5
Option Pricing under a Nonlinear and Nonnormal GARCH

This chapter is based in a paper written together with both my advisor and my co-advisor, Professors Álvaro Veiga and Ken Siu respectively. We investigate the pricing of options in a class of discrete-time Flexible Coefficient Generalized Autoregressive Conditional Heteroskedastic (FC-GARCH) models with non-normal innovations. A conditional Esscher transform was used to select a price kernel for valuation in the incomplete market. This choice of the pricing kernel can be justified by an economic equilibrium argument based on maximizing the expected power utility. We provide a numerical study on the pricing results when the GARCH innovations have a normal distribution or a shifted-Gamma distribution and identify some key features of the pricing results.

The conditional Esscher transform provides a convenient and flexible way to determine a price kernel under nonlinear time series models. Here we exploit this important tool in actuarial science to determine a price kernel for option valuation.

5.1
Flexible Coefficient Generalized Autoregressive Conditional Heteroskedastic (FC-GARCH) models for Asset Returns

We consider a discrete-time economy with a bond B and a share S. Let $\mathcal{T}$ denote the time index set $\{0, 1, 2, \ldots, T\}$ of the economy. To model uncertainty, we fix a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ where $\mathcal{P}$ is a real-world probability measure. To simplify our analysis, we assume that the continuously compounded rate of interest from the bond is a constant, say r per period. Consequently, the bond-price process $\{B_t | t \in \mathcal{T}\}$ evolves over time as:

$$B_t \;=\; B_{t-1} e^{r} \;, \quad B_0 = 1 \;. \tag{5-1}$$

Let $\epsilon = \{\epsilon_t\}_{t \in \mathcal{T}}$ be the return innovations of the share S, where we take $\epsilon_0 = 0$ by convention. Suppose $\{\epsilon_t | t \in \mathcal{T} \backslash \{0\}\}$ are independent and identically distributed, (i.i.d.), with common distribution $D(0,1)$, where $D(0,1)$ represents a general distribution with zero mean and unit variance.

Let $S := \{S_t\}_{t\in\mathcal{T}}$ be the price process of the share S. Let $Y_t := \ln(S_t/S_{t-1})$, which is the continuously compounded rate of return from the share S from time $t-1$ and time t. Then we assume that the return process $Y := \{Y_t | t \in \mathcal{T}\}$ follows a first-order Flexible Coefficient Generalized Autoregressive Conditional Heteroscedastic model with $m = H+1$ limiting regimes, henceforth, FC-GARCH $(m, 1, 1)$:

$$\begin{aligned} Y_t &= \mu_t + h_t^{1/2}\epsilon_t \ , \\ h_t &= G(w_t;\psi) \ . \end{aligned} \tag{5-2}$$

Here $G(w_t;\psi)$ is a nonlinear function of a vector of variables $w_t := (Y_{t-1}, h_{t-1}, s_t)'$, (i.e. "$\prime$" represents the transpose of a matrix, or in particular a vector), defined by:

$$G(w_t;\psi) := \alpha_0 + \beta_0 h_{t-1} + \lambda_0 Y_{t-1}^2 + \sum_{i=1}^{H}[\alpha_i + \beta_i h_{t-1} + \lambda_i Y_{t-1}^2] f(s_t;\gamma_i, c_i) \ ,$$

where

1. for each $i = 1, 2, \ldots, H$, the logistic function

$$f(s_t, \gamma_i, c_i) := \frac{1}{1 + e^{-\gamma_i(s_t - c_i)}} \ ;$$

2. the vector of parameters

$$\psi := (\alpha_0, \beta_0, \lambda_0, \alpha_1, \cdots, \alpha_H, \beta_1, \cdots, \beta_H, \lambda_1, \cdots, \lambda_H, \gamma_1, \cdots, \gamma_H, c_1, \cdots, c_H)' \in \mathbb{R}^{3+5H} \ ;$$

3. for each $i = 1, 2, \cdots, N$, the parameter γ_i is the slope parameter and is considered positive. When $\gamma_i \to \infty$, the function becomes a step function. Please refer to Medeiros and Veiga(2009)(36), for a complete list of assumptions made on the parameters. Note that their restrictions are for a simpler FC-GARCH in which the conditional mean is constant. Here, we consider a simple case that $s_t = Y_{t-1}$.

The class of FC-GARCH models provides the flexibility in incorporating the asymmetric effect of the sign and the size of the previous return Y_{t-1} on the current variance level h_t. It can also capture the heavy-tailedness of return's distribution and the slow decay of the autocorrelation of the squared returns process $\{Y_t^2 | t \in \mathcal{T}\}$. In addition, the FC-GARCH model can capture another important stylized empirical feature of returns data, namely, the Taylor effect, first documented by Taylor (1986)(44). The Taylor effect refers to the strong autocorrelation of absolute daily returns data. This also relates to the long-memory effect of volatility; that is, the decay of the autocorrelations of volatility is too slow to be described by any short memory autoregressive moving average time series models. In the empirical studies by Ding et al. (1993) (16), it has been documented that the realized volatility decays in a hyperparabolic rate.

When $\gamma_i = 0$, or $\alpha_i, \beta_i, \lambda_i = 0$, $i = 1, 2, \cdots, H$, the FC-GARCH model reduces to the GARCH(1,1) model. The FC-GARCH model also nests other important ARCH-type models in the literature. Some examples include the LST-GARCH(1,1) model, the GJR-GARCH(1,1) model, the VS-GARCH(1,1) model, the ANST-GARCH(1,1) model, the DT-ARCH(1,1) model, the DT-GARCH(1,1) model, and others. For detail, interested readers may refer to Medeiros and Veiga(2009)(36).

5.2 The Conditional Esscher Transform

In this section, we recall the method of the conditional Esscher transform described in Siu et al. (2004) (43) to determine a pricing kernel for option valuation. The method applies to determine a price kernel for a general FC-GARCH model in the next section.

For each $t \in \mathcal{T}$, write F_t for the $\mathcal{P}$-completed, σ-field generated by the share price process up to and including time t and write also $F := \{F_t | t \in \mathcal{T}\}$. We assume that under $\mathcal{P}$,

$$Y_t = \mu_t + \xi_t \ .$$

where ξ_t is an i.i.d innovation process having distribution $D(0, h_t)$ and μ_t and h_t are F_{t-1}-measurable.

We noticed while preparing that paper that the methodology is applied not only to the FC-GARCH but also for any model that has a μ_t and h_t structure being $F_{t-1}-$measurable and noises given by a infinitely divisible distribution having a moment generation function. As many of the GARCH especifications has those properties, the Siu et al. methodology can be used very broadly.

We now define the conditional Esscher transform. Let $\{\theta_t | t \in \mathcal{T}\backslash\{0\}\}$ be an F-predictable, real-valued, process on $(\Omega, \mathcal{F}, \mathcal{P})$. It means we know in time $t-1$ its value in time t. Denote, for each $t \in \mathcal{T}\backslash\{0\}$, the moment generating function of Y_t given F_{t-1} under $\mathcal{P}$ evaluated at $z \in \Re$ by $M_Y(t, z)$; that is,

$$M_Y(t, z) := E[e^{zY_t} | F_{t-1}] \ .$$

Here E is expectation under $\mathcal{P}$. Assume that, for each $t \in \mathcal{T}\backslash\{0\}$ and $z \in \Re$, $M_Y(t,z)$ exists and consider an F-adapted process $\{\Lambda_t | t \in \mathcal{T}\}$ on $(\Omega, \mathcal{F}, \mathcal{P})$ with $\Lambda_0 = 1$, $\mathcal{P}$-a.s., defined by:

$$\Lambda_t := \prod_{k=1}^{t} \frac{e^{\theta_k Y_k}}{M_Y(k,\theta_k)} \ , \quad t \in \mathcal{T}\backslash\{0\} \ .$$

Then, it is easy to check that $\{\Lambda_t\}_{t\in\mathcal{T}}$ is an $(F,\mathcal{P})$-martingale. So, $E[\Lambda_T] = 1$.

Now we define a new probability measure $\mathcal{P}^\theta$ equivalent to $\mathcal{P}$ on F_T by setting

$$\left.\frac{d\mathcal{P}^\theta}{d\mathcal{P}}\right|_{F_T} := \Lambda_T \ . \tag{5-3}$$

We call $\mathcal{P}^\theta$ the conditional Esscher transform associated with θ.

Let $M_Y^\theta(t,z)$ be the moment generating function of the return Y_t given F_{t-1} under the new measure $\mathcal{P}^\theta$. Write $E^\theta[\cdot]$ for expectation under $\mathcal{P}^\theta$. Then, by the Bayes' rule, it is easy to check that

$$M_Y^\theta(t,z) = \frac{M_Y(t,\theta_t+z)}{M_Y(t,\theta_t)} \ . \tag{5-4}$$

Indeed, by the Bayes' rule (Theorem 45 in the Appendix),

$$\begin{aligned} M_Y^\theta(t,z) &:= E^\theta[e^{zY_t}|F_{t-1}] \\ &= \frac{E[e^{zY_t}\Lambda_t|F_{t-1}]}{E[\Lambda_t|F_{t-1}]} \\ &= E\left[\frac{\Lambda_t}{\Lambda_{t-1}} e^{zY_t}|F_{t-1}\right] \\ &= \frac{E[e^{(z+\theta_t)Y_t}|F_{t-1}]}{M_Y(t,\theta_t)} \\ &= \frac{M_Y(t,\theta_t+z)}{M_Y(t,\theta_t)} \ . \end{aligned} \tag{5-5}$$

According to the fundamental theorem of asset pricing (see Harrsion and Kreps(1979)(33) and Harrsion and Pliska (1981, 1983))(34) and (35), the absence of arbitrage opportunities is “essentially” equivalent to the existence of an equivalent martingale measure under which discounted price processes are martingales. We call the latter a martingale condition. Please refer to Siu et al.(43) for the economic argument to select the measure. A similar argument will be shown in chapter 6.

Now we write $\tilde{S}_t := e^{-rt}S_t$, which is the discounted asset price at time t, for each $t \in \mathcal{T}$. Then in our case, the martingale condition is:

$$\tilde{S}_u = E^{\theta}[\tilde{S}_t | F_u] \ , \quad \text{for all } u, t \in \mathcal{T} \text{ with } u \leq t \ . \tag{5-6}$$

Here E^{θ} is expectation under $\mathcal{P}^{\theta}$.

The following proposition gives the necessary and sufficient condition for the martingale condition. It is in Siu et al. (2004)(43).

Proposition 33 *The martingale condition is satisfied if and only if there exists an F-predictable process $\{\theta_t | t \in \mathcal{T} \backslash \{0\}\}$ such that*

$$r = \ln M_Y(t, \theta_t + 1) - \ln M_Y(t, \theta_t) \ . \tag{5-7}$$

Proof:

Let $Y_t = \ln\left(\frac{S_t}{S_{t-1}}\right)$, such that $e^{Y_t} = \frac{S_t}{S_{t-1}}$.

($\Leftarrow$) First we prove that

$\tilde{S}_{t-1} = \mathbb{E}^{\theta}[\tilde{S}_t | F_{t-1}]$

Indeed, if $r = \ln\left(\frac{M_Y(t,\theta_t+1)}{M_Y(t,\theta_t)}\right)$, then

$$e^r = \frac{M_Y(t, \theta_t + 1)}{M_Y(t, \theta_t)} = M_Y(t, 1) = E^{\theta}[e^{Y_t} | F_{t-1}]. \tag{5-8}$$

Then,

$$\begin{aligned}
\mathbb{E}^{\theta}[e^{-rt} S_t | F_{t-1}] &= S_{t-1} e^{-rt} \mathbb{E}^{\theta}[e^{Y_t} | F_{t-1}] && \text{(5-9)} \\
&= S_{t-1} e^{-rt} \mathbb{E}^{\theta}[e^{Y_t} | F_{t-1}] && \text{(5-10)} \\
&= S_{t-1} e^{-rt} \frac{\mathbb{E}[e^{Y_t(\theta_t+1)} | F_{t-1}]}{\mathbb{E}[e^{Y_t \theta_t} | F_{t-1}]} && \text{(5-11)} \\
&= S_{t-1} e^{-rt} \frac{M_Y(t, \theta_t + 1)}{M_Y(t, \theta_t)} && \text{(5-12)} \\
&= S_{t-1} e^{-r(t-1)} && \text{(5-13)}
\end{aligned}$$

Now, we will show that for any $u, t \in \tau$ with $u < t$,

$$\mathbb{E}^{\theta}[e^{-rt} S_t | F_{t-1}] = e^{-ru} S_u, \ \text{a.s. with respect to } \mathbb{P}.$$

$$\begin{aligned}
\mathbb{E}^{\theta}[e^{-rt}S_t|F_u] &= \mathbb{E}^{\theta}[e^{-rt}S_{t-1}e^{Y_t}|F_u] && \text{(5-14)}\\
&= E^{\theta}[e^{-rt}S_{t-1}E^{\theta}[e^{Y_t}|F_{t-1}]|F_u] && \text{(5-15)}\\
&= E^{\theta}[e^{-r(t-1)}S_{t-1}e^{-r}E^{\theta}[e^{Y_t}|F_{t-1}]|F_u] && \text{(5-16)}\\
&= \mathbb{E}^{\theta}[e^{-r(t-1)}S_{t-1}|F_u] && \text{(5-17)}\\
&= ... && \text{(5-18)}\\
&= \mathbb{E}^{\theta}[e^{-r(u+1)}S_{u+1}|F_u] && \text{(5-19)}\\
&= e^{-ru}S_u && \text{(5-20)}
\end{aligned}$$

almost surely with respect to $\mathbb{P}$ as desired.

($\Rightarrow$) Now we are going to prove that if

$$\tilde{S}_u = \mathbb{E}^{\theta}[\tilde{S}_t|F_u] \quad \forall u \leq t \tag{5-21}$$

then

$$r = \ln\left(\frac{M_Y(t, \theta_t + 1)}{M_Y(t, \theta_t)}\right).$$

In particular, the hypothesis is true at $u = t - 1$. Then

$$S_{t-1}e^{-r(t-1)} = \mathbb{E}^{\theta}[S_t e^{-rt}|F_{t-1}]$$

$\Rightarrow$

$$e^{-r(t-1)} = \mathbb{E}^{\theta}[\frac{S_t}{S_{t-1}}|F_{t-1}]$$

$\Rightarrow$

$$e^{r} = E^{\theta}[e^{Y_t}|F_{t-1}]$$

$\Rightarrow$

$$r = \ln E^{\theta}[e^{Y_t}|F_{t-1}]$$

$\Rightarrow$

$$r = \ln\left(\frac{\mathbb{E}[e^{Y_t(\theta_t+1)}|F_{t-1}]}{\mathbb{E}[e^{Y_t\theta_t}|F_{t-1}]}\right)$$

$\Rightarrow$

$$r = \ln\left(\frac{M_Y(t, \theta_t + 1)}{M_Y(t, \theta_t)}\right).$$

The existence and uniqueness of the process θ can be established using some standard arguments.

$\square$

Consider an European-style option with payoff $V(S_T)$ at maturity T. Then, a conditional price of the option at time t given F_t is determined as:

$$V_t = e^{-r(T-t)} E^{\theta}[V(S_T)|F_t] \ . \tag{5-22}$$

The expected value of $V(S_T)$ is calculated via Monte Carlo simulation. To obtain the results we also use some variance reduction techniques: control variate (the Black and Scholes option price as the benchmark) and antithetic variables (Normal case only).

5.3
Some Parametric Cases

In this section, we consider some parametric cases of our model when the GARCH innovations have a normal distribution and a shifted gamma distribution. The development in this section follows that of Siu et al. (2004)(43). In this section two of the main theoretical results of our research will appear, viz., Theorems 35 and 36.

5.3.1
Normal innovations

Firstly, under $\mathcal{P}$, consider some $\mathcal{F}_{t-1}-$measurable conditional mean μ_t an the model below.

$$\begin{aligned} Y_t &= \mu_t + \xi_t \\ \xi_t|\mathcal{F}_{t-1} &= N(0, h_t) \\ h_t &= \alpha_0 + \beta_0 h_{t-1} + \lambda_0 \xi_{t-1}^2 + \sum_{i=1}^{H} (\alpha_i + \beta_i h_{t-1} + \lambda_i \xi_{t-1}^2) f(s_t; \gamma_i, c_i), \end{aligned}$$

where

$$f(s_t, \gamma_i, c_i) := \frac{1}{1 + e^{-\gamma_i (s_t - c_i)}} \ .$$

Then, under $\mathcal{P}$, $Y_t|F_{t-1} \sim N(\mu_t, h_t)$, as μ_t depends only on information available in $\mathcal{F}_{t-1}$.

In order to find the Esscher parameter:

$$r = \ln(M_{Y_t|\mathcal{F}_{t-1}}(1,\theta_t)) = \ln\left(\frac{M_{Y_t|\mathcal{F}_{t-1}}(1+\theta_t)}{M_{Y_t|\mathcal{F}_{t-1}}(\theta_t)}\right) \tag{5-23}$$

$$= \ln\left(\frac{e^{\mu_t(1+\theta_t)+\frac{(1+\theta_t)^2h_t}{2}}}{e^{\mu_t\theta_t+\frac{\theta_t^2h_t}{2}}}\right) = \ln(e^{\mu_t+h_t\theta_t+\frac{h_t}{2}}) = \mu_t + h_t\theta_t + \frac{h_t}{2} \tag{5-24}$$

i.e.,

$$r = \mu_t + h_t\theta_t + \frac{h_t}{2}$$

or

$$r - \mu_t - \frac{h_t}{2} = h_t\theta_t$$

which gives us $\theta_t = \frac{r-\mu_t-\frac{h_t}{2}}{h_t}$. (The Esscher parameter)

Using that $\theta_t = \frac{r-\left(\mu_t+\frac{h_t}{2}\right)}{h_t}$ in the relation

$$M_{Y_t|\mathcal{F}_{t-1}}(z,\theta_t) = \frac{M_{Y_t|\mathcal{F}_{t-1}}(z+\theta_t)}{M_{Y_t|\mathcal{F}_{t-1}}(\theta_t)},$$

we have

$$M_{Y_t|\mathcal{F}_{t-1}}(z,\theta_t) = \frac{e^{\mu_t(z+\theta_t)+\frac{(z+\theta_t)^2h_t}{2}}}{e^{\mu_t\theta_t+\frac{\theta_t^2h_t}{2}}} \tag{5-25}$$

$$= e^{\mu_t z + z\theta_t h_t + \frac{z^2h_t}{2}} \tag{5-26}$$

$$= e^{\mu_t z + z\left(r-\left(\mu_t+\frac{h_t}{2}\right)\right)+\frac{z^2h_t}{2}} \tag{5-27}$$

$$= e^{z\left(r-\frac{h_t}{2}\right)+\frac{z^2h_t}{2}}, \tag{5-28}$$

which is the mgf of a normal, i.e.,

$$Y_t|\mathcal{F}_{t-1} \sim N_{\mathcal{P}^\theta}\left(r - \frac{h_t}{2}, h_t\right)$$

under $\mathcal{P}^\theta$. By the dynamics

$$Y_t = \mu_t + \xi_t; \qquad \xi_t = h_t^{1/2}\epsilon_t,$$

we have

$$E^\theta_{\mathcal{P}}[\xi_t|\mathcal{F}_{t-1}] = E^\theta_{\mathcal{P}}[Y_t|\mathcal{F}_{t-1}] - \mu_t = r - \mu_t - \frac{h_t}{2}.$$

$$Var^{\theta}_{\mathcal{P}}[\xi_t|\mathcal{F}_{t-1}] = Var^{\theta}_{\mathcal{P}}[Y_t|\mathcal{F}_{t-1}] = h_t = Var_{\mathbb{P}}[\xi_t|\mathcal{F}_{t-1}].$$

Note that the variance does not change but the mean does and we want a zero mean variable. So make $\epsilon_t := \xi_t - r + \mu_t + \frac{h_t}{2}$ such that $\epsilon_t|\mathcal{F}_{t-1} \sim N^{\theta}_{\mathcal{P}}(0, h_t)$. Then, we can write the model under measure $\mathcal{P}^{\theta}$ as

$$Y_t = r - \frac{h_t}{2} + \epsilon_t, \tag{5-29}$$

where

$$\begin{aligned} h_t &= \alpha_0 + \beta_0 h_{t-1} + \lambda_0 \left(\epsilon_{t-1} + r - \mu_t - \frac{h_{t-1}}{2}\right)^2 && (5\text{-}30)\\ &+ \sum_{i=1}^{H} \left[\alpha_i + \beta_i h_{t-1} + \lambda_i \left(\epsilon_{t-1} + r - \mu_t - \frac{h_{t-1}}{2}\right)^2\right] f(s_t; \gamma_i, c_i) && (5\text{-}31) \end{aligned}$$

If we take $\mu_t = r + \lambda\sqrt{h_t} - \frac{1}{2}h_t$ to be the conditional mean as Duan(1995), we would have to consider

$$\begin{aligned} \epsilon_t :&= \xi_t - r + r + \lambda\sqrt{h_t} - \frac{1}{2}h_t + \frac{h_t}{2} && (5\text{-}32)\\ &= \xi_t + \lambda\sqrt{h_t} && (5\text{-}33) \end{aligned}$$

so that $\epsilon_t|\mathcal{F}_{t-1} \sim N^{\theta}_{\mathcal{P}}(0, h_t)$. And then we would conclude for this particular case that under $\mathcal{P}^{\theta}$:

$$Y_t = r - \frac{h_t}{2} + \epsilon_t, \tag{5-34}$$

where

$$\begin{aligned} h_t &= \alpha_0 + \beta_0 h_{t-1} + \lambda_0(\epsilon_{t-1} - \lambda\sqrt{h_t})^2 && (5\text{-}35)\\ &+ \sum_{i=1}^{H} \left[\alpha_i + \beta_i h_{t-1} + \lambda_i(\epsilon_{t-1} - \lambda\sqrt{h_t})^2\right] f(s_t; \gamma_i, c_i) && (5\text{-}36) \end{aligned}$$

We summarize the discussion above in the following Theorem:

Theorem 34 *Let*

$$\begin{aligned} Y_t &= r + \lambda\sqrt{h_t} - \frac{1}{2}h_t + \xi_t \\ \xi_t|\mathcal{F}_{t-1} &= N(0, h_t) \\ h_t &= \alpha_0 + \beta_0 h_{t-1} + \lambda_0 \xi_{t-1}^2 + \sum_{i=1}^{H}(\alpha_i + \beta_i h_{t-1} + \lambda_i \xi_{t-1}^2) f(s_t; \gamma_i, c_i) \ , \end{aligned}$$

be the model under $\mathcal{P}$, where

$$f(s_t, \gamma_i, c_i) := \frac{1}{1 + e^{-\gamma_i(s_t - c_i)}} \ .$$

Then, under the risk neutral measure the model is

$$\begin{aligned} Y_t &= r - \frac{1}{2}h_t + \epsilon_t \\ \epsilon_t|\mathcal{F}_{t-1} &\sim N_{\mathcal{P}}^{\theta}(0, h_t) \\ h_t &= \alpha_0 + \beta_0 h_{t-1} + \lambda_0(\epsilon_{t-1} - \lambda\sqrt{h_t})^2 \\ &+ \sum_{i=1}^{H}\left[\alpha_i + \beta_i h_{t-1} + \lambda_i(\epsilon_{t-1} - \lambda\sqrt{h_t})^2\right] f(s_t; \gamma_i, c_i) \end{aligned}$$

5.3.2
Shifted-Gamma Innovations

Some authors (Gerber and Shiu(1994) (31) and Siu et al. (2004)(43)) have been using shifted-gamma innovations to model log-returns in order to handle the skewness that real financial series usually exhibits. However, the skewness of the Gamma distribution is strictly positive whilst financial time series can present both signs. In practice, before adopting the shifted-gamma model, one may check if there is any skewness to be modeled. Otherwise, the Normal model should suffice. Then, one should check for the sign of the skewness so as to select an appropriate formulation of the shifted-gamma innovations. The positive case is similar and for the GARCH case it has already documented in Siu et al.(2004) (43). In the following, we develop a model to incorporate negative skewness.

Suppose that for each $t \in \mathcal{T}\backslash\{0\}$, $X_t \sim Ga(a, b)$, where $Ga(a, b)$ represents a Gamma distribution with shape parameter a and scale parameter b. We now suppose that under $\mathcal{P}$ the innovation at time t is given by:

$$\xi_t := -\sqrt{h_t}\left(\frac{X_t - a/b}{\sqrt{a/b^2}}\right) \ , \tag{5-37}$$

so we write $\xi_t|F_{t-1} \sim -SGa(0, h_t)$.

Then, under $\mathcal{P}$,

$$\begin{aligned} Y_t &= r + \lambda\sqrt{h_t} - \frac{1}{2}h_t + \xi_t \ , && \text{(5-38)} \\ h_t &= \alpha_0 + \beta_0 h_{t-1} + \lambda_0 \xi_{t-1}^2 + \sum_{i=1}^{H}(\alpha_i + \beta_i h_{t-1} + \lambda_i \xi_{t-1}^2) f(s_t; \gamma_i, c_i) \ , && \text{(5-39)} \end{aligned}$$

where

$$f(s_t, \gamma_i, c_i) := \frac{1}{1 + e^{-\gamma_i(s_t - c_i)}} \ .$$

The return process Y can be expressed as:

$$Y_t = r + \lambda\sqrt{h_t} - \frac{1}{2}h_t + \sqrt{ah_t} - b\sqrt{\frac{h_t}{a}}X_t \ .$$

Note that $b\sqrt{\frac{h_t}{a}}X_t \sim Ga(a, \sqrt{\frac{a}{h_t}})$, and that if $W \sim Ga(a,b)$ is a Gamma random variable, the moment generation function of $-W$ is $M_{-W}(t) = \left(\frac{b}{b+\theta}\right)^a$. Then the moment generation function of $Y_t|\mathcal{F}$ is given by

$$M_{Y_t|F_{t-1}}(\theta_t) = \left(\frac{\sqrt{\frac{a}{h_t}}}{\sqrt{\frac{a}{h_t}} + \theta_t}\right)^a e^{(r+\lambda\sqrt{h_t} - \frac{1}{2}h_t + \sqrt{ah_t})\theta_t} \qquad \text{(5-40)}$$

provided that $\theta_t + \sqrt{\frac{a}{h_t}} > 0$.

Applying the formula

$$M_Y^{\theta_t}(z, \theta_t) = \frac{M_Y(t, \theta_t + z)}{M_Y(t, \theta_t)} \ , \qquad \text{(5-41)}$$

we have

$$\begin{aligned} M_{Y_t|\mathcal{F}_{t-1}}(z, \theta_t) &= \frac{\left(\frac{\sqrt{\frac{a}{h_t}}}{\sqrt{\frac{a}{h_t}} + \theta_t + z}\right)^a e^{(r+\lambda\sqrt{h_t} - \frac{1}{2}h_t + \sqrt{ah_t})(\theta_t + z)}}{\left(\frac{\sqrt{\frac{a}{h_t}}}{\sqrt{\frac{a}{h_t}} + \theta_t}\right)^a e^{(r+\lambda\sqrt{h_t} - \frac{1}{2}h_t + \sqrt{ah_t})\theta_t}} && \text{(5-42)} \\ &= \left(\frac{\sqrt{\frac{a}{h_t}} + \theta_t}{\sqrt{\frac{a}{h_t}} + \theta_t + z}\right)^a e^{(r+\lambda\sqrt{h_t} - \frac{1}{2}h_t + \sqrt{ah_t})z} && \text{(5-43)} \end{aligned}$$

as long as $z < \sqrt{\frac{a}{h_t}} + \theta_t$.

By this formula and the relation

$$r = \ln M_{Y_t|\mathcal{F}_{t-1}}(1, \theta_t^q)$$

we have:

$$e^{r} = e^{r+\lambda\sqrt{h_t}-\frac{1}{2}h_t+\sqrt{ah_t}} \left(\frac{\sqrt{\frac{a}{h_t}} + \theta_t^q}{\sqrt{\frac{a}{h_t}} + \theta_t^q + 1} \right)^{a} \tag{5-44}$$

or equivalently

$$1 = e^{\frac{\lambda\sqrt{h_t}-\frac{1}{2}h_t+\sqrt{ah_t}}{a}} \left(\frac{\sqrt{\frac{a}{h_t}} + \theta_t^q}{\sqrt{\frac{a}{h_t}} + \theta_t^q + 1} \right) \tag{5-45}$$

Note that

$$\frac{\sqrt{\frac{a}{h_t}} + \theta_t^q}{\sqrt{\frac{a}{h_t}} + \theta_t^q + 1} = 1 - \frac{1}{\sqrt{\frac{a}{h_t}} + \theta_t^q + 1} \tag{5-46}$$

Then,

$$1 - e^{\frac{\lambda\sqrt{h_t}-\frac{1}{2}h_t+\sqrt{ah_t}}{a}} = -\frac{e^{\frac{\lambda\sqrt{h_t}-\frac{1}{2}h_t+\sqrt{ah_t}}{a}}}{\sqrt{\frac{a}{h_t}} + \theta_t^q + 1} \tag{5-47}$$

$\Longleftrightarrow$

$$\sqrt{\frac{a}{h_t}} + \theta_t^q + 1 = -\frac{e^{\frac{\lambda\sqrt{h_t}-\frac{1}{2}h_t+\sqrt{ah_t}}{a}}}{1 - e^{\frac{\lambda\sqrt{h_t}-\frac{1}{2}h_t+\sqrt{ah_t}}{a}}} \tag{5-48}$$

$\Longleftrightarrow$

$$\theta_t^q = -\frac{e^{\frac{\lambda\sqrt{h_t}-\frac{1}{2}h_t+\sqrt{ah_t}}{a}}}{1 - e^{\frac{\lambda\sqrt{h_t}-\frac{1}{2}h_t+\sqrt{ah_t}}{a}}} - \sqrt{\frac{a}{h_t}} - 1 \tag{5-49}$$

$\Longleftrightarrow$

$$\theta_t^q = -\frac{1}{1 - e^{\frac{\lambda\sqrt{h_t}-\frac{1}{2}h_t+\sqrt{ah_t}}{a}}} - \sqrt{\frac{a}{h_t}} \tag{5-50}$$

$\Longleftrightarrow$

$$\theta_t^q = \frac{1}{e^{\frac{\lambda\sqrt{h_t}-\frac{1}{2}h_t+\sqrt{ah_t}}{a}} - 1} - \sqrt{\frac{a}{h_t}} \tag{5-51}$$

Consequently, the martingale condition implies that

$$\theta_t^q = \frac{1}{e^{\frac{\lambda\sqrt{h_t}-\frac{1}{2}h_t+\sqrt{ah_t}}{a}} - 1} - \sqrt{\frac{a}{h_t}} \tag{5-52}$$

Now if we take $b_t := \sqrt{\frac{a}{h_t}}$ and $b_t^\theta := \frac{1}{e^{\frac{\lambda\sqrt{h_t}-\frac{1}{2}h_t+\sqrt{ah_t}}{a}}-1}$, then

$$b_t^\theta = \theta_t + b_t \ .$$

Let $\sim$ be "equal in distribution". Under $\mathcal{P}^\theta$,

$$Y_t|F_{t-1} \sim -SGa(a, b_t^\theta, -r - \lambda\sqrt{h_t} + \frac{1}{2}h_t - \sqrt{ah_t}) \ . \tag{5-53}$$

Then, we can write

$$Y_t \sim r + \lambda\sqrt{h_t} - \frac{1}{2}h_t + \sqrt{ah_t} + X_t^\theta \ .$$

Here $X_t^\theta \sim -\frac{1}{b_t^\theta}Ga(a,1)$ [1], and

$$\begin{aligned} h_t &= \alpha_0 + \beta_0 h_{t-1} + \lambda_0(X_{t-1}^\theta + \sqrt{ah_{t-1}})^2 \\ &+ \sum_{i=1}^{H}[\alpha_i + \beta_i h_{t-1} + \lambda_i(X_{t-1}^\theta + \sqrt{ah_{t-1}})^2]f(s_t;\gamma_i,c_i) \ . \end{aligned}$$

From the discussion above we have achieved:

Theorem 35 *Let the model under* $\mathcal{P}$ *be*

$$\begin{aligned} Y_t &= r + \lambda\sqrt{h_t} - \frac{1}{2}h_t + \xi_t, \\ \xi_t|F_{t-1} &\sim -SGa(0,h_t), \\ h_t &= \alpha_0 + \beta_0 h_{t-1} + \lambda_0\xi_{t-1}^2 + \sum_{i=1}^{H}(\alpha_i + \beta_i h_{t-1} + \lambda_i\xi_{t-1}^2)f(s_t;\gamma_i,c_i), \end{aligned}$$

where

$$f(s_t,\gamma_i,c_i) := \frac{1}{1+e^{-\gamma_i(s_t-c_i)}} \ .$$

[1] We write $Gamma(a,1)$ a Gamma random variable with shape parameter a and the scalar parameter 1.

Then, under the risk neutral measure, the model is

$$
\begin{aligned}
Y_t &\sim r + \lambda\sqrt{h_t} - \frac{1}{2}h_t + \sqrt{ah_t} + X_t^\theta \\
X_t^\theta &\sim -\frac{1}{b_t^\theta}Ga(a,1) \\
h_t &= \alpha_0 + \beta_0 h_{t-1} + \lambda_0(X_{t-1}^\theta + \sqrt{ah_{t-1}})^2 \\
&+ \sum_{i=1}^{H}[\alpha_i + \beta_i h_{t-1} + \lambda_i(X_{t-1}^\theta + \sqrt{ah_{t-1}})^2]f(s_t;\gamma_i,c_i)\ .
\end{aligned}
$$

5.4 Simulation Studies

In this section we conduct simulation experiments and compare both Call and Put option prices arising from different processes for the log-returns of the underlying asset.

We consider five pricing schemes for options with 90 days to maturity: the classical Black and Scholes *formulae* assuming a GBM process and the conditional Esscher transform method for GARCH and FC-GARCH processes, each one with Normal and shifted-Gamma innovations.

In the Esscher transform approach, as noted in section 3, the expected value in equation (3.20) is computed by Monte Carlo simulation. In simulation experiments, we used a sample of size of 10000 for the Gamma models and 20000 for the Normal ones, due to the antithetic variables.

The five pricing schemes are applied to two artificial series produced by a FC-GARCH model with Normal and Shifted-Gamma innovations with 3200 data points, obtained after a warm-up period of 1000 observations. The FC-GARCH parameters used in the simulations are given in table 1. These parameters, except for the a parameter and the risk-premium, are in Medeiros and Veiga (2009)(36).

FC-GARCH Parameters	
α	$[9.77\times10^{-16}, 5.14\times10^{-7}, 1.81\times10^{-5}]$
β	$[1.21, -0.32, -0.25]$
λ	$[0.06, -0.01, -0.04]$
γ	$[2.52, 2.85]$
c	$[-0.72, 1.56]$
Risk Premium	0.0349
a (Gamma case)	100

Table 5.1: Parameters for the FC-GARCH including the values of α, β and λ in the three different regimes.

To evaluate the Black and Scholes prices we estimate the volatility parameters by the sample variances, 4.0812×10^{-5}, for the Normal data and 3.8244×10^{-5} for the Gamma data.

For the estimation of the GARCH parameters we use an iterated two-stage method. Initially, we suppose h_t a constant equal to the sample variance. Then, we estimate the risk premium by weighted least squares (WLS). Then we fit a GARCH(1,1) model to the residuals of the WLS by performing a Quasi Maximum Likelihhod. We iterate these two steps until convergence is attained. The estimated parameters are shown in table 2.

GARCH Parameters in Normal Case	
α	7.4079×10^{-7}
β	0.9375
λ	0.0445
Risk Premium	0.0288

Table 5.2: Estimated GARCH Parameters in the Normal Case

We estimated the parameters using the two stage procedure as described before and then for finding a we used the method of moments estimate as in Siu et al. (2004) (43) as follows:

$$\hat{a} = \left[\frac{2 \sum_{t=1}^{T} h_t^{3/2}}{\sum_{t=1}^{T} \xi_t^3} \right]^2 , \tag{5-54}$$

which led us to the estimated parameters shown in table 3.

GARCH Parameters in Shifted-Gamma Case	
α	5.4959×10^{-7}
β	0.9461
λ	0.0397
Risk Premium	0.0363
a	79.4022

Table 5.3: Estimated GARCH Parameters in the Gamma Case

As discussed in section 4.2, one must check for the presence and the sign of the skewness before selecting the Normal or Shifted-Gamma approaches.

The resulting prices are presented in the tables that follows. IV stands for the initial volatility, here the ratio between the initial variance used for simulation and the sample variance. For the case where $IV = 1.0$, after each option price table there is another table with the ratios between the (FC)-GARCH prices and the Black-Scholes prices for each model. A graph of these comparative tables is also shown.

Call Prices with IV=1.0 for Artificial Normal FCGARCH Series

K/S_0	BS	FC-Normal	GARCH-Normal	FC-Gamma	GARCH-Gamma
0.80	20.0002	20.0009	19.9978	20.0149	20.0103
0.85	15.0063	15.0110	15.0079	15.0292	15.0238
0.90	10.0957	10.0939	10.1022	10.1216	10.1227
0.95	5.6537	5.5895	5.6292	5.6255	5.6565
1.00	2.4175	2.2812	2.3714	2.2757	2.3762
1.05	0.7397	0.6704	0.7163	0.6449	0.6845
1.10	0.1575	0.1515	0.1667	0.1362	0.1440
1.15	0.0234	0.0298	0.0310	0.0223	0.0294
1.20	0.0025	0.0049	0.0045	0.0032	0.0050

Table 5.4: Call Prices with IV=1.0 for Artificial Normal FCGARCH Series and T=90. The parameters used for the FCGARCH are in table 1, and the table with GARCH parameters are in table 2

K/S_0	BS	FC-Normal	GARCH-Normal	FC-Gamma	GARCH-Gamma
0.8	1	1.0000	0.9999	1.0007	1.0005
0.85	1	1.0003	1.0001	1.0015	1.0012
0.9	1	0.9998	1.0006	1.0026	1.0027
0.95	1	0.9886	0.9957	0.9950	1.0005
1	1	0.9436	0.9809	0.9413	0.9829
1.05	1	0.9063	0.9684	0.8718	0.9254
1.1	1	0.9619	1.0584	0.8648	0.9143
1.15	1	1.2735	1.3248	0.9530	1.2564
1.2	1	1.9600	1.8000	1.2800	2.0000

Table 5.5: Call Prices ratios with IV=1.0 for Artificial Normal FCGARCH Series and T=90

Call Prices with IV=1.2 for Artificial Normal FCGARCH Series

K/S_0	BS	FC-Normal	GARCH-Normal	FC-Gamma	GARCH-Gamma
0.80	20.0006	20.0044	20.0013	20.1102	20.0202
0.85	15.0144	15.0151	15.0164	15.1265	15.0386
0.90	10.1509	10.1177	10.1298	10.2467	10.1584
0.95	5.8159	5.6726	5.7121	5.7923	5.7428
1.00	2.6481	2.4260	2.4961	2.4920	2.5059
1.05	0.9156	0.7588	0.8015	0.7778	0.8111
1.10	0.2359	0.1828	0.2016	0.1688	0.2072
1.15	0.0454	0.0354	0.0478	0.0267	0.0367
1.20	0.0066	0.0053	0.0096	0.0043	0.0078

Table 5.6: Call Prices with IV=1.2 for Artificial Normal FCGARCH Series and T=90. The parameters used are in table 1, and the table with GARCH parameters are in table 2

Call Prices with IV=1.0 for Artificial Shifted-Gamma FCGARCH Series

K/S_0	BS	FC-Gamma	GARCH-Gamma	FC-Normal	GARCH-Normal
0.80	20.0001	20.0149	19.8948	20.0009	20.0079
0.85	15.0045	15.0292	14.9062	15.0110	15.0176
0.90	10.0803	10.1216	9.9955	10.0939	10.1031
0.95	5.6017	5.6255	5.5110	5.5895	5.6136
1.00	2.3402	2.2757	2.2413	2.2812	2.3280
1.05	0.6831	0.6449	0.6171	0.6704	0.6907
1.10	0.1349	0.1362	0.1209	0.1515	0.1537
1.15	0.0180	0.0223	0.0227	0.0298	0.0314
1.20	0.0017	0.0032	0.0041	0.0049	0.0049

Table 5.7: Call Prices with IV=1.0 for Artificial Shifted-Gamma FCGARCH Series and T=90. The parameters used are in table 1, and the table with GARCH parameters are in table 3

K/S_0	BS	FC-Gamma	GARCH-Gamma	FC-Normal	GARCH-Normal
0.80	1.0000	1.0007	0.9947	1.0000	1.0004
0.85	1.0000	1.0016	0.9934	1.0004	1.0009
0.90	1.0000	1.0041	0.9916	1.0013	1.0023
0.95	1.0000	1.0042	0.9838	0.9978	1.0021
1.00	1.0000	0.9724	0.9577	0.9748	0.9948
1.05	1.0000	0.9441	0.9034	0.9814	1.0111
1.10	1.0000	1.0096	0.8962	1.1231	1.1394
1.15	1.0000	1.2389	1.2611	1.6556	1.7444
1.20	1.0000	1.8824	2.4118	2.8824	2.8824

Table 5.8: Call Prices with IV=1.0 for Artificial Shifted-Gamma FCGARCH Series

Call Prices with IV=1.2 for Artificial Shifted-Gamma FCGARCH Series

K/S_0	BS	FC-Gamma	GARCH-Gamma	FC-Normal	GARCH-Normal
0.80	20.0004	20.1102	20.0946	20.0044	20.0052
0.85	15.0109	15.1265	15.1095	15.0151	15.0163
0.90	10.1290	10.2467	10.2206	10.1177	10.1205
0.95	5.7553	5.7923	5.7780	5.6726	5.6910
1.00	2.5635	2.4920	2.4712	2.4260	2.4512
1.05	0.8499	0.7778	0.7554	0.7588	0.7742
1.10	0.2052	0.1688	0.1839	0.1828	0.1851
1.15	0.0362	0.0267	0.0375	0.0354	0.0340
1.20	0.0047	0.0043	0.0073	0.0053	0.0069

Table 5.9: Call Prices with IV=1.2 for Artificial Shifted-Gamma FCGARCH Series and T=90. The parameters used are in table 1, and the table with GARCH parameters are in table 3

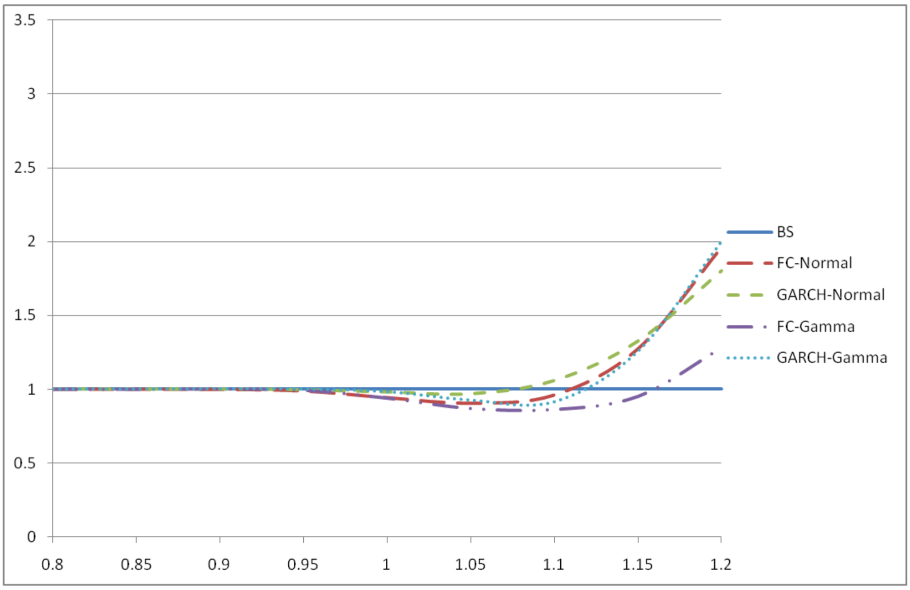

Figure 5.1: Graphs of Call Prices ratios with IV=1.0 for Artificial Normal FCGARCH Series

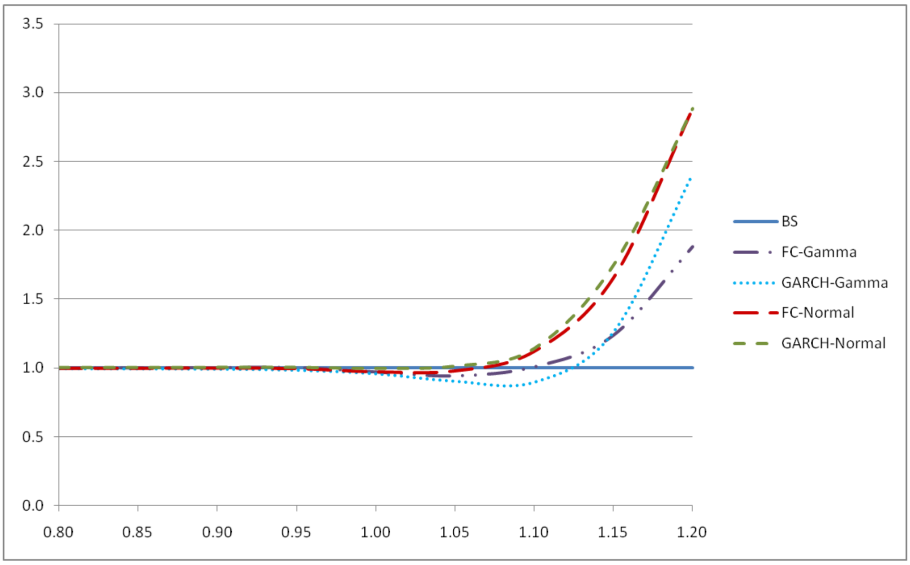

Figure 5.2: Graphs of Call Prices ratios with IV=1.0 for Artificial Shifted-Gamma FCGARCH Series

Put Prices with IV=1.0 for Artificial Normal FCGARCH Series

K/S_0	BS	FC-Normal	GARCH-Normal	FC-Gamma	GARCH-Gamma
0.80	0.0002	0.0001	0.0008	0.0024	0.0010
0.85	0.0063	0.0083	0.0097	0.0169	0.0107
0.90	0.0957	0.0893	0.0940	0.1095	0.1041
0.95	0.6537	0.5753	0.6230	0.6088	0.6523
1.00	2.4175	2.2835	2.3685	2.3003	2.3666
1.05	5.7397	5.6568	5.7088	5.5981	5.6728
1.10	10.1575	10.1466	10.1593	10.0596	10.1118
1.15	15.0234	15.0281	15.0358	14.9438	14.9701
1.20	20.0025	20.0059	20.0135	19.9292	19.9468

Table 5.10: Put Prices with IV=1.0 for Artificial Normal FCGARCH Series and T=90. The parameters used are in table 1, , and the table with GARCH parameters are in table 2

K/S_0	BS	FC-Normal	GARCH-Normal	FC-Gamma	GARCH-Gamma
0.8	1	0.5	4	12	5
0.85	1	1.31746	1.539683	2.68254	1.698413
0.9	1	0.933124	0.982236	1.144201	1.087774
0.95	1	0.880067	0.953037	0.931314	0.997858
1	1	0.944571	0.979731	0.95152	0.978945
1.05	1	0.985557	0.994616	0.97533	0.988344
1.1	1	0.998927	1.000177	0.990362	0.995501
1.15	1	1.000313	1.000825	0.994702	0.996452
1.2	1	1.00017	1.00055	0.996335	0.997215

Table 5.11: Put Prices ratios with IV=1.0 for Artificial Normal FCGARCH Series

Put Prices with IV=1.2 for Artificial Normal FCGARCH Series

K/S_0	BS	FC-Normal	GARCH-Normal	FC-Gamma	GARCH-Gamma
0.80	0.0006	0.0015	0.0015	0.0045	0.0016
0.85	0.0144	0.0147	0.0162	0.0248	0.0174
0.90	0.1509	0.1143	0.1325	0.1369	0.1444
0.95	0.8159	0.6534	0.7193	0.6856	0.7842
1.00	2.6481	2.4055	2.4884	2.4509	2.6057
1.05	5.9156	5.7375	5.8160	5.8001	5.9466
1.10	10.2359	10.1780	10.2118	10.2156	10.3430
1.15	15.0454	15.0418	15.0502	15.0689	15.1735
1.20	20.0066	20.0087	20.0134	20.0372	20.1402

Table 5.12: Put Prices with IV=1.0 for Artificial Normal FCGARCH Series and T=90. The parameters used are in table 1, and the table with GARCH parameters are in table 2

Put Prices with IV=1.0 for Artificial Shifted-Gamma FCGARCH Series

K/S_0	BS	FC-Gamma	GARCH-Gamma	FC-Normal	GARCH-Normal
0.80	0.0001	0.0024	0.0009	0.0001	0.0002
0.85	0.0045	0.0169	0.0099	0.0083	0.0085
0.90	0.0803	0.1095	0.0946	0.0893	0.0948
0.95	0.6017	0.6088	0.5760	0.5753	0.6149
1.00	2.3402	2.3003	2.2122	2.2835	2.3398
1.05	5.6831	5.5981	5.4999	5.6568	5.6952
1.10	10.1349	10.0596	9.9490	10.1466	10.1505
1.15	15.0180	14.9438	14.8426	15.0281	15.0213
1.20	20.0017	19.9292	19.8236	20.0059	19.9995

Table 5.13: Put Prices with IV=1.0 for Artificial Shifted-Gamma FCGARCH Series and T=90. The parameters used are in table 1, and the table with GARCH parameters are in table 3

K/S_0	BS	FC-Gamma	GARCH-Gamma	FC-Normal	GARCH-Normal
0.8	1.0000	24.0000	9.0000	1.0000	2.0000
0.85	1.0000	3.7556	2.2000	1.8444	1.8889
0.9	1.0000	1.3636	1.1781	1.1121	1.1806
0.95	1.0000	1.0118	0.9573	0.9561	1.0219
1	1.0000	0.9830	0.9453	0.9758	0.9998
1.05	1.0000	0.9850	0.9678	0.9954	1.0021
1.1	1.0000	0.9926	0.9817	1.0012	1.0015
1.15	1.0000	0.9951	0.9883	1.0007	1.0002
1.2	1.0000	0.9964	0.9911	1.0002	0.9999

Table 5.14: Put Prices ratios with IV=1.0 for Artificial Shifted-Gamma FC-GARCH Series

Put Prices with IV=1.2 for Artificial Shifted-Gamma FCGARCH Series

K/S_0	BS	FC-Gamma	GARCH-Gamma	FC-Normal	GARCH-Normal
0.80	0.0004	0.0045	0.0008	0.0015	0.0008
0.85	0.0109	0.0248	0.0184	0.0147	0.0121
0.90	0.1290	0.1369	0.1357	0.1143	0.1086
0.95	0.7553	0.6856	0.7007	0.6534	0.6632
1.00	2.5635	2.4509	2.4390	2.4055	2.4242
1.05	5.8499	5.8001	5.7436	5.7375	5.7507
1.10	10.2052	10.2156	10.1765	10.1780	10.1735
1.15	15.0362	15.0689	15.0415	15.0418	15.0329
1.20	20.0047	20.0372	20.0196	20.0087	20.0058

Table 5.15: Put Prices with IV=1.2 for Artificial Shifted-Gamma FCGARCH Series and T=90. The parameters used are in table 1, and the table with GARCH parameters are in table 3

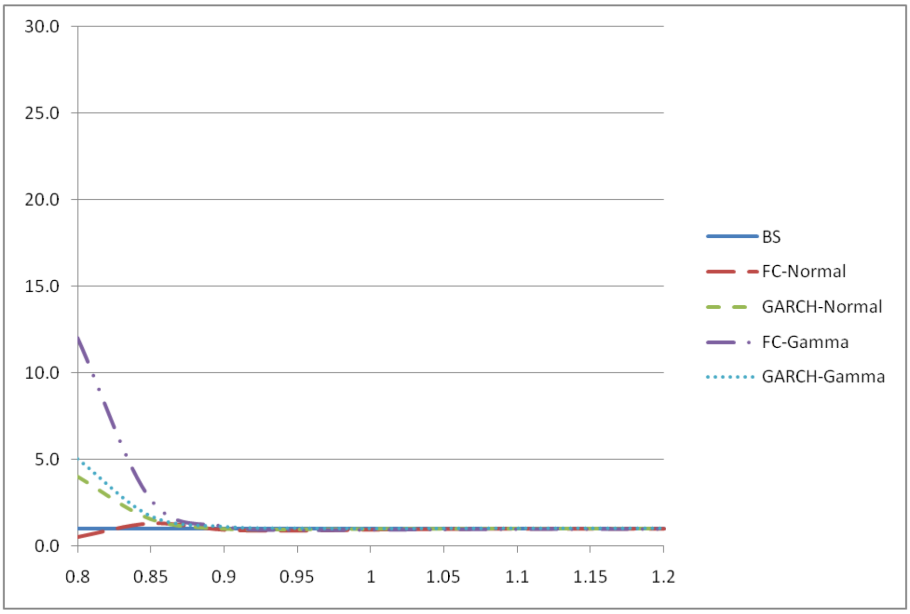

Figure 5.3: Graphs of Put Prices ratios with IV=1.0 for Artificial Normal FCGARCH Series

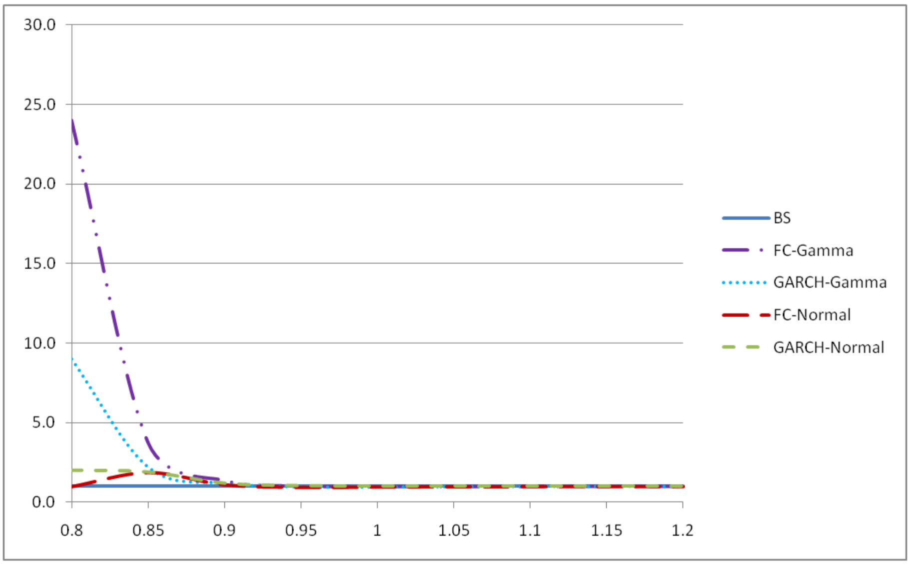

Figure 5.4: Graphs of Put Prices ratios with IV=1.0 for Artificial Shifted-Gamma FCGARCH Series

We can see that Calls and Puts have a different behavior. In tables 5.11 and 5.14, the put option price ratios have their largest values deep in the money. The more pronounced effect is in the FC-Gamma scheme. The values are larger than those in other schemes. Tables 5.5 and 5.8, on the other hand, show their largest values deep out the money, although its effect is not as significant as in the put case. For call options, the Normal models overprice the other models whilst in the put options, the Shifted-Gamma models do. This behavior may be explained by the negative asymmetry we introduced changing the signs of the innovations, in the Shifted-Gamma case.

To illustrate the changes in the option prices when we change the measures, we simulated prices under both the physical and risk neutral measures in all schemes. Then we checked for the proportions of scenarios where the options were exercised, which then give an estimate of the real-world probability of exercising an option. We chose $S_0 = 100$ and $K = 100$ to perform this exercise. We notice that in all schemes presented in table 5.16, the prices under the risk neutral measure are less likely to exceed the strike price than the prices under the physical measure.

Table 5.16: Average rate of exercising

Model/rate	Risk Neutral Measure	Physical Measure
FCGARCH Normal	0.4881	0.6197
FCGARCH Gamma	0.5032	0.6198
Gamma GARCH	0.4940	0.6338
Normal Garch	0.4885	0.5939
GARCH-Gamma with Normal data	0.4805	0.5994
GARCH-Normal with Gamma data	0.4870	0.6238

Then we focus on the GARCH-Gamma case using parameters estimated from artificial FC-GARCH Gamma data to produce the numerical results presented in the table below. The choice of this scheme is justified by its largest difference in the estimates above between the two measures. We considered $S_0 = 100$ and varied the strike price. Note that for every strike price, the risk neutral prices have a smaller chance of exceeding the strike price than the prices in the physical measure.

Average rate of exercising (Gamma GARCH)

K/S_0	Risk Neutral measure	Physical measure
0.80	0.9991	1.0000
0.85	0.9932	0.9978
0.90	0.9491	0.9784
0.95	0.7922	0.8885
1.00	0.4908	0.6336
1.05	0.2004	0.2991
1.10	0.0552	0.0862
1.15	0.0095	0.0162
1.20	0.0015	0.0023

Table 5.17: Average rate of exercising for Artificial Shifted-Gamma GARCH Series and T=90. The parameters used are in table 1, and the table with GARCH parameters are in table 3

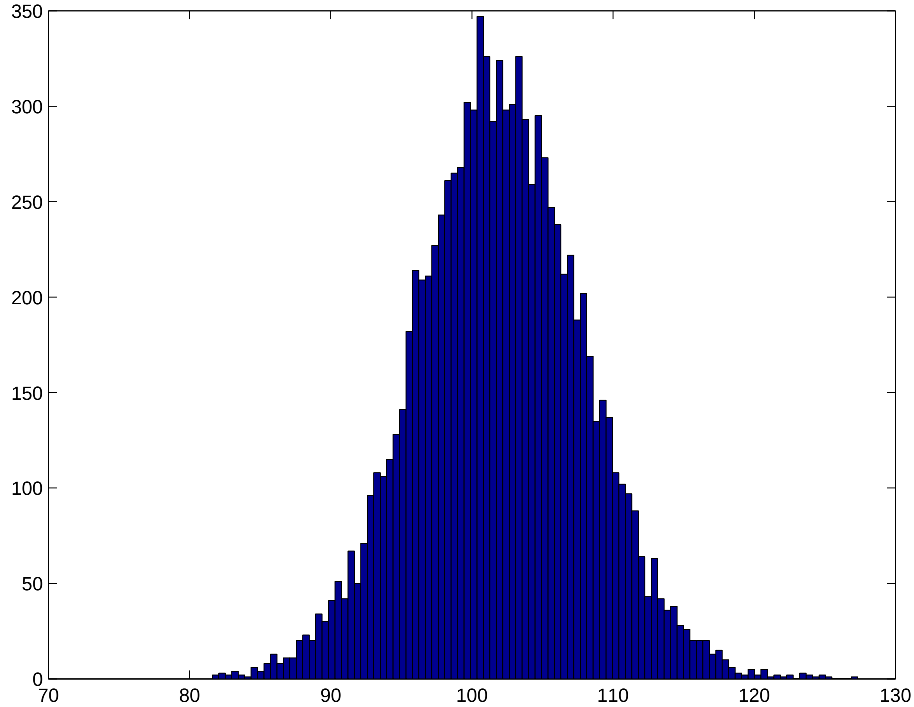

Figure 5.5: Histogram of S_T in the physical measure

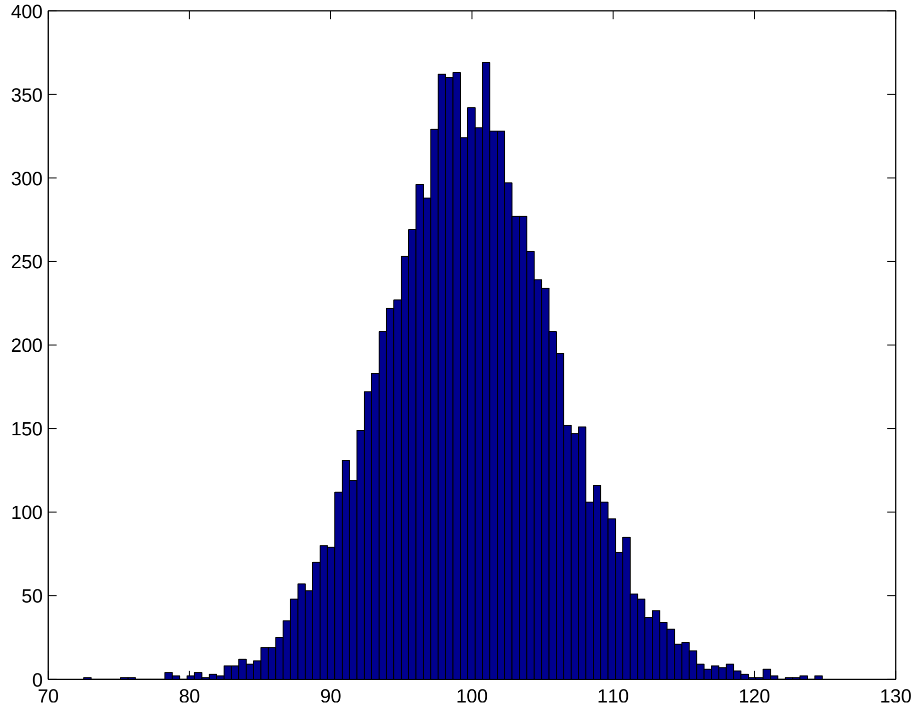

Figure 5.6: Histogram of S_T in the risk neutral measure

5.5 Sensitivity Analysis

Now we are going to check how the option prices change when some of the parameters are perturbed. We performed simulations imposing a small variation around the values of the parameters.

In some cases, it is important to bear in mind the stationarity condition. We note that putting the parameters so that the stationarity condition is close to 2 makes the effect on the option prices more pronounced. The stationarity condition for the Normal case, what is going to be used as a benchmark, once we don't have a stability condition for the Gamma case, is given by:

$$2\lambda_1 + 2\beta_1 + \beta_2 + \lambda_2 + \beta_3 + \lambda_3 < 2.$$

Proceeding with this exercise we capture the importance of each parameter in the option prices. We perform this analysis with the FC-GARCH models having Normal and Shifted-Gamma innovations. Graphs are shown in the appendix to illustrate such analysis.

5.5.1 Normal innovations

As we increase the value β_1 by steps of 0.05, the option price also increases. Note that a larger value of β_1 has a deeper impact than the others. This is because the last β_1 are closer to the stationarity condition. Then we increase the value of β_2 by the same increment, we have the same effect. After that, we notice that changing β_3 doesn't affect much the option price. When we repeated the experiment with steps of 0.1, even then the graph was the same, but we can see a slightly increasing pattern in the output numbers.

For the λ_1 and λ_2 graphs we can see a slightly increase in the option prices. For the λ_3, although a slight increase occurs it is not obviously shown in the graph even when we performed with increments of 0.05. Running $\gamma * \sigma = 1, 6, 11, 16$ and 21, no changes are seen in both cases γ_1 and γ_2.

For the effect of c_1, we noticed that the option value increases with an increase in c_1. On the other hand there is no clear effect of changing c_2.

The risk premium didn't show any clear effect. The graphic shows five variations of the risk premium with steps of 0.05. It seems having an increasing trend, but it is not clear even performing steps of 0.1.

5.5.2
Shifted Gamma innovations

For the Gamma case, we chose 0.05 to be the increment of β_1. The option value increases with an increase in the value of β_1 as in the Normal case. Note that in the last curve, when the stationarity condition exceeds 2, the increase in option prices has a more pronounced effect. Again, with steps of 0.05 we perturbed β_2 and the option values were also increased. We can see that as β_2 is closer to 2 the bigger its impact on the option price. In the analysis of β_3 we can not see the same behavior. It may be attributed to the insignificant effect of the last regime.

The a parameter doesn't have a clear effect on the option prices. It varied from 10 to 190 in steps of 45.

An increase in λ_1 and λ_2 with increments of 0.01 increases the option prices. We performed the analysis of λ_3 with increments 0.01,0.05 and 0.1. No significant effect is noticed in any of the cases.

For the c_1, with 0.1 increments, we see a slight increase of the price of the option. The c_2 has no significant effect on prices. For the Gamma parameter, running $\gamma * \sigma = 1, 6, 11, 16$ and 21 no changes are seen in both cases γ_1 and γ_2.

The risk premium has no clear monotonic pattern influence on the option prices. We simulated with increments of 0.05 and also steps of 0.1, that are relatively large relative to the initial value 0.0363.

5.6
Discussion of the results

In the tables and graphs of section 5, we noticed that the prices obtained from the FC-GARCH models are slightly higher than the Black Scholes prices, but the FC-GARCH option prices are lower than the GARCH option prices in both the Normal and Gamma cases for the calls, but for the puts this effect is only noticed in the normal case.

In the sensitivity analysis, we noticed that the GARCH parameters for the regime zero and the first regime were the most sensitive to perturbations of these model parameters while the GARCH parameters of the third regime, the logistic parameter γ, and the risk premium λ have little or no impact on option prices.

5.7
Conclusions

In this chapter we adopted the method of Siu et al. (2004) (43) to find a pricing kernel for the FC-GARCH models with two different parametric distributions for innovations, the Normal and the shifted-Gamma cases. We also performed simulations and showed tables comparing the Black Scholes prices and the GARCH prices to our simulation results of the FC-GARCH models as well as we performed a sensitivity analysis to understand how changes in some parameters affect the option valuation results.

The FC-GARCH models can capture features that some other models cannot, like the high kurtosis with low first-order autocorrelation of the squared observations, so that the option prices are more precise if calculated in the way we did in this chapter. Here we performed simulations with 3 regimes but the model can mimic an economy with many regimes. A further research would be developing tests to find the optimal number of regimes for each situation.