3
GARCH Models

We will here, define the basic econometric models that will be the basis for our work. In chapters 5 and 6 we are going to adapt the methodoly developed in Siu et al. (43) to the FC-GARCH model of Álvaro and Medeiros (36) and for the Mixture of GARCHs.

To start, we should first go to the basics and define the ARCH and GARCH models.

3.1
ARCH Models

When we want to express the volatility in terms of past returns, we should use an Autoregresive Conditional Heterokesdastic model (henceforth ARCH) model, which is defined by a model with:

$$y_t = \mu_t + h_t^{1/2}\epsilon_t \tag{3-1}$$

where

$$h_t = \alpha_0 + \sum_{i=1}^{p} \alpha_i y_{t-i}^2 \tag{3-2}$$

that can be rewritten as

$$h_t = \alpha_0 + \sum_{i=1}^{p} \alpha_i\prime\epsilon_{t-i}^2 \tag{3-3}$$

where ϵ_t is a noise usually a $N(0,1)$ random variable and μ_t is the conditional mean that we will be always considering conditionally deterministic, or more precisely, $\mathcal{F}_{t-1}$ measurable, sometimes even being considered to be zero for stationarity purposes.

When we want to model the conditional volatility taking in account not only the past errors but also the past volatilities we should use a General Autoregresive Conditional Heterokesdastic model (henceforth GARCH) which is modeled as below:

$$y_t = \mu_t + h_t^{1/2}\epsilon_t \tag{3-4}$$

with

$$h_t = \alpha_0 + \sum_{i=1}^{p} \alpha_i y_{t-i}^2 + \sum_{j=1}^{q} \beta_j h_{t-j} \tag{3-5}$$

Considering the same observations for μ_t and ϵ_t as above. For simplicity let´s suppose it zero.

Another way of writing the same thing (but note the different alpha) is:

$$h_t = \alpha_0 + \sum_{i=1}^{p} \alpha_i\prime \epsilon_{t-i}^2 + \sum_{j=1}^{q} \beta_j h_{t-j} \tag{3-6}$$

In the next subsection we are going to show a more general GARCH-like model, viz., the FC-GARCH by Medeiros and Álvaro, that poorly speaking, gives a GARCH to each regime of the economy.

3.2 Flexible Coefficient Generalized Autoregressive Conditional Heteroskedastic (FC-GARCH)

First, we consider a discrete-time financial model consisting of one risk-free bond B and one risky stock S. We consider a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with $\mathcal{P}$ being the real-world probability measure. Let $\mathcal{T}$ denote the time index set $\{0, 1, 2, \ldots, T\}$ of the financial model. Let r denote a constant continuously compounded risk-free rate of interest or force of interest. Under $\mathcal{P}$, the dynamics of the bond-price process $\{B_t\}_{t\in\mathcal{T}}$ satisfy, for each $t \in \mathcal{T}\backslash\{0\}$:

$$B_t \quad = \quad B_{t-1}e^r \ , \quad B_0 = 1 \ . \tag{3-7}$$

Let $\epsilon = \{\epsilon_t\}_{t\in\mathcal{T}}$ denote the innovations of the returns from the risky stock S, where $\epsilon_0 = 0$. We assume that $\{\epsilon_t\}_{t\in\mathcal{T}}$ are i.i.d. with distribution $D(0,1)$, where $D(0,1)$ represents a general distribution with mean zero and unit variance.

Let $S := \{S_t\}_{t\in\mathcal{T}}$ denote the price process of the risky asset S. Let $Y_t := \ln(S_t/S_{t-1})$, which represents the continuously compounded rate of return from the risky asset S over the time horizon $[0,t]$. We suppose that the return process $Y := \{Y_t\}_{t\in\mathcal{T}}$ follows a first-order flexible coefficient Generalized

Autoregressive Conditional Heteroskedastic model with $m = H + 1$ limiting regimes, henceforth, FC-GARCH$(m, 1, 1)$ as below:

$$\begin{aligned} Y_t &= \mu_t + h_t^{1/2}\epsilon_t \\ h_t &= G(w_t; \psi) \ . \end{aligned} \qquad (3\text{-}8)$$

Here $G(w_t; \psi)$ is a nonlinear function of a vector of variables $w_t = [Y_{t-1}, h_{t-1}, s_t]^T$ (i.e. "T" represents the transpose of a vector or a matrix) defined by:

$$G(w_t; \psi) := \alpha_0 + \beta_0 h_{t-1} + \lambda_0 Y_{t-1}^2 + \sum_{i=1}^{H} [\alpha_i + \beta_i h_{t-1} + \lambda_i Y_{t-1}^2] f(s_t; \gamma_i, c_i), \quad t = 1, ..., T.$$

where

1. for each $i = 1, 2, \ldots, H$, the logistic function

$$f(s_t, \gamma_i, c_i) := \frac{1}{1 + e^{-\gamma_i (s_t - c_i)}} \ ;$$

2. The vector of parameters

$$\psi := [\alpha_0, \beta_0, \lambda_0, \alpha_1, ..., \alpha_H, \beta_1, ..., \beta_H, \lambda_1, ..., \lambda_H, \gamma_1, ..., \gamma_H, c_1, ..., c_H]^T \in \mathbb{R}^{3+5H} \ ;$$

3. The parameter $\gamma_i, i = 1, ..., H$ is called the slope parameter. When $\gamma_i \to \infty$, the function becomes a step function. Here, we consider the case when $s_t = Y_{t-1}$.

The rationale of introducing the FC-GARCH model is to provide a useful and practical way to incorporate the asymmetric effect of the sign and the size of the previous return Y_{t-1} on the current variance level h_t. It can also capture the heavy-tailedness of return's distribution and the slow decaying of the autocorrelation of the squared returns process $\{Y_t^2\}_{t \in \mathcal{T}}$. The FC-GARCH model can also incorporate another important stylized empirical feature of returns data, namely, the Taylor effect, first documented by Taylor (1986)(44). The Taylor effect refers to the strong autocorrelation of absolute daily returns data. This also relates to the long-memory effect of volatility; that is, the decay of the autocorrelations of volatility is too slow to be described by any short memory autoregressive moving average time series models. We are going to get back to this model in Chapter 5.