8
Surfaces invariant by one-parameter group of hyperbolic isometries having constant mean curvature in $\widetilde{\mathbf{PSL_2}}(\mathbb{R},\tau)$

On (3) Ricardo Sa Earp gave explicit formulas for parabolic and hyperbolic screw motions surfaces immersed in $\mathbb{H}^2 \times \mathbb{R}$. There, they gave several examples.

In this chapter we only consider surfaces invariant by one-parameter group of hyperbolic isometries having constant mean curvature immersed in $\widetilde{PSL}_2(\mathbb{R},\tau)$. Since for $\tau \equiv 0$ we are in $\mathbb{H}^2\times\mathbb{R}$ then we have generalized the result obtained by Ricardo Sa Earp when the surface is invariant by one-parameter group of hyperbolic isometries having constant mean curvature.

In this chapter we focus our attention on surfaces invariant by one-parameter group of hyperbolic isometries. To study this kind of surfaces, we take $M^2 = \mathbb{H}^2$ that is the half plane model for the hyperbolic space. Thus, the metric of M^2 is given by:

$$d\sigma^2 = \lambda^2(dx^2 + dy^2), \qquad \lambda = \frac{1}{y}.$$

From Proposition 5.1.1, we know that to obtain a hyperbolic motion on $\widetilde{PSL}_2(\mathbb{R},\tau)$, it is necessary consider a hyperbolic motion (hyperbolic isometry) on $\mathbb{H}^2$.

8.1
Surfaces invariant by one-parameter group of hyperbolic isometries main lemma

The idea to obtain a surface invariant by one-parameter group of hyperbolic isometries is simple, we will take a curve in the xt plane and we will apply one-parameter group Γ of hyperbolic isometries on $\widetilde{PSL}_2(\mathbb{R},\tau)$. We denote by $\alpha(x) = (x, 1, u(x))$ the curve in the xt plane and by $S = \Gamma(\alpha)$, the surface invariant by one-parameter group of hyperbolic isometries generate by α.

Since the most simple hyperbolic isometry on $\mathbb{H}^2$ is the homotetia from the origin, we re-parameterized the hyperbolic plane whit coordinates R^+ and

$\theta \in (0,\pi)$, so that

$$\begin{aligned} x &= R\cos(\theta) \\ y &= R\sin(\theta) \end{aligned}$$

where $R \in \mathbb{R}$ and $\theta \in (0,\pi)$.

Thus, the surface S is parameterized by,

$$\varphi(\rho,\theta) = (R\cos(\theta), R\sin(\theta), u(\theta))$$

Now, we rewrite all the expressions in this news coordinates, thus we obtain the next lemma.

Lemma 8.1.1. *By considering the above re-parametrization to the hyperbolic plane, we may rewrite all terms in the form,*

- $\partial_x = \cos(\theta)\partial_R - \dfrac{\sin(\theta)}{R}\partial_\theta$
- $\partial_y = \sin(\theta)\partial_R + \dfrac{\cos(\theta)}{R}\partial_\theta$
- $R_x = \cos(\theta)$
- $R_y = \sin(\theta)$
- $\theta_x = -\dfrac{\sin(\theta)}{R}$
- $\theta_y = \dfrac{\cos(\theta)}{R}$
- $\lambda = \dfrac{1}{R\sin(\theta)}$
- $d\sigma^2 = \dfrac{1}{R^2\sin^2(\theta)}\left(dR^2 + R^2 d\theta^2\right)$

Proof. From,

$$\begin{cases} x = R\cos(\theta), \\ y = R\sin(\theta), \end{cases} \tag{8-1}$$

we obtain,

$$\begin{cases} dx = \cos(\theta)dR - R\sin(\theta)d\theta, \\ dy = \sin(\theta)dR + R\cos(\theta)d\theta, \end{cases} \tag{8-2}$$

since,

$$\begin{aligned} dx^2 + dy^2 &= dR^2 + R^2 d\theta^2, \\ \lambda &= \frac{1}{R\sin(\theta)} \end{aligned}$$

then $\lambda^2 = \dfrac{1}{R^2\sin^2(\theta)}$, thus,

$$\begin{aligned} d\sigma^2 &= \lambda^2(dx^2+dy^2) \\ &= \frac{1}{R^2\sin^2(\theta)}\left(dR^2+R^2d\theta^2\right) \end{aligned}$$

Furthermore, setting

$$\partial_x = a\partial_R + b\partial_\theta$$

and evaluating in equation (8-2). We obtain,

$$\begin{cases} 1 = \cos(\theta)a - R\sin(\theta)b, \\ 0 = \sin(\theta)a + R\cos(\theta)b, \end{cases}$$

hence,

$$a = \cos(\theta), \qquad b = -\frac{\sin(\theta)}{R}$$

that is,

$$\partial_x = \cos(\theta)\partial_R - \frac{\sin(\theta)}{R}\partial_\theta.$$

By considering,

$$\partial_y = a\partial_R + b\partial_\theta$$

and evaluating in equation (6-2). We obtain,

$$\begin{cases} 0 = \cos(\theta)a - R\sin(\theta)b, \\ 1 = \sin(\theta)a + R\cos(\theta)b, \end{cases}$$

so

$$a = \sin(\theta), \qquad b = \frac{\cos(\theta)}{R}$$

that is,

$$\partial_y = \sin(\theta)\partial_R + \frac{\cos(\theta)}{R}\partial_\theta.$$

From $x^2+y^2=R^2$, we obtain,

$$\begin{aligned} 2x &= 2RR_x \\ 2y &= 2RR_y \end{aligned}$$

Considering the equation (8-1), we obtain,

$$\begin{aligned} R_x &= \cos(\theta) \\ R_y &= \sin(\theta) \end{aligned}$$

Derivation of equation (8-1) with respect to x and y respectively gives,

$$1 = \mathbb{R}_x \cos(\theta) - R\sin(\theta)\theta_x$$

$$1 = R_y \sin(\theta) + R\cos(\theta)\theta_y$$

Now, by using R_x and R_y, we obtain,

$$\begin{aligned}\theta_x &= -\frac{\sin(\theta)}{R}\\ \theta_y &= \frac{\cos(\theta)}{R}\end{aligned}$$

This completes the proof.

□

The next Lemma is crucial for our study. We follow ideas of Appendix A of (11). Consider the graph $t = u(\theta)$ in the plane xt, and denote by $S = grap(\Gamma u)$. Supposing that S has constant mean curvature H, we have the lemma:

Lemma 8.1.2. *The function u satisfies*

$$u(\theta) = \int \frac{(d - 2H\cot(\theta))\sqrt{1 + 4\tau^2\cos^2(\theta)}}{\sqrt{1 - \sin^2(\theta)(d - 2H\cot(\theta))^2}}d\theta - 2\tau\theta \tag{8-3}$$

where d is a real number.

Proof. Since S has mean curvature H, then by lemma 5.2.1 the function u satisfies the equation

$$2H = div_{\mathbb{H}^2}\left(\frac{\alpha}{W}e_1 + \frac{\beta}{W}e_2\right), \tag{8-4}$$

where $W = \sqrt{1 + \alpha^2 + \beta^2}$, $\alpha = \dfrac{u_x}{\lambda} - 2\tau$, and $\beta = \dfrac{u_y}{\lambda}$.

By abuse of notation we consider $u(x, y) = u(R(x, y), \theta(x, y))$, since S is a surface invariant by one-parameter group of hyperbolic isometries, then the function u does not depend of R, so,

$$\begin{cases} u_x = u_R R_x + u_\theta \theta_x = u_\theta \theta_x, \\ u_y = u_R R_y + u_\theta \theta_y = u_\theta \theta_y, \end{cases}$$

By considering the lemma 8.1.1, this implies,

$$\begin{aligned}\frac{u_x}{\lambda} &= -u_\theta \sin^2(\theta)\\ \frac{u_y}{\lambda} &= u_\theta \sin(\theta)\cos(\theta)\end{aligned}$$

Thus

$$\begin{aligned}\alpha &= -u_\theta \sin^2(\theta) - 2\tau\\ \beta &= u_\theta \sin(\theta)\cos(\theta)\end{aligned}$$

this implies,

$$\begin{aligned}\alpha^2+\beta^2 &= \sin^2(\theta)(u_\theta+2\tau)^2 + 4\tau^2\cos^2(\theta)\\ W^2 &= 1+\sin^2(\theta)(u_\theta+2\tau)^2 + 4\tau^2\cos^2(\theta)\end{aligned}$$

Setting $X_u = \frac{\alpha}{W}e_1 + \frac{\beta}{W}e_2 = \frac{1}{W}\left(\alpha\frac{\partial_x}{\lambda} + \beta\frac{\partial_y}{\lambda}\right)$.

We need express X_u in coordinates R and θ. Observe that,

$$\frac{\partial_x}{\lambda} = R\sin\theta\cos\theta\partial_R - \sin^2\theta\partial_\theta$$

and

$$\frac{\partial_y}{\lambda} = R\sin^2\theta\partial_R + \sin\theta\cos\theta\partial_\theta$$

Furthermore,

$$\begin{aligned}\alpha\frac{\partial_x}{\lambda} + \beta\frac{\partial_y}{\lambda} &= -(\sin^2\theta u_\theta + 2\tau)(R\sin\theta\cos\theta\partial_R - \sin^2\theta\partial_\theta) +\\ &\quad + \sin\theta\cos\theta u_\theta(R\sin^2\theta\partial_R + \sin\theta\cos\theta\partial_\theta)\\ &= -[R\sin^3\theta\cos\theta u_\theta + 2\tau R\sin\theta\cos\theta]\partial_R + [\sin^4\theta u_\theta + 2\tau\sin^2\theta]\partial_\theta\\ &\quad + R\sin^3\theta\cos\theta u_\theta\partial_R + \sin^2\theta\cos^2\theta u_\theta\partial_\theta\\ &= -2\tau R\sin\theta\cos\theta\partial_R + \sin^2\theta(u_\theta+2\tau)\partial_\theta\end{aligned}$$

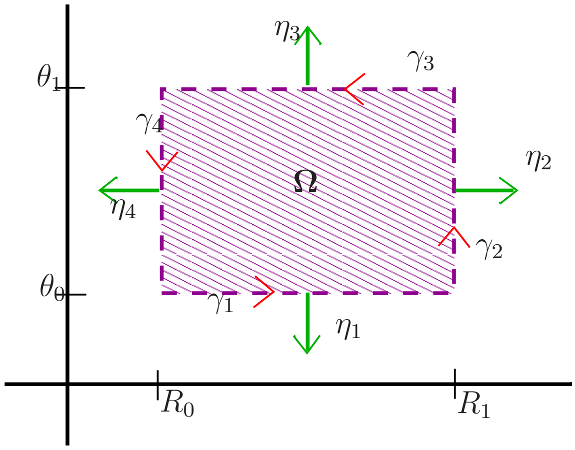
So, we conclude that,

$$X_u = \frac{1}{W}\left[\sin^2\theta(u_\theta+2\tau)\partial_\theta - 2\tau R\sin\theta\cos\theta\partial_R\right]$$

and

$$W = \sqrt{1+4\tau^2 cos^2\theta + \sin^2\theta(u_\theta+2\tau)^2}$$

Let $\theta_0, \theta_1 \in (0,\pi)$ with $\theta_0 < \theta_1$ and $R_0, R_1 \in \mathbb{R}^+$ with $R_0 < R_1$ and consider the domain $\Omega = [\theta_0, \theta_1] \times [R_0, R_1]$ in the plane $R^+\theta$.



By integrating the equation (8-4), we obtain

$$\int_{\partial(\Omega)} \langle X_u, \eta\rangle = 2H\,Area([\theta_0,\theta_1]\times[R_0,R_1])$$

where η is the outer co-normal.

Since

$$\int_{\partial\Omega} \langle X_u, \eta\rangle = \int_{\gamma_1} \langle X_u, \eta_1\rangle + \int_{\gamma_2} \langle X_u, \eta_2\rangle + \int_{\gamma_3} \langle X_u, \eta_3\rangle + \int_{\gamma_4} \langle X_u, \eta_4\rangle$$

we compute each integral:

For the fist integral: Observe that,

$$\gamma_1(s) = (s, \theta_0), \qquad R_0 \le s \le R_1.$$

This implies,

$$\gamma_1' = \partial_R,$$

thus

$$|\gamma_1'| = \frac{1}{R\sin\theta},$$

and

$$\eta_1 = -\sin\theta\partial_\theta.$$

Furthermore

$$\langle X_u, \eta_1\rangle = -\frac{\sin^3\theta(u_\theta + 2\tau)(\theta_0)}{W}\frac{1}{\sin^2\theta} = -\frac{\sin\theta(u_\theta+2\tau)}{W},$$

hence
$$\int_{\gamma_1}\langle X_u,\eta_1\rangle=\int_{R_0}^{R_1}-\frac{u_\theta+2\tau}{RW}(\theta_0)dR$$

For the third integral: Observe that,

$$\gamma_3(s)=(R_1-s,\theta_1),\quad 0\leq s\leq R_1-R_0.$$

This implies,
$$\gamma_3'=-\partial_R,$$
thus
$$|\gamma_3'|=\frac{1}{R\sin\theta},$$
and
$$\eta_3=\sin\theta\partial_\theta.$$
Furthermore
$$\langle X_u,\eta_3\rangle=\frac{\sin^3\theta(u_\theta+2\tau)(\theta_1)}{W}\frac{1}{\sin^2\theta}=\frac{\sin\theta(u_\theta+2\tau)}{W},$$
hence,
$$\int_{\gamma_3}\langle X_u,\eta_3\rangle=\int_0^{R_1-R_0}\frac{\sin\theta(u_\theta+2\tau)}{W}\frac{1}{(R_1-s)\sin\theta}ds=\int_{R_0}^{R_1}\frac{(u_\theta+2\tau)}{RW}(\theta_1)dR$$

For the second integral: Observe that,

$$\gamma_2(s)=(R_1,s),\quad \theta_0\leq s\leq\theta_1.$$

This implies,
$$\gamma_2'=\partial_\theta,$$
thus
$$|\gamma_2'|=\frac{1}{\sin\theta},$$
and
$$\eta_2=R\sin\theta\partial_R.$$
Furthermore
$$\langle X_u,\eta_2\rangle=-\frac{2\tau R^2\sin^2\theta\cos\theta}{W}\frac{1}{R^2\sin^2\theta}=-\frac{2\tau\cos\theta}{W},$$

hence,

$$\int_{\gamma_2}\langle X_u,\eta_2\rangle=\int_{\theta_0}^{\theta_1}-\frac{2\tau\cos\theta}{W\sin\theta}d\theta$$

For the four integral: Observe that,

$$\gamma_4(s)=(R_0,\theta_1-s),\quad 0\le s\le\theta_1-\theta_0.$$

This implies,

$$\gamma_4'=-\partial_\theta,$$

thus

$$|\gamma_4'|=\frac{1}{\sin\theta},$$

and

$$\eta_4=-R\sin\theta\partial_R.$$

Furthermore

$$\langle X_u,\eta_4\rangle=\frac{2\tau R^2\sin^2\theta\cos\theta}{W}\frac{1}{R^2\sin^2\theta}=\frac{2\tau\cos\theta}{W},$$

hence,

$$\int_{\gamma_4}\langle X_u,\eta_4\rangle=\int_0^{\theta_1-\theta_0}\frac{2\tau\cos(\theta_1-s)}{W}\frac{1}{\sin(\theta_1-s)}ds=\int_{\theta_0}^{\theta_1}\frac{2\tau\cos\theta}{W\sin\theta}d\theta$$

Taking into account this four integrals, we obtain

$$\int_{\partial\Omega}\langle X_u,\eta\rangle=\int_{R_0}^{R_1}\frac{(u_\theta+2\tau)}{W}(\theta_1)\frac{dR}{R}-\int_{R_0}^{R_1}\frac{(u_\theta+2\tau)}{W}(\theta_0)\frac{dR}{R}$$

Observe that,

$$\begin{aligned}Area(\Omega)&=\int_{\theta_0}^{\theta_1}\int_{R_0}^{R_1}\sqrt{\det(g_{ij})}dRd\theta\\&=\int_{\theta_0}^{\theta_1}\int_{R_0}^{R_1}\frac{1}{R\sin^2\theta}dRd\theta\end{aligned}$$

So, we conclude that

$$\int_{R_0}^{R_1}\frac{(u_\theta+2\tau)}{W}(\theta_1)\frac{dR}{R}-\int_{R_0}^{R_1}\frac{(u_\theta+2\tau)}{W}(\theta_0)\frac{dR}{R}=\int_{\theta_0}^{\theta_1}\int_{R_0}^{R_1}\frac{1}{R\sin^2\theta}dRd\theta$$

which we can write in the form:

$$\int_{\theta_0}^{\theta_1} \partial_\theta \left(\frac{(u_\theta + 2\tau)}{W}\right) d\theta = 2H \int_{\theta_0}^{\theta_1} \frac{1}{\sin^2\theta} d\theta$$

By considering that Ω is any domain in the plane $R\theta$, and taking the derivative with respect to ρ we obtain:

$$\partial_\theta \left(\frac{u_\theta + 2\tau}{W}\right) = \frac{2H}{\sin^2\theta}$$

by integrating this expression:

$$\frac{u_\theta + 2\tau}{\sqrt{1 + 4\tau^2\cos^2\theta + \sin^2\theta(u_\theta + 2\tau)^2}} = -2H\cot\theta + d$$

where $d \in \mathbb{R}$. This implies:

$$(u_\theta + 2\tau)^2[1 - \sin^2\theta(d - 2H\cot\theta)^2] = (d - 2H\cot\theta)^2[1 + 4\tau^2\cos^2\theta]$$

so the function u satisfies:

$$u(\theta) = \int \frac{(d - 2H\cot\theta)\sqrt{1 + 4\tau^2\cos\theta}}{\sqrt{1 - \sin^2\theta(d - 2H\cot\theta)^2}} d\theta - 2\tau\theta$$

$\square$

8.2 Examples of surfaces invariant by one-parameter group of hyperbolic isometries in $\widetilde{\mathbf{PSL}_2}(\mathbb{R},\tau)$

Now, we explore the Lemma 8.1.2. By considering $\tau = -1/2$ we obtain the next consequences.

Lemma 8.2.1. *Making* $d \equiv 0$ *and* $\tau = -1/2$*, then the integral*

$$u = -\int \frac{2H\cot(\theta)\sqrt{1 + \cos^2(\theta))}}{\sqrt{1 - \sin^2(\theta)(2H\cot(\theta))^2}} d\theta + \theta$$

has the next solution,

$$\begin{aligned} u(\theta) &= \frac{1}{2}\arctan\left(\frac{\sin^2(\theta)-1+(1-4H^2)/8H^2}{\sqrt{(\sin^2(\theta)+(1-4H^2)/4H^2)(2-\sin^2(\theta))}}\right)+ \\ &+\frac{\sqrt{2}H}{\sqrt{1-4H^2}}\ln\{\frac{1-4H^2}{H^2}+\frac{(4H^2-1)\sin^2\theta}{4H^2}+ \\ &+\frac{\sqrt{2(1-4H^2)}}{H}\sqrt{\left((1-4H^2)/4H^2+\sin^2\theta\right)(2-\sin^2(\theta))}\}+ \\ &-\frac{\sqrt{2}H}{\sqrt{1-4H^2}}\ln\{\sin^2\theta\}+\theta \end{aligned}$$

where $0 < H < 1/2$.

Proof. Taking $\tau = -1/2$, $d \equiv 0$, the expression

$$f(\theta) = -2H\int\frac{\cot(\theta)\sqrt{1+\cos^2(\theta))}}{\sqrt{1-\sin^2(\theta)(2H\cot(\theta))^2}}d\theta$$

can be write as:

$$f(\theta) = -\int\frac{\sqrt{1+\cos^2(\theta))}}{\sin\theta\sqrt{(1-4H^2)/4H^2+\sin^2\theta}}\cos\theta d\theta$$

making $b = (1-4H^2)4H^2$, $v = \sin\theta$, then $dv = \cos\theta d\theta$, so the integral becomes:

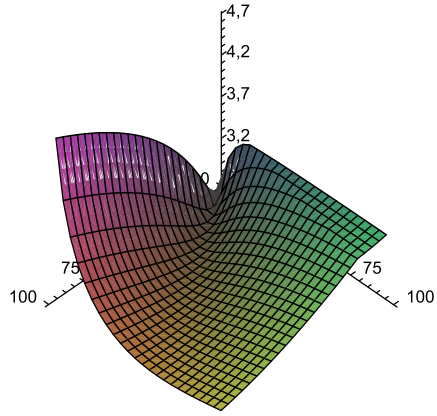
$$f \equiv -\int\frac{\sqrt{2-v^2}}{v\sqrt{b+v^2}}dv$$

whose solution is given by:

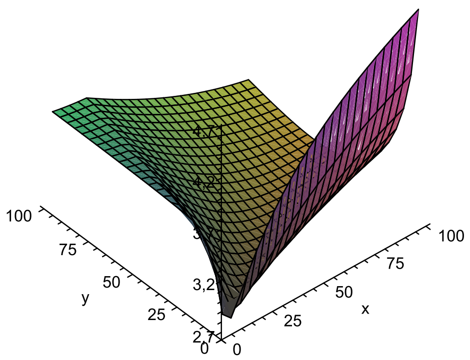
$$\begin{aligned} f &= \frac{1}{2}\arctan\left(\frac{(v^2-1)+b/2}{\sqrt{2-v^2}\sqrt{b+v^2}}\right)+ \\ &+\frac{\sqrt{2}}{2\sqrt{b}}\ln\left(\frac{4b+v^2(2-b)+2\sqrt{2b}\sqrt{(2-v^2)(b+v^2)}}{v^2}\right) \end{aligned}$$

by substitution the values of v and b, we obtain the result. □

Example 8.2.1. *Making* $H = 0.3$ *in the lemma 8.2.1, since* $\theta = \arctan\left(\frac{y}{x}\right)$, *so in coordinates* XYT, *the graph has two sheet. The first sheet is given by* $x > 0$ *(here we have two views to the first sheet),* H Surfaces, Invariantby Hyperbolic Isometries, with x>0

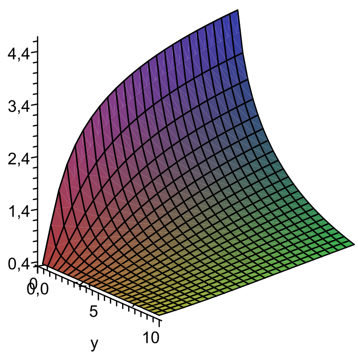


H Surfaces, Invariantby Hyperbolic Isometries, with x>0

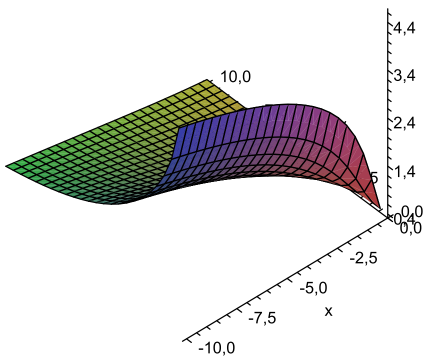


Now, we consider the sheet given by $x < 0$*,*

H Surfaces, Invariantby Hyperbolic Isometries, with x<0

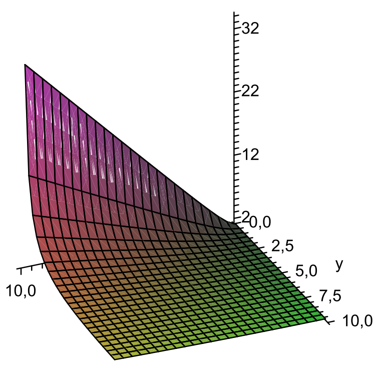


H Surfaces, Invariantby Hyperbolic Isometries, with x<0

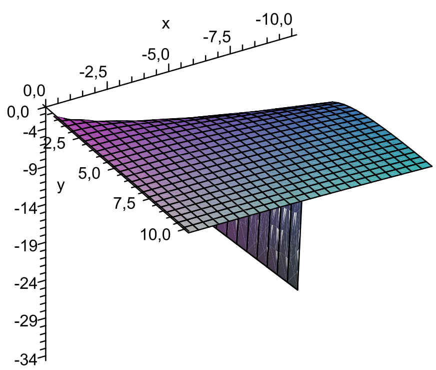


Example 8.2.2. *Making* $H = 1/2$*,* $\tau = -1/2$*,* $d \equiv 0$*, and by following the ideas of the proof of the Lemma 8.2.1, we obtain a surface invariant by one-parameter group of hyperbolic isometries having constant mean curvature* $H = 1/2$*. We plot the surface with Maple's help.*

1/2 Surface, Invariantby Hyperbolic Isometries, with x>0



1/2 Surface, Invariantby Hyperbolic Isometries, with x<0



Example 8.2.3. *Making $H \equiv 0$, $d \equiv 1$, we obtain a minimal surface invariant by one-parameter group of hyperbolic isometries. To see this, observe that*

$$u = \int \frac{(d - 2H\cot(\theta))\sqrt{1+4\tau^2\cos^2(\theta))}}{\sqrt{1-\sin^2(\theta)(d-2H\cot(\theta))^2}} d\theta - 2\tau\theta$$

becomes:

$$u = \int \frac{\sqrt{1+4\tau^2\cos^2(\theta))}}{\cos\theta} d\theta - 2\tau\theta$$

Making:

$$f(\theta) = \int \frac{\sqrt{1+4\tau^2\cos^2(\theta))}}{\cos\theta} d\theta$$

then,

$$f(\theta) = \int \frac{\sqrt{1+4\tau^2\cos^2(\theta)}}{\cos^2\theta} \cos\theta d\theta$$

let $w = \sin\theta$ so $dw = \cos\theta d\theta$. Setting $a = \dfrac{1+4\tau^2}{4\tau^2}$, the last integral becomes:

$$f(\theta) = |2\tau| \int \frac{\sqrt{a-w^2}}{1-w^2} dw$$

which has the next solution:

$$\begin{aligned} f &= \arctan\left(\frac{w}{\sqrt{a-w^2}}\right) + \frac{1}{2}\sqrt{-1+a}\ln\left(\frac{a-w+\sqrt{-1+a}\sqrt{a-w^2}}{-1+w}\right) + \\ &\quad - \frac{\sqrt{-1+a}}{2}\ln\left(\frac{a+w+\sqrt{-1+a}\sqrt{a-w^2}}{w+1}\right) \end{aligned}$$

which can be write in the form:

$$f = \arctan\left(\frac{w}{\sqrt{a-w^2}}\right) + \frac{\sqrt{a-1}}{2}\ln\left(\frac{(a-w+\sqrt{-1+a}\sqrt{a-w^2})(w+1)}{(a+w+\sqrt{-1+a}\sqrt{a-w^2})(w-1)}\right)$$

so,

$$\begin{aligned} f &= \arctan\left(\frac{|2\tau|\sin\theta}{\sqrt{1+4\tau^2\cos^2\theta}}\right) + \\ &\quad + \frac{1}{2|2\tau|}\ln\left(\frac{(1+4\tau^2-4\tau^2\sin\theta+\sqrt{1+4\tau^2\cos^2\theta})(\sin\theta+1)}{(1+4\tau^2+4\tau^2\sin\theta+\sqrt{1+4\tau^2\cos^2\theta})(\sin\theta-1)}\right) \end{aligned}$$

So, our integral has the solution:

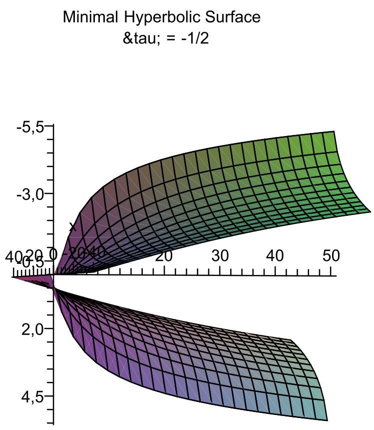
$$\begin{aligned} u(\theta) &= \arctan\left(\frac{|2\tau|\sin\theta}{\sqrt{1+4\tau^2\cos^2\theta}}\right) - 2\tau\theta + \\ &+ \frac{1}{2|2\tau|}\ln\left(\frac{(1+4\tau^2-4\tau^2\sin\theta+\sqrt{1+4\tau^2\cos^2\theta})(\sin\theta+1)}{(1+4\tau^2+4\tau^2\sin\theta+\sqrt{1+4\tau^2\cos^2\theta})(\sin\theta-1)}\right) \end{aligned}$$

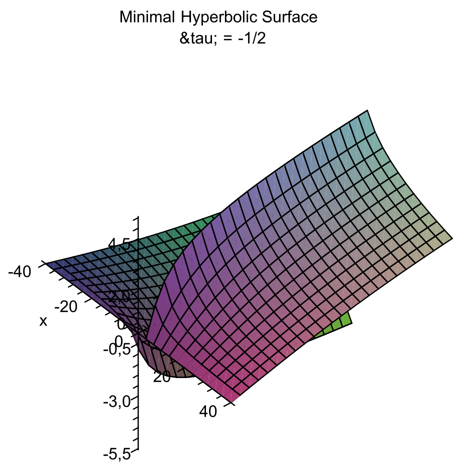
Observe o domain.

Observe that, by considering $\tau = -1/2$ *we obtain (following the same idea):*

$$u = \arcsin\left(\frac{\sin\theta}{\sqrt{2}}\right) - \frac{1}{2}arctanh\left(\frac{2+\sin\theta}{\sqrt{2-\sin^2\theta}}\right) - \frac{1}{2}arctanh\left(\frac{\sin\theta-2}{\sqrt{2+\sin^2\theta}}\right) + \theta$$

This surface goes to $x = 0$ *when* y *goes to* 0*, and when* $x \to 0^+$ *the surface goes to* $-\infty$*, if* $x \to 0^-$ *the surface goes to* $-\infty$*. See figure:*

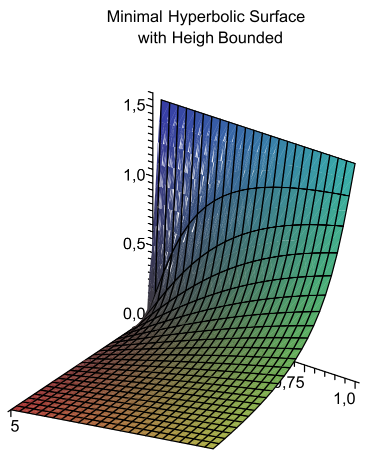


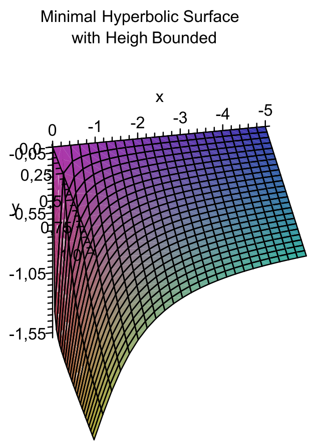


Example 8.2.4. *Making* $H \equiv 0$, $\tau = -1/2$, $d \equiv 0$*, we obtain a minimal surfaces which consist of two sheet (see figure). When* y *goes to* 0 *the surface goes to* $x = 0$*. When* $x \to 0^-$ *the function* u *goes to a negative finite value, and when* $x \to^+$ *the function goes to a positive finite value. Observe that:*

$$u(x,y) = \arctan\left(\frac{y}{x}\right)$$

We plot the surface with Maple's help.


Minimal Hyperbolic Surface
with Heigh Bounded
1,5
1,0
0,5
0,0
0,75
1,0
5


Minimal Hyperbolic Surface
with Heigh Bounded
x
0
-1
-2
-3
-4
-5
-0,05
-1,05
-1,55