4
Option Pricing

This Chapter explains basically how should we price an option.

In a first Finance Mathematics course we are thaught the concept of present value, which consists in discounting by a constant risk free interest rate some value in the future so that we can find the price of it in the present.

It is correct if we think of loans and bank accounts with fixed interest rates because we know the evolution of these values, so we can track it and use it in a reverse way to find the present value, but not in general. For risky assets we don't know the evolution of the price so we cannot discount by a proper rate.

On the other hand Probability Theory have some answers to solve that problem. Under some assumptions, if we change properly the measure in which we take expectations we can think of any discounted portfolio as a risk neutral one in that measure. That's why such measures are more usually called Risk Neutral Measures than Equivalent Martingale Measures. The latter name is because under that new equivalent measure, the discounted portfolio becomes a martingale.

4.1
Pricing Formula in the Risk-Neutral Measure

For this section, we follow the chapter 5 in Shreve (41) for the background and to show the Black-Scholes example.

Let $D_t = e^{-\int_0^t r(u)du}$ be a discount factor process and X_t be a self-financing portfolio made of fixed interest investment and risky assets where the amount invested into risky assets in time t is Δ_t. It means that changes in the value of the portfolio are due to changes in the price of the risk asset and changes in the safe investment and not for putting more money into this portfolio. Mathematically we ask that

$$dX_t = \Delta_t dS_t + r_t(X_t - \Delta_t S_t)dt$$

Note that according to the Itô product rule, asking for being self-financing is very restrictive because we lose some terms in the equation above. Suppose also that the asset follows

$$dS_t = \alpha_t S_t dt + \sigma_t S_t dW_t \tag{4-1}$$

or as in example 25 of the use of Itô's Lemma:

$$S_t = S_0 e^{\int_0^t \sigma d\widetilde{W}_s + \int_0^t \left(r - \frac{1}{2}\sigma^2\right) ds} \tag{4-2}$$

By the Itô Lemma, we get first that

$$dD_t = e^{-\int_0^t r(u)du} r(t)dt = -D_t r_t dt$$

Then

$$\begin{aligned} dX_t &= \Delta_t dS_t + r_t(X_t - \Delta_t S_t)dt & \text{(4-3)}\\ &= \Delta_t \alpha_t S_t dt + \Delta_t \sigma_t S_t dW_t + r_t(X_t - \Delta_t S_t)dt & \text{(4-4)}\\ &= (\Delta_t \alpha_t S_t dt + r_t X_t - r_t \Delta_t S_t)dt + \Delta_t \sigma_t S_t dW_t & \text{(4-5)}\\ &= (r_t X_t + (\alpha_t - r_t)\Delta_t S_t)dt + \Delta_t \sigma_t S_t dW_t & \text{(4-6)}\\ &= r_t X_t dt + \sigma_t \Delta_t S_t(\theta_t dt + dW_t) & \text{(4-7)}\\ &= r_t X_t dt + \sigma_t \Delta_t S_t d\widetilde{W}_t. & \text{(4-8)} \end{aligned}$$

where $\widetilde{W}_t$ is a Brownian Motion by Girsanov's Theorem and $\theta_t = \frac{(\alpha_t - r_t)}{\sigma_t}$.

Proposition 30 *$D_t X_t$ is a martingale under $\mathbb{Q}$*

Proof: We will proceed as Shreve (41). By the product rule (example 23 of the use of Itô's Lemma) we have

$$\begin{aligned} d(D_t X_t) &= X_t dD_t + D_t dX_t + dD_t dX_t & \text{(4-9)}\\ &= X_t(-D_t r_t)dt + D_t dX_t + 0 & \text{(4-10)}\\ &= -X_t D_t r_t dt + D_t(r_t X_t dt + \sigma_t \Delta_t S_t d\widetilde{W}_t) & \text{(4-11)}\\ &= D_t \sigma_t \Delta_t S_t d\widetilde{W}_t. & \text{(4-12)} \end{aligned}$$

The Girsanov's Theorem garantee us that $\widetilde{W}_t$ is a Brownian Motion and so, as a stochastic integral with respect to a Brownian Motion is a martingale, $D_t X_t$ is a martingale. □

Let the value of some derivative V_T be $\mathcal{F}_T$-measurable. We want to know what is the initial money X_0 and which process Δ_t are required to be in a portfolio so that an investor can hedge a position in this derivative, i.e., such that

$$X_T = V_T$$

If we can do that, given that D_tX_t is a martingale under $\mathbb{Q}$ we have

$$D_tX_t = \mathbb{E}_{\mathbb{Q}}[D_TX_T|\mathcal{F}_t] = \mathbb{E}_{\mathbb{Q}}[D_TV_T|\mathcal{F}_t]$$

Then, D_tX_t is the amount required in t to make the hedge of this derivative with payoff V_T. We can then say that the price of the derivative in t is V_t where

$$D_tV_t = \mathbb{E}_{\mathbb{Q}}[D_TV_T|\mathcal{F}_t] \quad 0 \leq t \leq T \tag{4-13}$$

or

$$V_t = \mathbb{E}_{\mathbb{Q}}[e^{-\int_t^T r(u)du}V_T|\mathcal{F}_t] \quad 0 \leq t \leq T \tag{4-14}$$

□

4.2
Black-Scholes-Merton Formula

The most famous finance formula is the Black-Scholes formula. It is easy to find in many books the proof that consists in deriving the Black-Scholes PDE and then turning it in a heat equation problem. Here we present it under the risk-neutral framework as in Shreve (41).

The Black-Scholes-Merton Model supposes that the interest rate and the volatility are fixed. For an European Call option, we have the payoff

$$V_T = (S_T - K)^+ \tag{4-15}$$

Using the pricing formula just discussed we need to calculate

$$V_t = \mathbb{E}_{\mathbb{Q}}[e^{-r(T-t)}(S_T - K)^+|\mathcal{F}_t] \tag{4-16}$$

Let's think of the formula above as a function of t and S_t:

$$C(t, S_t) = \mathbb{E}_{\mathbb{Q}}[e^{-r(T-t)}(S_T - K)^+|\mathcal{F}_t] \tag{4-17}$$

The equation

$$S_t = S_0 e^{\int_0^t \sigma d\widetilde{W}_s + \int_0^t \left(r - \frac{1}{2}\sigma^2\right)ds} \tag{4-18}$$

can be written as

$$\begin{aligned} S_T &= S_t e^{\int_t^T \sigma d\widetilde{W}_s + \int_t^T \left(r - \frac{1}{2}\sigma^2\right) ds} && (4\text{-}19) \\ &= S_t e^{\sigma(\widetilde{W}_T - \widetilde{W}_t) + \left(r - \frac{1}{2}\sigma^2\right)\tau} && (4\text{-}20) \end{aligned}$$

where $\tau = T - t$. If we further define $Y = -\frac{\widetilde{W}_T - \widetilde{W}_t}{\sqrt{T-t}}$ we can write

$$S_T = S_t e^{-\sigma\sqrt{\tau}Y + \left(r - \frac{1}{2}\sigma^2\right)\tau} \quad (4\text{-}21)$$

Then, as S_t is $\mathcal{F}_t$-measurable and $e^{-\sigma\sqrt{\tau}Y + \left(r - \frac{1}{2}\sigma^2\right)\tau}$ is independent of $\mathcal{F}_t$:

$$\begin{aligned} C(t,x) &= \mathbb{E}_{\mathbb{Q}}[e^{-r\tau}(S_T - K)^+] && (4\text{-}22) \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-r\tau}(xe^{-\sigma\sqrt{\tau}Y + \left(r - \frac{1}{2}\sigma^2\right)\tau} - K)^+ e^{-\frac{1}{2}y^2} dy && (4\text{-}23) \end{aligned}$$

Now, note that $(xe^{-\sigma\sqrt{\tau}Y + \left(r - \frac{1}{2}\sigma^2\right)\tau} - K)^+$ is positive if and only if

$$y < d_- \equiv \frac{1}{\sigma\sqrt{\tau}} \left[\ln\left(\frac{x}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)\tau \right] \quad (4\text{-}24)$$

Indeed,

$$\begin{aligned} e^{-\sigma\sqrt{\tau}y + \left(r - \frac{1}{2}\sigma^2\right)\tau} > K/x &\iff e^{-\sigma\sqrt{\tau}y} > (K/x)e^{-\left(r - \frac{1}{2}\sigma^2\right)\tau} \\ &\iff -\sigma\sqrt{\tau}y > \ln(K/x) - \left(r - \frac{1}{2}\sigma^2\right)\tau \\ &\overset{*(-1)}{\iff} y < \frac{1}{\sigma\sqrt{\tau}} \left[\ln\left(\frac{x}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)\tau \right] \end{aligned}$$

We don't have to consider the integral when it is zero. This leads us to:

$$\begin{aligned} C(t,x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_-} e^{-r\tau}(xe^{-\sigma\sqrt{\tau}y + \left(r - \frac{1}{2}\sigma^2\right)\tau} - K) e^{-\frac{1}{2}y^2} dy \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_-} xe^{-\frac{1}{2}y^2 - \sigma\sqrt{\tau}y + \left(-\frac{1}{2}\sigma^2\right)\tau} dy - \int_{-\infty}^{d_-} e^{-r\tau} K e^{-\frac{1}{2}y^2} dy \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_-} xe^{-\frac{(y+\sigma\sqrt{\tau})^2}{2}} dy - \Phi(d_-)e^{-r\tau}K \\ &\overset{z=y+\sigma\sqrt{\tau}}{=} x\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_- + \sigma\sqrt{\tau}} e^{-\frac{(z)^2}{2}} dz - \Phi(d_-)e^{-r\tau}K \\ &= x\Phi(d_+) - \Phi(d_-)e^{-r\tau}K \end{aligned}$$

where

$$\begin{aligned} d_+ &= d_- + \sigma\sqrt{\tau} \\ &= \frac{1}{\sigma\sqrt{\tau}}\left[\ln\left(\frac{x}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)\tau\right] + \sigma\sqrt{\tau} \\ &= \frac{\ln\left(\frac{x}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)\tau + \sigma^2\tau}{\sigma\sqrt{\tau}} \\ &= \frac{\ln\left(\frac{x}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}} \end{aligned}$$

□

4.3 Put-Call Parity

Now, we will see that who can solve the problem of pricing the European Call option, can also find the price of an Europena Put Option.

Build a portfolio made of:

- Buy a stock.
- Buy a put option with this stock bought as the underlying.
- Sell a call option also based in the stock bought.

Then, if we call this portfolio Π, we can write:

$$\Pi_t = S_t + P_t - C_t.$$

But then at time T we will have:

$$\begin{aligned} \text{payoff}_T(\Pi) &= S_T + (K - S_T)^+ - (S_T - K)^+ \\ &= \begin{cases} S_T + K - S_T & \text{, if } S_T \leq K \\ S_T - (S_T - K) & \text{, if } S_T > K \end{cases} \\ &= K. \end{aligned}$$

This portfolio always gives a return K, the exercise price. Then, because it is a deterministic portfolio, we can think of its present value being that of K discounted:

$$\Pi_t = Ke^{-r(T-t)}$$

Then we have:

$$Ke^{-r(T-t)} = S_t + P_t - C_t \;\Rightarrow\; P_t = Ke^{-r(T-t)} + C_t - S_t.$$

□

4.4 Incomplete Markets

Here we are going to show two well known techniques for pricing in incomplete markets, viz. the Davis method and the Esscher method.

When we can not find a hedge for some asset in the economy we say that the market is incomplete and then it is known that there are infinite equivalent martingale measures. The risk neutral way of pricing do not involve requirements concerning the risk preferences of the investors, but in order to justify the choice of one particular risk neutral measure among many, we will have to think about a reasonable way to select it.

Besides the two methodologies that we are going to discuss briefly here, there are other approaches to hedging and pricing in incomplete markets, such as the quadratic hedging approach. They are mainly interested in minimizing risk, Maximizing expected utility or minimize loss. For a comprehensible reading see Bingham and Kiesel (2004)(5).

4.4.1 Davis Method

We start by the method of Davis. Let Φ_a be the set of all self-financing strategies (i.e. in which you can not inject money in the portfolio. For details see (21)and (26)) and write

$$\tilde{U}(x) = \sup_{\varphi \in \Phi_a} E[U(V_{\varphi,x}(T))]$$

for the maximum expected utility of an investor and suppose p is the price of the asset X.

Let

$$W(\delta, x, p) = \sup_{\varphi \in \Phi_a} E[U(V_{\varphi,x-\delta}(T) + \frac{\delta}{p}X)]$$

be a disturb created for being able to sense the effect of changing the strategy. If the maximum utility does not get affected by this "tilt", $\hat{p}$ is the fair price for the option.

Under some conditions, $\hat{p}$ is the unique solution of

$$\frac{\partial W}{\partial \delta} W(0, p, x) = 0$$

which is the fair price of the option in $t = 0$. Furthermore, this value is given by:

$$\hat{p} = \frac{E[U'(V_{\varphi^*,x}(T))X]}{\tilde{U}'(x)}$$

For more details see Davis(1994)(13) and Bingham and Kiesel(2004) (5).

4.4.2 Esscher Method

Now let's see the Esscher method. Let $S_t = S_0 e^{X_t}$ be the model for the asset prices where X_t has independent and stationary increments. Assume that

$$M(h,t) = M(h,1)^t = E[e^{hX_t}] \tag{4-25}$$

Then, having that in mind, let's define

$$\Lambda_t = \frac{e^{hX_t}}{E[e^{hX_t}]} = e^{hX_t} M(h,1)^{-t} = \frac{S_t^h}{E[S_t^h]}; \quad t \geq 0 \tag{4-26}$$

that is a positive martingale and will be used for the change of measure.

Now, by the Baye's rule,

$$E^h[\Psi(Y)] = E[\Psi(Y); h] = \frac{E[\Psi(Y) e^{hY}]}{E[e^{hY}]} \tag{4-27}$$

$$= E[\Lambda_t \Psi(Y)] \tag{4-28}$$

where Y is a random variable and Ψ is a measurable function. For example $\Psi(Y)$ could be the logreturn e^{X_t}.

We will call $\mathbb{Q}$ the Esscher measure of parameter $h = h^*$ if $\{e^{-rt} S_t\}_{t \geq 0}$ is a martingale.

By the condition on the asset prices we have the following equivalences:

$$
\begin{aligned}
E[e^{-rt}S_t, h^*] = S_0 &\iff E\left[\frac{S_t}{S_0}, h^*\right] = e^{rt} && \text{(4-29)}\\
&\iff E[e^{X_t}, h^*] = e^{rt} && \text{(4-30)}\\
&\iff E\left[e^{X_t}\frac{e^{h^*X_t}}{M(h^*,1)^t}\right] = e^{rt} && \text{(4-31)}\\
&\iff \frac{E[e^{X_t(h^*+1)}]}{M(h^*,1)^t} = e^{rt} && \text{(4-32)}\\
&\iff \frac{M(h^*+1,1)^t}{M(h^*,1)^t} = e^{rt} && \text{(4-33)}\\
&\iff \left[\frac{M(h^*+1,1)}{M(h^*,1)}\right]^t = e^{rt} && \text{(4-34)}\\
&\iff \frac{M(h^*+1,1)}{M(h^*,1)} = e^{r} && \text{(4-35)}\\
& && \text{(4-36)}
\end{aligned}
$$

The equation

$$
\frac{M(h^*+1,1)}{M(h^*,1)} = e^{r} \tag{4-37}
$$

determines the parameter h^* uniquely. We will be interested in this parameter for finding the risk neutral measure to be used in the option pricing.

There is a useful result that we should show here. It is known as the factorization formula:

Theorem 31 *Let g be a measurable function and h, k and t be real values with $t \geq 0$, then*

$$
E[S_t^k g(S_t); h] = E[S_t^k; h]E[g(S_t); k+h]. \tag{4-38}
$$

Proof:

$$
\begin{aligned}
E[S_t^k g(S_t); h] &= E[S_t^k g(S_t) e^{hX_t} M(h,1)^{-t}] && \text{(4-39)}\\
&= E\left[S_t^k g(S_t)\frac{S_t^h}{E[S_t^h]}\right] && \text{(4-40)}\\
&= \frac{E[S_t^{k+h}]E[S_t^{k+h} g(S_t)]}{E[S_t^h]E[S_t^{k+h}]} && \text{(4-41)}\\
&= E[S_t^k; h]E[g(S_t); k+h] && \text{(4-42)}
\end{aligned}
$$

□

For more details see Gerber and Shiu (1994)(31) and Bingham and Kiesel(2004)(5). Some of the points discussed above will be revisited in the conditional Esscher Transform section.

4.5 Duan's breakthrough

In 1995, Duan developped a method for pricing options under GARCH processes. Although there are economic restrictions and is valid only for normal noises, it was a milestone in Finance. He also developped some papers concerning the relationship between the diffusions and the econometric models and their risk neutral versions. This discussion doesn't help us in this thesis and it can be found in Duan(1997)(18). Here we show the main results from the paper published in 1995 (17).

Consider the model:

$$ln\frac{X_t}{X_{t-1}} = r + \lambda\sqrt{h_t} - \frac{1}{2}h_t + \xi_t$$

where

$$\xi_t|\phi_{t-1} \sim N(0, h_t) \text{ under mesure } \mathbb{P}$$

$$h_t = \alpha_0 + \sum_{i=1}^{q} \alpha_i \xi_{t-i}^2 + \sum_{j=1}^{p} \beta_j h_{t-j}$$

Basically the paper raises the question: What would the risk-neutral GARCH be?

For answering this question, he defines the concept of Local Risk Neutral Valuation Relationship hereafter LRNVR, as below:

A measure $\mathbb{Q}$ satisfies the LRNVR if:

1 $\mathbb{Q}$ is mutually absolutely continuous with respect to measure P.(Equivalents)

2 $\frac{X_t}{X_{t-1}}$ is lognormally distributed (under $\mathbb{Q}$)

3 $E^{\mathbb{Q}}[\frac{X_t}{X_{t-1}}|\phi_{t-1}] = e^r$

4 $Var^{\mathbb{Q}}(ln\frac{X_t}{X_{t-1}}|\phi_{t-1}) = Var^{\mathbb{P}}(ln\frac{X_t}{X_{t-1}}|\phi_{t-1})$ a.s.

Having in mind this concept we can state the main Theorem of the paper:

Theorem 32 *Supposing the LRNVR, under $\mathbb{Q}$ we have:*

$$ln\frac{X_t}{X_{t-1}} = r - \frac{1}{2}h_t + \epsilon_t$$

where:

$$\epsilon_t|\phi_{t-1} \sim N(0, h_t)$$
$$h_t = \alpha_0 + \sum_{i=1}^{q} \alpha_i(\epsilon_{t-i} - \lambda\sqrt{h_{t-i}})^2 + \sum_{j=1}^{p} \beta_j h_{t-j}$$

Proof: It follows exactly as in Duan (17). Since $\frac{X_t}{X_{t-1}}|\phi_{t-1}$ lognormally distributed under measure $\mathbb{Q}$, it can be written as:

$$\ln \frac{X_t}{X_{t-1}} = v_t + \epsilon_t,$$

where v_t is the conditional mean and ϵ_t is a $\mathbb{Q}$-normal random variable. The conditional mean of ϵ_t is zero and its conditional variance is to be determined. The proof is in two parts.
First we prove that $v_t = r - \frac{h_t}{2}$. Indeed,

$$E^{\mathbb{Q}}\left[\frac{X_t}{X_{t-1}}|\phi_{t-1}\right] = E^{\mathbb{Q}}[e^{v_t+\epsilon_t}|\phi_{t-1}] = e^{v_t+\frac{h_t}{2}}$$

where $h_t = Var^{\mathbb{P}}[\frac{X_t}{X_{t-1}}|\phi_{t-1}] = Var^{\mathbb{Q}}[\frac{X_t}{X_{t-1}}|\phi_{t-1}]$ by the LRNVR.
Since $E^{\mathbb{Q}}[\frac{X_t}{X_{t-1}}|\phi_{t-1}] = e^r$ also by the LRNVR, it follows that $v_t = r - \frac{h_t}{2}$.

Now we prove the second part. It remains to prove that h_t can indeed be expressed as stated in the Theorem. By the preceding result and the model under $\mathbb{P}$:

$$\ln \frac{X_t}{X_{t-1}} = r + \lambda\sqrt{h_t} - \frac{h_t}{2} + \xi_t, \tag{4-43}$$

we have, comparing the logs:

$$r + \lambda\sqrt{h_t} - \frac{h_t}{2} + \xi_t = r - \frac{h_t}{2} + \epsilon_t. \tag{4-44}$$

so that $\xi_t = \epsilon_t - \lambda\sqrt{h_t}$. Substituting ϵ_t into the conditional variance equation yields the desired result:

$$h_t = \alpha_0 + \sum_{i=1}^{q} \alpha_i(\epsilon_{t-i} - \lambda\sqrt{h_{t-i}})^2 + \sum_{j=1}^{p} \beta_j h_{t-j}$$

□

The result just proved can be found as a particular case of the method described in the next section. This verification can be found in section 3.1

in the paper by Siu et al. (2004)(43). The method below doesn't require the noises to be normal, they just have to have a moment generation function.