2 Continuous Time Finance

This Chapter will cover the basics of Probability Theory used in Finance and then the use of it in Finance. We deal, in particular, with the topics concerning Continuous Time Finance.

The work here is done with discrete time models but as the continuous case account for the discrete case also as its particular case we will do all discussion in continuous time for mental exercising and generality.

Most of what is here was made initially as beamer slides presented to my co-advisor as a weekly seminar based from the Elliot and Kopp book (21) but I also did a mixing of contents from other books such as the Lecture Notes of Evans version 1.2 (26), Billingsley (4), Oksendal (38), Shreve (41) and Bingham and Kiesel (5). We refer to these for some proofs and additional details.

2.1 Initial Definitions

This subsection deals with continuous time stochastic processes, so we are going to consider that t lies in $\mathcal{T}$, where $\mathcal{T}$ is one of the following sets: $[0,T], [0,\infty)$ or $[0,\infty]$. And we will be always considering a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition 1 *(Filtration) $\mathbb{F} = \{\mathcal{F}_t\}_{t\in\mathcal{T}}$ is called a filtration if it is a increasing family of sub-σ-algebras, i.e., if it is a family of σ-algebras such that for $s \leq t$:*

$$\mathcal{F}_t \subset \mathcal{F}, \quad \mathcal{F}_s \subset \mathcal{F}_t$$

We also suppose that the filtration is complete and is right continuous in the folowing sense: (Information now comes continuously.)

$$\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s, t \in \mathcal{T}$$

Definition 2 *(Stochastic Processes) A continuous-time stochastic process X taking values in a measurable space $(E, \mathcal{E})$ is a family of r.v's $\{X_t\}$ defined in $(\Omega, \mathcal{F}, \mathbb{P})$, indexed by t taking values in $(E, \mathcal{E})$. Notice that:*

- *$X_t(\cdot)$ is a random variable.*
- *$X_.(\omega)$ is a path of the process X.*

2.2 Equivalences among Processes and other Definitions

We are going to define different ways of comparing stochastic processes as well as clarify the differences among those concepts. Besides that, we are going to give some common definitions concerning the measurability of stochastic processes.

Definition 3 *(Equivalent Processes) Let*

$$\phi^X_{t_1,t_2,...,t_n}(A) = \mathbb{P}(\{\omega \in \Omega : (X_{t_1}(\omega), X_{t_2}(\omega), ..., X_{t_n}(\omega)) \in A\})$$

be a measure in $\mathbb{R}^n$. X and Y are equivalent if their families of finite distributions coincides and we denote by $X \sim Y$.

Definition 4 *(Modification of a process) Suppose $(X_t)_{t\geq 0}$ and $(Y)_{t\geq 0}$ two processes defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and taking values in $(E, \mathcal{E})$. The process $\{Y_t\}$ is said to be a modification of $\{X_t\}$ if*

$$X_t = Y_t \ a.s. \quad \forall t \in \mathcal{T};$$

i.e.,

$$\mathbb{P}(X_t = Y_t) = 1 \quad \forall t \in \mathcal{T}.$$

Remark: Note that for each t, it is possible to have a null set associated to it.

Definition 5 *(Indistinguishable) Suppose $(X_t)_{t\geq 0}$ and $(Y)_{t\geq 0}$ two processes defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and taking values in $(E, \mathcal{E})$. The process $\{Y_t\}$ is said to be indistinguishable from $\{X_t\}$ if for almost every $\omega \in \Omega$,*

$$X_t(\omega) = Y_t(\omega) \quad \forall t \in \mathcal{T};$$

i.e.,

$$\mathbb{P}(\{\omega \in \Omega : X_t(\omega) = Y_t(\omega), \ \forall t \in \mathcal{T}\}) = 1.$$

Remark: Note that in this case, there is only one null set.

Now we are going to present a concept concerning sets but that is related to indistinguishability:

Definition 6 *(Evanescent) $A \subset [0,\infty] \times \Omega$ is evanescent if*

$$1_A(t,\omega) = \begin{cases} 1, & \text{if } (t,\omega) \in A \\ 0, & \text{if } (t,\omega) \notin A \end{cases} \tag{2-1}$$

is indistinguishable from the zero process, i.e.,

$$\mathbb{P}(\{\omega \in \Omega : 1_A(t,\omega) = 0,\ \forall t \in \mathcal{T}\}) = 1. \tag{2-2}$$

what is equivalent to

$$\mathbb{P}(\{\omega \in \Omega : \exists t \in \mathcal{T} \text{ with } (t,\omega) \in A\}) = 0. \tag{2-3}$$

which means that the probability of having an omega that allows the existence of a t such that the pair (t,ω) is in A is null.

Definition 7 *(Adapted Process) X is said to be adapted to $(\mathcal{F}_t)_{t\geq 0}$ if X_t is $\mathcal{F}_t$-measurable $\forall t \in \mathcal{T}$.*

It means that information comes according to time. No future information is known. Here we can for each t, check the measurability of X as a function only of omega.

Definition 8 *(Progressively Measurable) The process X defined in $([0,T] \times \Omega, \mathcal{B}([0,T] \times \mathcal{F}))$ to a measurable space $(E,\mathcal{E})$ is said to be "progressively measurable" (or simply "progressive") if, for every time $t \in [0,T]$, the map*

$$\begin{aligned} X : [0,T] \times \Omega &\to E \\ (t,\omega) &\mapsto X_t(\omega) \end{aligned}$$

is $\mathcal{B}([0,T]) \otimes \mathcal{F}_t$-Measurable function. This implies that X is $\mathcal{F}_t$-adapted.

This property asks a jointly measurability condition that concerns not only the space but also time.

2.3 Martingales

One of the most important concepts in Finance is the process property of being a martingale. Here we are going to define, give examples and see the main related results shortly.

2.3.1
Definition

Definition 9 *(Martingale) A real valued adapted process (M_t) is said to be a martingale with respect to the filtration $\{\mathcal{F}_t\}_{t\in\mathcal{T}}$ if $\mathbb{E}|M_t| < \infty$ $\forall t$ and $\forall s \leq t$:*

$$\mathbb{E}[M_t|\mathcal{F}_s] = M_s. \ a.s.$$

If the equality is replaced by $\leq$ then (M_t) is said to be a supermartingale. If the equality is replaced by $\geq$ then (M_t) is said to be a submartingale.

The martingale condition can be regarded as $\mathbb{E}[X_t|\mathcal{F}_s]$ being a version of the process X_t:

$$\int_A \mathbb{E}[X_t|\mathcal{F}_s]d\mathbb{P} = \int_A X_s d\mathbb{P} \quad A \in \mathcal{F}_s \tag{2-4}$$

but by the definition of conditional expectation we have:

$$\int_A \mathbb{E}[X_t|\mathcal{F}_s]d\mathbb{P} = \int_A X_t d\mathbb{P} \quad A \in \mathcal{F}_s \tag{2-5}$$

so that for $s \leq t$:

$$\int_A X_s d\mathbb{P} = \int_A X_t d\mathbb{P} \quad A \in \mathcal{F}_s. \tag{2-6}$$

The martingale can be seen as a model for "fair games" or a "pure random process" because given the information available about the process until now, the expected value is the present value.

Remark: A martingale is a process "constant in mean", in the sense that

$$\mathbb{E}[M_t] = \mathbb{E}[M_0] \quad \forall t \geq 0$$

Indeed,

$$\mathbb{E}[M_t|\mathcal{F}_s] = M_s \quad a.s. \quad s \leq t$$

implies

$$\mathbb{E}[\mathbb{E}[M_t|\mathcal{F}_s]] = \mathbb{E}[M_s]$$

so that by the iterated expectation property:

$$\mathbb{E}[M_t] = \mathbb{E}[M_s] \quad \forall s \leq t.$$

Theorem 10 (Levy) *Let $(B_t)_{t\geq 0}$ be a standard Brownian Motion with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$. Then:*

a) $(B_t)_{t\geq 0}$ *is an* $\mathcal{F}_t$*-martingale.*

b) $(B_t^2 - t)_{t\geq 0}$ *is an* $\mathcal{F}_t$*-martingale.*

c) $(e^{\sigma B_t - \frac{\sigma^2 t}{2}})_{t\geq 0}$ *is an* $\mathcal{F}_t$*-martingale.*

Also, the converse holds true (stated as Characterization of a Brownian Motion in section 3.6). Besides that, there is a curious property of the Brownian Motion, viz., its paths although continuous a.s. are non-differentiable almost everywhere.

For the proof of this curiosity and the Theorem see Evans (26) and Elliot (21) respectivelly.

2.4 Stochastic Integrals

Here we build the definition of the stochastic integral starting from simple functions and finishing with a wider range of functions.

Definition 11 *(Simple processes)Consider* (W_t) *a* $(\mathcal{F}_t)$*-Brownian motion defined on* $(\Omega, \mathcal{F}, \mathbb{P})$ *. A real-valued simple process on* $[0,T]$ *is a function* H *for which*

a) There is a partition $0 = t_0 < t_1 < ... < t_n = T$ *; and*

b) $H_{t_0} = H_0(\omega)$ *and* $H_t = H_i(\omega)$ *for* $t \in (t_i, t_{i+1}]$*, where* $H_i(\cdot)$ *is* $\mathcal{F}_{t_i}$*-measurable and square integrable. That is,*

$$H_t = H_0(\omega) + \sum_{i=0}^{n-1} H_i(\omega) 1_{(t_i,t_{i+1}]}, t \in [0,T].$$

Definition 12 *(Stochastic Integral of a simple process) If* H *is a simple process, the stochastic integral of* H *with respect to the Brownian Motion* (W_t) *is the process defined for* $t \in (t_k, t_{k+1}]$*, by*

$$\int_0^t H_s dW_s = \sum_{i=0}^{k-1} H_i (W_{t_{i+1}} - W_{t_i}) + H_k (W_t - W_{t_k}).$$

This can be written as a martingale transform:

$$\int_0^t H_s dW_s = \sum_{i=0}^{n} H_i (W_{t_{i+1}\wedge t} - W_{t_i \wedge t}).$$

Theorem 13 *Suppose* H *is a simple process. Then:*

a) $\int_0^t H_s dW_s$ *is a continuous* $\mathcal{F}_t$*-martingale.*

b) $\mathbb{E}\left[(\int_0^t H_s dW_s)^2\right] = \mathbb{E}\left[\int_0^t H_s^2 ds\right]$ *(Itô Isometry).*

c) $\mathbb{E}\left[\sup_{0\leq t\leq T} |\int_0^t H_s dW_s|^2\right] \leq 4\mathbb{E}\left[\int_0^T H_s^2 ds\right].$

Lemma 14 *Let* $\mathcal{H}$ *be the space of processes adapted to* $(\mathcal{F}_t)$ *that satisfy* $\mathbb{E}[\int_0^T H_s^2 ds] < \infty$*. Suppose* $H_s \in \mathcal{H}$*. Then there is a sequence* $\{H_s^n\}$ *of simple processes such that*

$$\lim_{n\to\infty} \mathbb{E}\left[\int_0^T |H_s - H_s^n|^2 ds\right] = 0.$$

i.e., we can now define the stochastic integral to a broader space taking this special L^2 *limit of simple processes. Simple processes are dense in* $\mathcal{H}$ *if we consider this convergence.*

There is a broader class of processes that the integral can be defined keeping the properties in the Theorem above. Please refer to Elliott (21) for this generalization.

2.5
Itô Processes, Differentiation Rule and Solution of a SDE

In this section we are going to present the Itô Processes, the so called Itô Lemma and see through some examples the usual uses of it in solving Stochastic Differential Equations.

2.5.1
Initial Definitions

Definition 15 *(Itô Processes) Suppose* $(\Omega, \mathcal{F}, \mathbb{P})$ *is a probability space with a filtration* $(\mathcal{F}_t)_{t\geq 0}$ *and* (W_t) *is a standard* $(\mathcal{F}_t)$*-Brownian Motion. A real valued Itô process* $(X_t)_{t\geq 0}$ *is a process of the form:*

$$X_t = X_0 + \int_0^t K_s ds + \int_0^t H_s dW_s,$$

where:

(a) X_0 *is* $\mathcal{F}_0$*-measurable,*

(b) K *and* H *are adapted to* $\mathcal{F}_t$

(c) $\int_0^T |K_s| ds < \infty$ *a.s. and* $\int_0^T |H_s|^2 ds < \infty$ *a.s.*

Definition 16 *(Quadratic Variation) Given a partition* $0 = t_0 < t_1 < ... < t_n = t$ *of the interval* $[0, t]$ *and writing* $|\pi| = \max_i(t_{i+1} - t_i)$, *the quadratic variation of a continuous martingale* $(M_t)_{t\geq 0}$, *denoted by* $\langle M \rangle_t$, *is defined by:*

$$\langle M \rangle_t = \lim_{|\pi|\to 0} \sum_{i=0}^{n} (M_{t_{i+1}} - M_{t_i})^2$$

For the Brownian Motion, $\langle W \rangle_t = t$

2.5.2
Itô Formula and SDEs

Let's start with the main result in Stochastic Calculus:

Theorem 17 *(Itô Formula) Suppose* $\{X_t\}_{t\geq 0}$ *is an Itô process of the form*

$$X_t = X_0 + \int_0^t K_s ds + \int_0^t H_s dW_s$$

Suppose f *twice differentiable. Then,*

$$f(X_t) = f(X_0) + \int_0^t f^{'}(X_s) dX_s + \frac{1}{2} \int_0^t f^{''}(X_s) d\langle X \rangle_s$$

Here, by definition, $\langle X \rangle_t = \int_0^t H_s^2 ds$ *; that is the (predictable) quadratic variation of* X *is the quadratic variation of the martingale component* $\int_0^t H_s dW_s$ *Also,*

$$\int_0^t f^{'}(X_s) dX_s = \int_0^t f^{'}(X_s) K_s ds + \int_0^t f^{'}(X_s) H_s dW_s$$

Definition 18 *(Solution of a SDE)*

A process X_t , $0 \leq t \leq T$ *is a solution of the stochastic differential equation*

$$dX_t = f(X_t, t)dt + \sigma(X_t, t)dW_t$$

with initial condition $X_0 = \xi$ *if for all* t *the integrals*

$$\int_0^t f(X_s, s)ds \ \ and \ \int_0^t \sigma(X_s, s)dW_s$$

are well defined and

$$X_t = \xi + \int_0^t f(X_s, s)ds + \int_0^t \sigma(X_s, s)dW_s \ \ a.s.$$

Note that the symbols "dt", "dW" have no meaning alone they just make sense when they are in a equation. Even then the meaning is that of a notation for the integral equation of its solution. Now that we have introduced the differential notation, for pencil and paper compuations, we prefer to write the result in Itô Lemma as:

$$df(X_t) = f^{'}(X_t)dX_t + \frac{1}{2}f^{''}(X_t)d\langle X\rangle_t,$$

together with the fact that $\langle X\rangle_t = \int_0^t H_s^2 ds$ *we have ultimately:*

$$\begin{aligned} df(X_t) &= f^{'}(X_t)dX_t + \frac{1}{2}f^{''}(X_t)H_s^2 ds && \text{(2-7)} \\ &= f^{'}(X_t)K_t dt + f^{'}(X_t)H_t dW_t + \frac{1}{2}f^{''}(X_t)H_t^2 dt && \text{(2-8)} \\ &= \left(f^{'}(X_t)K_t + \frac{1}{2}f^{''}(X_t)H_t^2\right)dt + f^{'}(X_t)H_t dW_t && \text{(2-9)} \end{aligned}$$

It is worth writing the more general result although later we are going to show a sufficient general particular case.

Theorem 19 *(Multidimensional Itô Lemma)*

Suppose $X_t = (X_t^1, ..., X_t^N)$ *is a n-dimensional Itô process with*

$$dX_t^i = K_t^i dt + \sum_{j=1}^m H_t^{ij} dW_t^j, \tag{2-10}$$

and suppose $f : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ *is* $C^{1,2}$. *Then*

$$\begin{aligned} df(t, X_t^1, ..., X_t^n) &= \frac{\partial f}{\partial t}(t, X_0^1, ..., X_0^n)dt + \sum_{i=1}^n \frac{\partial f}{\partial x_i}(t, X_0^1, ..., X_0^n)dX_t^i \\ &+ \frac{1}{2}\sum_{i,j=1}^n \frac{\partial f^2}{\partial x_i \partial x_j}(t, X_0^1, ..., X_0^n)\left(\sum_{r=1}^m H_t^{i,r}H_t^{j,r}\right)dt \end{aligned}$$

Note that if $m = 1$ *we have:*

$$\begin{aligned} df(t, X_t^1, ..., X_t^n) &= \frac{\partial f}{\partial t}(t, X_0^1, ..., X_0^n)dt + \sum_{i=1}^n \frac{\partial f}{\partial x_i}(t, X_0^1, ..., X_0^n)dX_t^i \\ &+ \frac{1}{2}\sum_{i,j}^n \frac{\partial f^2}{\partial x_i \partial x_j}(t, X_0^1, ..., X_0^n)H_t^i H_t^j dt \end{aligned}$$

2.5.3 Existence and Uniqueness

The result we are going to present assure us about the existence and uniquiness of a solution in some Stochastic Differential Equations.

Theorem 20 *(Existence and Uniqueness of solutions) Suppose the usual assumptions and that ξ,f and σ satisfy:*

$$|f(x,t) - f(x',t)| + |\sigma(x,t) - \sigma(x',t)| \leq K|x - x'|, \tag{2-11}$$

$$|f(x,t)|^2 + |\sigma(x,t)|^2 \leq K_0^2(1 + |x|^2), \tag{2-12}$$

$$\mathbb{E}[|\xi|^2] < \infty \tag{2-13}$$

for all $0 \leq t \leq T, x, \hat{x} \in \mathbb{R}^n$ and some constant K. Then there is a unique solution X (up to indistinguishability) given by

$$X_t = \xi + \int_0^t f(X_s, s)ds + \int_0^t \sigma(X_s, s)dW_s \ a.s.$$

such that

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |X_t|^2\right] < C\left(1 + \mathbb{E}[|\xi|^2]\right).$$

Proof:

Please refer to (21) and (26).

2.6 Examples of Itô Formula

Example 21 *Lets Calculate the integral of W_t w.r.t. W_t , i.e.,*

$$\int_s^t W_u dW_u$$

For that, we consider $dX_t = dW_t$ so that $K_u = 0$ and $H_u = 1$ and the fuction $y = x^m$.

By the Itô formula,

$\langle X\rangle_u = \langle W\rangle_u = \int_0^u 1dv = u$

and

$$d(W_t^m) = mW_u^{m-1}dW_u + \frac{1}{2}m(m-1)W^{m-2}du.$$

or equivalently

$$W_t^m - W_s^m = m\int_s^t W_u^{m-1}dW_u + \frac{1}{2}\int_s^t m(m-1)W^{m-2}du.$$

Substituting $m = 2$, we have

$$W_t^2 - W_s^2 = 2\int_s^t W_u dW_u + \frac{1}{2}\int_s^t 2du$$

so that

$$W_t^2 - W_s^2 = 2\int_s^t W_u dW_u + \int_s^t du$$

Then

$$\frac{W_t^2 - W_s^2}{2} - \frac{(t-s)}{2} = \int_s^t W_u dW_u.$$

instead of the expected

$$\int_s^t W_u dW_u = \frac{W_t^2 - W_s^2}{2}$$

that we used to have in calculus.

Lemma 22 *(Two-dimensional formula)*

For,

$$dX_s = K_s^x dt + H_s^x dW_s$$

$$dY_s = K_s^y dt + H_s^y dW_s$$

We have

$$\begin{aligned} f(t, X_t, Y_t) &= f(0, X_0, Y_0) + \int_0^t \frac{\partial f}{\partial s}(s,x,y)ds + \int_0^t \frac{\partial f}{\partial x}(s,x,y)dX + \\ &+ \int_0^t \frac{\partial f}{\partial y}(s,x,y)dY + \int_0^t \frac{\partial f^2}{\partial x \partial y}(s,x,y)H_s^x H_s^y ds \\ &+ \frac{1}{2}\int_0^t \frac{\partial f^2}{\partial x^2}(s,x,y)H_s^{x2}ds + \frac{1}{2}\int_0^t \frac{\partial f^2}{\partial y^2}(s,x,y)H_s^{y2}ds \end{aligned}$$

Example 23 *(Itô Product Rule) Suppose*

$$dX_s = K_s^x dt + H_s^x dW_s$$

$$dY_s = K_s^y dt + H_s^y dW_s$$

The Itô Formula applied to $f(x,y) = x.y$ we have:

$$\begin{aligned} f(t,X,Y) &= f(0,X_0,Y_0) + \int_0^t Y dX + \int_0^t X dY \\ &+ \int_0^t 1 H_s^x H_s^y ds \end{aligned}$$

or

$$d(X_t.Y_t) = Y_t dX_t + X_t dY_t + H_t^x H_t^y dt$$

Example 24 *(Solution of a Lognormal SDE)*

Suppose

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t, \quad S_0 = X_0.$$

or equivalently

$$S_t = X_0 + \int_0^t S_s \mu ds + \int_0^t S_s \sigma dW_s.$$

This is a Itô Process if we think $K_s = S_s\mu$ and $H_s = S_s\sigma$ satisfying the usual conditions.

Then $\langle X \rangle_t = \int_0^t \sigma^2 S_s^2 ds$.

If $S_t > 0$, by the Itô Formula with $f(x) = log(x)$ we have

$$\begin{aligned} logS_t &= logX_0 + \int_0^t \frac{1}{S_s} dS_s + \frac{1}{2}\int_0^t -\frac{1}{S_s^2}\sigma^2 S_s^2 ds && \text{(2-14)} \\ &= logX_0 + \int_0^t \left(\mu - \frac{\sigma^2}{2}\right) ds + \int_0^t \sigma dW_s && \text{(2-15)} \\ &= logX_0 + \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t && \text{(2-16)} \end{aligned}$$

so that

$$S_t = X_0.e^{\left\{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right\}}.$$

Example 25 *(Testing a Solution of a Lognormal SDE) Consider*

$$F(t,x) = X_0 e^{\left\{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma x\right\}}.$$

Show that $S_t = F(t, W_t)$ satisfies the lognormal SDE of the example using Itô Formula.

By Itô Formula,

$$
\begin{aligned}
F(t,W_t) &= F(0,W_0) + \int_0^t \frac{\partial F}{\partial s}(s,W_s)ds \\
&+ \int_0^t \frac{\partial F}{\partial W_s}(s,W_s)dW_s + \int_0^t \frac{\partial^2 F}{\partial W_s^2}(s,W_s)d\langle W\rangle_s
\end{aligned}
$$

Equivalently

$$
\begin{aligned}
X_0 e^{\left\{\left(\mu-\frac{\sigma^2}{2}\right)t+\sigma W_t\right\}} &= X_0 + \int_0^t \left(\mu - \frac{\sigma^2}{2}\right) X_0 e^{\left\{\left(\mu-\frac{\sigma^2}{2}\right)s+\sigma W_s\right\}} ds \\
&+ \int_0^t \sigma X_0 e^{\left\{\left(\mu-\frac{\sigma^2}{2}\right)s+\sigma W_s\right\}} dW_s \\
&+ \frac{1}{2}\int_0^t \sigma^2 e^{\left\{\left(\mu-\frac{\sigma^2}{2}\right)s+\sigma W_s\right\}} X_0 ds
\end{aligned}
$$

Then

$$
\begin{aligned}
X_0 e^{\left\{\left(\mu-\frac{\sigma^2}{2}\right)t+\sigma W_t\right\}} &= X_0 + \int_0^t \mu X_0 e^{\left\{\left(\mu-\frac{\sigma^2}{2}\right)s+\sigma W_s\right\}} ds \\
&+ \int_0^t \sigma X_0 e^{\left\{\left(\mu-\frac{\sigma^2}{2}\right)s+\sigma W_s\right\}} dW_s.
\end{aligned}
$$

Now, by the definition of S_t

$$S_t = X_0 + \int_0^t \mu S_s ds + \int_0^t \sigma S_s dW_s.$$

□

Example 26 *(Quotient Rule) Suppose B a Brownian Motion and*

$$dX_t = X_t(\mu_X dt + \sigma_X dB_t)$$

$$dY_t = Y_t(\mu_Y dt + \sigma_Y dB_t)$$

Define Z by $Z_t = Y_t/X_t$. Show that Z is lognormal with dynamics

$$dZ_t = Z_t(\mu_Z dt + \sigma_Z dB_t)$$

and determine μ_Z and σ_Z in terms of the coefficients of X and Y.

The Itô Formula applied to $f(x,y) = y/x$ gives us:

$$
\begin{aligned}
d\left(\frac{Y_t}{X_t}\right) &= \frac{-Y_t}{X_t^2}dX_t + \frac{1}{X_t}dY_t + \frac{1}{2}\frac{2Y_t}{X_t^3}X_t^2\sigma_X^2 dt - \frac{1}{X_t^2}X_tY_t\sigma_X\sigma_Y = \\
&= \frac{-Y_t}{X_t^2}X_t\mu_X dt + \frac{-Y_t}{X_t^2}X_t\sigma_X dB_t + \frac{1}{X_t}Y_t\mu_Y dt + \frac{1}{X_t}Y_t\sigma_Y dB_t \\
&+ \frac{Y_t}{X_t}\sigma_X^2 dt - \frac{Y_t}{X_t}\sigma_X\sigma_Y dt \\
&= \frac{Y_t}{X_t}(\mu_Y - \mu_X + \sigma_X^2 - \sigma_X\sigma_Y)dt + \frac{Y_t}{X_t}(\sigma_Y - \sigma_X)dB_t
\end{aligned}
$$

so that

$$d\left(\frac{Y_t}{X_t}\right) = \frac{Y_t}{X_t}(\mu_Y - \mu_X + \sigma_X^2 - \sigma_X\sigma_Y)dt + \frac{Y_t}{X_t}(\sigma_Y - \sigma_X)dB_t$$

means by the definition of Z_t that

$$dZ_t = Z_t(\mu_Z dt + \sigma_Z dB_t)$$

with

$$\mu_Z = (\mu_Y - \mu_X + \sigma_X^2 - \sigma_X\sigma_Y)$$

$$\sigma_Z = (\sigma_Y - \sigma_X)$$

□

2.7
Change of Measure

Let $\mathbb{Q}$ be a second probability measure on $(\Omega, \mathcal{F})$ that is absolutely continuous with respect to $\mathbb{P}$ ($\mathbb{Q} << \mathbb{P}$) such that

$$M_t = \left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t}, \quad \text{and} \quad \mathbb{Q}(A) = \int_A M_t dP, \quad \text{for} \quad A \in \mathcal{F}_t$$

Lemma : (X_tM_t) is a martingale under $\mathbb{P}$ iff X_t is a martingale under $\mathbb{Q}$.

Proof : Suppose $s \leq t$ and $A \in \mathcal{F}_s$. Then

$$\int_A X_t d\mathbb{Q} = \int_A X_t M_t d\mathbb{P} = \int_A X_s M_s d\mathbb{P} = \int_A X_s d\mathbb{Q}.$$

Theorem 27 *(Characterization of a Brownian Motion) Suppose $(W_t)_{t\geq 0}$ is a continuous (scalar) martingale on the filtered space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$, such that $(W_t^2 - t)_{t\geq 0}$ is a martingale. Then (W_t) is a Brownian Motion.*

So this Theorem implies that a real process $(B_t)_{t\geq 0}$ is a standard Brownian Motion if:

a) $t \rightarrow B_t(\omega)$ *is continuous a.s.,*

b) B_t *is a martingale*

c) $B_t^2 - t$ *is a martingale.*

Definition 28 *(History of the Brownian Motion Process) Write $\mathcal{F}_t^0 = \sigma(B_s : s \leq t)$ for the σ-algebra on Ω generated by the history of the Brownian Motion up to time t. Then, $(\mathcal{F}_t)_{t\geq 0}$ will denote the right-continuous complete filtration generated by the $\mathcal{F}_t^0$.*

The idea of the Girsanov's Theorem is to show how (B_t) behaves under a change of measure.

Theorem 29 *(Girsanov's Theorem) Suppose $(\theta_t)_{0\leq t\leq T}$ is an adapted, measurable process such that $\int_0^T \theta_s^2 ds < \infty$ a.s. and also so that the process*

$$\Lambda_t = e^{\left\{-\int_0^t \theta_s dB_s - \frac{1}{2}\int_0^t \theta_s^2 ds\right\}} \tag{2-17}$$

is an $(\mathcal{F}_t, \mathbb{P})$ martingale. Define a new measure $\mathbb{Q}_\theta$ on $\mathcal{F}_T$ by

$$\left.\frac{d\mathbb{Q}_\theta}{d\mathbb{P}}\right|_{\mathcal{F}_T} = \Lambda_T \tag{2-18}$$

Then the process

$$W_t = B_t + \int_0^t \theta_s ds$$

is a standard Brownian motion on $(\mathcal{F}_t, \mathbb{Q}_\theta)$.

Proof: By Itô differentiation rule and by the definition of Λ. For details please refer to (21).