Pontifícia Universidade Católica
do Rio de Janeiro

Carlos Diosdado Espinoza Peñafiel

Surfaces of Constant Mean Curvature in Homogeneous Three Manifolds with Emphasis in $\widetilde{\mathbf{PSL}_2}(\mathbb{R}, \tau)$

Tese de Doutorado

Thesis presented to the Postgraduate Program in Matematics of the Departamento de Matemática, PUC–Rio as partial fulfillment of the requirements for the degree of Doutor em Matemática

Advisor : Prof. Ricardo Sá Earp
Co–Advisor: Prof. Harold Rosenberg

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Abstract

Espinoza Peñafiel , Carlos Diosdado; Sá Earp, Ricardo; Rosenberg, Harold. **Surfaces of Constant Mean Curvature in Homogeneous Three Manifolds with Emphasis in $\widetilde{\mathbf{PSL}}_{\mathbf{2}}(\mathbb{R}, \tau)$**. Rio de Janeiro, 2010. 138p. Tese de Doutorado — Departamento de Matemática, Pontifícia Universidade Católica do Rio de Janeiro.

In this thesis we study H-surfaces, that is, surfaces having constant mean curvature, immersed in homogeneous simply connected 3-manifold. We focus our attention in the study of existence of H multigraphs. We also study the H-surfaces invariant by one-parameter group of isometries which are immersed in the space $\widetilde{PSL}_2(\mathbb{R}, \tau)$.

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Keywords

Constant Mean Curvature. Homogeneous Manifolds. Invariant Surfaces. One-parameter Group of Isometries. Multigraphs.

Resumo

Espinoza Peñafiel , Carlos Diosdado; Sá Earp, Ricardo; Rosenberg, Harold. **Superfícies de Curvatura Media Constante em Variedades Homogéneas de Dimensão 3 com Enfâse em $\widetilde{\mathbf{PSL}}_{\mathbf{2}}(\mathbb{R}, \tau)$.** Rio de Janeiro, 2010. 138p. Tese de Doutorado — Departamento de Matemática, Pontifícia Universidade Católica do Rio de Janeiro.

Nesta teses, nós estudamos H-superfícies, isto é, superfícies tendo curvatura media constante, imersas em variedades homogéneas simplesmente conexas de dimensão 3. Nós focamos nossa atenção no estudo de existência de H multigráficos. Também estudamos a H-superfícies invariantes por um grupo a um parâmetro de isometrias que estão imersas no espaço $\widetilde{PSL}_2(\mathbb{R}, \tau)$.

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Palavras–chave

Curvatura Media Constante. Variedades Homogéneas. Superfícies Invariantes. Grupo a Um-parâmetro de Isometrias. Multigraficos.

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Inclina hoy la cabeza ante los libros para que
mañana no la inclines ante los hombres.

Mi madre Rosa y mi abuela Delia, *Enseñanzas que valen oro.*