1 Introduction

Option valuation is one of the most important topics in financial economics. The path-breaking work by Black and Scholes (1973)(6) and Merton (1973)(37) provides the cornerstone for an explosive growth in the literature describing the theory and practice of option pricing models. The key economic insights behind the Black-Scholes-Merton model are the concept of perfect hedging of an option, by constructing a replicating portfolio via trading the underlying assets continuously, and pricing by the no-arbitrage principle. Cox, Ross and Rubinstein (1979) (12) were the first to establish the relationship between the risk-neutral valuation and the no-arbitrage principle. Harrison and Kreps (1979)(33), and Harrison and Pliska (1981, 1983)(34) (35) established a solid mathematical foundation for the relationship between the no-arbitrage principle and the notion of risk-neutral valuation using the language of probability theory. They also provide a solid theoretical foundation to the concept of market incompleteness. If the securities market is complete, there is a unique equivalent martingale measure, i.e., a unique risk neutral measure and, hence, the unique price of any contingent claim is given by its expected discounted payoff at expiry under the martingale measure. However, in an incomplete market, there are infinitely many equivalent martingale measures and, so, a range of no-arbitrage prices for a contingent claim can be found, and this complicates the pricing and hedging issues.

Föllmer and Sondermann (1986)(29), Föllmer and Schweizer (1991)(28) and Schweizer (1996)(40) determined the equivalent martingale pricing measure by minimizing the quadratic utility of the losses due to imperfect hedging. Davis (1997)(14) adopted a traditional valuation approach in economics, namely the marginal rate of substitution, to determine a pricing measure by solving a utility maximization problem. The seminal work by Gerber and Shiu (1994)(31) provided a pertinent solution to the option pricing problem in an incomplete market by using the Esscher transform, a time-honored tool in actuarial science introduced by Esscher (1932)(25). Their model provided a convenient and flexible way to price options under different parametric assumptions on the stock returns within the class of infinitely divisible distributions. They can justify the pricing result by considering a utility maximization problem with respect to a power utility function. Their significant contributions highlighted the interplay between the financial and insurance pricing in incomplete markets and its importantance is mentioned in Bühlmann et al. (1996) (9) and Embrechts (2001) (24). Bühlmann et al. (1996) (9) developed the conditional Esscher transform by generalizing the classical Esscher transform to stochastic processes in order to incorporate the richer theory of semi-martingales under the no-arbitrage condition in the Gerber-Shiu option-pricing model. Bühlmann et al. (1998) (10) investigated the use of Esscher transforms in discrete finance models and established a solid foundation for its use based on economic arguments. Siu, Tong and Yang (2001) (42) introduced the concept of a random Esscher transform with a random Esscher parameter and adopted the random Esscher transform to incorporate the uncertainty of the probability measures for risk measurement. Elliott, Chan and Siu (2004) (22) provided a modification of the random Esscher transform and developed the regime switching random Esscher transform to identify a pricing measure for option valuation under a Markov-modulated Geometric Brownian Motion (MMGBM). Yao (2002) (46) adopted the Esscher Transform to specify the forward-risk-adjusted measure and provided a general and consistent framework for pricing derivatives on stocks, interest rates and currency rates.

Autoregressive conditional heteroskedastic (ARCH) models were proposed by the Nobel Laureate Robert Engle as a tool to describe time-varying volatility dynamics and other stylised empirical facts of many financial time series. Bollerslev (1986) (7) and Taylor (1986) (44) generalized the idea of the ARCH models and developed the generalized ARCH (GARCH) models independently by assuming that the current level of the conditional variance not only depends on the past values of the innovations but also the past values of the conditional variances. For an excellent overview of ARCH-type models, see Bollerslev, Chou and Kroner (1992)(8).

There has been a considerable interest in option valuation under GARCH models in the finance literature. The seminal work by Duan (1995) (17) was the pioneer to provide a solid theoretical foundation for option valuation in the context of GARCH models. He generalized the concept of risk-neutral valuation and introduced the notion of locally risk-neutral valuation relationship (LRNVR) which provides a sound economic argument to choose a particular equivalent martingale measure in the GARCH model with a conditionally normal stock innovation. Under the preference assumptions and distributional assumptions, Duan (1995) provided a rigorous theoretical foundation and economic justification of the validity of LRNVR. Duan, Popova and Ritcken (2002)

(19) developed a family of option pricing models where the underlying stock price dynamics are modelled by a regime switching process in which the prices stay in one volatility regime for a random amount of time before switching over into a new regime. Barone-Adesi, Engle and Mancini (2008)(3) proposed a new approach to compute option prices under the GARCH models in an incomplete market framework. Their model allows the actual volatility of asset returns to be different from the volatility of asset returns under the pricing probability measure. Siu, Tong and Yang (2004) (43) proposed an alternative method to price options under the GARCH models with infinitely divisible innovations by using the conditional Esscher transform proposed by Bühlmann et al. (1996). Elliott, Siu and Chan (2004) introduced the use of a modified version of the conditional Esscher transform, namely the Markov switching conditional Esscher transform (MSCET), to determine an equivalent martingale pricing measure under a Markov switching GARCH model. They justified their pricing results by considering the stochastic power utility function with Markov switching risk-aversion parameters.

The thesis extends the literature in two main points. First, we deal with more general processes to perform option pricing than in Duan(1995) (17) and Siu et al.(2004) (43), viz. the Flexible Coefficient GARCH model (FC-GARCH) which is a nonlinear model and nests several well-known GARCH specifications in the literature and the mixture of GARCHs. Second, we can treat those two models in a variety of innovation distributions, not being restricted to the normal case as in Duan(1995) (17). In particular in this thesis we perform calculations and simulations for the Normal and Shifted-Gamma cases. A minor contribution is that we include the possibility of a negative innovation in the Shifted-Gamma case. This allow us to mimic the small and negative skewness usually found in the empirical literature. This contribution give the practitioners flexibility in choosing among the Normal, the positive and negative Shifted-Gamma innovation cases according to the sign and magnitude of the skewness.

Although the non-linearity of the FC-GARCH affects little the prediction of the volatility, it considerably affects the option prices. We notice both in the FC-GARCH and in the Mixture of GARCHs that the option prices vary a lot in our experiments. We also noticed that the choice of the innovation distribution is important for better describing the skewness of the series. We deal with negative shifted-Gamma innovations to treat negative skewness data. When using a different innovation, a significant difference in the option price is also detected.

The document is structured as follows. First we make a review on the mathematical background knowledge, then in chapter three we talk about the simplest ARCH models. In chapter 4, we briefly discuss the option pricing methodologies. Chapters 5 and 6 contain the main results of the research. They consist of two risk neutral option pricing papers, one assuming the FC-GARCH as the underlying log-return process and the other a mixture of GARCHs. In chapter 5, we develop the methodology in Siu et al.(2004) (43) to the FC-GARCH. In chapter 6, we perform a similar methodology to the Mixture of GARCHs. The theoretical results we achieved are the Theorems 35, 36, 41 and 42. In both these chapters we discuss the model, the methodology, the calculations and simulation experiments. We finish the thesis with the conclusions, future intent of work and a small appendix.