d\omega = \langle \Phi f, \Phi f\rangle_2 = \|\Phi f\|_2^2$$

is continuous in $H_+^{2,-m}$ if and only if $m \le k$. It is coercive in $H_+^{2,-m}$ (i. e., there is a real number $\alpha > 0$ such that $\langle \Phi f, \Phi f\rangle_2 \ge \alpha^2 \| f\|^2_{2,-m}$ for all $f \in H_+^{2,-m}$) if and only if $m = k$. Moreover, it is strictly convex and $\|\Phi f\|_2$ defines a norm in $H_+^{2,-k}$ equivalent to $\| f\|_{2,-k}$. □

3. Optimal H^2 unconstrained control problems. This section presents the mathematical extension of the usual H^2 unconstrained optimal control problem on the mathematical framework developed in the last section. New conditions about its solution will be obtained, clarifying the ones in [8], [9]. This extension will be used in the next section to solve the H^2/H^∞ optimal control problem.

The unconstrained H^2 problem can be defined as follows: find a function $\breve{K}(s)$ solution to:

$$(3.1)\quad \inf_K \{ \int_{-\infty}^{\infty} [K^*(i\omega)\Gamma(i\omega)K(i\omega) - 2K^*(i\omega)\gamma(i\omega)]d\omega\} \equiv \inf_K J[K],$$

where the functions $K(s)$ belong to some $H_+^{2,-k}$ space, or, formally, to $(S_+)'$, a space containing $H_+^{2,-k}$ for all integer k. $\Gamma(s)$ and $\gamma(s)$ are given real-rational functions. Remember that $K(s)$ is the parameter describing the set of stabilizing controllers (or the set of controllers solving a given servomechanism problem), initially a free real-rational stable and proper function. To define the functional $J[.]$ some assumptions are needed:

A1) $\Gamma(s) = \Phi^*(s)\Phi(s)$ is a para-hermitian real-rational function in R_{2k} without poles or zeroes in $i\mathbb{R}$, $\Phi(s)$ being a real rational stable function in R_k^+ with all its zeroes in C_+^0 (i.e., minimum-phase);

A2) $\gamma(s)$ is a real-rational function without poles in $i\mathbb{R}$, $\partial_r(\gamma) = p$.

The functional $J[.]$ will be finite only for a meager parameter subset if $\Gamma(s)$ or $\gamma(s)$ have poles on the imaginary axis, as both are rational functions. Indeed, if such happens, $J[K]$ will be finite only for $K(s)$ with zeroes on those imaginary poles. The other conditions on assumption A1 are natural for quadratic functional on $H_+^{2,-k}$ spaces, according to Theorem 5 above. Indeed, it is possible to represent all integral quadratic real functional on $H_+^{2,-k}$ spaces as an integral quadratic operator with a para-hermitian kernel by a procedure similar to the auto-adjoint representation for integral quadratic functional on L^2 spaces. Moreover, $\Gamma(s)$ is assumed with no zeroes on the imaginary axis because this allows unstable solutions (see Remark 8). Finally, the Wiener-Hopf factorization $\Gamma = \Phi^*\Phi$ is a consequence of the known Youla factorization theorem cited above as Theorem 5a [27].

LEMMA 1. Under assumptions A1, A2, let $m = \min\{k,p\text{-}1\}$. Then the functional $J[.]$ is continuous in $H_+^{2,-m}$ and it is not well-defined in larger spaces, i. e., the integrals in $J[K]$ diverge for $K\in (S_+)' \text{-} H_+^{2,-k}$ (the complement of $H_+^{2,-k}$ in $(S_+)'$). □

Proof. The first statement follows from continuity conditions in Theorems 4 and 5b. For the second statement, if $K\in (S_+)' \text{-} H_+^{2,-k}$ is a rational function, $J[K]$ is not defined because $\partial_r(K^*\Gamma K) \le 1$ or $\partial_r(K^*\gamma) \le 1$. ■

DEFINITION 2. The space $H_+^{2,-m}$ in Lemma 1 will be called the "effective domain" of the functional $J[.]$. This terminology is inherited from convex analysis and adapted to the chain of spaces defined here. □

Now, note that, for $K\in H_+^{2,-m}$ and m as in Lemma 1, $\Phi K\in H_+^2$. Then the functional $J[K]$ can be written as

$$(3.2)\quad J[K] = \| \Phi K \|_2^2 - 2\int_{-\infty}^{\infty} \{[\Phi(i\omega)K(i\omega)]^* [\Phi^*(i\omega)]^{-1}\gamma(i\omega)d\omega.$$

As $(\Phi^*)^{-1}\gamma\in L^2_{p-k-1}(i\mathbb{R})$, it can be factorized as a sum of a function in $H_+^{2,p-k-1}$ with a function in $H_-^{2,p-k-1}$, according to Theorem 1c.

If $p \le k$, this factorization is not unique because p-k-1 ≤ -1. As $(\Phi^*)^{-1}\gamma$ is rational, it is possible to choice a factorization where the polynomial part of $(\Phi^*)^{-1}\gamma$ is taken on the unstable factor. This factorization will be denoted by:

$$(\Phi^*)^{-1}\gamma = [(\Phi^*)^{-1}\gamma]_+ + [(\Phi^*)^{-1}\gamma]_-,$$

with $\partial_r([(\Phi^*)^{-1}\gamma]_+) \ge 1$, $[(\Phi^*)^{-1}\gamma]_+ \in H_+^{2,p-k-1}$, $[(\Phi^*)^{-1}\gamma]_- \in H_-^{2,p-k-1}$.

Actually, $[(\Phi^*)^{-1}\gamma]_+ \in H_+^2$ because it is a stable strictly proper rational function with all its poles in C_-^0.

If $p+1 \ge k$, $L_{p-k-1}^2(i\mathbb{R}) \subset L^2(i\mathbb{R})$, p-k-1 ≥ 0. The above factorization will be interpreted as $[(\Phi^*)^{-1}\gamma]_+ \in H_+^{2,p-k-1} \subset H_+^2$, $[(\Phi^*)^{-1}\gamma]_- \in H_-^{2,p-k-1} \subset H_-^2$, because $\partial_r([(\Phi^*)^{-1}\gamma]_+) \ge p-k \ge 1$. Note that all the stable projections are denoted by $[.]_+$, but the different spaces will be clear from the context.

With this notation, the linear part of $J[K]$ becomes

$$-2\langle \Phi K, [(\Phi^*)^{-1}\gamma]_+\rangle_2 - 2\int_{-\infty}^{\infty} K^*(i\omega)\Phi^*(i\omega)[(\Phi^*(i\omega))^{-1}\gamma(i\omega)]_- \, d\omega$$

$$= -2\langle \Phi K, [(\Phi^*)^{-1}\gamma]_+\rangle_2,$$

the integral being zero because all the integrand poles are in C_+^0 and its relative degree is less or equal than 2 (the residue theorem applied to a circuit involving C_-^0 proves the statement – see [8]). In other words, the unstable term $[(\Phi^*)^{-1}\gamma]_-$ is orthogonal to the stable function ΦK. Then, completing the square in (3.2), we get

$$J[K] = \|\Phi K\|_2^2 - 2\langle \Phi K, [(\Phi^*)^{-1}\gamma]_+\rangle_2 + \|[(\Phi^*)^{-1}\gamma]_+\|_2^2 - \|[(\Phi^*)^{-1}\gamma]_+\|_2^2$$

$$= \|\Phi K - [(\Phi^*)^{-1}\gamma]_+\|_2^2 - \|[(\Phi^*)^{-1}\gamma]_+\|_2^2.$$

Therefore, the minimum of $J[.]$ is attained at $\breve{K}$ such that $\Phi\breve{K} - [(\Phi^*)^{-1}\gamma]_+ = 0$, if $J[\breve{K}] < \infty$, i.e., if $\breve{K}$ belongs to $J[.]$ effective domain, $H_+^{2,-m}$. These conclusions are collected in the next theorem.

THEOREM 6. Let assumptions A1, A2 be verified, $\breve{K}$ a rational function given by:

(3.3) $\quad \breve{K} = \Phi^{-1}[(\Phi^*)^{-1}\gamma]_+,$

where $[(\Phi^*)^{-1}\gamma]_+$ denote the stable strictly proper part of $(\Phi^*)^{-1}\gamma$, $m = \min\{k, p-1\}$. If $\breve{K} \in H_+^{2,-m}$ (the $J[.]$ effective domain) then $\inf\{J[K]\} = J[\breve{K}]$ in $H_+^{2,-m}$. □

As it was commented on above, in common H^2/H^∞ problems, $K(s)$ is a proper stable real-rational function, which means $\partial_r(K) \ge 0$. In the mathematical framework presented here, this implies $K \in H_+^{2,-1}$. This situation is explored in the next corollary, easily proved from Theorem 6 and the calculations above. Note that the condition $H_+^{2,-m} \supset H_+^{2,-1}$ is not necessary, but only the condition $\breve{K} \in H_+^{2,-q} \subset H_+^{2,-1}$ with $H_+^{2,-q} \subset H_+^{2,-m}$, for some $q \le 1$.

COROLLARY 1. Under the same assumptions as in Theorem 6, $\partial_r(\breve{K}) \ge 0$ if and only if $\partial_r([(\Phi^*)^{-1}\gamma]_+) \ge k$. Sufficient conditions for this conclusion are $p = \partial_r(\gamma) \ge 2k$ or $k = \partial_r(\Phi) \le 1$. □

REMARK 6. The sufficient conditions in Corollary 1 are not necessary. Indeed, for any Φ with $\partial_r(\Phi) = k$ and for any $q \le k$, it is possible to find a function $\gamma(s)$ as in (3.1) such that $\partial_r(\breve{K}) = 1-q$ and $J[\breve{K}] < \infty$. For that, let $\gamma = \Phi^* B$, $B \in L^2(i\mathbb{R})$ such that

$\partial_r([B]_+) = 1+k-q$, which is always possible if $q \le k$. Note that $k \le p-1 = \partial_r(\gamma) - 1$, which implies that $J[.]$ is well defined in $H_+^{2,-k}$. Then

$$\partial_r(\breve{K}) = \partial_r(\Phi^{-1}[(\Phi^*)^{-1}\Phi^* B]_+) = \partial_r(\Phi^{-1}[B]_+) = 1-q.$$

Also, $\breve{K} \in H_+^{2,-k}$, then $J[\breve{K}] < \infty$. □

In the control context, criteria as (3.1) usually appear from the sum of functional in the form

(3.4) $\| AK + B \|_2^2 =$

$$\int_{-\infty}^{\infty} \{K^*(i\omega)A^*(i\omega)A(i\omega)K(i\omega) - 2K^*(i\omega)A^*(i\omega)B(i\omega) + B^*(i\omega)B(i\omega)\}d\omega,$$

where $B(i\omega) \in L^2(i\mathbb{R})$ is a real-rational strictly proper function with $\partial_r(B) \ge 1$. For each simple functional, by direct verification, $\Phi(s) = A(s)$, $\gamma(s) = A^*(s)B(s)$. Therefore $\partial_r(\gamma) = \partial_r(\Phi)+\partial_r(B)$, which implies the condition:

(3.5) $\partial_r(\gamma) \ge \partial_r(\Phi)+1$.

Condition (3.5) is inherited by a sum of quadratic functional as (3.4) and will strongly simplify the use of Theorem 6. Indeed, under such condition, the function $[\Phi^*(i\omega)]^{-1}\gamma(i\omega) \in L^2(i\mathbb{R})$ because condition (3.5) is exactly the condition $p+1 \ge k$. Then the decomposition used to prove Theorem 6 will be the usual $L^2(i\mathbb{R}) = H_+^2 \oplus H_-^2$, which no need to consider larger spaces. In other words, $[(\Phi^*)^{-1}\gamma]_+$ is the usual projection on H_+^2. Moreover, $m = \min\{k, p-1\} = k$. Therefore, $\breve{K} = \Phi^{-1}[(\Phi^*)^{-1}\gamma]_+$ is a rational function with $\partial_r(\breve{K}) \ge \partial_r([(\Phi^*)^{-1}\gamma]_+) - \partial_r(\Phi) = 1-k$, which implies $J[\breve{K}] < \infty$ and $\breve{K} \in H_+^{2,-m} = H_+^{2,-k}$ with no further condition. In the other sense, if $H_+^{2,-m} = H_+^{2,-k}$, $m = k \le p-1$, which implies (3.5).

These remarks are collected in the next corollary.

COROLLARY 2. Let assumptions A1, A2 hold. Then condition (3.5) is equivalent to the effective domain of $J[.]$ be $H_+^{2,-k}$. In this case the function $\breve{K}(s)$ given by (3.3) is such that $\inf\{J[K]\} = J[\breve{K}]$ in $H_+^{2,-k}$, $[.]_+$ denoting the usual orthogonal projection on H_+^2. □

REMARK 7. The conditions found in the literature about the unconstrained problem are particular cases of assumptions in Corollaries 1 and 2 [8]. In special, in [9] a well-motivated criterion is presented such that these conditions are naturally verified. □

REMARK 8. If $\Gamma(s)$ has zeroes on the imaginary axis, $\Phi(s)$ will have the same zeroes if it is used the generalized Wiener-Hopf factorization as in [27]. Then $\breve{K}$ given in (3.2) will have these zeroes as poles, being unstable. In other words, the completion of R_0 in the norm induced by the quadratic part of $J[.]$ will contain, in this case, unstable rational functions, the minimum being attained in such a function. □

Remark 6 shows that $\partial_r(\breve{K})$ can be different from $1-m$, where the $J[.]$ effective domain is $H_+^{2,-m}$. This possibility will be essential to the algorithm convergence regularity, see section 5 above. Corollary 1 gives conditions for $\partial_r(\breve{K}) \ge 0$ if $m \ge 1$. The same considerations used in its proof can be generalized to any relative degree for the optimal solution. Actually, much of the work founded in the literature can be linked with this search of regularity. It was essential in the existence proofs in [8], [9] and in some seminal but unclear comments in [2]. Moreover, a lot of work was needed in [9] to define a natural criterion such that $\partial_r(\breve{K}) \geq 0$ for all linear systems for which the servomechanism problem proposed there is solvable. This natural criteria verify assumptions A1, A2 and condition (3.5) with $k = p = 1$. Then, by Corollary 2, $m = k$, the effective domain is exactly $H_+^{2,-1}$, which eases considerably the application of the methodology proposed therein.

4. Optimal $\mathbf{H^2/H^\infty}$ control problems. This section presents the mathematical extension of the usual H^2/H^∞ control problem on the mathematical framework developed in section 2. The optimal solution existence and uniqueness will be proved in the following and regularity results will be presented.

In the H^2/H^∞ optimal control problem the goal is to find a function $\hat{K}(s)$ solution to :

$$(4.1)\qquad \inf_{K\in\Omega\cap\Theta}\{\int_{-\infty}^{\infty}[K^*(i\omega)\Gamma(i\omega)K(i\omega)-2K^*(i\omega)\gamma(i\omega)]d\omega\} \equiv \inf_{K\in\Omega\cap\Theta} J[K],$$

where Ω is a bounded closed convex subset of H_+^∞ and Θ is a bounded closed convex subset of H_+^2. The usual examples of sets Ω and Θ arising from performance, filtering and robustness specifications are

$\Omega = \bigcap_{m=1}^{M}\Omega_m$; $\Omega_m = \{K\in H_+^\infty$ *such that* $\|A_mK+B_m\|_\infty\leq\lambda_m\}$, A_m and B_m functions in H_+^∞;

$\Theta = \bigcap_{n=1}^{N}\Theta_n$; $\Theta_n = \{K\in H_+^2$ *such that* $\|C_nK+D_n\|_2\leq\mu_n\}$, $C_n\in H_+^\infty$ and $D_n\in H_+^2$;

λ_m and μ_n positive real numbers so that the set Ω is nonempty.

Now, under the assumptions of Lemma 1, the criterion functional in (4.1) is strictly convex and continuous in its effective domain, $H_+^{2,-m}$. By Theorem 3a, the set Θ is convex, bounded and closed in $H_+^{2,-m}$ for $m \geq 0$, as a convex, bounded and closed subset of H_+^2. By Theorem 3b, Ω is convex, bounded and closed in $H_+^{2,-1}$, as a convex, bounded and closed subset of H_+^∞. Then, Ω is convex, bounded and closed in $H_+^{2,-m}$ for $m \geq 1$, by Theorem 3a. Therefore, we are in conditions to apply a well-known theorem to show the existence and uniqueness of the optimal solution for problem (4.1).

THEOREM 7. Let assumptions A1, A2 with $\partial_r(\Gamma) \geq 2$, $\partial_r(\gamma) \geq 2$ be verified. Then,

(a) if the constraint set $\Omega\cap\Theta$ is nonempty, the optimal control problem (4.1) has one and only one solution in $H_+^{2,-1}$;

(b) if Ω is nonempty, the optimal solution is in H_+^∞; if Θ is nonempty, the optimal solution is in H_+^2; □

Proof. By the comments above, Lemma 1 and the assumptions, the functional in (4.1) is strictly convex and continuous in $H_+^{2,-m}$, for some $m \geq 1$. Also, the constraint set is convex, closed and bounded in $H_+^{2,-1}$. Then Theorem 1.4.1, [28], page 9 applies, proving the first statement. The second statement is clear. ■

Naturally, it is possible to add $H_+^{2,-1}$ closed convex subsets as new constraints without changing the above conclusions.

REMARK 9. A direct consequence of this last theorem is the convergence of the approximating sequence generated by the algorithm proposed in [1] to the optimal solution of problem (4.1). In the same paper it is showed that the optimal control, if it exists, belongs to the H^∞ constraint boundary. Also, it is explicitly solved the H^2 optimal control problem with only H^2 constraints by duality, a key to the method proposed therein. □

Before the presentation of numerical methods to solve the optimal control problem (4.1) it will be interesting to rewrite this problem as a minimal norm problem, a step in the strong convergence proof. Assume that $\breve{K} \in H_+^{2,-q} \subset H_+^{2,-m}$ for some $q \le m = \inf\{k,p\text{-}1\}$. Then $\Phi \breve{K} \in H_+^2$. Now, the calculations used to prove Theorem 6 give

(4.2) $\mathcal{J}[K] = \| \Phi\{K - \Phi^{-1}[(\Phi^*)^{-1}\gamma]_+\} \|_2^2 - \| \Phi\{\Phi^{-1}[(\Phi^*)^{-1}\gamma]_+\} \|_2^2$
$= \| \Phi(K - \breve{K}) \|_2^2 - \| \Phi\breve{K} \|_2^2$.

NOTATION. Let $\| f \|_\Gamma = \| \Phi f \|_2$, a norm associated to $\mathcal{J}[.]$ quadratic term, $\langle f, g\rangle_\Gamma = \langle \Phi f, \Phi g\rangle_\Gamma$ the associated internal product. □

Theorem 5b says that if assumption A1 is verified $\| f \|_\Gamma$ defines a norm on $H_+^{2,-k}$ equivalent to the norm $\| . \|_{2,-k}$. Then

(4.3) $\mathcal{J}[K] = \| K - \breve{K} \|_\Gamma^2 - \| \breve{K} \|_\Gamma^2$.

Therefore, under the assumptions of Theorem 7, the optimal control problem (4.1) is equivalent to find a function $\breve{K}$ solution to

(4.4) $\inf\limits_{\Omega\cap\Theta} \| K - \breve{K} \|_\Gamma^2$,

a best approximation problem in $H_+^{2,-k}$. Note that, if condition (3.5) is verified, $H_+^{2,-k} = H_+^{2,-m}$, but here it is needed only that $\breve{K} \in H_+^{2,-q} \subset H_+^{2,-m} \subset H_+^{2,-k}$.

COROLLARY 3. Let assumptions A1, A2 hold with $\partial_r(\Gamma) \ge 2$, $\partial_r(\gamma) \ge 2$. Problems (4.1) and (4.4) are equivalent if and only if $\breve{K} \in H_+^{2,-q} \subset H_+^{2,-m} \subset H_+^{2,-k}$, i.e., $\partial_r(\breve{K}) \ge 1\text{-}m$, $m = \min\{k,p\text{-}1\}$. Moreover, assumptions A1, A2 with $\partial_r(\Gamma) \ge 2$, $\partial_r(\gamma) \ge 2$ and condition (3.5) are sufficient for the same conclusion. □

Proof. According to the above comments, the first statement is a consequence of Theorem 7 and the second statement is a consequence of Theorem 7 and Corollary 2. ■

Optimal control problem (4.1) can be rewritten as a minimal norm problem in $H_+^{2,-k}$ if this space is translated by $\breve{K}$. For that, redefine $G = K - \breve{K}$, $\Omega' = \Omega - \breve{K}$, $\Theta' = \Theta - \breve{K}$. Note that Ω' and Θ' are convex, bounded and closed in $H_+^{2,-k}$ because these properties are not changed by translations in a Hilbert space. In these notations the optimization problem (4.4) can be translated as the new problem: find $\hat{G} \in H_+^{2,-k}$ solution to

(4.5) $\inf\limits_{G\in\Omega'\cap\Theta'} || G ||_\Gamma^2$,

a minimal norm problem. Note that $\breve{K}$ could not belong to $H_+^{2,-1}$. Thus problem (4.5) shall be solved carefully from a numerical point of view.

Regularity now is essential. If the optimal control problem needs to be solved in some $H_+^{2,-q}$ as a minimal norm problem, beyond the existence conditions in Theorem 7, it will be also needed the condition $H_+^{2,-q} = H_+^{2,-k} \subset H_+^{2,-m}$. This means $q = k \le m$, with $\breve{K} \in H_+^{2,-q}$, or, more exactly, $\breve{K} \in H_+^{2,-r}$, for some $r \le q$. For that, Corollary 1 (and its extensions) and Corollary 2 are useful. The usual setting is $q = 1$, as in [8], [20], [21], or, in a more restrict way, $q = k = 1$, as in [9]. In the present paper this setting was generalized to well understanding weak and strong convergence of the algorithm proposed in the next section.

5. Gallerkin method. If $\{\beta_n,\ n\in N\}$ is a generator set for $H_+^{2,-1}$, not necessarily orthogonal, denote by H_n the finite-dimensional subspace generated by the n first vectors in the generator set. If $\Omega_n = \Omega\cap\Theta\cap H_n$, it is possible to project the optimal control problem (2.3) in H_n, which defines the following finite-dimensional optimization problem: find $\hat{K}_n$ in H_n solution to

(5.1) $$\inf_{K\in\Omega_n} \{\| K \|_\Gamma^2 - 2\langle K,\gamma\rangle_2\}.$$

As Ω_n is a bounded closed convex subset of H_n and the criterion is strictly convex, this optimal control problem has one and only one solution $\hat{K}_n$ in H_n, for each $n\in\mathbb{N}$ (see [28], page 9). The Gallerkin method consists in approximating the optimal solution $\hat{K}$ to the optimal control problem (3) by $\hat{K}_n$, if the sequence $\{\hat{K}_n\}$ converges to the optimal solution $\hat{K}$. This is the content of the next theorem.

THEOREM 8. Let the assumptions in Theorem 7 and assume that $\breve{K}$ does not belong to $\Omega\cap\Theta$ (otherwise the optimal solution will be $\breve{K}$). Then the sequence $\{\hat{K}_n\}$ generated by Gallerkin method converges weakly in $H_+^{2,-1}$ to the unique optimal solution $\hat{K}$ to the optimal control problem (2.2). □

Under the assumptions of Corollary 3, including (3.5), the optimal control problem (4.1) can be rewritten as minimal norm problems (4.4) and (4.5), which will allow to show the strong convergence of the sequence $\{\hat{K}_n\}$ in suitable spaces. For that, let $\{\beta_n, n\in\mathbb{N}\}$ be a generator set for $H_+^{2,-k}$ and $\|.\|_\Gamma$ the norm defined in section 4. Thus we can define the projection of the minimal norm problem (4.4) in H_n as find $\hat{K}_n$ in H_n solution to

(5.2) $$\inf_{K\in\Omega_n} \| K - \breve{K}_n \|_\Gamma^2,$$

where $\breve{K}_n$ is the projection of $\breve{K}$ in H_n. Analogously, translating H_n by $\breve{K}_n$, the minimal norm problem (4.5) can be projected, the minimal norm problem becomes how to find $\hat{G}_n$ solution of:

(5.3) $$\inf_{G\in\Omega_n'} \| G \|_\Gamma^2,$$

where $\Omega_n' = \Omega_n - \breve{K}_n$. As Ω_n and Ω_n' are bounded closed convex sets, the optimization problems (5.2) and (5.3) have one and only one solution, defining sequences of functions approximating the optimal solution to optimal norm problems (4.4) and (4.5), for $n\in\mathbb{N}$.

THEOREM 9. Let assumptions A1, A2, and condition (3.5) be verified. Also, assume that $\partial_r(\Gamma) \geq 2$, $\partial_r(\gamma) \geq 2$, $\breve{K}$ not belonging to $\Omega\cap\Theta$. Then, the sequences $\{\hat{K}_n\}$

and $\{\hat{G}_n\}$ of solutions to problems (5.2) and (5.3) for all $n\in\mathbb{N}$, converge strongly in $H_+^{2,-k}$ to the optimal solutions to problems (4.4) and (4.5), respectively. □

REMARK 10. Note that the strong convergence in $H_+^{2,-1}$ arrives only if k = 1 and condition (3.5) is verified, as in [9]. □

In Theorem 8 proof, (5.1) and (5.2) are characterized by linear variational inequalities on $H_+^{2,-k}$. Gallerkin methods are powerful to solve this type of inequality in functional spaces [29], which give linear matrix inequalities (LMI) after the choice of a basis for $H_+^{2,-k}$. Another approach to problems (4.4) and (4.5) is the one presented under the name of best approximation, using convex projections or proximinal maps (the mapping from $\breve{K}$ to $\hat{K}$). This approach is interesting for minimum norm problems in Hilbert spaces, as in the present paper, where the proximinal map is continuous ([30], pages 157, 164). The same reference shows the difficulties when the problem is considered in H_+^∞, a not reflexive Banach space (see [30], page 77).

Theorem 8, 9 speak about convergence in $H_+^{2,-1}$, not in H_+^∞. In general, strong $H_+^{2,-1}$ convergence does not imply H_+^∞ strong convergence. It allows spikes in sequences converging to zero, as in $f_n(s) = (ns+1)^{-1}$ (see Remark 5 proof in Appendix 1). Actually, $\hat{K}_n \to \hat{K}$ strongly in $H_+^{2,-1}$ implies $\Phi_{-1}\hat{K}_n \to \Phi_{-1}\hat{K}$ in measure on the imaginary axis and $\hat{K}_n \to \hat{K}$ in measure on any finite measure subset of the imaginary axis (in this case the $H_+^{2,-1}$ and H_+^2 strong topologies coincide). From Theorem 7.11, page 73, in [38], this implies the almost uniform convergence on the finite measure subset. But this result does not imply the H_+^∞ strong convergence even in those subsets. In spite of these difficulties, the next theorem and remark show some relevant results in H_+^∞.

THEOREM 10. If the sequence $\hat{K}_n$ converges to $\hat{K}$ strongly in $H_+^{2,-1}$, as in Theorem 9, it then converges to $\hat{K}$ in the weak topology of H_+^∞. □

REMARK 11. If the sequence $\hat{K}_n$ converges to $\hat{K}$ weakly in $H_+^{2,-1}$, as in Theorem 8, then it is possible to prove, after some identifications, that $\hat{K}_n$ converges to $\hat{K}$ in the weak-star topology of (H_+^∞)'. □

To end the theoretical presentation of Gallerkin methods some generator set for $H_+^{2,-1}$ and for $H_+^{2,-k}$ must be presented. Due to the density of H_+^2 in $H_+^{2,-k}$, $k \geq 1$, any one of the basis obtained from the Runge Theorem [31] for the space of analytic functions on C_+^0 can be used. Note that the topology used in Runge Theorem (the topology of the uniform convergence in all compacts in C_+^0) is finer than the $L^2(i\mathbb{R})$ topology. An example, already used in [15], is the Laguerre orthonormal basis in H_+^2:

$\{L_n = \dfrac{1}{\sqrt{2a}}\dfrac{(s-a)^{n-1}}{(s+a)^n}, n\in\mathbb{N}\}$, for each positive real number a.

The numerical experiments in [21] show the interest in the use of redundant sets of generators, as

$$\{L_0 = 1,\ L_n = \frac{1}{\sqrt{2a}}\frac{(s-a)^{n-1}}{(s+a)^n},\ n\in\mathbb{N}\},$$ for each positive real number a,

which capture more quickly the asymptotic behavior of the optimal solutions. The proofs of Theorems 8 and 9 apply to these redundant sets without changes.

An orthonormal basis for $H_+^{2,-1}$, in relation to the inner product $\langle..,..\rangle_{\Gamma_{-1}}$, is given by

$$\{M_0 = (\|\ \Phi\ \|_2)^{-1},\ M_n = \frac{(1-s)}{\sqrt{2a}}\frac{(s-a)^{n-1}}{(s+a)^n},\ n\in\mathbb{N}\}.$$

Note that $\partial_r(M_n) = 0$, differently from Laguerre basis. Reference [21] presents other orthonormal basis for $H_+^{2,-k}$ built under the same principle, with the poles of Φ^* as the zeroes of the basis functions. The numerical solution of problem (5.2) needs some mathematical programming developments [21], which will be presented in a future paper. Some comments about the numerical procedure following the developments in [21] will be presented next.

After the choice of a redundant generator set, say $\{1,\beta_n,\ n\in\mathbb{N}\}$, and the choice of the number of poles of $K_n(s)$, say n, the functions $K_n(s)$ in the finite-dimensional space H_n can be represented as:

$$K_n(s) = \sum_{m=0}^{n}\alpha_m\beta_m(s),$$

where the constant function is represented by $\beta_0(s)$. By substitution of this last expression in (5.1) or (5.2) it is defined a $(n+1)$-dimensional programming problem which variable is the $(n+1)$-vector $\vec{\alpha} = (\alpha_0,\alpha_1,...,\alpha_n)^T$. The integrals in the quadratic functional calculation can be performed analytically, being this functional quadratic in $\vec{\alpha}$. The quadratic constraints are calculated by the same methods, but not the H^∞ constraints. Actually, there is no need to explicitly calculate these hard constraints, but only a generalized gradient. The reason is that the finite-dimensional constrained optimization problem was solved by a penalty method coupled with the known BFGS algorithm, where the position of the H^∞ constraint gradient (which do not exist) was filled by a generalized gradient. If this constraint is represented by

$$\sup|A(i\omega)K_n(i\omega)+B(i\omega)|-\lambda\leq 0,$$

it is proved in [21] that the derivative of $|A(i\omega)K_n(i\omega)+B(i\omega)|$ for $\omega = \omega_0$, ω_0 one of the values where this function assumes its maximum, is a generalized gradient for the constraint. The ω_0 calculation uses the tools of H^∞ theory, as showed in [6]. Note that the procedure should consider also the case where $\omega_0 = \infty$. The convergence of this procedure was proved in [21], and the authors did not find significant problems to obtain the optimal parameters $\hat{K}_n$ after perform the functional calculations through state variable and Lyapunov equation tools.

5. Numerical example. The example shown here was developed in [21], where a more complete discussion can be found. It represents the pitch optimal control of a fight airplane described in [32] to exemplify LQG/LTR design and it is used in [33] to exemplify the dual method from Corrêa [1]. In this example the transfer function from the elevation angle to the aptitude angle is

$$P(s) = \frac{-(948{,}12s^3 + 30325s^2 + 56482s + 1215.3)}{s^6 + 64.554s^5 + 1167s^4 + 372.86s^3 - 5495.4s^2 + 1102s + 708.1},$$

the quadratic criterion being the one defined in [9] with weighting filters and weighting coefficients given by

$$\phi_w(s) = 0,\ \phi_d(s) = \frac{1}{s^2+2s+2},\ \phi_v(s) = \frac{1}{s+10},\ \rho_v = \rho_d^u = \rho_v^u = 1.$$

After some calculations, the optimal control problem criterion can be transformed in

$$J_2[K(s)] = \| A(s) + B(s)K(s) \|_2^2 + J_F,$$

where A(s) and B(s) are 14th and 10th-order rational functions (presented in the Appendix), both with unitary relative degree, J_F = 0.30612 and $K(s)$ is the rational proper and stable Youla parameter. The stability margin functional for the control problem, after some transformations to put it in Nehari form [21] is given by

$$J_\infty[K(s)] = \| K(s) - F_0(s) \|_\infty,$$

$F_0(s)$ a second order unstable proper rational function (also presented in Appendix 2) with unitary relative degree. The minimum value for $J_\infty[K(s)]$, i.e., the optimal stability margin, is 0.610513.

If we define the robustness constraint allowing a 10% degradation of the optimal stability margin, the H^2/H$^\infty$ problem to be solved becomes:

"find $K(s)$ minimizing $J_2[K(s)]$ subject to $J_\infty[K(s)] \le \gamma = 0.6715643$".

Assumptions A1, A2, with k = p = 1, condition (3.5) and the others conditions on Theorem 7 are verified. Then, by Theorem 7 this problem has one and only one solution in $H_+^{2,-1}$, belonging to H_+^∞ Also, by Theorem 9 the sequence of functions generated by Gallerkin method, as exposed in section 5, converges strongly to the H^2/H$^\infty$ problem optimal solution, for any basis or redundant generator set in $H_+^{2,-1}$.

Table 1 presents some characteristics of controllers solving related optimal control problems, where $K(s)$ is the optimization parameter used to obtain a controller by the Youla parameterization. There,

- $K_{H2}(s)$ represents the Youla parameter corresponding to the controller minimizing the quadratic criterion $J_2[K(s)]$ without constraints (the H^2 optimal controller),
- $K_{H\infty}(s)$ represents the Youla parameter corresponding to the controller minimizing the stability margin (the H$^\infty$ optimal controller),
- $K_{SPQ}(s)$ represents the Youla parameter corresponding to a nonfeasible controller approximating the H^2/H$^\infty$ problem solution (with γ = 0.6715643) calculated by the dual method from Corrêa [1],
- $K_{SPQR}(s)$ represents the Youla parameter corresponding to a reduced order controller generated from $K_{SPQ}(s)$ by truncation of a balanced realization.

TABLE 1. *Characteristics of some related controllers.*

$K(s)$	$J_2[K(s)]$	$J_\infty[K(s)]$	order
$K_{H2}(s)$	0.306120137	2.07804793	17
$K_{H\infty}(s)$	3.964188309	0.61051297	1
$K_{SPQ}(s)$	2.141469573	0.67180700	29
$K_{SPQR}(s)$	2.141470588	0.67193138	14

Note that $K_{SPQ}(s)$ and $K_{SPQR}(s)$ do not verify the stability margin constraint, as expected, i.e., they are not feasible.

Table 2 presents the same characteristics for the controllers obtained by Gallerkin method, $n = 1,...,9$, using the redundant generator set based on Laguerre functions as in section 4, $\gamma = 0.6715643$:

TABLE 2. *Characteristics of optimal Gallerkin controllers for extended Laguerre functions.*

$K(s)$	$J_2[K(s)]$	$J_\infty[K(s)]$	order
$K_1(s)$	2.436117	0.6715643	1
$K_2(s)$	2.367955	0.6715643	2
$K_3(s)$	2.346453	0.6715643	3
$K_4(s)$	2.250182	0.6715643	4
$K_5(s)$	2.209556	0.6715643	5
$K_6(s)$	2.207430	0.6715643	6
$K_7(s)$	2.206113	0.6715643	7
$K_8(s)$	2.191661	0.6715643	8
$K_9(s)$	2.175038	0.6715643	9

First, all solutions are feasible, as expected. Second, the greater the order, the smaller the quadratic criterion value. Third, comparing this value for $K_9(s)$ and $K_{SPQ}(s)$ and using the dual solutions properties, we verify that

$$J_2[K_{SPQ}(s)] = 2.141469573 < J_2[\hat{K}(s)] < 2.175038 = J_2[K_9(s)],$$

$\hat{K}(s)$ being the H^2/H^∞ problem optimal solution. Therefore, the difference between the quadratic criterion values error of $K_9(s)$ and the quadratic criterion values error of the optimal solution is less than 1.54%.

Table 3 presents the same characteristics for the optimal controllers obtained by Gallerkin method using a $H_+^{2,-1}$ basis obtained step by step by minimization of the quadratic criterion (under the H^∞ constraint) as a function of both of the basis coefficients and the basis poles [21]. The optimization problem to be solved for each dimension n is not convex. Then, the usual optimization algorithms give only H_n locally optimal solutions, depending on the algorithm initialization. The BFGS method extended for generalized gradients was used to solve the finite-dimensional optimization problems, the constraints considered by a Lagrangean method [21]. As above, $\gamma = 0.6715643$.

TABLE 3. *Characteristics of optimal Gallerkin controllers for "optimal step-by-step" basis.*

$K(s)$	$J_2[K(s)]$	$J_\infty[K(s)]$	order
$K_{0A}(s)$	2.651499	0.6715643	0
$K_{1A}(s)$	2.417010	0.6715643	1
$K_{2A}(s)$	2.412348	0.6715643	2
$K_{3A}(s)$	2.278789	0.6715643	3
$K_{4A}(s)$	2.195134	0.6715643	4
$K_{5A}(s)$	2.195134	0.6715643	5
$K_{6A}(s)$	2.164122	0.6715643	6

Note that $K_{4A}(s)$ and $K_{5A}(s)$ present the same characteristics, the new coefficients calculated for $K_{5A}(s)$ being zero: the new dimension did not allow a smaller criterion value for the chosen initialization vector. The local character of the n-dimensional numerical optimization and its dependence on the initialization vector is showed by the worst behavior of $K_{2A}(s)$ in relation to $K_2(s)$. In spite of those difficulties, the 6th-order controller attains a smaller criterion value than $K_9(s)$, which allows us to find a best estimation for the criterion optimal value and a best approximation for the optimal controller (corresponding to $K_{6A}(s)$):

$$J_2[K_{SPQ}(s)] = 2.141469573 < J_2[\hat{K}(s)] < 2.164122 = J_2[K_{6A}(s)],$$

with a relative error smaller than 1.05%.

For the sake of comparison, Figures 6.1, 6.2 and 6.3 show the Bode diagrams for the functions $K_{SPQ}(s) - F_0(s)$, $K_9(s) - F_0(s)$, and $K_{6A}(s) - F_0(s)$, respectively. It was verified in [21] that Bode diagrams for the Gallerkin approximations do not suffer significant changes after a sufficiently great dimension n and they do not present "spikes", in spite of the discussion just before Theorem 10.

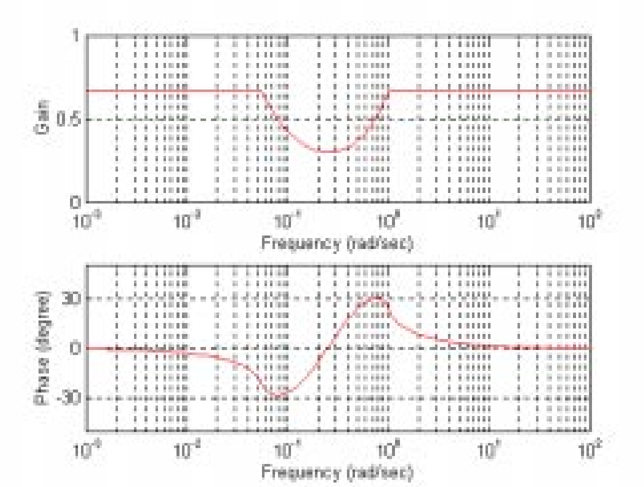

FIG. 6.1. *Bode diagrams for the function $K_{SPQ}(s) - F_0(s)$.*

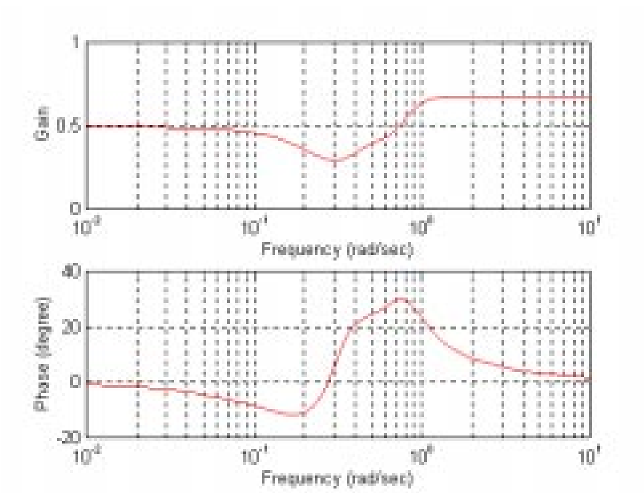

FIG. 6.2. *Bode diagrams for the function $K_9(s) - F_0(s)$.*

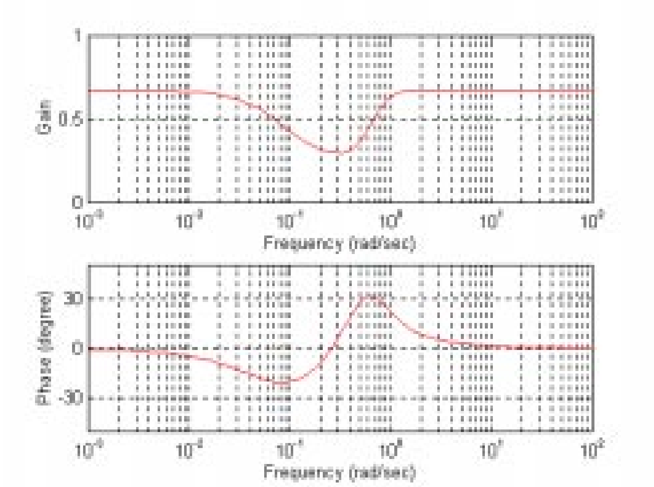

FIG. 6.3. *Bode diagrams for the function $K_{6A}(s) - F_0(s)$.*

Numerical calculations where performed in a PC computer using MATLAB programming.

7. Conclusions and comments. In this paper the H^2/H^∞ was studied in the context of weighted Hardy spaces, allowing the proof of the existence and uniqueness of its solution and the proof of the convergence of Gallerkin method. The extension of these results to the multivariable case is straightforward but tedious, in view of the existing techniques presented, for example, in [8] and [34].

Essentially, if $M[A]$ denotes the set of matrices with entries in A and the dimensions established by the context, K^T the transpose of the matrix K, $[K(s)]^* = [K(-s)]^T$, $\Phi(s)$ a maximal rank real-rational matrix in $M[R_{1\text{-}k}]$ with all its poles and zeroes in C_+^0, $\Gamma(s) = \Phi^*(s)\Phi(s)$ a maximal rank real-rational para-hermitian matrix,

$$\langle K,G\rangle_\Gamma = \int_{-\infty}^{\infty} Trace\{K^*(i\omega)\Gamma(i\omega)G(i\omega)d\omega,\ \| K \|_\Gamma = [\langle K,K\rangle_\Gamma]^{1/2},$$

$\| K \|_\infty = \bar{\sigma}\{\| K_{jk} \|_\infty\}$ (the greatest singular value of the matrix which entries are the H^∞ norm of the K-entries), the results presented in this paper can be rewritten *ipsis literis* on the spaces $M[L^2_{-k}(i\mathbb{R})]$, $M[H_+^{2,-k}]$, $M[H_-^{2,-k}]$, $M[H_+^\infty]$ and $M[H_-^\infty]$, with the use of $M[R_k]$, $M[R_k^+]$ and $M[R_k^-]$ and the obvious adaptations in notations and proofs. A serious problem not considered in this paper is the great number of entries in multivariable basis, which increases dramatically the number of parameters in the optimization problems (4.1), (4.2) and (4.3). The most parcimonious basis, with the same poles in all the entries of the rational matrix $K(s)$, uses as much parameters as the product of the entries number by the number of parameters in the one-dimensional problem (i.e., for a m-dimensional problem on H_n, we have a nm-dimensional optimization problem).

The algorithms presented in section 5 and 6 do not explore all the theoretical possibilities. The freedom in the choice of a generator set allowed by Runge Theorem linked to a convenient use of model order reduction algorithms by balanced realizations can be used to build an algorithm optimizing, in a certain sense, the generator set used in each step of Gallerkin method. The numerical behavior and the convergence of such algorithm are better than the simpler algorithms proposed in this paper as it is shown in [21]. The presentation of the "optimized basis" methodology will be the subject of a next paper, where the mathematical programming algorithms to be used will be carefully developed.

There is a point to stress: the relevance of the dual algorithm presented in [1], giving inferior bounds to the optimal solution, allowing a good control of the approximation error, undoubtedly a very useful tool not always present in Gallerkin methods.

Now, the possibility of extensions of some results to other problems will be considered. First, sometimes there is interest in weighted H_+^∞ spaces where the norm is defined as $\| K \|_{\infty,\Phi}$ = *ess. sup.* $|\Phi(i\omega)K(i\omega)|$, Φ as in assumption A1. Note that closed balls in this new space are closed in $H_+^{2,-(k+1)}$ if $\partial_r(\Phi) = k$, which allows to easily extend the theory exposed in this paper to this new context. Second, the assumption that the weighting filters $\Phi(s)$ are rational functions is natural in finite-dimensional control problems, but it can be generalized to assume the weighting filters as $H_+^{2,k}$ functions, with no restrictions to rational ones. The only real constraints are the integrability conditions, the non-existence of zeroes on $i\mathbb{R}$ and the asymptotic order at $s = \pm\infty$. With these changes and the extension of Youla Theorem to this new setting, the theory developed here can be applied to strongly stabilizable infinite-dimensional linear systems, a problem presently in study.

A last comment is about the so-called L^1 problem [35]. Denoting by A the algebra of the stable impulse functions [36],

$$A = \{F : f(t) + \sum_{n=1}^{\infty} a_n \delta(t - t_n), f(.) \in L^1[0,\infty), t_n \geq 0, \sum_{n=1}^{\infty} | a_n | < \infty\}$$

and $\hat{A}$ its Laplace transform, it can be shown (with the appropriate identifications) that

$$\hat{A} \subset C^0(i\mathbb{R}) \cap H_+^\infty \subset H_+^{2,-1},$$

$C^0(i\mathbb{R})$ denoting the continuous functions in $i\mathbb{R}$ and $L^1[0,\infty)$ the usual space of integrable functions. Moreover, if A is normed with the sum of the L^1-norm of f plus the l_1-norm of $\{a_n\}$, bounded sets in A have theirs Laplace images bounded in H_+^∞ and in $H_+^{2,-1}$. In spite of that, it is possible to show bounded sequences in $L^1[0,\infty)$ which Laplace transform converges to a discontinuous function on $i\mathbb{R}$ in the $H_+^{2,-1}$ topology. Therefore, bounded closed sets in A are not transformed in bounded closed sets in $H_+^{2,-1}$, which shows that the mathematical construction presented here does not fit the L^1 problem: it is not possible to consider "L^1 constraints" in the H^2/H^∞ problem with the methods developed in this paper.

Appendix 1. In this appendix we provide the proofs not presented in the main text.

PROOF OF COMMENTS AFTER DEFINITION 1. As $\langle f, g\rangle_{2,-k} = \langle \Phi_{-k} f, \Phi_{-k} g\rangle_2$ and $\Phi_{-k}^*(i\omega)\Phi_{-k}(i\omega) > 0$ for all ω, the announced properties are inherited from the inner product and the norm in $L^2(i\mathbb{R})$, if the integrals are finite. If $f\in R_{k-1}$, this last property is a consequence of f being a rational function without poles in $i\mathbb{R}$ and

$$\partial_r(f^*\Phi^*_{-k}\Phi_{-k}f) \geq (1-k)+2k+(1-k) = 2$$

(then integrable on $i\mathbb{R}$). ■

THEOREM 1 PROOF.

(a) The function f belongs to $L^2_{-k}(i\mathbb{R})$ if and only if $\Phi_{-k}f$ belongs to $L^2(i\mathbb{R})$, by definition. As R_1 is dense in $L^2(i\mathbb{R})$ [25], $\Phi^{-1}_{-k}R_1$ is dense in $L^2_{-k}(i\mathbb{R})$. But $\Phi^{-1}_{-k}R_1 = R_{1-k}$,. Indeed, if $f\in R_{1-k}$, $g = (\Phi_{-k})f$ belongs to R_1 because $\partial_r(\Phi_{-k}f) \geq 1$ and g has no poles in $i\mathbb{R}$. In the reverse direction, if f belongs to R_1, then $\Phi^{-1}_{-k}f$ belongs to R_{1-k} because $\partial_r(\Phi_{-k}f) \geq 1-k$ and f has no poles in $i\mathbb{R}$. Therefore, R_{1-k} is dense in $L^2_{-k}(i\mathbb{R})$. The same argument when applied to R^+_1, H^2_+, to R^+_{1-k}, $H^{2,-k}_+$, to R^-_1, H^2_-, and to R^-_{1-k}, $H^{2,-k}_-$, proves the announced densities. The final statement in (a) is a consequence of $L^2_{-k}(i\mathbb{R})$ be the completion of R_{1-k} in the norm $\|\,.\,\|_{2,-k}$, the same arriving to H^2_+ in relation to R^+_1, to $H^{2,-k}_+$ in relation to R^+_{1-k}, etc.

(b) $H^{2,-k}_+$ and $H^{2,-k}_-$, as closures of R^+_{1-k} and R^-_{1-k} in $L^2_{-k}(i\mathbb{R})$, are closed subspaces.

(c) Straightforward from (a). Note that $e^{s\Delta}\in H^\infty_+$ for Δ a real number because $e^{(a+\omega i)\Delta}$ is bounded on each vertical straight line in C^0_+, for each real a > 0. Then $e^{s\Delta}$ belongs to $H^{2,-1}_+$ ■

THEOREM 2 PROOF.

(a) As $k < m$, $\Phi_{-m} = \Phi\Phi_{-k}$ for some real-rational stable and minimum phase function with $\partial_r(\Phi) = m\text{-}k > 0$. Then a function f belongs to $L^2_{-m}(i\mathbb{R})$ if and only if Φf belongs to $L^2_{-k}(iR)$, as a consequence of Definition 3 and as a consequence of $\|f\|_{2,-m} = \|\Phi_{-m}f\|_2 = \|\Phi\Phi_{-k}f\|_2 = \|\Phi f\|_{2,-k}$. Therefore, the operator $f\to\Phi f$ is an isometry from $L^2_{-m}(i\mathbb{R})$ to $L^2_{-k}(i\mathbb{R})$, the inverse isometry being $g\to\Phi^{-1}g$. By the Cauchy-Schwarz inequality applied in $L^2(i\mathbb{R})$,

$$\|f\|_{2,-m} = \|\Phi\Phi_{-k}f\|_2 \leq \|\Phi\|_2\ \|\Phi_k f\|_2 = \|\Phi\|_2\ \|f\|_{2,-k},$$

and as $\|\Phi\|_2 < \infty$, $L^2_{-k}(i\mathbb{R})\subset L^2_{-m}(i\mathbb{R})$. Then the isometry from $L^2_{-k}(i\mathbb{R})$ to $L^2_{-m}(i\mathbb{R})$ is an injective mapping and its inverse is a surjective mapping.

(b) A direct consequence of Theorem 1b and Theorem 2a.

(c) Let $k \geq 0$. First we will prove that R_1 is dense in R_0 in the $L^2_{-1}(i\mathbb{R})$ topology. Actually, we only need to show that the constant function $f(s)\equiv 1$ is a limit of R_1-functions in this topology. Defining $f_n(s) = n(s+n)^{-1}$,

$$\|f_n - 1\|^2_{2,-1} = \int_{-\infty}^{\infty}\frac{\omega^2}{n^2+\omega^2}\frac{1}{1+\omega^2}d\omega = \frac{\pi}{n+1},$$

which converges to zero if n goes to ∞, showing the desired convergence and the stated density.

Second, as R_0 is dense in $L^2_{-1}(i\mathbb{R})$, R_1 is also dense in $L^2_{-1}(i\mathbb{R})$ in its topology.

Third, as $R_1 \subset L^2(i\mathbb{R}) \subset L^2_{-1}(i\mathbb{R})$, the density of R_1 in $L^2_{-1}(i\mathbb{R})$ implies the density of $L^2(i\mathbb{R})$ in $L^2_{-1}(i\mathbb{R})$.

Fourth, more generally, let M be a total set in $L^2_{-k}(i\mathbb{R})$, $k < m$, and assume f(s)$\in L^2_{-m}(i\mathbb{R})$. Set:

$$\int_{-\infty}^{\infty} f^*(i\omega)\Phi^*_{-m}(i\omega)\Phi_{-m}(i\omega)g(i\omega)d\omega = 0, \text{ for all } g(s)\in \mathrm{M},$$

which has a sense because g(s)$\in L^2_{-k}(i\mathbb{R})\subset L^2_{-m}(i\mathbb{R})$. Then,

$$f\Phi_{-m}\Phi^*_{-m} \in L^2_{-m}(i\mathbb{R})\subset L^2_{k}(i\mathbb{R}) \approx [\,L^2_{-k}(i\mathbb{R})]'$$

(where the symbol $\approx$ denotes the identification to be showed in Theorem 4a below, which will be proved independently from the present theorem), implying that $f\Phi_{-m}\Phi^*_{-m}$ can be taken as the zero function. This implies that $f(s) \equiv 0$ because $\Phi_{-m}(i\omega)\Phi^*_{-m}(i\omega)$ is strictly positive for all real ω. Therefore, as $g(s)$ is any function in a total set, the set M is also total in $L^2_{-m}(i\mathbb{R})$, by a known corollary of Hahn-Banach Theorem. From k < m, $L^2_{-k}(i\mathbb{R})\subset L^2_{-m}(i\mathbb{R})$, proving the density of the first in the second.

Analogous arguments can be used for $H_+^{2,-k}$ and $H_-^{2,-k}$.

(d) Assume that f_n converges to f in H_+^{∞}. Then

$$\| f_n - f \|^2_{2,-1} = \int_{-\infty}^{\infty} | f_n(i\omega) - f(i\omega)|^2 |\Phi_{-1}(i\omega)|^2 \, d\omega$$

$$\le \operatorname*{ess.sup}_{\omega\in R}\{| f_n(i\omega) - f(i\omega)|^2\} \int_{-\infty}^{\infty} \Phi^*_{-1}(i\omega)\Phi_{-1}(i\omega)d\omega$$

$$\le \| f_n - f \|^2_{\infty} \; \| \Phi_{-1} \|^2_2 .$$

and because $\| \Phi_{-1} \|^2_2$ is finite, f_n converges to f in $L^2_{-1}(i\mathbb{R})$. The stability of f is assured because $f \in H_+^{\infty}$. To complete the proof, let sus exhibit a function in $H_+^{2,-1}$ that do not belongs to H_+^{∞}. First, note that there are unbounded functions in $L^2(i\mathbb{R})$, as $g(i\omega) = |i\omega|^{-1/4}\chi_{[-1,1]}$, where $\chi_{[-1,1]}$ denotes the characteristic function of the closed interval [-1,1]. Straightforward calculations show that $\| g \|_2 = 2$ and that $|g(i\omega)|$ diverges when ω goes to zero. As a $L^2(i\mathbb{R})$ function, $g = g_+ + g_-$, where $g_+\in \mathrm{H}^2_+$ and $g_-\in \mathrm{H}^2_-$. Both functions cannot be simultaneously bounded, because g is not bounded. If g_+ is unbounded, it is the example completing the proof, because $g_+\in H^2_+\subset H_+^{2,-1}$ but $g_+\notin H_+^{\infty}$. If g_+ is bounded, g_- is unbounded, and $g^*_-(s) = g_-(-s)\in H^2_+\subset H_+^{2,-1}$ and is unbounded because $|g^*_-(s)| = |g_-(s)|$, $g^*_-(s)$ being the example, and completing the proof. ■

REMARK 2 PROOF. Remark 2 is proved in (c) above, if we note that

$$\| f_n \|^2_2 = \int_{-\infty}^{\infty} \frac{n^2}{n^2+\omega^2} d\omega = n\pi,$$

which implies that the sequence $\{f_n\}$ does not converge in $L^2(i\mathbb{R})$ when n goes to ∞, in spite of its convergence in $H_+^{2,-1}$. ■

THEOREM 3 PROOF.

(a) As $k \le m$, $\Phi_{-m} = \Phi\Phi_{-k}$ for some real-rational stable and minimum phase function with $\partial_r(\Phi) = m\text{-}k \ge 0$. First, if $\| f \|_{2,-k} \le M$, by the Cauchy-Schwarz inequality

$$\| f \|_{2,-k} \le \| f \|_{2,-m} \; \| \Phi \|_2 \le M \| \Phi \|_2 ,$$

prooving the first part of the statement. Second, the closed balls of $L^2_{-k}(i\mathbb{R})$ are closed in $L^2_{-m}(i\mathbb{R})$ as inverse image of closed sets by an isometric isomorphism (see Lemma 2)[1]. Third, if Ω is a bounded closed set in $L^2_{-k}(i\mathbb{R})$, it is contained in a closed ball in $L^2_{-k}(i\mathbb{R})$, which is a closed subset of $L^2_{-m}(i\mathbb{R})$. As Ω is closed in a closed subset of a metric subspace of $L^2_{-m}(i\mathbb{R})$, Ω is also closed in $L^2_{-m}(i\mathbb{R})$ (see Theorem 2,II,9,2,b, in [24], page 27). Fourth, as $H^{2,-k}_+$ is a closed subspace of $L^2_{-k}(i\mathbb{R})$, the last property is inherited by $H^{2,-k}_+$.

(b) First, if $\|f\|_\infty \le M$,

$$\| f \|^2_{2,-1} = \int_{-\infty}^{\infty} | f(i\omega) |^2 | \Phi_{-1}(i\omega) |^2 \; d\omega \le \| f \|^2_\infty \; \| \Phi_{-1} \|^2_2 \le M^2 \| \Phi_{-1} \|^2_2 < \infty,$$

which shows that bounded subsets of H^∞_+ are bounded in the $L^2_{-1}(i\mathbb{R})$ metric.

Second, it will be shown that the closed balls in H^∞_+ are closed in $H^{2,-1}_+$. For that, let $\{f_n\}$ a sequence in a closed ball of H^∞_+ with radius M, i.e., $\| f_n \|_\infty \le M$ for all $n \in \mathbb{N}$. Let $f \in H^\infty_+$ with $\| f \|_\infty > M$. Thus there is a positive real number ε so that $\| f \|_\infty$ is strictly greater than M+2ε. The definition of "essential supremum" implies that there exists a set $E \subset \mathbb{R}$ with strictly positive measure so that $| f(i\omega) | > M+\varepsilon$ for all $\omega \in E$. Therefore, f_n does not converge to f in $H^{2,-1}_+$ because

$$\begin{aligned}
\| f_n - f \|^2_{2,-1} &= \int_{-\infty}^{\infty} | f_n(i\omega - f(i\omega) |^2 | \Phi_{-1}(i\omega) |^2 \; d\omega \\
&\ge \int | f_n(i\omega) - f(i\omega) |^2 | \Phi_{-1}(i\omega) |^2 \; d\omega \\
&\ge \int_E | f(i\omega) |^2 | \Phi_{-1}(i\omega) |^2 \; d\omega - \int_E | f_n(i\omega) |^2 | \Phi_{-1}(i\omega) |^2 \; d\omega \\
&> [(M+\varepsilon) - M] \int_E | \Phi_{-1}(i\omega) |^2 \; d\omega = \varepsilon \int_E | \Phi_{-1}(i\omega) |^2 \; d\omega > 0 ,
\end{aligned}$$

---

[1] A more direct proof uses the weak continuity of the multiplication by Φ. Indeed, reasoning on $L^2(i\mathbb{R})$, $L^2_{-1}(i\mathbb{R})$, if $f_n \to f$ weakly in $L^2(i\mathbb{R})$, $| \int (f_n\text{-}f)\Phi g d\omega| \le \|\Phi\|_\infty \int |(f_n\text{-}f)g| d\omega \to 0$ for all g in $L^2(i\mathbb{R})$. Then, if, for all $n \in \mathbb{N}$, $\| f_n \|_2 \le M$, there is a subsequence, say $\{f_m\}$, converging weakly in $L^2(i\mathbb{R})$ to a limit f_w such that $\|f_w\|_2 \le M$ (see [27], page 26). The weak continuity proved above implies that Φf_m converges weakly in $L^2(i\mathbb{R})$ to Φf_w. But, as f_n converges to f strongly in $L^2_{-1}(i\mathbb{R})$, Φf_n converges strongly to Φf in $L^2(i\mathbb{R})$, and then Φf_n converges weakly to Φf in $L^2(i\mathbb{R})$. As $\{f_m\}$ represents a subsequence of $\{f_n\}$, $\Phi f_w = \Phi f$ (which implies $f_w = f$ in $L^2(i\mathbb{R})$ because $\Phi(.)$ is a continuous bounded function with no zeroes on $i\mathbb{R}$), it follows that $\| f \|_2 \le M$, proving that the closed ball with radius M in $L^2(i\mathbb{R})$ is also closed in $L^2_{-1}(i\mathbb{R})$. The same reasoning applies to $L^2_{-k}(i\mathbb{R})$ for any integer k.

the last integral being strictly positive from because $\Phi_{-1}(i\omega)$ is continuous and strictly positive on the real axis. The contrapositive proposition is:

"if $f_n \to f$ in $H_+^{2,-1}$ then $\|f\|_\infty \le M$",

implying that closed balls of H_+^∞ are also closed in the $H_+^{2,-1}$ topology.

Third, if Ω is a bounded closed set in H_+^∞, it is contained in a closed ball in H_+^∞, which is a closed set in $H_+^{2,-1}$. As Ω is closed in a closed subset of a metric subspace of $H_+^{2,-1}$, Ω is also closed in $H_+^{2,-1}$ (see Theorem 2,II,9,2,b, in [24], page 27). ■

REMARK 5 PROOF. Let $f_n(s) = (ns+1)^{-1}$, $n \in \mathbb{N}$. These functions belongs to H_+^∞ with $\| f_n \|_\infty = f_n(0) = 1$. Also

$$\| f_n \|_2^2 = \int_{-\infty}^{\infty} \frac{(1/n)^2}{(1/n)^2 + \omega^2} d\omega = \frac{\pi}{n}.$$

Therefore, the sequence f_n(s) converges to zero in H_+^2 and, *a fortiori*, in $H_+^{2,-1}$. But it does not converges to zero in in H_+^∞. Now, let g(s)$\in H_+^\infty$ any function such that $\| g \|_\infty \le 1$. Then $g_n(s) = g(s) + 3f_n(s)$ converges to $g(s)$ in $H_+^{2,-1}$ but does not converge in H_+^∞ because $\| g_n \|_\infty > 2$, for all n. Therefore, any function in the closed unit ball in H_+^∞ can be strongly approximated in $H_+^{2,-1}$ by functions in the exterior of this ball: all the functions in this set are in its $H_+^{2,-1}$ boundary. The H_+^∞ closed balls have an empty interior in the $H_+^{2,-1}$ topology. ■

THEOREM 4 PROOF.

(a) As continuous linear functional on Hilbert spaces are uniformly continuous, we need to proof the statement only on R_{1-k}^+, a dense subset of $H_+^{2,-k}$ ([37], page 98). In this case, as $\gamma(s)$ and $f(s)$ have no poles on the imaginary axis, the integral will be finite if and only if $\partial_r(f)+\partial_r(\gamma) \ge 2$. This arrives for all $f \in R_{1-k}^+$ if and only if $\partial_r(\gamma) \ge 2-(1-k) = k+1$. Also, $(\Phi_{-k}^*)^{-1}\gamma \in L^2(i\mathbb{R})$ and $\Phi_{-k}f \in L^2(i\mathbb{R})$. Then, by the Cauchy-Schwarz inequality, $F(f)$ is continuous on R_{1-k}^+, because

$$\left| \int_{-\infty}^{\infty} f^*(i\omega)\gamma(i\omega)d\omega \right| = \left| \int_{-\infty}^{\infty} (f\Phi_{-k})^*(i\omega)[(\Phi_{-k}^*)^{-1}\gamma](i\omega)d\omega \right|$$
$$\le \| (\Phi_{-k}^*)^{-1}\gamma \|_2 \; \| f \|_{2,-k}.$$

(b) As a consequence of (a), a rational function $g(s)$ is in the dual space of $H_+^{2,-k}$ if and only if $\partial_r(g) \ge$ k+1, i. e, $g \in H_+^{2,k}$. As the dual of $H_+^{2,-k}$ is a Hilbert space, the completion argument proves the statement. ■

THEOREM 5 PROOF.

(a) The statement is an adaptation of the known Youla Theorem, see [27].

(b) We need to proof the statement only on R_{1-k}^+, a dense subset of $H_+^{2,-k}$ (see [37], page 100). Now, if $f(s)$ is a rational function without poles on $i\mathbb{R}$,

$$\int_{-\infty}^{\infty} f^*(i\omega)\Gamma(i\omega)f(i\omega)d\omega = \int_{-\infty}^{\infty} [\Phi f(i\omega)]^*[\Phi f(i\omega)]d\omega = \| \Phi f \|_2^2 < \infty$$

if and only if $\partial_r(f\Phi) \ge 1$, i.e., $\partial_r(f) \ge 1-k$ or $f \in H_+^{2,-k}$.

Also, as $\Phi(s)$ and $\Phi_{-k}(s)$ are rational functions with no poles or zeroes on $i\mathbb{R}$ and those functions have the same relative degree, there are real numbers α and β such that:

$$0 < \alpha \le |\,\Phi(i\omega)\Phi_{-k}^{-1}(i\omega)\,| \le \beta < \infty.$$

This implies that, if $f \in R_{1-k}^{+}$, then

$$\alpha \| f \|_{2,-k}^{2} \le \int_{-\infty}^{\infty} f^{*}(i\omega)\Gamma(i\omega) f(i\omega) d\omega = \| \Phi f \|_{2}^{2} \le \beta \| f \|_{2,-k}^{2}.$$

Thus $\| \Phi f \|_{2}$ defines a norm equivalent to $\| f \|_{2,-k}$, the quadratic functional being continuous on R_{1-k}^{+} as the square of an equivalent norm. Finally, if $m < k$ and $f \in H_{+}^{2,-m}$, $| f\Phi_{-m}(i\omega)|^{2}$ and $|\Phi\Phi_{-m}^{-1}(i\omega)|^{2}$ belong to $L^{2}(i\mathbb{R})$, then

$$\int_{-\infty}^{\infty} f^{*}(i\omega)\Gamma(i\omega) f(i\omega) d\omega = \int_{-\infty}^{\infty} [\Phi_{-m} f(i\omega)]^{*}[\Phi_{-m} f(i\omega)][\Phi\Phi_{-m}^{-1}(i\omega)]^{*}[\Phi\Phi_{-m}^{-1}(i\omega)] d\omega$$

$$\le \| \Phi\Phi_{-m}^{-1} \|_{2}^{2} \| \Phi_{-m} f \|_{2}^{2} = \| \Phi\Phi_{-m}^{-1} \|_{2}^{2} \| f \|_{2,-m}^{2},$$

by the Cauchy-Schwarz inequality. Therefore, the quadratic functional is continuous in $H_{+}^{2,-m}$ at the origin, then continuous in $H_{+}^{2,-m}$ for $m < k$.

The coerciveness on $H_{+}^{2,-k}$ was shown above, where α is the coerciveness constant. For $m < k$ and $f \in H_{+}^{2,-m}$, let

$$f_{n}(s) = \Phi_{-m}^{-1}(s)[s\sqrt{n}(s+n)^{-2}],\ \Phi(s)\Phi_{-m}^{-1}(s) = g(s)(s+1)^{-1},$$

where $|g(i\omega)|^{2} \le \beta^{2} < \infty$ for some real number β because $g(s)$ is a proper rational function without poles on the imaginary axis. Straightforward calculations show that:

$$0 \le \| \Phi f_{n} \|_{2}^{2} \le \beta^{2} \| \frac{1}{s+1}\frac{s\sqrt{n}}{(s+n)^{2}} \|_{2}^{2} = \beta^{2}\frac{(2n^{2}-2n+1)\pi}{2(n^{2}-1)^{2}},$$

which converges to zero when n goes to ∞. But $\| f_{n} \|_{-m}^{2} = \pi/2$ for all n. Then there is no real number α such that $\alpha^{2} \| f_{n} \|_{-m}^{2} \le \| \Phi f_{n} \|_{2}^{2}$ for all n, which shows that the quadratic functional is not coercive on $H_{+}^{2,-m}$ for $m < k$.

The proof of the strictly convexity is straightforward. ■

COROLLARY 1 PROOF. As $\partial_{r}(\breve{K}) = \partial_{r}([(\Phi^{*})^{-1}\gamma]_{+})$ - k, $\partial_{r}(\breve{K}) \ge 0$ if and only if $\partial_{r}([(\Phi^{*})^{-1}\gamma]_{+}) \ge k$. If $\partial_{r}(\gamma) \ge 2k$, $\partial_{r}((\Phi^{*})^{-1}\gamma) \ge k$, implying $\partial_{r}([(\Phi^{*})^{-1}\gamma]_{+}) \ge k$, which proves the sufficiency of the condition. If $k = \partial_{r}(\Phi) \le 1$, as $\partial_{r}([(\Phi^{*})^{-1}\gamma]_{+}) \ge 1$, $\partial_{r}(\breve{K}) \ge 0$. ■

PROOF OF COMMENTS ABOUT CONDITION (3.5). We need to prove that condition (3.5) is inherited by a finite sum of quadratic functional as in (3.4). For that, denote the functional as

$$J[K] = \sum J_{n}[K], \qquad J_{n}[K] = \int_{-\infty}^{\infty} \{K^{*}\Gamma_{n}K - 2K^{*}\gamma_{n}\} d\omega, \qquad \Gamma_{n} = \Phi_{n}^{*}\Phi_{n}.$$

Then,

$$J[K] = \int_{-\infty}^{\infty} \{K^{*}\Gamma K - 2K^{*}\gamma\} d\omega \qquad \text{for } \Gamma = \Phi^{*}\Phi = \sum \Phi_{n}^{*}\Phi_{n},\ \gamma = \sum \gamma_{n}.$$

Let $\partial_{r}(\gamma_{n}) \ge \partial_{r}(\Phi_{n})+1$ and assumptions A1, A2 hold for each n. Then

$\partial_r(\sum \gamma_n) \geq \min\{\partial_r(\gamma_n)\}$,

as usual, but:

$\partial_r(\sum \Gamma_n) = \partial_r(\sum \Phi_n^* \Phi_n) = \min\{\partial_r(\Gamma_n)\}$

because the numerator of the first term is a sum of para-hermitian functions, each one strictly positive on the imaginary axis, which implies that its degree is the maximum degree of the parcels. See [9] for a complete development of this argument. Therefore,

$\partial_r(\sum \gamma_n) \geq \min\{\partial_r(\gamma_n)\} \geq \partial_r(\Phi_n)+1 = (½)\min\{\partial_r(\Gamma_n)\}+1$

$= (½)\min\{\partial_r(\sum \Gamma_n)\}+1 = \partial_r(\Phi_n)+1$,

completing the proof. ■

THEOREM 8 PROOF. Here it will be used the notations $\|.\|_\Gamma$ and $\langle .,.\rangle_\Gamma$ from section 4. The strictly convex criterion in problem (5.1) is a continuous function because H_n is finite-dimensional. Ω_n is a closed convex set as the interception of the closed convex sets Ω, Θ and H_n. It will be nonempty if the dimension "n" is sufficiently big because $\Omega \cap \Theta$ is nonempty by assumption and $\bigcup_{n=1}^{\infty} H_n$ is $H_+^{2,-1}$. Therefore, if "n" is sufficiently big, problem (5.1) has one and only one solution $\hat{K}_n$.

(a) For all $V \in H_1$, $\|\hat{K}_n\|_\Gamma \leq \|V\|_\Gamma$ because $\Omega_1 \subset \Omega_2 \subset \ldots \subset \Omega_n \subset \ldots \subset \Omega \cap \Theta$. Then the sequence $\{\hat{K}_n, n \in \mathbb{N}\}$ is bounded, which implies the existence of a weakly convergent subsequence that converges weakly in $H_+^{2,-1}$ to a function, denoted here by $\hat{K}_w$ (Bolzano-Weierstrass Theorem, [28], page 26). This subsequence will be denoted by $\{\hat{K}_m, m \in \mathbb{N}\}$. Note that $\hat{K}_w$ depends on the chosen subsequence.

(b) As $\Omega \cap \Theta$ is convex and strongly closed, it is also weakly closed (Mazur Theorem, [28] page 20). Then $\hat{K}_w \in \Omega \cap \Theta$.

(c) $\hat{K}_n$ is a solution of problem (5.1) if and only if it verifies the following variational inequality:

$\langle V_m, V_m - \hat{K}_m \rangle_\Gamma \geq 0$, for all $V_m \in \Omega_m$,

([29], pages 9-11 or, in a more general setting, [30], page 76). The weakly convergence of $\hat{K}_m$ implies the convergence of the inequality above to the condition:

$\langle V_m, V_m - \hat{K}_w \rangle_\Gamma \geq 0$, for all $V_m \in \Omega_m$,

for each m used in the subsequence. As the sequence of spaces $\{H_n\}$ increases, then $\bigcup_{m=1}^{\infty} H_m$ is a dense subspace of $H_+^{2,-1}$ and $\bigcup_{m=1}^{\infty} \Omega_m \cup \Omega_m$ is a dense subset of $\Omega \cap \Theta$. Taking the limit in the last inequality, we arrive to

$\langle V, V - \hat{K}_w \rangle_{2,-1} \geq 0$, for all $V \in \Omega \cap \Theta$,

a necessary and sufficient condition to $\hat{K}_w$ be the solution of problem (2.3). Then $\hat{K}_w$ equals $\hat{K}$, the solution of problem (4.1), for any subsequence $\hat{K}_m$ of the sequence $\hat{K}_n$ generated by Gallerkin method, which implies the weakly convergence of this sequence to the optimal solution to problem (4.1). ■

THEOREM 9 PROOF. First, it will be considered the situation where $\breve{K}_n = \breve{K}$, when problems (5.2) and (5.3) are essentially the same. Second, note that Theorem 8 can be generalized to the space $H_+^{2,-k}$ without changes, which proves the weak convergence (in $H_+^{2,-k}$) of the sequences generated by problems (5.2) and (5.3), when $n \in \mathbb{N}$, it means, $\hat{K}_n$ converges weakly to $\hat{K}$ and $\hat{G}_n + \breve{K}$ converges weakly to $\hat{G} + \breve{K} = \hat{K}$ in $H_+^{2,-k}$. Under the assumptions of Theorem 9 these sequences will converge strongly in $H_+^{2,-k}$, for the same limit. Indeed, the density of $\bigcup_{n=1}^{\infty} H_n$ in $H_+^{2,-k}$ and the fact that $\hat{G} \in \Omega' \cap \Theta'$, a closed convex set, imply that, for all positive real number ε, there is an integer N such that $\| \hat{G} - G_n \|_\Gamma < \varepsilon$ for all $n > N$ and $G_n \in \Omega_n'$. Thus, by the triangle inequality,

$$\| G_n \|_\Gamma = \| G_n + \hat{G} - \hat{G} \|_\Gamma \le \| G_n - \hat{G} \|_\Gamma + \| \hat{G} \|_\Gamma < \| \hat{G} \|_\Gamma + \varepsilon.$$

Squaring this expression and remembering the minimizing property of $\hat{G}$ in $\Omega' \cap \Theta' \supset \Omega_n'$, we have

$$(\| \hat{G} \|_\Gamma + \varepsilon)^2 = \| \hat{G} \|_\Gamma^2 + \varepsilon(2 \| \hat{G} \|_\Gamma + \varepsilon) > \| G_n \|_\Gamma^2 \ge \| \hat{G}_n \|_\Gamma^2 \ge \| \hat{G} \|_\Gamma^2.$$

Taking ε to zero, it is proved that $\| \hat{G}_n \|_\Gamma^2$ converges to $\| \hat{G} \|_\Gamma^2$.

Now, an argument due to Frederic Riesz shows the strong convergence of $\hat{G}_n$ to $\hat{G}$:

$$\| \hat{G}_n - \hat{G} \|_\Gamma^2 = \langle \hat{G}_n - \hat{G}, \hat{G}_n - \hat{G} \rangle_\Gamma = \| \hat{G}_n \|_\Gamma^2 - 2\langle \hat{G}_n, \hat{G} \rangle_\Gamma + \| \hat{G} \|_\Gamma^2,$$

which goes to $\| \hat{G} \|_\Gamma^2 - 2\langle \hat{G}, \hat{G} \rangle_\Gamma + \| \hat{G} \|_\Gamma^2 = 0$ as n goes to ∞ by the weak convergence of $\hat{G}_n$ to $\hat{G}$ and by the norms convergence (showed above). This ends this part of the proof.

The strong convergence of $\hat{K}_n = \hat{G}_n + \breve{K}$ to $\hat{K} = \hat{G} + \breve{K}$ is a consequence of the continuity of the sum in Hilbert spaces.

Now, if $\breve{K}_n$ is the projection of $\breve{K}$ in Ω_n, $\hat{K}_n = \hat{G}_n + \breve{K}_n$, where $\{ \hat{G}_n \}$ is exactly the sequence considered just above. As $\hat{G}_n$ converges strongly to $\hat{G}$ and $\breve{K}_n$ converges strongly to $\breve{K}$ (by the continuity of convex projections in Hilbert spaces [30], pages 157-158), $\hat{K}_n$ converges strongly to $\hat{K}$ in $H_+^{2,-k}$.

The strong convergence of $\hat{G}_n$ to $\hat{G}$, in the case where $\breve{K}_n$ is the projection of $\breve{K}$ in Ω_n, is now a consequence of the equivalence between problems (4.2) and (4.3).

To end the proof, note that $\Omega_n \subset H_+^{2,-1}$, which implies that $\hat{K}_n$ belongs to $H_+^{2,-1}$. Then, the convergence of $\hat{K}_n$ in $H_+^{2,-k}$ implies the convergence in $H_+^{2,-1}$ to the same limit by the inverse isometry of Theorem 2a. ■

REMARK 11 AND THEOREM 10 PROOF. First, note that $H_+^{\infty} \subset H_+^{2,-1} \approx (H_+^{2,-1})' = H_+^{2,1} \subset (H_+^{\infty})'$, H_+^{∞} being dense in $H_+^{2,-1}$ and $(H_+^{2,-1})'$ being weak-star dense in $(H_+^{\infty})'$ (apply the Corollary, page 298 and T2,XIX,7;5, page 299 [24]). Then $\hat{K}_n$, $\hat{K}$ above can be identified with functions in $(H_+^{2,-1})' \subset (H_+^{\infty})'$ by $K \approx G_K(f) = \langle \Phi_{-1}K, \Phi_{-1}f \rangle_2$, for

$f \in H_+^\infty$. Second, "$F_n \in (H_+^\infty)$' converges to $F \in (H_+^\infty)$' in the weak-star topology" means $F_n(g) \to F(g)$ for all $g \in H_+^\infty$.

If $\hat{K}_n \to \hat{K}$ weakly in $H_+^{2,-1}$, $\langle \Phi_{-1}\hat{K}_n, \Phi_{-1}g\rangle_2 \to \langle \Phi_{-1}\hat{K}, \Phi_{-1}g\rangle_2$ for each $g \in H_+^\infty$, which proves the sequence weak-star convergence in (H_+^∞)' and Remark 11.

Now, for Theorem 10, let $\hat{K}_n$ converges to $\hat{K}$ strongly in $H_+^{2,-1}$. For each functional $G \in (H_+^\infty)$', let F_m a functional sequence in $(H_+^{2,-1})$' approaching G in the (H_+^∞)' weak-star topology, i. e., $F_m(g) \to G(g)$ for each $g \in H_+^\infty$. By the Banach-Steinhaus theorem [24], the set $\{F_m\}$ is equicontinuous in $H_+^{2,-1}$. Thus $F_n(\hat{K}_n)$ converges to $G(\hat{K})$. Indeed,

$$|G(\hat{K}) - F_n(\hat{K}_n)| \le |F_n(\hat{K}_n) - F_n(\hat{K})| + |F_n(\hat{K}) - G(\hat{K})|,$$

the first term in the right going to zero because $\{F_m\}$ is equicontinuous and $\hat{K}_n \to \hat{K}$ strongly in $H_+^{2,-1}$, the second term in the right going to zero because $F_m(g) \to G(g)$ for each $g \in H_+^\infty$. As a consequence $G(\hat{K}_n)$ converges to $G(\hat{K})$ for each functional $G \in (H_+^\infty)$', proving the weak convergence in H_+^∞. ■

PROOF OF THE LAST COMMENT IN SECTION 7. If $F(t) = f(t) + \sum_{j=1}^{\infty} a_j \delta(t - t_j)$ with $\sum_{j=1}^{\infty} |a_j| < \infty$ and f belongs to $L^1[0, \infty)$ (it means, F is a distribution in the algebra A), $\hat{F}(s) = \hat{f}(s) + \sum a_j e^{st}$, its Laplace transform (a function in $\hat{A}$), where $\hat{f}(i\omega)$ is a continuous function going to zero at $\pm\infty$ ($\hat{F}$ and $\hat{f}$ representing the Fourier transforms of F and f). From the Fourier transform properties (see [25]) with $\Gamma_{-1} = (1-s^2)^{-1}$,

$$\|\hat{F}(s)\|_{2,-1}^2 = \int_{-\infty}^{\infty} \hat{f}(-i\omega)\Gamma_{-1}(i\omega)\hat{f}(i\omega)d\omega + \int_{-\infty}^{\infty} \sum_{j=1}^{\infty} a_j e^{-i\omega t_j}\Gamma_{-1}(i\omega)\sum_{k=1}^{\infty} a_k e^{i\omega t_l} d\omega$$

$$\le \|\hat{f}\|_\infty^2 \ \|\Gamma_{-1}\|_2^2 + [\sum_{j=1}^{\infty} |a_j|^2]^2 \ \|\Gamma_{-1}\|_2^2$$

$$\le [\ \|f\|_1 + \sum_{j=1}^{\infty} |a_j|^2\ ]^2.$$

Therefore, $\hat{F}$ belongs to $H_+^{2,-1}$ and bounded sets in A are also bounded in $H_+^{2,-1}$. But closed bounded sets in A are not bounded closed in $H_+^{2,-1}$.

Indeed, if $g(i\omega) = 1$ for $|\omega| < 1$ and $g(i\omega) = 0$ for $|\omega| > 1$, g belongs to $L^2(i\mathbb{R})$. Then, g can be decomposed in its stable and unstable components as $g = g_+ + g_-$. Without loss of generality, assume that g_+ is discontinuous. As a H_+^2 function, g_+ is a limit of a sequence of continuous functions $G_n = \sum_{k=1}^{n} b_k \beta_k$, where $\beta_k(s) = (s-a)^{k-1}/(s+a)^k$, $a > 0$, the Laguerre functions, and $\sum_{k=1}^{\infty} |b_k|^2 < \infty$. As the Laguerre functions inverse

Fourier transforms are in $L^1[0,\infty)$, a function that is not in $\hat{A}$, g_+, is a limit in H_+^2 (then in $H_+^{2,-1}$) of a sequence of functions in $\hat{A}$. Moreover, this sequence is bounded in $\hat{A}$ because

$$\| G_n \|_\infty \le \sqrt{a^{-1}\sum_{k=1}^{\infty} | b_k |^2} < \infty \text{ for all } n \in \mathbb{N}.$$

This last inequality is a consequence of

$$|G_n(i\omega)|^2 = \sum_{k=1}^{n} | b_k |^2 | \beta_k(i\omega) |^2$$

and

$$\| G_n \|_\infty^2 = sup\{|G_n(i\omega)|^2\} = sup\{ \sum_{k=1}^{n} | b_k |^2 | \beta_k(i\omega) |^2 \}$$

$$\le sup\{|\beta_k(i\omega)|^2\} \sum_{k=1}^{\infty} | b_k |^2 \le a^{-1}\sum_{k=1}^{\infty} | b_k |^2 ,$$

where it was used that a set of functions in $L^1[0,\infty)$, which Fourier transforms are bounded in $\hat{A}$, are also bounded in A, as a consequence of $\| \hat{f} \|_\infty \le \| f \|_1$ (see [25], page 26) and $\sum_{k=1}^{\infty} | b_k |^2 \le [\sum_{k=1}^{\infty} | b_k |]^2$. ■

Appendix 2. In this appendix we provide the numerical data for the example in section 5. Rational functions described below have been calculated from the data $P(s)$, $\phi_w(s)$, $\phi_d(s)$ and $\phi_v(s)$ given in Section 5. For that some diophantine equations (arising from the parameterization of stabilizing controllers) was solved, a variable change to reduce the robustness condition to Nehari form was applied, some multiplications of rational functions and cancellations of coincident poles and zeroes were made. The two first calculations were performed by state variable methods, as exposed in [6], the cancellation being performed by model order reduction using the Hankel singular value technique. The use of double precision calculations was impositive.

Rational functions	Degree	Numerator coefficients	Denominator coefficients
A(s)	s^{14}	0	1.000000000000000e+000
	s^{13}	1.381024580093296e+000	1.211595042867158e+002
	s^{12}	1.686008580615162e+000	5.458713034337957e+003
	s^{11}	7.703019538462904e+003	1.125395291285975e+005
	s^{10}	1.634508577704982e+005	1.076974629099070e+006
	s^{9}	1.671129690051401e+006	4.927775329734212e+006
	s^{8}	8.766081844981248e+006	1.233200905234427e+007
	s^{7}	2.566570185137830e+007	1.843165614381086e+007
	s^{6}	4.288376514277657e+007	1.679171067887532e+007
	s^{5}	3.638011973651214e+007	8.937537473511269e+006
	s^{4}	1.128209738056106e+007	2.752841097866921e+006
	s^{3}	1.881317702823051e+006	4.943826116024184e+005
	s^{2}	1.129883884492416e+005	4.781899120202310e+004
	s^{1}	2.747279969974730e+003	1.733418128119080e+003
	s^{0}	2.330461985378970e+001	1.960830911271398e+001
B(s)	s^{10}	0	1.000000000000000e+000
	s^{9}	-1.000012371305996e+000	1.863782215050616e+001
	s^{8}	-1.562399952832152e+001	1.181106597540888e+002
	s^{7}	-1.161078545444695e+002	3.834756797815088e+002
	s^{6}	-4.837125168666809e+002	7.437879887932461e+002
	s^{5}	-1.205831279245822e+003	9.129112097836384e+002
	s^{4}	-1.706884574538163e+003	7.095696057600543e+002
	s^{3}	-1.220645462582847e+003	3.376552213167196e+002
	s^{2}	-3.711128214145705e+002	9.277136443643682e+001
	s^{1}	-3.777832594378928e+001	1.316517110219557e+001
	s^{0}	-7.343714254348321e-001	7.250551615217723e-001
$F_0(s)$	s^{2}	0	1.000000000000000e+000
	s^{1}	-2.488088793672762e+000	-2.263724821234260e+000
	s^{0}	8.620956412513727e-001	8.843128062448000e-001

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