3
The 2-dimensional space forms

We denote by $M^2(\kappa)$ the space form having constant Gauss curvature κ. Throughout this thesis, we only consider the values of κ in the set $\{-1, 0, 1\}$, that is, $\kappa \in \{-1, 0, 1\}$. We will see that, there is a natural orthonormal frame on $M^2(\kappa)$, which, will denote by $\{e_1, e_2\}$, and such that:

$$\begin{aligned} e_1 &= \lambda^{-1}\partial_x \\ e_2 &= \lambda^{-1}\partial_y \end{aligned}$$

where $\{\partial_x, \partial_y\}$ is the natural frame of $M^2(\kappa)$.

3.1
The 2-dimensional Euclidean space

In this section, we discuss briefly about the properties of the Euclidean space. The simplest 2-dimensional Riemannian manifold is given by the Euclidean space $\mathbb{R}^2$, that is, the set,

$$\mathbb{R}^2 = \{(x, y); x \in \mathbb{R}, y \in \mathbb{R}\}$$

endowed with the canonic metric ds^2. If $u = (u_1, u_2) \in T_p\mathbb{R}^2 \approx \mathbb{R}^2$ and $v = (v_1, v_2) \in T_p\mathbb{R}^2 \approx \mathbb{R}^2$, where $T_p\mathbb{R}^2$ denote the tangent space at p of $\mathbb{R}^2$, then,

$$ds(u, v) = u_1v_1 + u_2v_2.$$

The orthonormal frame is given by $\{\partial_x, \partial_y\}$, that is, $e_1 = \partial_x$, $e_2 = \partial_y$ and $\lambda \equiv 1$. Here, the Gauss curvature is identically null, that is, $\kappa \equiv 0$.

So, $M^2(\kappa) \equiv M^2(0) \equiv \mathbb{R}^2$. We denote by $Isom(\mathbb{R}^2)$ the isometries group of $\mathbb{R}^2$. It is well-know the following result:

Proposition 3.1.1. *The isometries group of the Euclidean space is given by,*

$$Isom(\mathbb{R}^2) = \{F : \mathbb{R}^2 \to \mathbb{R}^2; F(x) = v + Ax, v \in \mathbb{R}^2, A \in O(2)\}$$

where $O(2)$ is the group of orthogonal matrices of order 2×2.

3.2 The 2-dimensional Euclidean sphere

In this section, we discuss briefly the properties of the 2-dimensional Euclidean sphere. The Euclidean sphere is the set,

$$\mathbb{S}^2 = \{(x,y,z) \in \mathbb{R}^3; x^2+y^2+z^2=1\}$$

endowed with the metric induced of the 3-dimensional Euclidean space. In this case $\kappa \equiv 1$, and $M^2(\kappa) \equiv M^2(1) \equiv S^2$

By considering the stereographic projection, we can write $\mathbb{S}^2$ in intrinsic form (that is, without mention of the ambient space) as:

$$\mathbb{S}^2 = \mathbb{R}^2 \cup \infty,$$

endowed with the metric:

$$ds^2 = \lambda^2(dx^2+dy^2), \qquad \lambda = \frac{2}{1+\kappa(x^2+y^2)}.$$

The natural frame is given by $\{\partial_x, \partial_y\}$ and the orthonormal frame is given by $\{e_1 = \lambda^{-1}\partial_x, e_2 = \lambda^{-1}\partial_y\}$.

We denote by $Isom(\mathbb{S}^2)$ the isometries group of $\mathbb{S}^2$. It is well-know the following result:

Proposition 3.2.1. *The isometries group of the Euclidean sphere is given by,*

$$Isom(\mathbb{S}^2) = O(3)$$

where $O(3)$ *is the group of orthogonal matrices of order* 3×3.

3.3 The 2-dimensional hyperbolic space

Now we focus our attention on the properties of the hyperbolic space. In this section we will follow the ideas of Eric Toubiana and Ricardo Sa Earp, that is our main reference will be (16).

When $\kappa < 0$, $M^2(\kappa)$ denote the hyperbolic space having constant Gauss curvature κ.

There are several model of the hyperbolic space, we will consider the half-plane, and the disc model for the hyperbolic space. We begin with the half-plane model.

3.3.1 The half-plane model for the hyperbolic space

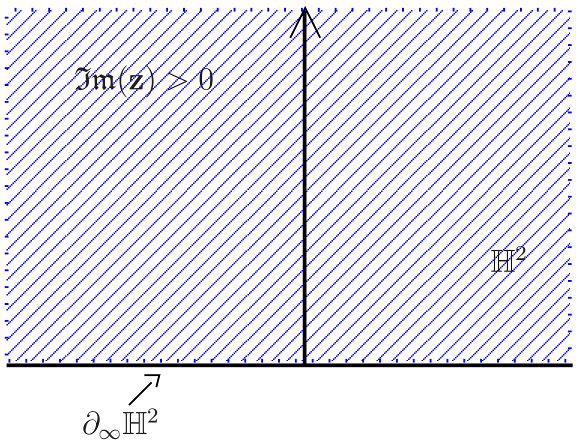
Consider the set

$$\mathbb{H}^2 = \{z \in \mathbb{C}; \mathfrak{Im}(z) > 0\}$$

endowed with the metric,

$$ds^2_{\mathbb{H}^2} = \frac{|dz|^2}{\mathfrak{Im}^2(z)}$$

where $|dz|^2$ denote the Euclidean metric on $\mathbb{C}$.



The set $\mathbb{H}^2$ endowed with the metric $ds^2_{\mathbb{H}^2}$ is a 2-dimensional Riemannian manifold called the half-plane model for the hyperbolic space. We will call the real axis together with the infinity point, the asymptotic boundary of $\mathbb{H}^2$ and we will denote it by $\partial_\infty \mathbb{H}^2$. That is

$$\partial_\infty \mathbb{H}^2 = \{z \in \mathbb{R}\} \cup \{\infty\}$$

We denote by $Isom(\mathbb{H}^2)$ the isometries group of the hyperbolic space.

Proposition 3.3.1. *(16, Chapter 2) The isometries group $Isom(\mathbb{H}^2)$ of the hyperbolic space $\mathbb{H}^2$ is given by*

$$Isom\{\mathbb{H}^2\} = \left\{ z \longrightarrow \frac{az+b}{cz+d}; ad - bc = 1 \right\} \bigcup \left\{ z \longrightarrow \frac{-a\overline{z} - b}{c\overline{z} + d}; ad - bc = 1 \right\}$$

where $a, b, c, d \in \mathbb{R}$.

3.3.2
Behavior of the positive isometries of $\mathbb{H}^2$

There are three kind of special curves in $\mathbb{H}^2$, namely: geodesic, circles and horocycles. Maybe the most mysterious are the horocycles, but this are simply horizontal line or tangent circle to the asymptotic boundary. It is possible to show that, every horocycle is taking to other horocycle by a isometry of $\mathbb{H}^2$.

Let T a positive isometry of $(\mathbb{H}^2, ds^2_{\mathbb{H}^2})$, different from the identity,

$$T(z) = \frac{az+b}{cz+d}$$

where $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$. One point $z \in \mathbb{H}^2 \cup \partial_\infty \mathbb{H}^2$ is fixed by T if and only if $T(z) = z$. So

$$T(z) = z \Longleftrightarrow z = \frac{a - d \pm \sqrt{(a+d)^2 - 4}}{2c}$$

Thus, T has at most two fixed points. If $c = 0$, then $T(z) = az + b$, and ∞ is a fixed point for T. Furthermore, if $a = 1$, there is not other fixed point different of ∞. If $a \neq 1$, then the other fixed point is a real number. This give:

Definition 3.3.1. *(16, Chapter 2) There are three kind of positive isometries, namely*

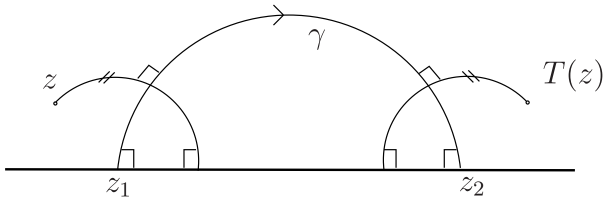
1. *If $c \neq 0$ and $(a+d)^2 - 4 > 0$, T has two distinct fixed points, $x_1, x_2 \in \mathbb{R}$. If $T(z) = az + b$, with $a \neq 1$, T has a real fixed point and the other fixed point is ∞. In both cases T has two distinct fixed points on $\partial_\infty \mathbb{H}^2$. In this case we say that, T is a hyperbolic isometry.*

2. *If $c \neq 0$ and $(a+d)^2 - 4 = 0$, T has a double real fixed point. If $T(z) = z + b$, $b \neq 0$, ∞ is the single fixed point of T on $\partial_\infty \mathbb{H}^2$. So T has a single fixed point on $\partial_\infty \mathbb{H}^2$. In this case we say that, T is a parabolic isometry.*

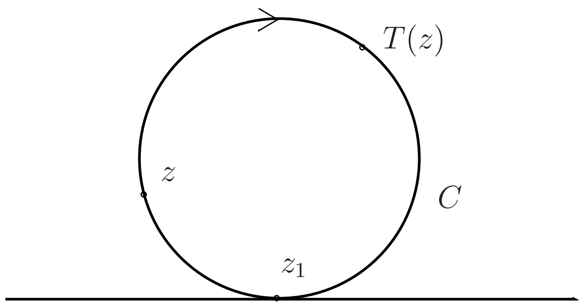
3. *If $c \neq 0$ and $(a+d)^2 - 4 < 0$, T has two fixed points on $\mathbb{C}$, and a single fixed point on $\mathbb{H}^2$. In this case we say that, T is a elliptic isometry.*

Now we discuss briefly the behavior of the positive isometries, again we following the ideas of Eric Toubiana and Ricardo Sa Earp, see (16, section 2.4,pag. 73).

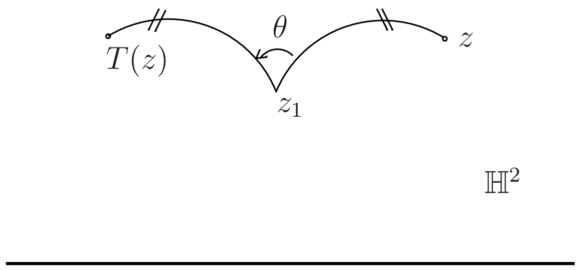
A Hyperbolic isometry. Let $z_1, z_2 \in \partial_\infty \mathbb{H}^2$ be two fixed point of T. Denotes by γ a complete geodesic on $\mathbb{H}^2$ joint the point z_1 to the point z_2, so γ is fixed by T. Furthermore, T acts like a translation along γ. Observe that γ divides $\mathbb{H}^2$ in two connected components. Let $z \in \mathbb{H}^2$ and let β be the complete geodesic passing by z and orthogonal to γ. Set $w_1 = \gamma \cap \beta$, and denote by $\beta_1 = T(\beta)$, then β_1 is the complete geodesic, passing by $T(w_1)$ and orthogonal to γ at the point $T(w_1)$. The image $T(z)$ stay in β_1, and in the same connected component that contains z. Since T is an isometry, we have $d(z, z_1) = d(T(z), T(w_1))$, where $d(.,.)$ denote the distance on $\mathbb{H}^2$.



B Parabolic isometry. Let $z_1 \in \partial_\infty \mathbb{H}^2$, and $z \in \mathbb{H}^2$. Denotes by C the unique horocycle that contain z and passing by z_1, then C is fixed by T, so $T(z) \in C$.



C Elliptic isometry. Let $z_1 \in \mathbb{H}^2$ the unique fixed point of T on $\mathbb{H}^2$. Denotes by γ the unique complete geodesic connecting the points z_1 and z, and denote by $\beta = T(\gamma)$ the complete geodesic connecting the points $T(z)$ and $T(z_1) = z_1$, that is T acts like a rotation around z_1.



Observe that, since T is a positive isometry, then T preserves orientation. This observation completes the brief study of the behavior of positive isometries.

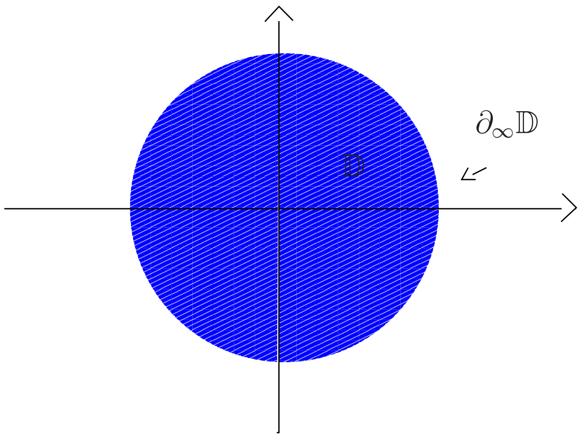
3.3.3
The disc model for the hyperbolic space

Consider the set,

$$\mathbb{D} = \{w \in \mathbb{C}, |w| < 1\}$$

endowed with the metric

$$ds^2_{\mathbb{D}} = \frac{4}{(1-|w|^2)^2}|dw|^2$$



The set $\mathbb{D}$ endowed with the metric $ds^2_{\mathbb{D}}$ is call the disc model for the hyperbolic space. The asymptotic boundary is denoted and defined by

$$\partial_\infty \mathbb{D} = \{w \in \mathbb{C}; |w| = 1\}$$

Denoting by $Isom(\mathbb{D})$ the isometries group of $\mathbb{D}$, we have:

Proposition 3.3.2. *(16, Chapter 2) The isometries group $Isom\{\mathbb{D}\}$ of the hyperbolic disc is given by,*

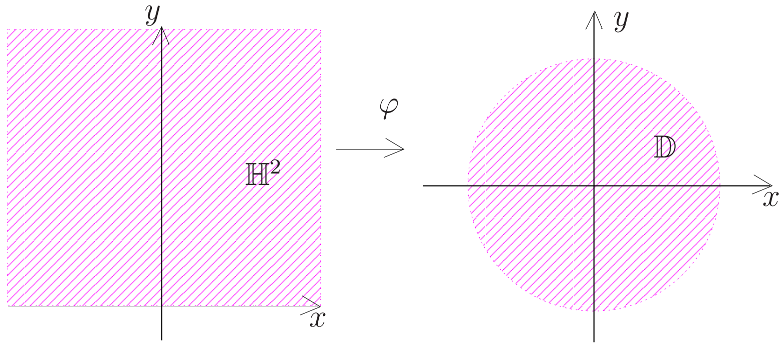
$$Isom\{\mathbb{D}\} = \left\{w \longrightarrow \frac{aw+\overline{c}}{cw+\overline{a}}; a, c \in \mathbb{C}, a\overline{a} - c\overline{c} = 1\right\}$$

Observe that, there is an isometry

$$\varphi : \mathbb{H}^2 \longrightarrow \mathbb{D}$$

given by,

$$\varphi(z) = \frac{z-i}{z+i}$$



This isometry let us rewrite each property of the hyperbolic half-space in the disk model, that is the behavior of positive isometries, the geometry of the important curves as well as the study of the itself hyperbolic space.