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5
Appendixes

5.1.
Appendix of Chapter 1

5.1.1.
Aggregate Price Level

In this appendix, we show how to express the (equilibrium) aggregate price level in terms of the high order beliefs. First, we replace (1.1) in (1.3) to obtain:

$$\begin{aligned} P_t &= \sum_{j=0}^{\infty} \int_{\Lambda_{t-j}} E[rP_t + (1-r)\theta_t \mid \Im_{t-j}(z)]dz \\ &= r\sum_{j=0}^{\infty} \int_{\Lambda_{t-j}} E[P_t \mid \Im_{t-j}(z)]dz + (1-r)\sum_{j=0}^{\infty} \int_{\Lambda_{t-j}} E[\theta_t \mid \Im_{t-j}(z)]dz. \end{aligned}$$

From the definition of the average 1^{st} order belief in (1.4):

$$P_t = r\bar{E}[P_t] + (1-r)\bar{E}[\theta_t].$$

If we iterate one time, we obtain:

$$\begin{aligned} P_t &= r\bar{E}[r\bar{E}[P_t] + (1-r)\bar{E}[\theta_t]] + (1-r)\bar{E}[\theta_t] \\ &= r^2\bar{E}[\bar{E}[P_t]] + r(1-r)\bar{E}[\bar{E}[\theta_t]] + (1-r)\bar{E}[\theta_t] \\ &= r^2\bar{E}^2[P_t] + r(1-r)\bar{E}^2[\theta_t] + (1-r)\bar{E}[\theta_t]. \end{aligned}$$

If we iterate N times:

$$P_t = r^N\bar{E}^N[P_t] + (1-r)\sum_{k=1}^{N} r^{k-1}\bar{E}^k[\theta_t].$$

Taking the limit as $N \to \infty$, we obtain expression (1.5):

$$P_t = (1-r)\sum_{k=1}^{\infty} r^{k-1}\bar{E}^k[\theta_t],$$

which proves the result.

5.1.2.
Expectations

In this appendix, we show how a firm z that updated its information set a period $t-j$ computes its expectation about the fundamental θ_{t-m}, $E\left[\theta_{t-m} \mid \Im_{t-j}(z)\right]$. First, we calculate the distribution of the fundamental θ_{t-j} given that the firm updated its information set at period $t-j$. We can compute $f\left(\theta_{t-j} \mid \Theta_{t-j-1}, x_{t-j}\right)$ as

$$\begin{aligned} f(\theta_{t-j} \mid \theta_{t-j-1}, x_{t-j}) &= \frac{f(\theta_{t-j}, \theta_{t-j-1}, x_{t-j})}{\int_{-\infty}^{\infty} f(\theta_{t-j}, \theta_{t-j-1}, x_{t-j}) d\theta_{t-j}} \\ &= \frac{f(\theta_{t-j-1}, x_{t-j} \mid \theta_{t-j}) f(\theta_{t-j})}{\int_{-\infty}^{\infty} f(\theta_{t-j}, \theta_{t-j-1}, x_{t-j}) d\theta_{t-j}} \\ &= \frac{f(\theta_{t-j-1} \mid \theta_{t-j}) f(x_{t-j} \mid \theta_{t-j}) f(\theta_{t-j})}{\int_{-\infty}^{\infty} f(\theta_{t-j}, \theta_{t-j-1}, x_{t-j}) d\theta_{t-j}} \end{aligned}$$

where the last equality holds due to the independence of $\xi_t(z)$ and ε_{t-j}. As

$$\begin{aligned} x_{t-j}(z) &= \theta_{t-j} + \xi_{t-j}(z), \\ \theta_{t-j-1} &= \theta_{t-j} - \varepsilon_{t-j}, \end{aligned}$$

where $\xi_t(z) \sim N\left(0, \beta^{-1}\right)$ and $\varepsilon_{t-j} \sim N\left(0, \alpha^{-1}\right)$, we have that $f\left(x_{t-j} \mid \theta_{t-j}\right) = N\left(\theta_{t-j}, \beta^{-1}\right)$ and $f\left(\theta_{t-j-1} \mid \theta_{t-j}\right) = N\left(\theta_{t-j}, \alpha^{-1}\right)$. If the dynamics of θ_t was $\theta_{t-j-1} = \rho\theta_{t-j} - \varepsilon_{t-j}$, we would have

$$\begin{aligned} E[\theta_{t-j}] &= E[\theta_t] = \frac{E[\varepsilon_t]}{1-\rho} = 0, \\ Var[\theta_{t-j}] &= Var[\theta_t] = \frac{Var[\varepsilon_t]}{1-\rho^2} = \frac{\alpha^{-1}}{1-\rho^2}. \end{aligned}$$

Therefore, the distribution of θ_{t-j} would be given by $f\left(\theta_{t-j}\right) = N\left(0, \Psi^{-1}\right)$ where $\Psi = \alpha\left(1-\rho^2\right)$. Thus, we would obtain

$$\begin{aligned} f(\theta_{t-j}, \theta_{t-j-1}, x_{t-j}) &= c \times \exp\left\{-\frac{1}{2}\left[\frac{(x_{t-j}(z) - \theta_{t-j})^2}{\beta^{-1}} + \frac{(\theta_{t-j-1} - \rho^{-1}\theta_{t-j})^2}{(\rho^2\alpha)^{-1}} + \frac{\theta_{t-j}^2}{\Psi^{-1}}\right]\right\} \\ &= c \times \exp\left\{-\frac{1}{2}\left[(\beta + \alpha + \Psi)\theta_{t-j}^2 - 2(\beta x_{t-j}(z) + \alpha\rho\theta_{t-j-1})\theta_{t-j}\right]\right\} \\ &\times \exp\left\{-\frac{1}{2}\left[\beta x_{t-j}^2(z) + \alpha\rho^2\theta_{t-j-1}^2\right]\right\} \\ &= c \times d \times \frac{1}{\sqrt{2\pi}\sigma\Sigma} \times \exp\left\{-\frac{1}{2}\frac{(\theta_{t-j} - \mu)^2}{\Sigma^2}\right\}, \end{aligned}$$

where

$$c = (2\pi)^{-3/2}(\beta\alpha\Psi)^{1/2}, \qquad d = \sqrt{2\pi}\sigma \exp\left\{-\tfrac{1}{2}\left[-\mu^2\Sigma^{-2} + \beta x_{t-j}^2(z) + \alpha\rho^2\theta_{t-j-1}^2\right]\right\},$$

$$\mu = \left[\Delta x_{t-j}(z) + (1-\Delta) z_{t-j-1}\right], \qquad \Delta = \beta(\beta+\alpha+\Psi)^{-1},$$

$$z_{t-j-1} = \Lambda\rho\theta_{t-j-1}, \qquad \Lambda = \alpha(\beta+\alpha)^{-1}.$$

$$\Sigma^2 = (\beta+\alpha+\Psi)^{-1},$$

As $\rho \to 1$, we have $\Psi \to 0$, $\Delta \to \delta$, and $\Sigma^2 \to (\beta+\alpha)^{-1}$. Thus $f(\theta_{t-j} \mid \theta_{t-j-1}, x_{t-j}) = N(\mu, \sigma^2)$ where $\mu = \left[\delta\, x_{t-j}(z) + (1-\delta)\,\theta_{t-j-1}\right]$, and $\sigma^2 = (\beta+\alpha)^{-1}$.

5.1.3. High order beliefs

In this Appendix we derive the general formula of the k -th order average expectation

$$\bar{E}^k[\theta_t] = \lambda \sum_{m=0}^{\infty} (1-\lambda)^m [\kappa_{m,k}\theta_{t-m} + \delta_{m,k}\theta_{t-m-1}]$$

with the weights $(\kappa_{m,k}, \delta_{m,k})$ recursively defined for $k \geq 1$

$$\begin{bmatrix} \kappa_{m,k+1} \\ \delta_{m,k+1} \end{bmatrix} = \begin{bmatrix} (1-\delta) \\ \delta \end{bmatrix} [1-(1-\lambda)^m]^k + A_m \begin{bmatrix} \kappa_{m,k} \\ \delta_{m,k} \end{bmatrix},$$

where the matrix A_m is given by

$$A_m \equiv \begin{bmatrix} \left[(1-\delta)\left[1-(1-\lambda)^{m+1}\right] + \delta[1-(1-\lambda)^m]\right] & 0 \\ \delta\left[\left[1-(1-\lambda)^{m+1}\right] - [1-(1-\lambda)^m]\right] & \left[1-(1-\lambda)^{m+1}\right] \end{bmatrix},$$

and the initial weights are $(\kappa_{1,k}, \delta_{1,k}) \equiv (1-\delta, \delta)$. We start by computing $\bar{E}^1[\theta_t]$ as

$$\begin{aligned} \bar{E}^1[\theta_t] &= \sum_{j=0}^{\infty} \int_{\Lambda_j} E[\bar{E}^0[\theta_t] \mid \Im_{t-j}(z)] dz \\ &= \sum_{j=0}^{\infty} \int_{\Lambda_j} E[\theta_t \mid \Im_{t-j}(z)] dz \\ &= \sum_{j=0}^{\infty} \int_{\Lambda_j} [(1-\delta) x_{t-j}(z) + \delta\theta_{t-j-1}] dz \\ &= \lambda \sum_{j=0}^{\infty} (1-\lambda)^j [(1-\delta)\theta_{t-j} + \delta\theta_{t-j-1}]. \end{aligned}$$

We can use this result to obtain $\bar{E}^2[\theta_t]$ as

$$\bar{E}^2[\theta_t] = \sum_{m=0}^{\infty} \int_{\Lambda_m} E[\bar{E}^1[\theta_t] \mid \Im_{t-m}(z)]dz$$
$$= \lambda \sum_{m=0}^{\infty} \int_{\Lambda_m} \sum_{j=0}^{\infty} (1-\lambda)^j E[(1-\delta)\theta_{t-j} + \delta\theta_{t-j-1} \mid \Im_{t-m}(z)]dz.$$

We know that

$$E[\theta_{t-j} \mid \Im_{t-m}(z)] = \begin{cases} (1-\delta)x_{t-m}(z) + \delta\theta_{t-m-1} & : m \geq j, \\ \theta_{t-j} & : m < j. \end{cases}$$

Thereafter,

$$\bar{E}^2[\theta_t] = \lambda \sum_{m=0}^{\infty} \int_{\Lambda_m} \sum_{j=0}^{m-1} (1-\lambda)^j \{(1-\delta)E[\theta_{t-j} \mid \Im_{t-m}(z)] + \delta E[\theta_{t-j-1} \mid \Im_{t-m}(z)]\}dz$$
$$+ \lambda \sum_{m=0}^{\infty} \int_{\Lambda_m} (1-\lambda)^m \{(1-\delta)E[\theta_{t-m} \mid \Im_{t-m}(z)] + \delta\theta_{t-m-1}\}dz$$
$$+ \lambda \sum_{m=0}^{\infty} \int_{\Lambda_m} \sum_{j=m+1}^{\infty} (1-\lambda)^j [(1-\delta)\theta_{t-j} + \delta\theta_{t-j-1}]dz$$
$$= \lambda \sum_{m=0}^{\infty} \int_{\Lambda_m} \sum_{j=0}^{m-1} (1-\lambda)^j [(1-\delta)x_{t-m}(z) + \delta\theta_{t-m-1}]dz$$
$$+ \lambda \sum_{m=0}^{\infty} \int_{\Lambda_m} (1-\lambda)^m [(1-\delta)[(1-\delta)x_{t-m}(z) + \delta\theta_{t-m-1}] + \delta\theta_{t-m-1}]dz$$
$$+ \lambda \sum_{m=0}^{\infty} \int_{\Lambda_m} \sum_{j=m+1}^{\infty} (1-\lambda)^j [(1-\delta)\theta_{t-j} + \delta\theta_{t-j-1}]dz$$
$$= \lambda^2 \sum_{m=0}^{\infty} (1-\lambda)^m [(1-\delta)\theta_{t-m} + \delta\theta_{t-m-1}] \sum_{j=0}^{m-1} (1-\lambda)^j$$
$$+ \lambda^2 \sum_{m=0}^{\infty} (1-\lambda)^{2m} \left[(1-\delta)^2\theta_{t-m} + \left[1-(1-\delta)^2\right]\theta_{t-m-1}\right]$$
$$+ \lambda^2 \sum_{j=1}^{\infty} (1-\lambda)^j [(1-\delta)\theta_{t-j} + \delta\theta_{t-j-1}] \sum_{m=0}^{j-1} (1-\lambda)^m$$
$$= \lambda \sum_{m=0}^{\infty} (1-\lambda)^m [(1-\delta)\theta_{t-m} + \delta\theta_{t-m-1}][1-(1-\lambda)^m]$$
$$+ \lambda^2 \sum_{m=0}^{\infty} (1-\lambda)^{2m} \left[(1-\delta)^2\theta_{t-m} + \left[1-(1-\delta)^2\right]\theta_{t-m-1}\right]$$
$$+ \lambda \sum_{j=1}^{\infty} (1-\lambda)^j [(1-\delta)\theta_{t-j} + \delta\theta_{t-j-1}]\left[1-(1-\lambda)^j\right]$$
$$= \lambda \sum_{m=0}^{\infty} (1-\lambda)^m 2[1-(1-\lambda)^m][(1-\delta)\theta_{t-m} + \delta\theta_{t-m-1}]$$
$$+ \lambda^2 \sum_{m=0}^{\infty} (1-\lambda)^{2m} \left[(1-\delta)^2\theta_{t-m} + \left[1-(1-\delta)^2\right]\theta_{t-m-1}\right].$$

We can write this expression as

$$\bar{E}^2[\theta_t] = \lambda \sum_{j=0}^{\infty} (1-\lambda)^j [\kappa_{j,2}\theta_{t-j} + \delta_{j,2}\theta_{t-j-1}],$$

where

$$\kappa_{j,2} = (1-\delta^2)\left[1-(1-\lambda)^j\right] + (1-\delta)^2\left[1-(1-\lambda)^{j+1}\right]$$
$$= \left[1-(1-\lambda)^{j+1}\right]\kappa_{j,1}^2 + \left[1-(1-\lambda)^j\right]\left(1-\delta_{j,1}^2\right),$$
$$\delta_{j,2} = \delta^2\left[1-(1-\lambda)^j\right] + \left[1-(1-\delta)^2\right]\left[1-(1-\lambda)^{j+1}\right]$$
$$= \left[1-(1-\lambda)^{j+1}\right]\left(1-\kappa_{j,1}^2\right) + \left[1-(1-\lambda)^j\right]\delta_{j,1}^2.$$

Note that

$$\kappa_{j,2} + \delta_{j,2} = \sum_{n=0}^{1}\Big[1-(1-\lambda)^{j}\Big]^{n}\Big[1-(1-\lambda)^{j+1}\Big]^{1-n}.$$

We use induction to obtain the general case. Suppose that (1.11) holds for $k-1$. Then

$$\bar{E}^{k-1}[\theta_t] = \lambda\sum_{m=0}^{\infty}(1-\lambda)^{m}[\kappa_{m,k-1}\theta_{t-m} + \delta_{m,k-1}\theta_{t-m-1}],$$

where

$$\sum_{j=0}^{m-1}(1-\lambda)^{j}(\kappa_{j,k-1}+\delta_{j,k-1}) = \frac{1}{\lambda}[1-(1-\lambda)^{m}]^{k-1}.$$

As a result,

$$\begin{aligned}
\bar{E}^{k}[\theta_t] &= \sum_{m=0}^{\infty}\int_{\Lambda_m} E[\bar{E}^{k-1}[\theta_t] \mid \Im_{t-m}(z)]dz\\
&= \sum_{m=0}^{\infty}\int_{\Lambda_m} E\Big[\lambda\sum_{j=0}^{\infty}(1-\lambda)^{j}[\kappa_{j,k-1}\theta_{t-j}+\delta_{j,k-1}\theta_{t-j-1}] \mid \Im_{t-m}(z)\Big]dz\\
&= \lambda\sum_{m=0}^{\infty}\int_{\Lambda_m}\sum_{j=0}^{m-1}(1-\lambda)^{j}\{\kappa_{j,k-1}E[\theta_{t-j} \mid \Im_{t-m}(z)]+\delta_{j,k-1}E[\theta_{t-j-1} \mid \Im_{t-m}(z)]\}dz\\
&+ \lambda\sum_{m=0}^{\infty}\int_{\Lambda_m}(1-\lambda)^{m}\{\kappa_{m,k-1}E[\theta_{t-m} \mid \Im_{t-m}(z)]+\delta_{m,k-1}\theta_{t-m-1}\}dz\\
&+ \lambda\sum_{m=0}^{\infty}\int_{\Lambda_m}\sum_{j=m+1}^{\infty}(1-\lambda)^{j}[\kappa_{j,k-1}\theta_{t-j}+\delta_{j,k-1}\theta_{t-j-1}]dz\\
&= \lambda\sum_{m=0}^{\infty}\int_{\Lambda_m}\sum_{j=0}^{m-1}(1-\lambda)^{j}(\kappa_{j,k-1}+\delta_{j,k-1})[(1-\delta)x_{t-m}(z)+\delta\theta_{t-m-1}]dz\\
&+ \lambda\sum_{m=0}^{\infty}\int_{\Lambda_m}(1-\lambda)^{m}[\kappa_{m,k-1}[(1-\delta)x_{t-m}(z)+\delta\theta_{t-m-1}]+\delta_{m,k-1}\theta_{t-m-1}]dz\\
&+ \lambda\sum_{m=0}^{\infty}\int_{\Lambda_m}\sum_{j=m+1}^{\infty}(1-\lambda)^{j}[\kappa_{j,k-1}\theta_{t-j}+\delta_{j,k-1}\theta_{t-j-1}]dz\\
&= \lambda^{2}\sum_{m=0}^{\infty}(1-\lambda)^{m}[(1-\delta)\theta_{t-m}+\delta\theta_{t-m-1}]\sum_{j=0}^{m-1}(1-\lambda)^{j}(\kappa_{j,k-1}+\delta_{j,k-1})\\
&+ \lambda^{2}\sum_{m=0}^{\infty}(1-\lambda)^{2m}[\kappa_{m,k-1}(1-\delta)\theta_{t-m}+[\kappa_{m,k-1}\delta+\delta_{m,k-1}]\theta_{t-m-1}]\\
&+ \lambda^{2}\sum_{j=1}^{\infty}(1-\lambda)^{j}[\kappa_{j,k-1}\theta_{t-j}+\delta_{j,k-1}\theta_{t-j-1}]\sum_{m=0}^{j-1}(1-\lambda)^{m}\\
&= \lambda\sum_{m=0}^{\infty}(1-\lambda)^{m}[1-(1-\lambda)^{m}]^{k-1}[(1-\delta)\theta_{t-m}+\delta\theta_{t-m-1}]\\
&+ \lambda^{2}\sum_{m=0}^{\infty}(1-\lambda)^{2m}[\kappa_{m,k-1}(1-\delta)\theta_{t-m}+[\kappa_{m,k-1}\delta+\delta_{m,k-1}]\theta_{t-m-1}]\\
&+ \lambda\sum_{m=0}^{\infty}(1-\lambda)^{m}[1-(1-\lambda)^{m}][\kappa_{m,k-1}\theta_{t-m}+\delta_{m,k-1}\theta_{t-m-1}].
\end{aligned}$$

We can rewrite the last three lines above as

$$\bar{E}^{k}[\theta_t] = \lambda\sum_{m=0}^{\infty}(1-\lambda)^{m}[\kappa_{m,k}\theta_{t-m}+\delta_{m,k}\theta_{t-m-1}],$$

where

$$\kappa_{m,k} \equiv (1-\delta)[1-(1-\lambda)^m]^{k-1} + [(1-\delta)\lambda(1-\lambda)^m + [1-(1-\lambda)^m]]\kappa_{m,k-1}$$
$$= (1-\delta)[1-(1-\lambda)^m]^{k-1}$$
$$+ \left[(1-\delta)\left[1-(1-\lambda)^{m+1}\right] + \delta[1-(1-\lambda)^m]\right]\kappa_{m,k-1},$$
$$\delta_{m,k} \equiv \delta[1-(1-\lambda)^m]^{k-1} + \delta\lambda(1-\lambda)^m\kappa_{m,k-1} + [\lambda(1-\lambda)^m + [1-(1-\lambda)^m]]\delta_{m,k-1}$$
$$= \delta[1-(1-\lambda)^m]^{k-1}$$
$$+ \delta\left[\left[1-(1-\lambda)^{m+1}\right] - [1-(1-\lambda)^m]\right]\kappa_{m,k-1} + \left[1-(1-\lambda)^{m+1}\right]\delta_{m,k-1},$$

since $\lambda(1-\lambda)^m = \left[1-(1-\lambda)^{m+1}\right] - \left[1-(1-\lambda)^m\right]$.

Rewriting these weights in matrix format, we obtain

$$\begin{bmatrix} \kappa_{m,k+1} \\ \delta_{m,k+1} \end{bmatrix} = \begin{bmatrix} (1-\delta) \\ \delta \end{bmatrix} [1-(1-\lambda)^m]^k + A_m \begin{bmatrix} \kappa_{m,k} \\ \delta_{m,k} \end{bmatrix},$$

where the matrix A_m is given by

$$A_m \equiv \begin{bmatrix} \left[(1-\delta)\left[1-(1-\lambda)^{m+1}\right] + \delta[1-(1-\lambda)^m]\right] & 0 \\ \delta\left[\left[1-(1-\lambda)^{m+1}\right] - [1-(1-\lambda)^m]\right] & \left[1-(1-\lambda)^{m+1}\right] \end{bmatrix},$$

which is exactly our result.

5.2.
Appendix of Chapter 2

5.2.1.
Expectations

At this appendix we show that

$$E[\varepsilon_{t-i} \mid y_{t-i}] = \left(\frac{\gamma}{\gamma+\alpha}\right) y_{t-i}$$

and that

$$E[\varepsilon_{t-j} \mid y_{t-j}, v_{t-j}] = \left[\left(\frac{\gamma}{\gamma+\beta+\alpha}\right) y_{t-j} + \left(\frac{\beta}{\gamma+\beta+\alpha}\right) v_{t-j}\right].$$

In order to derive $E[\varepsilon_{t-i} \mid y_{t-i}]$, we have to find $f(\varepsilon_{t-i} \mid y_{t-i})$. Using Bayes theorem, we have

$$f(\varepsilon_{t-i} \mid y_{t-i}) = \frac{f(\varepsilon_{t-i}, y_{t-i})}{f(y_{t-i})} = \frac{f(y_{t-i} \mid \varepsilon_{t-i}) f(\varepsilon_{t-i})}{\int f(y_{t-i} \mid \varepsilon_{t-i}) f(\varepsilon_{t-i}) d\varepsilon_{t-i}}$$

But

$$\begin{aligned} &f(y_{t-i} \mid \varepsilon_{t-i})f(\varepsilon_{t-i}) \\ &= \frac{1}{2\pi(\alpha\gamma)^{-1/2}} \exp -\frac{1}{2}\left[\left(\frac{(y_{t-i}-\varepsilon_{t-i})^2}{\gamma^{-1}}\right)+\frac{\varepsilon_{t-i}^2}{\alpha^{-1}}\right] \\ &= \frac{1}{2\pi(\alpha\gamma)^{-1/2}} \exp -\frac{1}{2}[\gamma y_{t-i}^2 - 2\gamma\varepsilon_{t-i}y_{t-i} + \gamma\varepsilon_{t-i}^2 + \alpha\varepsilon_{t-i}^2] \\ &= \frac{1}{2\pi(\alpha\gamma)^{-1/2}} \exp -\frac{1}{2}\left[(\gamma+\alpha)\left(\varepsilon_{t-i} - \left(\frac{\gamma}{\gamma+\alpha}\right)y_{t-i}\right)^2 - \left(\frac{\gamma^2}{\gamma+\alpha}\right)y_{t-i}^2 + \gamma y_{t-i}^2\right] \\ &= c\frac{1}{\sqrt{2\pi}(\alpha+\gamma)^{-1/2}} \exp -\frac{1}{2}\left[\left(\frac{\varepsilon_{t-i} - \left(\frac{\gamma}{\gamma+\alpha}\right)y_{t-i}}{(\gamma+\alpha)^{-1}}\right)\right] \end{aligned}$$

where

$$c = \sqrt{\frac{1}{2\pi}\left(\frac{\alpha\gamma}{\alpha+\gamma}\right)} \exp -\frac{1}{2}\left[\left(\frac{\gamma\alpha}{\gamma+\alpha}\right)y_{t-i}^2\right]$$

So

$$\begin{aligned} f(\varepsilon_{t-i} \mid y_{t-i}) &= \frac{f(y_{t-i} \mid \varepsilon_{t-i})f(\varepsilon_{t-i})}{\int f(y_{t-i} \mid \varepsilon_{t-i})f(\varepsilon_{t-i})d\varepsilon_{t-i}} \\ &= N\left(\left(\frac{\gamma}{\gamma+\alpha}\right)y_{t-i}, (\gamma+\alpha)^{-1}\right) \end{aligned}$$

Therefore

$$E[\varepsilon_{t-i} \mid y_{t-i}] = \left(\frac{\gamma}{\gamma+\alpha}\right)y_{t-i}, \forall i \geq 1$$

We could obtain this result computing

$$\begin{aligned} E[\varepsilon_{t-i} \mid y_{t-i}] &= \frac{cov(\varepsilon_{t-i}, y_{t-i})}{var(y_{t-i})}y_{t-i} \\ &= \frac{cov(\varepsilon_{t-i}, \varepsilon_{t-i}+\eta_{t-i})}{var(\varepsilon_{t-i}+\eta_{t-i})}y_{t-i} \\ &= \left(\frac{\alpha^{-1}}{\alpha^{-1}+\gamma^{-1}}\right)y_{t-i} = \left(\frac{\gamma}{\gamma+\alpha}\right)y_{t-i} \end{aligned}$$

We also use Bayes theorem to obtain $E[\varepsilon_{t-j} \mid y_{t-j}, v_{t-j}]$.

$$\begin{aligned} f(\varepsilon_{t-j} \mid y_{t-j}, v_{t-j}) &= \frac{f(y_{t-j}, v_{t-j}, \varepsilon_{t-j})}{f(y_{t-j}, v_{t-j})} \\ &= \frac{f(y_{t-j}, v_{t-j} \mid \varepsilon_{t-j})f(\varepsilon_{t-j})}{\int f(y_{t-j}, v_{t-j} \mid \varepsilon_{t-j})f(\varepsilon_{t-j})d\varepsilon_{t-j}} \\ &= \frac{f(y_{t-j} \mid \varepsilon_{t-j})f(v_{t-j} \mid \varepsilon_{t-j})f(\varepsilon_{t-j})}{\int f(y_{t-j}, v_{t-j} \mid \varepsilon_{t-j})f(\varepsilon_{t-j})d\varepsilon_{t-j}} \end{aligned}$$

where the last equality holds due to the independence of $\xi_{t-j}(z)$ and η_{t-j}. So

$$
\begin{aligned}
& f(y_{t-j} \mid \varepsilon_{t-j}) f(v_{t-j} \mid \varepsilon_{t-j}) f(\varepsilon_{t-j}) \\
&= d \exp -\frac{1}{2}\left[\left(\frac{(y_{t-j}-\varepsilon_{t-j})^2}{\gamma^{-1}}\right)+\left(\frac{(v_{t-j}-\varepsilon_{t-j})^2}{\beta^{-1}}\right)+\frac{\varepsilon_{t-j}^2}{\alpha^{-1}}\right] \\
&= d \exp -\frac{1}{2}\left[\gamma y_{t-j}^2 - 2\gamma\varepsilon_{t-j} y_{t-j} + \gamma\varepsilon_{t-j}^2 + \beta v_{t-j}^2 - 2\beta\varepsilon_{t-j} v_{t-j} + \beta\varepsilon_{t-j}^2 + \alpha\varepsilon_{t-j}^2\right] \\
&= d \exp -\frac{1}{2}\left[\gamma y_{t-j}^2 + \beta v_{t-j}^2 - 2\varepsilon_{t-j}(\gamma y_{t-j} + \beta v_{t-j}) + (\gamma+\beta+\alpha)\varepsilon_{t-j}^2\right] \\
&= d \exp -\frac{1}{2}\left[\gamma y_{t-j}^2 + \beta v_{t-j}^2 - \frac{(\gamma y_{t-j}+\beta v_{t-j})^2}{\gamma+\beta+\alpha} + (\gamma+\beta+\alpha)\left(\varepsilon_{t-j} - \frac{\gamma y_{t-j}+\beta v_{t-j}}{\gamma+\beta+\alpha}\right)^2\right] \\
&= e \frac{1}{2\pi^{1/2}(\gamma+\beta+\alpha)^{-1/2}} \exp -\frac{1}{2}\left[\frac{\left(\varepsilon_{t-j} - \frac{\gamma y_{t-j}+\beta v_{t-j}}{\gamma+\beta+\alpha}\right)^2}{(\gamma+\beta+\alpha)^{-1}}\right]
\end{aligned}
$$

where

$$
\begin{aligned}
d &= \frac{1}{2\pi^{3/2}(\alpha\beta\gamma)^{-1/2}} \\
e &= \frac{1}{2\pi}\sqrt{\frac{\alpha\beta\gamma}{\gamma+\beta+\alpha}} \exp -\frac{1}{2}\left[\gamma y_{t-j}^2 + \beta v_{t-j}^2 - \frac{(\gamma y_{t-j}+\beta v_{t-j})^2}{\gamma+\beta+\alpha}\right]
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
f(\varepsilon_{t-j} \mid y_{t-j}, v_{t-j}) &= \frac{f(y_{t-j} \mid \varepsilon_{t-j}) f(v_{t-j} \mid \varepsilon_{t-j}) f(\varepsilon_{t-j})}{\int f(y_{t-j}, v_{t-j} \mid \varepsilon_{t-j}) f(\varepsilon_{t-j}) d\varepsilon_{t-j}} \\
&= N\left(\frac{\gamma y_{t-j}+\beta v_{t-j}}{\gamma+\beta+\alpha}, (\gamma+\beta+\alpha)^{-1}\right)
\end{aligned}
$$

and, consequently,

$$
E[\varepsilon_{t-i} \mid y_{t-i}] = \frac{\gamma y_{t-j}+\beta v_{t-j}}{\gamma+\beta+\alpha}
$$

5.2.2. Beliefs

In this appendix we prove Lemma 1. That is, we wants to derive the general formula of the k -th order average expectation $\bar{E}^k[\theta_t]$.

$$
\bar{E}^k[\theta_t] = \lambda \sum_{m=0}^{\infty}(1-\lambda)^m [a_{m,k}\theta_{t-m} + b_{m,k}\theta_{t-m-1}] + \kappa \sum_{m=0}^{\infty}(1-\lambda)^m c_{m,k} y_{t-m}
$$

considering that the weights $(a_{m,k}, b_{m,k}, c_{m,k})$ are recursively defined, for $k \geq 1$, by

$$\begin{bmatrix} a_{m,k+1} \\ b_{m,k+1} \\ b_{m,k+1} \end{bmatrix} = \begin{bmatrix} (1-\delta) \\ \delta \\ \rho \end{bmatrix} [1-(1-\lambda)^m]^k + A_m \begin{bmatrix} a_{m,k} \\ b_{m,k} \\ c_{m,k} \end{bmatrix},$$

where the matrix A_m is given by

$$A_m \equiv \begin{bmatrix} (1-\delta)\left[1-(1-\lambda)^{m+1}\right]+\delta[1-(1-\lambda)^m] & 0 & 0 \\ \delta\left[\left[1-(1-\lambda)^{m+1}\right]-[1-(1-\lambda)^m]\right] & \left[1-(1-\lambda)^{m+1}\right] & 0 \\ \lambda\rho(1-\lambda)^m & 0 & 1 \end{bmatrix} \quad (4.1)$$

and the initial weights are $(a_{m,1}, b_{m,1}, c_{m,1}) \equiv (1-\delta, \delta, \rho)$, and $\rho \equiv 1-\lambda(1-\delta)$.

We start by computing $\bar{E}^1[\theta_t]$ as

$$\begin{aligned}
&\sum_{j=0}^{\infty}\int_{\Lambda_j} E[\bar{E}^0[\theta_t] \mid \Im_{t-j}(z)]dz \\
&= \sum_{j=0}^{\infty}\int_{\Lambda_j} E[\theta_t \mid \Im_{t-j}(z)]dz \\
&= \sum_{j=0}^{\infty}\int_{\Lambda_j}\left[(1-\delta)x_{t-j}(z)+\delta\theta_{t-j-1}+\delta\kappa y_{t-j}+\kappa\textstyle\sum_{i=0}^{j-1} y_{t-i}\right]dz \\
&= \lambda\sum_{j=0}^{\infty}(1-\lambda)^j\left[(1-\delta)\theta_{t-j}+\delta\theta_{t-j-1}+\delta\kappa y_{t-j}+\kappa\textstyle\sum_{i=0}^{j-1} y_{t-i}\right] \\
&= \lambda\sum_{j=0}^{\infty}(1-\lambda)^j[(1-\delta)\theta_{t-j}+\delta\theta_{t-j-1}] \\
&+\kappa\left[\delta\lambda\sum_{j=0}^{\infty}(1-\lambda)^j y_{t-j}+\lambda\sum_{i=0}^{\infty}y_{t-i}\sum_{j=i+1}^{\infty}(1-\lambda)^j\right] \\
&= \lambda\sum_{j=0}^{\infty}(1-\lambda)^j[(1-\delta)\theta_{t-j}+\delta\theta_{t-j-1}]+\rho\kappa\sum_{i=0}^{\infty}(1-\lambda)^i y_{t-i}.
\end{aligned}$$

This expression shows that $(a_{m,1}, b_{m,1}, c_{m,1}) \equiv (1-\delta, \delta, \rho)$. We can use this result to obtain $\bar{E}^2[\theta_t]$ as

$$\begin{aligned}
&\sum_{m=0}^{\infty}\int_{\Lambda_m} E[\bar{E}^1[\theta_t] \mid \Im_{t-m}(z)]dz \\
&= \lambda\sum_{m=0}^{\infty}\int_{\Lambda_m}\sum_{j=0}^{\infty}(1-\lambda)^j E[(1-\delta)\theta_{t-j}+\delta\theta_{t-j-1} \mid \Im_{t-m}(z)]dz+\rho\kappa\sum_{k=0}^{\infty}(1-\lambda)^k y_{t-k}.
\end{aligned}$$

This last equality holds because y_{t-k} belongs to the information set

$\Im_{t-m}(z)$, $\forall k, m$. We know that

$$E[\theta_{t-j} \mid \Im_{t-m}(z)] = \begin{cases} (1-\delta)x_{t-m}(z) + \delta\theta_{t-m-1} + \delta\kappa y_{t-m} + \kappa\sum_{i=j}^{m-1} y_{t-i} & : j \le m \\ \theta_{t-j} & : j > m \end{cases}$$

Using this expression, we can write $\overline{E}^2[\theta_t]$ as

$$\lambda\sum_{m=0}^{\infty}\int_{\Lambda_m}\sum_{j=0}^{m-1}(1-\lambda)^j\{(1-\delta)E[\theta_{t-j} \mid \Im_{t-m}(z)] + \delta E[\theta_{t-j-1} \mid \Im_{t-m}(z)]\}dz$$
$$+\lambda\sum_{m=0}^{\infty}\int_{\Lambda_m}(1-\lambda)^m\{(1-\delta)E[\theta_{t-m} \mid \Im_{t-m}(z)] + \delta\theta_{t-m-1}\}dz$$
$$+\lambda\sum_{m=0}^{\infty}\int_{\Lambda_m}\sum_{j=m+1}^{\infty}(1-\lambda)^j[(1-\delta)\theta_{t-j} + \delta\theta_{t-j-1}]dz + \rho\kappa\sum_{k=0}^{\infty}(1-\lambda)^k y_{t-k}$$
$$= \lambda\sum_{m=0}^{\infty}\int_{\Lambda_m}\sum_{j=0}^{m-1}(1-\lambda)^j\{(1-\delta)x_{t-m}(z) + \delta\theta_{t-m-1}\}dz$$
$$+\lambda\kappa\sum_{m=0}^{\infty}\int_{\Lambda_m}\sum_{j=0}^{m-1}(1-\lambda)^j\left[\sum_{i=j}^{m-1}[(1-\delta)y_{t-i} + \delta y_{t-i-1}]\right]dz$$
$$+\lambda\sum_{m=0}^{\infty}\int_{\Lambda_m}(1-\lambda)^m[(1-\delta)[(1-\delta)x_{t-m}(z) + \delta\theta_{t-m-1} + \delta\kappa y_{t-m}] + \delta\theta_{t-m-1}]dz$$
$$+\lambda\sum_{m=0}^{\infty}\int_{\Lambda_m}\sum_{j=m+1}^{\infty}(1-\lambda)^j[(1-\delta)\theta_{t-j} + \delta\theta_{t-j-1}]dz + \rho\kappa\sum_{k=0}^{\infty}(1-\lambda)^k y_{t-k}$$
$$= \lambda^2\sum_{m=0}^{\infty}(1-\lambda)^m[(1-\delta)\theta_{t-m} + \delta\theta_{t-m-1}]\sum_{j=0}^{m-1}(1-\lambda)^j$$
$$+\kappa\lambda^2\sum_{m=0}^{\infty}(1-\lambda)^m\sum_{i=0}^{m-1}[(1-\delta)y_{t-i} + \delta y_{t-i-1}]\sum_{j=0}^{i}(1-\lambda)^j$$
$$+\lambda^2\sum_{m=0}^{\infty}(1-\lambda)^{2m}\left[(1-\delta)^2\theta_{t-m} + \left[1-(1-\delta)^2\right]\theta_{t-m-1} + \delta(1-\delta)\kappa y_{t-m}\right]$$
$$+\lambda^2\sum_{j=1}^{\infty}(1-\lambda)^j[(1-\delta)\theta_{t-j} + \delta\theta_{t-j-1}]\sum_{m=0}^{j-1}(1-\lambda)^m + \rho\kappa\sum_{k=0}^{\infty}(1-\lambda)^k y_{t-k}$$
$$= \lambda\sum_{m=0}^{\infty}(1-\lambda)^m[(1-\delta)\theta_{t-m} + \delta\theta_{t-m-1}][1-(1-\lambda)^m]$$
$$+\kappa\lambda\sum_{i=0}^{\infty}[(1-\delta)y_{t-i} + \delta y_{t-i-1}]\left[1-(1-\lambda)^{i+1}\right]\sum_{m=i+1}^{\infty}(1-\lambda)^m$$
$$+\lambda^2\sum_{m=0}^{\infty}(1-\lambda)^{2m}\left[(1-\delta)^2\theta_{t-m} + \left[1-(1-\delta)^2\right]\theta_{t-m-1} + \delta(1-\delta)\kappa y_{t-m}\right]$$
$$+\lambda\sum_{j=1}^{\infty}(1-\lambda)^j[(1-\delta)\theta_{t-j} + \delta\theta_{t-j-1}]\left[1-(1-\lambda)^j\right] + \rho\kappa\sum_{k=0}^{\infty}(1-\lambda)^k y_{t-k}$$

$$
\begin{aligned}
&= \lambda \sum_{m=0}^{\infty} (1-\lambda)^m 2[1-(1-\lambda)^m][(1-\delta)\theta_{t-m} + \delta\theta_{t-m-1}] \\
&\quad + \lambda^2 \sum_{m=0}^{\infty} (1-\lambda)^{2m} \left[(1-\delta)^2 \theta_{t-m} + \left[1-(1-\delta)^2 \right] \theta_{t-m-1} \right] \\
&\quad + \kappa \sum_{i=0}^{\infty} (1-\lambda)^i y_{t-i} \left\{ \delta\left[1-\rho(1-\lambda)^i \right] + (1-\delta)\left[1-\rho(1-\lambda)^{i+1} \right] + 2\rho - 1 \right\}
\end{aligned}
$$

We can write this expression as

$$
\bar{E}^2[\theta_t] = \lambda \textstyle\sum_{j=0}^{\infty} (1-\lambda)^j [a_{j,2}\theta_{t-j} + b_{j,2}\theta_{t-j-1}] + \kappa \textstyle\sum_{j=0}^{\infty} (1-\lambda)^j c_{j,2} y_{t-j}
$$

where

$$
\begin{aligned}
\alpha_{j,2} &= (1-\delta^2)\left[1-(1-\lambda)^j \right] + (1-\delta)^2 \left[1-(1-\lambda)^{j+1} \right] \\
b_{j,2} &= \delta^2 \left[1-(1-\lambda)^j \right] + \left[1-(1-\delta)^2 \right]\left[1-(1-\lambda)^{j+1} \right] \\
c_{j,2} &= \delta\left[1-\rho(1-\lambda)^j \right] + (1-\delta)\left[1-\rho(1-\lambda)^{j+1} \right] + 2\rho - 1
\end{aligned}
$$

These expressions shows that we have

$$
\begin{bmatrix} a_{j,2} \\ b_{j,2} \\ c_{j,2} \end{bmatrix} = \begin{bmatrix} (1-\delta) \\ \delta \\ \rho \end{bmatrix} [1-(1-\lambda)^m] + A_m \begin{bmatrix} (1-\delta) \\ \delta \\ \rho \end{bmatrix}
$$

where the matrix A_m is given by (4.1). Note also that we can write $a_{j,2} + b_{j,2}$ as

$$
a_{j,k} + b_{j,k} = \sum_{n=0}^{k-1} \left[1-(1-\lambda)^j \right]^n \left[1-(1-\lambda)^{j+1} \right]^{k-1-n}, \tag{4.2}
$$

for $k = 2$. We use induction to prove this formula and to obtain the general case. Suppose that (2.8) holds for k. Assume that

$$
\bar{E}^k[\theta_t] = \lambda \sum_{j=0}^{\infty} (1-\lambda)^j [a_{j,k}\theta_{t-j} + b_{j,k}\theta_{t-j-1}] + \kappa \sum_{j=0}^{\infty} (1-\lambda)^j c_{j,k} y_{t-j}
$$

Then

$$
a_{j,k+1} + b_{j,k+1} = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_{j,k+1} \\ b_{j,k+1} \\ c_{j,k+1} \end{bmatrix}
$$

$$= \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} (1-\delta) \\ \delta \\ \rho \end{bmatrix} \left[1-(1-\lambda)^j \right] + \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} A_j \begin{bmatrix} a_{j,k} \\ b_{j,k} \\ c_{j,k} \end{bmatrix}$$

$$= \left[1-(1-\lambda)^j \right]^k + \left[1-(1-\lambda)^{j+1} \right](a_{j,k}+b_{j,k})$$

$$= \left[1-(1-\lambda)^j \right]^k + \sum_{n=0}^{k-1} \left[1-(1-\lambda)^j \right]^n \left[1-(1-\lambda)^{j+1} \right]^{k-n}$$

$$= \sum_{n=0}^{k} \left[1-(1-\lambda)^j \right]^n \left[1-(1-\lambda)^{j+1} \right]^{k-n}$$

This proves that (4.2) holds forn any k. Therefore, we have that

$$\sum_{j=0}^{m-1} (1-\lambda)^j (a_{j,k}+b_{j,k})$$

$$= \sum_{j=0}^{m-1} (1-\lambda)^j \sum_{n=0}^{k-1} \left[1-(1-\lambda)^j \right]^n \left[1-(1-\lambda)^{j+1} \right]^{k-1-n}$$

$$= \sum_{j=0}^{m-1} (1-\lambda)^j \left[1-(1-\lambda)^{j+1} \right]^{k-1} \sum_{n=0}^{k-1} \left[\frac{1-(1-\lambda)^j}{1-(1-\lambda)^{j+1}} \right]^n$$

$$= \frac{1}{\lambda} \sum_{j=0}^{m-1} \left\{ \left[1-(1-\lambda)^{j+1} \right]^k - \left[1-(1-\lambda)^j \right]^k \right\}$$

$$= \frac{1}{\lambda} \left\{ [1-(1-\lambda)^m]^k - \left[1-(1-\lambda)^0 \right]^k \right\} = \frac{1}{\lambda} [1-(1-\lambda)^m]^k$$

With this result, we are now able to obtain $\bar{E}^{k+1}[\theta_t]$, assuming that $\bar{E}^k[\theta_t]$ is given by (2.6).

$$\bar{E}^{k+1}[\theta_t]$$

$$= \sum_{m=0}^{\infty} \int_{\Lambda_m} E[\bar{E}^k[\theta_t] \mid \Im_{t-m}(z)] dz$$

$$= \sum_{m=0}^{\infty} \int_{\Lambda_m} E\left[\lambda \sum\nolimits_{j=0}^{\infty} (1-\lambda)^j [a_{j,k}\theta_{t-j} + b_{j,k}\theta_{t-j-1}] + \kappa \sum_{j=0}^{\infty} (1-\lambda)^j c_{j,k} y_{t-j} \mid \Im_{t-m}(z) \right] dz$$

$$= \lambda \sum_{m=0}^{\infty} \int_{\Lambda_m} \sum_{j=0}^{m-1} (1-\lambda)^j \{ a_{j,k} E[\theta_{t-j} \mid \Im_{t-m}(z)] + b_{j,k} E[\theta_{t-j-1} \mid \Im_{t-m}(z)] \} dz$$

$$+ \lambda \sum_{m=0}^{\infty} \int_{\Lambda_m} (1-\lambda)^m \{ a_{m,k} E[\theta_{t-m} \mid \Im_{t-m}(z)] + b_{m,k}\theta_{t-m-1} \} dz$$

$$+ \lambda \sum_{m=0}^{\infty} \int_{\Lambda_m} \sum_{j=m+1}^{\infty} (1-\lambda)^j [a_{j,k}\theta_{t-j} + b_{j,k}\theta_{t-j-1}] dz + \kappa \sum_{j=0}^{\infty} (1-\lambda)^j c_{j,k} y_{t-j}$$

$$= \lambda \sum_{m=0}^{\infty} \int_{\Lambda_m} \sum_{j=0}^{m-1} (1-\lambda)^j (a_{j,k} + b_{j,k})[(1-\delta)x_{t-m}(z) + \delta\theta_{t-m-1} + \delta\kappa y_{t-m}]dz$$

$$+ \lambda \sum_{m=0}^{\infty} \int_{\Lambda_m} \sum_{j=0}^{m-1} (1-\lambda)^j \left\{ (a_{j,k} + b_{j,k})\kappa \sum_{i=j+1}^{m-1} y_{t-i} + \kappa a_{j,k} y_{t-j} \right\} dz$$

$$+ \lambda \sum_{m=0}^{\infty} \int_{\Lambda_m} (1-\lambda)^m [a_{m,k}[(1-\delta)x_{t-m}(z) + \delta\theta_{t-m-1} + \delta\kappa y_{t-m}] + b_{m,k}\theta_{t-m-1}]dz$$

$$+ \lambda \sum_{m=0}^{\infty} \int_{\Lambda_m} \sum_{j=m+1}^{\infty} (1-\lambda)^j [a_{j,k}\theta_{t-j} + b_{j,k}\theta_{t-j-1}]dz + \kappa \sum_{j=0}^{\infty} (1-\lambda)^j c_{j,k} y_{t-j}$$

$$= \lambda^2 \sum_{m=0}^{\infty} (1-\lambda)^m [(1-\delta)\theta_{t-m} + \delta\theta_{t-m-1} + \delta\kappa y_{t-m}] \sum_{j=0}^{m-1} (1-\lambda)^j (a_{j,k} + b_{j,k})$$

$$+ \kappa\lambda^2 \sum_{m=0}^{\infty} (1-\lambda)^m \sum_{i=1}^{m-1} y_{t-i} \sum_{j=0}^{i-1} (1-\lambda)^j (a_{j,k} + b_{j,k})$$

$$+ \kappa\lambda^2 \sum_{j=0}^{\infty} (1-\lambda)^j a_{j,k} y_{t-j} \sum_{m=j+1}^{\infty} (1-\lambda)^m$$

$$+ \lambda^2 \sum_{m=0}^{\infty} (1-\lambda)^{2m} [a_{m,k}[(1-\delta)\theta_{t-m} + \delta\kappa y_{t-m}] + (a_{m,k}\delta + b_{m,k})\theta_{t-m-1}]$$

$$+ \lambda^2 \sum_{j=1}^{\infty} (1-\lambda)^j [a_{j,k}\theta_{t-j} + b_{j,k}\theta_{t-j-1}] \sum_{m=0}^{j-1} (1-\lambda)^m + \kappa \sum_{j=0}^{\infty} (1-\lambda)^j c_{j,k} y_{t-j}$$

$$= \lambda \sum_{m=0}^{\infty} (1-\lambda)^m [(1-\delta)\theta_{t-m} + \delta\theta_{t-m-1} + \delta\kappa y_{t-m}][1 - (1-\lambda)^m]^k$$

$$+ \kappa\lambda \sum_{i=1}^{\infty} y_{t-i} \left[1 - (1-\lambda)^i \right]^k \sum_{m=i+1}^{\infty} (1-\lambda)^m$$

$$+ \kappa\lambda \sum_{j=0}^{\infty} (1-\lambda)^{2j+1} a_{j,k} y_{t-j}$$

$$+ \lambda^2 \sum_{m=0}^{\infty} (1-\lambda)^{2m} [a_{m,k}[(1-\delta)\theta_{t-m} + \delta\kappa y_{t-m}] + (a_{m,k}\delta + b_{m,k})\theta_{t-m-1}]$$

$$+ \lambda \sum_{j=1}^{\infty} (1-\lambda)^j [a_{j,k}\theta_{t-j} + b_{j,k}\theta_{t-j-1}] \left[1 - (1-\lambda)^j \right]$$

$$+ \kappa \sum_{j=0}^{\infty} (1-\lambda)^j c_{j,k} y_{t-j}$$

$$= \lambda \sum_{m=0}^{\infty} (1-\lambda)^m [1 - (1-\lambda)^m]^k [(1-\delta)\theta_{t-m} + \delta\theta_{t-m-1}]$$

$$+ \lambda^2 \sum_{m=0}^{\infty} (1-\lambda)^{2m} [a_{m,k}(1-\delta)\theta_{t-m} + (a_{m,k}\delta + b_{m,k})\theta_{t-m-1}]$$

$$+ \lambda \sum_{m=0}^{\infty} (1-\lambda)^m [a_{m,k}\theta_{t-m} + b_{m,k}\theta_{t-m-1}][1 - (1-\lambda)^m]$$

$$+ \kappa \sum_{m=0}^{\infty} (1-\lambda)^m y_{t-m} \left\{ \rho[1 - (1-\lambda)^m]^k + \lambda\rho(1-\lambda)^m a_{m,k} + c_{m,k} \right\}$$

We can rewrite this expression as

$$\bar{E}^{k+1}[\theta_t] = \sum_{m=0}^{\infty}(1-\lambda)^m \{\lambda[a_{m,k+1}\theta_{t-m} + b_{m,k+1}\theta_{t-m-1}] + \kappa c_{m,k+1} y_{t-j}\},$$

where

$$\begin{aligned}
a_{m,k+1} &\equiv (1-\delta)[1-(1-\lambda)^m]^k + [(1-\delta)\lambda(1-\lambda)^m + [1-(1-\lambda)^m]]a_{m,k} \\
&= (1-\delta)[1-(1-\lambda)^m]^k \\
&\quad + \left[(1-\delta)\left[1-(1-\lambda)^{m+1}\right] + \delta[1-(1-\lambda)^m]\right]a_{m,k} \\
b_{m,k+1} &\equiv \delta[1-(1-\lambda)^m]^k + \delta\lambda(1-\lambda)^m a_{m,k} + [\lambda(1-\lambda)^m + [1-(1-\lambda)^m]]b_{m,k} \\
&= \delta[1-(1-\lambda)^m]^k \\
&\quad + \delta\left[\left[1-(1-\lambda)^{m+1}\right] - [1-(1-\lambda)^m]\right]a_{m,k} + \left[1-(1-\lambda)^{m+1}\right]b_{m,k} \\
c_{m,k+1} &= \rho[1-(1-\lambda)^m]^k + \lambda\rho(1-\lambda)^m a_{m,k} + c_{m,k}
\end{aligned}$$

since $\lambda(1-\lambda)^m = \left[1-(1-\lambda)^{m+1}\right] - \left[1-(1-\lambda)^m\right]$.

Rewriting these weights in matrix format, we obtain

$$\begin{bmatrix} a_{m,k=1} \\ b_{m,k+1} \\ c_{m,k+1} \end{bmatrix} = \begin{bmatrix} (1-\delta) \\ \delta \\ \rho \end{bmatrix}[1-(1-\lambda)^m]^k + A_m \begin{bmatrix} a_{m,k} \\ b_{m,k} \\ c_{m,k} \end{bmatrix},$$

where the matrix A_m is given by (4.1), which is exactly our result.

5.2.3. Linear Equilibrium

In this appendix we prove that the linear equilibrium is the unique equilibrium of the game. We departure from the equilibrim expression for $P_t = (1-r)\sum_{k=1}^{\infty} r^{k-1}\bar{E}^k[\theta_t]$ to obtain $P_t = \sum_{k=0}^{\infty} c_k \theta_{t-k} + \sum_{k=1}^{\infty} d_k y_{t-k}$. Plugging (2.6) into (2.4), we obtain

$$\begin{aligned}
P_t &= (1-r)\sum_{k=1}^{\infty} r^{k-1}\bar{E}^k[\theta_t] \\
&= (1-r)\sum_{k=1}^{\infty} r^{k-1}\sum_{m=0}^{\infty}(1-\lambda)^m \begin{bmatrix} \lambda\theta_{t-m} & \lambda\theta_{t-m-1} & \kappa y_{t-m} \end{bmatrix} \begin{bmatrix} a_{m,k} \\ b_{m,k} \\ c_{m,k} \end{bmatrix}.
\end{aligned}$$

We write (2.6) as a function of the initial parameters, we write it

$$\begin{bmatrix} a_{m,k} \\ b_{m,k} \\ c_{m,k} \end{bmatrix} = \sum_{i=0}^{k-1} A_m^i B_m^{k-1-i} \begin{bmatrix} (1-\delta) \\ \delta \\ \rho \end{bmatrix} = [1-(1-\lambda)^m]^{k-1} \sum_{i=0}^{k-1} C_m^i \begin{bmatrix} (1-\delta) \\ \delta \\ \rho \end{bmatrix}$$

where $B_m = \left[1-(1-\lambda)^m\right] I$, $C_m = \left[1-(1-\lambda)^m\right]^{-1} A_m$, and I is the identity matrix of order three. Using this expression and defining the column vector of inicial parameters, $V_1 \equiv \left[(1-\delta) \quad \delta \quad \rho\right]^T$, we can express P_t as

$$\begin{aligned}
&(1-r)\sum_{m=0}^{\infty}\sum_{k=1}^{\infty}\sum_{i=0}^{k-1}(r[1-(1-\lambda)^m])^{k-1}(1-\lambda)^m \begin{bmatrix} \lambda\theta_{t-m} & \lambda\theta_{t-m-1} & \kappa y_{t-m} \end{bmatrix} C_m^i V_1 \\
&= (1-r)\sum_{m=0}^{\infty}\sum_{i=0}^{\infty}\left(\sum_{k=i+1}^{\infty}(r[1-(1-\lambda)^m])^{k-1}\right)(1-\lambda)^m \begin{bmatrix} \lambda\theta_{t-m} & \lambda\theta_{t-m-1} & \kappa y_{t-m} \end{bmatrix} C_m^i V_1 \\
&= (1-r)\sum_{m=0}^{\infty}\left(\frac{(r[1-(1-\lambda)^m])^i}{1-r[1-(1-\lambda)^m]}\right)(1-\lambda)^m \begin{bmatrix} \lambda\theta_{t-m} & \lambda\theta_{t-m-1} & \kappa y_{t-m} \end{bmatrix}\left(\sum_{i=0}^{\infty} C_m\right) V_1 \\
&= (1-r)\sum_{m=0}^{\infty}\left(\frac{(1-\lambda)^m}{1-r[1-(1-\lambda)^m]}\right)\begin{bmatrix} \lambda\theta_{t-m} & \lambda\theta_{t-m-1} & \kappa y_{t-m} \end{bmatrix}\left(\sum_{i=0}^{\infty}(rA_m)^i\right) V_1 \\
&= (1-r)\sum_{m=0}^{\infty}\left(\frac{(1-\lambda)^m}{1-r[1-(1-\lambda)^m]}\right)\begin{bmatrix} \lambda\theta_{t-m} & \lambda\theta_{t-m-1} & \kappa y_{t-m} \end{bmatrix}(I-rA_m)^{-1} V_1
\end{aligned}$$

Computing $(I-rA_m)^{-1}$, we obtain

$$\begin{bmatrix}
\frac{1}{1-r\left[(1-\delta)\left[1-(1-\lambda)^{m+1}\right]+\delta[1-(1-\lambda)^m]\right]} & 0 & 0 \\
\frac{r\delta\left[\left[1-(1-\lambda)^{m+1}\right]-[1-(1-\lambda)^m]\right]}{\left[1-r\left[(1-\delta)\left[1-(1-\lambda)^{m+1}\right]+\delta[1-(1-\lambda)^m]\right]\right]\left[1-r\left[1-(1-\lambda)^{m+1}\right]\right]} & \frac{1}{\left[1-r\left[1-(1-\lambda)^{m+1}\right]\right]} & 0 \\
\frac{r\lambda\rho(1-\lambda)^m}{1-r\left[(1-\delta)\left[1-(1-\lambda)^{m+1}\right]+\delta[1-(1-\lambda)^m]\right](1-r)} & 0 & \frac{1}{1-r}
\end{bmatrix}.$$

Therefore,

$$(I-rA_m)^{-1}V_1 = \begin{bmatrix}
\frac{(1-\delta)\left[1-r\left[1-(1-\lambda)^{m+1}\right]\right]}{\left[1-r\left[(1-\delta)\left[1-(1-\lambda)^{m+1}\right]+\delta[1-(1-\lambda)^m]\right]\right]\left[1-r\left[1-(1-\lambda)^{m+1}\right]\right]} \\
\frac{\delta[1-r[1-(1-\lambda)^m]]}{\left[1-r\left[(1-\delta)\left[1-(1-\lambda)^{m+1}\right]+\delta[1-(1-\lambda)^m]\right]\right]\left[1-r\left[1-(1-\lambda)^{m+1}\right]\right]} \\
\left[\frac{[1-r[1-(1-\lambda)^m]]\left[1-r\left[1-(1-\lambda)^{m+1}\right]\right]}{\left[1-r\left[(1-\delta)\left[1-(1-\lambda)^{m+1}\right]+\delta[1-(1-\lambda)^m]\right]\right]\left[1-r\left[1-(1-\lambda)^{m+1}\right]\right]}\right]\frac{\rho}{1-r}
\end{bmatrix}$$

Finally, we use this expression to compute P_t as

$$P_t = (1-r)\sum_{m=0}^{\infty}\left(\frac{(1-\lambda)^m}{1-r[1-(1-\lambda)^m]}\right)\left[\begin{array}{ccc}\lambda\theta_{t-m} & \lambda\theta_{t-m-1} & \kappa y_{t-m}\end{array}\right]$$

$$\times\left[\begin{array}{c}\frac{(1-\delta)\left[1-r\left[1-(1-\lambda)^{m+1}\right]\right]}{\left[1-r\left[(1-\delta)\left[1-(1-\lambda)^{m+1}\right]+\delta[1-(1-\lambda)^m]\right]\right]\left[1-r\left[1-(1-\lambda)^{m+1}\right]\right]}\\ \frac{\delta[1-r[1-(1-\lambda)^m]]}{\left[1-r\left[(1-\delta)\left[1-(1-\lambda)^{m+1}\right]+\delta[1-(1-\lambda)^m]\right]\right]\left[1-r\left[1-(1-\lambda)^{m+1}\right]\right]}\\ \left[\frac{[1-r[1-(1-\lambda)^m]]\left[1-r\left[1-(1-\lambda)^{m+1}\right]\right]}{\left[1-r\left[(1-\delta)\left[1-(1-\lambda)^{m+1}\right]+\delta[1-(1-\lambda)^m]\right]\right]\left[1-r\left[1-(1-\lambda)^{m+1}\right]\right]}\right]\frac{\rho}{1-r}\end{array}\right]$$

$$= \sum_{m=0}^{\infty} K_m[(1-\Delta_m)\theta_{t-m} + \Delta_m\theta_{t-m-1}] + \sum_{m=0}^{\infty} d_m y_{t-m}$$

where

$$K_m \equiv \frac{(1-r)\lambda(1-\lambda)^m}{(1-r[1-(1-\lambda)^m])\left(1-r\left[1-(1-\lambda)^{m+1}\right]\right)},$$

$$\Delta_m \equiv \frac{\delta[1-r[1-(1-\lambda)^m]]}{\left[1-r\left[(1-\delta)\left[1-(1-\lambda)^{m+1}\right]+\delta[1-(1-\lambda)^m]\right]\right]},$$

$$d_m \equiv \kappa\left[\frac{\rho(1-\lambda)^m}{\left[1-r\left[(1-\delta)\left[1-(1-\lambda)^{m+1}\right]+\delta[1-(1-\lambda)^m]\right]\right]}\right].$$

As

$$1-r\left[(1-\delta)\left[1-(1-\lambda)^{m+1}\right]+\delta[1-(1-\lambda)^m]\right] = 1-r[1-\rho(1-\lambda)^m]$$

and

$$c_0 = K_0(1-\Delta_0),$$
$$c_k = K_{m-1}\Delta_{m-1} + K_m(1-\Delta_m),\ m \geq 1,$$

we have our result.

5.2.4.
Matching coefficients

In this appendix we compute the equilibrium P_t, assuming that it is a linear function of (Θ_t, Y_t), i.e.

$$P_t = \textstyle\sum_{j=0}^{\infty} c_j\theta_{t-j} + \sum_{j=0}^{\infty} d_j y_{t-j}. \tag{4.3}$$

First, remember that

$$E[\theta_{t-m} \mid \Im_{t-j}(z)] = \begin{cases} (1-\delta)x_{t-j}(z) + \delta\theta_{t-j-1} + \delta\kappa y_{t-j} + \kappa \sum_{i=m}^{j-1} y_{t-i} & ,m \le j \\ \theta_{t-m} & ,m > j \end{cases} .$$

We combine (2.1) and (2.2) to obtain optimal price for a firm z that last updated information at $t-j$ as, $p_{j,t}(z)$, as

$$\begin{aligned} p_{j,t}(z) &= E[(1-r)\theta_t + rP_t \mid \Im_{t-j}(z)] \\ &= (1-r)E[\theta_t \mid \Im_{t-j}(z)] + r\textstyle\sum_{m=0}^{j} c_m E[\theta_{t-m} \mid \Im_{t-j}(z)] \\ &+ r\textstyle\sum_{m=j+1}^{\infty} c_m \theta_{t-m} + r\sum_{m=0}^{\infty} d_m y_{t-m}. \end{aligned}$$

This last equality holds because $\Im_{t-j}(z)$ encompasses Θ_{t-j-1} and Y_t, meaning that firm z knows θ_{t-m}, for $m > j$, and y_{t-m}. We use (2.5) to obtain $p_{j,t}(z)$ as

$$\begin{aligned} p_{j,t}(z) &= (1-r)\left[(1-\delta)x_{t-j}(z) + \delta\theta_{t-j-1} + \delta\kappa y_{t-j} + \kappa \sum_{i=0}^{j-1} y_{t-i} \right] \\ &+ r\sum_{m=0}^{j} c_m \left[(1-\delta)x_{t-j}(z) + \delta\theta_{t-j-1} + \delta\kappa y_{t-j} + \kappa \sum_{i=m}^{j-1} y_{t-i} \right] \\ &+ r\sum_{m=j+1}^{\infty} c_m \theta_{t-m} + r\sum_{m=0}^{\infty} d_m y_{t-m} \\ &= (1-r)\left[(1-\delta)x_{t-j}(z) + \delta\theta_{t-j-1} + \delta\kappa y_{t-j} + \kappa \sum_{i=0}^{j-1} y_{t-i} \right] \\ &+ r[(1-\delta)x_{t-j}(z) + \delta\theta_{t-j-1} + \delta\kappa y_{t-j}]\left(\sum_{m=0}^{j} c_m \right) + r\kappa \sum_{i=0}^{j-1} y_{t-i}\left(\sum_{m=0}^{i} c_m \right) \\ &+ r\sum_{m=j+1}^{\infty} c_m \theta_{t-m} + r\sum_{m=0}^{\infty} d_m y_{t-m} \\ &= [1 - r(1-C_j)][(1-\delta)x_{t-j}(z) + \delta\theta_{t-j-1} + \delta\kappa y_{t-j}] \\ &+ \kappa \sum_{k=0}^{j-1}[1 - r(1-C_k)]y_{t-k} + r\sum_{m=j+1}^{\infty} c_m \theta_{t-m} + r\sum_{m=0}^{\infty} d_m y_{t-m} \end{aligned}$$

where

$$C_m \equiv \textstyle\sum_{j=0}^{m} c_j.$$

As a result, the price level P_t is writen as

$$\int p_t(z)dz$$

$$= \lambda \sum_{m=0}^{\infty}(1-\lambda)^m \left[\begin{array}{c} (1-r)\left[(1-\delta)\theta_{t-m} + \delta\theta_{t-m-1} + \delta\kappa y_{t-m} + \kappa \sum_{i=1}^{m} y_{t-m+i} \right] \\ +r\sum_{j=0}^{m} c_j \left[(1-\delta)\theta_{t-m} + \delta\theta_{t-m-1} + \delta\kappa y_{t-m} + \kappa \sum_{i=1}^{m-j} y_{t-m+i} \right] \\ +r\sum_{j=m+1}^{\infty} c_j\theta_{t-j} + r\sum_{j=0}^{\infty} d_j y_{t-j} \end{array} \right]$$

Comparing this solution with the proposed one, we obtain

$$\begin{aligned} &\sum_{j=0}^{\infty} c_j\theta_{t-j} + \sum_{j=0}^{\infty} d_j y_{t-j} \\ &= (1-r)(1-\delta)\lambda \sum_{m=0}^{\infty}(1-\lambda)^m \theta_{t-m} + (1-r)\delta\lambda \sum_{m=0}^{\infty}(1-\lambda)^m \theta_{t-m-1} \\ &+ r(1-\delta)\lambda \sum_{m=0}^{\infty}(1-\lambda)^m C_m \theta_{t-m} \\ &+ \lambda r\delta \sum_{m=0}^{\infty}(1-\lambda)^m C_m \theta_{t-m-1} + r\sum_{j=1}^{\infty} c_j\theta_{t-j} - r\sum_{j=1}^{\infty} c_j(1-\lambda)^j\theta_{t-j} \\ &+ (1-r)\lambda\delta\kappa \sum_{m=0}^{\infty}(1-\lambda)^m y_{t-m} + (1-r)\kappa \sum_{i=0}^{\infty}(1-\lambda)^{i+1} y_{t-i} \\ &+ r\lambda\delta\kappa \sum_{m=0}^{\infty}(1-\lambda)^m C_m y_{t-m} + \kappa r\sum_{i=0}^{\infty} C_i(1-\lambda)^{i+1} y_{t-i} + r\sum_{j=0}^{\infty} d_j y_{t-j} \end{aligned}$$

Matching coefficients of θ_{t-j}, we obtain

$$c_0 = (1-r)(1-\delta)\lambda + r(1-\delta)\lambda c_0$$

and, $\forall j \geq 1$,

$$\begin{aligned} c_j &= (1-r)(1-\delta)\lambda(1-\lambda)^j + (1-r)\delta\lambda(1-\lambda)^{j-1} \\ &+ r(1-\delta)\lambda(1-\lambda)^j C_j + \lambda r\delta(1-\lambda)^{j-1} C_{j-1} \\ &+ rc_j - rc_j(1-\lambda). \end{aligned}$$

Solving recursively this equations we obtain

$$\begin{aligned} c_0 &\equiv \frac{(1-r)(1-\rho)}{1-r(1-\rho)} = \left(\frac{1-r}{r}\right)\left[\frac{1}{1-r(1-\rho)} - 1\right] \\ c_j &= \left(\frac{1-r}{r}\right)\left[\frac{1}{\left[1-r\left[1-\rho(1-\lambda)^j\right]\right]} - \frac{1}{\left[1-r\left[1-\rho(1-\lambda)^{j-1}\right]\right]}\right], j > 1 \end{aligned} \tag{4.4}$$

where $\rho = 1-\lambda(1-\delta)$.

This result show us that

$$\begin{aligned}C_m &\equiv c_0 + \textstyle\sum_{j=1}^{m} c_j \\ &= \frac{(1-r)(1-\rho)}{1-r(1-\rho)} + \left(\frac{1-r}{r}\right)\frac{1}{[1-r[1-\rho(1-\lambda)^m]]} - \left(\frac{1-r}{r}\right)\frac{1}{[1-r(1-\rho)]} \\ &= \frac{(1-r)}{[1-r(1-\rho)]}\left[\frac{r(1-\rho)-1}{r}\right] + \left(\frac{1-r}{r}\right)\left[\frac{1}{1-r[1-\rho(1-\lambda)^m]}\right] \\ &= \left(\frac{1-r}{r}\right)\left[\frac{1}{1-r[1-\rho(1-\lambda)^m]} - 1\right]\end{aligned}$$

Although this solution considers $m > 0$, it also holds for the case $m = 0$. Matching coefficients of y_{t-j}, we have, $\forall j \geq 0$,

$$\begin{aligned}d_j &= (1-r)\lambda\delta\kappa(1-\lambda)^j + (1-r)\kappa(1-\lambda)^{j+1} \\ &\quad + r\lambda\delta\kappa(1-\lambda)^j C_j + \kappa r C_j(1-\lambda)^{j+1} + r d_j.\end{aligned}$$

Therefore, we have that

$$(1-r)d_j = [\lambda\delta\kappa + \kappa(1-\lambda)](1-\lambda)^j[(1-r) + rC_j]$$

Using the solution we found for C_j, we obtain

$$d_j = (1-\lambda)^j\left[\frac{\lambda\delta\kappa + \kappa(1-\lambda)}{1-r\left[1-\rho(1-\lambda)^j\right]}\right]. \tag{4.5}$$

In summary, the equilibrium price level is given by (4.3), where the coefficients (c_j, d_j) is given by (4.4) and (4.5).

5.2.5. Prices

In this appendix we write $p_t\left(x_{t-j}, \Theta_{t-j-1}, Y_t\right)$ as a function of independent shocks.

$$\begin{aligned}&p_t(x_{t-j}, \Theta_{t-j-1}, Y_t) \\ &= \Omega_j\{(1-\delta)x_{t-j}(z) + \delta\theta_{t-j-1} + \delta\kappa y_{t-j}\} \\ &\quad + \kappa\textstyle\sum_{k=0}^{j-1}\Omega_k y_{t-k} \\ &\quad + \textstyle\sum_{m=j+1}^{\infty}(\Omega_m - \Omega_{m-1})\theta_{t-m} + r\textstyle\sum_{m=0}^{\infty} d_m y_{t-m} \\ &= \Omega_j\{(1-\delta)[\theta_{t-j-1} + \varepsilon_{t-j} + \xi_{t-j}(z)] + \delta\theta_{t-j-1} + \delta\kappa(\varepsilon_{t-j} + \eta_{t-j})\} \\ &\quad + \kappa\textstyle\sum_{k=0}^{j-1}\Omega_k(\varepsilon_{t-k} + \eta_{t-k}) \\ &\quad - \Omega_j\theta_{t-j-1} + \textstyle\sum_{m=j+1}^{\infty}\Omega_m\varepsilon_{t-m} + \kappa\textstyle\sum_{m=0}^{\infty}\frac{r\rho(1-\lambda)^m}{1-r}\Omega_m(\varepsilon_{t-m} + \eta_{t-m})\end{aligned}$$

$$= \Omega_j\{(1-\delta)[\varepsilon_{t-j} + \xi_{t-j}(z)] + \delta\kappa(\varepsilon_{t-j} + \eta_{t-j})\} + \frac{r\rho(1-\lambda)^j}{1-r}\Omega_j(\varepsilon_{t-j} + \eta_{t-j})$$
$$+ \kappa \sum_{m=0}^{\infty}\left[\frac{r\rho(1-\lambda)^m}{1-r} + 1\right]\Omega_m(\varepsilon_{t-m} + \eta_{t-m})$$
$$+ \sum_{m=j+1}^{\infty}\Omega_m\varepsilon_{t-m} + \kappa\sum_{m=j+1}^{\infty}\frac{r\rho(1-\lambda)^m}{1-r}\Omega_m(\varepsilon_{t-m} + \eta_{t-m})$$
$$= \Omega_j\{(1-\delta)[\varepsilon_{t-j} + \xi_{t-j}(z)] + \delta\kappa(\varepsilon_{t-j} + \eta_{t-j})\} + \left[\Omega_j^{-1} - 1\right]\Omega_j(\varepsilon_{t-j} + \eta_{t-j})$$
$$+ \kappa\sum_{m=0}^{\infty}\Omega_m^{-1}\Omega_m(\varepsilon_{t-m} + \eta_{t-m})$$
$$+ \sum_{m=j+1}^{\infty}\Omega_m\varepsilon_{t-m} + \kappa\sum_{m=j+1}^{\infty}[\Omega_m^{-1} - 1]\Omega_m(\varepsilon_{t-m} + \eta_{t-m})$$
$$= +\Omega_j(1-\delta)\xi_{t-j}(z) + [1 - \delta\Omega_j(1-\kappa)]\varepsilon_{t-j} + [1 - \Omega_j(1-\delta\kappa)]\eta_{t-j}$$
$$+ \kappa\sum_{m=0}^{\infty}(\varepsilon_{t-m} + \eta_{t-m})$$
$$+ \sum_{m=j+1}^{\infty}[\kappa + (\Omega_m - \kappa)\Omega_m]\varepsilon_{t-m} + \kappa\sum_{m=j+1}^{\infty}[1 - \Omega_m]\eta_{t-m}$$

5.2.6. Social Welfare and optimal pricing

We now introduce an efficiency benchmark that addresses whether higher welfare could be obtained if agents were to use their available information in a different way than they do in equilibrium. Following Angeletos and Pavan (2007), we adopt as our efficiency benchmark the strategy that maximizes ex ante utility subject to the sole constraint that information cannot be transferred from one agent to another. The Lagrangian for our problem is

$$E\Pi = -\lambda\int_{(\Theta_t,Y_t)}\left[\sum_{j=0}^{\infty}(1-\lambda)^j u(x_{t-j},\Theta_t,Y_t)^2 dF(x_{t-j} \mid \Theta_t,Y_t)\right]dF(\Theta_t,Y_t)$$
$$+\int_{(\Theta_t,Y_t)}\eta(\Theta_t,Y_t)h(\Theta_t,Y_t)dF(\Theta_t,Y_t) = 0$$

where $u(x_{t-j},\Theta_t,Y_t) \equiv p_t(x_{t-j},\Theta_{t-j-1},Y_t) - [(1-r)\theta_t + rP_t(\Theta_t,Y_t)]$ and $\eta(\Theta_t,Y_t)$ is a Lagrangian multiplier associated with the constraint

$$h(\Theta_t,Y_t) \equiv P_t(\Theta_t,Y_t) - \lambda\sum_{j=0}^{\infty}(1-\lambda)^j\int_{x_{t-j}}p_t(x_{t-j},\Theta_{t-j-1},Y_t)dF(x_{t-j} \mid \Theta_t,Y_t).$$

Because the program is concave, the solution is given by the first-order conditions relative to $P_t(\Theta_t,Y_t)$ and $p_t(x_{t-j},\Theta_{t-j-1},Y_t)$.

$$2r\lambda \sum_{j=0}^{\infty} (1-\lambda)^j \int_{x_{t-j}} u(x_{t-j}, \Theta_t, Y_t) dF(x_{t-j} \mid \Theta_t, Y_t) + \eta(\Theta_t, Y_t) = 0,$$

$$-2\int_{\Theta_t} u(x_{t-j}, \Theta_t, Y_t) dF(\Theta_t \mid x_{t-j}, \Theta_{t-j-1}, Y_t)$$
$$-\int_{\Theta_t} \eta(\Theta_t, Y_t) dF(\Theta_t \mid x_{t-j}, \Theta_{t-j-1}, Y_t) = 0$$

Consider the first condition. Noting that

$$P_t(\Theta_t, Y_t) = \lambda \sum_{j=0}^{\infty} (1-\lambda)^j \int_{x_{t-j}} p_t(x_{t-j}, \Theta_{t-j-1}, Y_t) dF(x_{t-j} \mid \Theta_t, Y_t),$$

we obtain an expression for the multiplier

$$\eta(\Theta_t, Y_t) = 2r(1-r)[\theta_t - P_t(\Theta_t, Y_t)].$$

If we substitute this expression in the secord condition we obtain

$$p_t(x_{t-j}, \Theta_{t-j-1}, Y_{t-j}) = E[(1-r^*)\theta_t + r^* P_t(\Theta_t, Y_t) \mid \Im_{t-j}(z)],$$

where

$$r^* \equiv 1-(1-r)^2.$$

5.2.7.
Welfare and communication

In this appendix, we show that the efficiency criterion $E\Pi$ can be expressed as

$$-\left(\frac{\lambda}{\alpha+\beta+\gamma} + \frac{1-\lambda}{\alpha+\gamma}\right) \sum_{j=0}^{\infty} (1-\lambda)^j \Omega_j^2.$$

The efficiency criterion is given by

$$E\Pi = -\lambda \int_{(\Theta_t, Y_t)} \left[\sum_{j=0}^{\infty} (1-\lambda)^j \int_{x_{t-j}} u(x_{t-j}, \Theta_{t-j-1}, Y_t)^2 dF(x_{t-j} \mid \Theta_t, Y_t) \right] dF(\Theta_t, Y_t)$$
$$+ \int_{(\Theta_t, Y_t)} \eta(\Theta_t, Y_t) h(\Theta_t, Y_t) dF(\Theta_t, Y_t)$$

First, we compute $u(x_{t-j}, \Theta_{t-j-1}, Y_t)$, considering that the equilibrium price level is given by (2.7) and a firm z that last updated its information set at period $t-j$ computes expectations using (2.5), we have that

$$
\begin{aligned}
&p_t(x_{t-j},\Theta_{t-j-1},Y_t)\\
&= E\Big[(1-r)\theta_t + r\sum\nolimits_{m=0}^{\infty} c_m\theta_{t-m} + r\sum\nolimits_{m=0}^{\infty} d_m y_{t-m} \;\mid\; \Im_{t-j}(z)\Big]\\
&= (1-r)E[\theta_t \;\mid\; \Im_{t-j}(z)] + r\sum\nolimits_{m=0}^{j} c_m E[\theta_{t-m} \;\mid\; \Im_{t-j}(z)]\\
&+ r\sum\nolimits_{m=j+1}^{\infty} c_m E[\theta_{t-m} \;\mid\; \Im_{t-j}(z)] + r\sum\nolimits_{m=0}^{\infty} d_m y_{t-m}\\
&= (1-r)\Big[(1-\delta)x_{t-j}(z) + \delta\theta_{t-j-1} + \delta\kappa y_{t-j} + \kappa\sum\nolimits_{i=0}^{j-1} y_{t-i}\Big]\\
&+ r\sum\nolimits_{m=0}^{j} c_m\Big[(1-\delta)x_{t-j}(z) + \delta\theta_{t-j-1} + \delta\kappa y_{t-j} + \kappa\sum\nolimits_{i=m}^{j-1} y_{t-i}\Big]\\
&+ r\sum\nolimits_{m=j+1}^{\infty} c_m\theta_{t-m} + r\sum\nolimits_{m=0}^{\infty} d_m y_{t-m}\\
&= (1-r)\Big[(1-\delta)x_{t-j}(z) + \delta\theta_{t-j-1} + \delta\kappa y_{t-j} + \kappa\sum\nolimits_{i=0}^{j-1} y_{t-i}\Big]\\
&+ r[(1-\delta)x_{t-j}(z) + \delta\theta_{t-j-1} + \delta\kappa y_{t-j}]\Big(\sum\nolimits_{m=0}^{j} c_m\Big) + r\kappa\sum\nolimits_{i=0}^{j-1} y_{t-i}\Big(\sum\nolimits_{m=0}^{i} c_m\Big)\\
&+ r\sum\nolimits_{m=j+1}^{\infty} c_m\theta_{t-m} + r\sum\nolimits_{m=0}^{\infty} d_m y_{t-m}\\
&= [1-r+rC_j]\{(1-\delta)x_{t-j}(z) + \delta\theta_{t-j-1} + \delta\kappa y_{t-j}\}\\
&+ \kappa\sum\nolimits_{k=0}^{j-1}[1-r+rC_k]y_{t-k}\\
&+ r\sum\nolimits_{m=j+1}^{\infty} c_m\theta_{t-m} + r\sum\nolimits_{m=0}^{\infty} d_m y_{t-m}
\end{aligned}
$$

As a result,

$$
\begin{aligned}
u(x_{t-j},\Theta_{t-j-1},Y_t) &= p_t(x_{t-j},\Theta_{t-j-1},Y_t) - [(1-r)\theta_t + rP_t]\\
&= [1-r+rC_j]\{(1-\delta)x_{t-j}(z) + \delta\theta_{t-j-1} + \delta\kappa y_{t-j}\}\\
&+ \kappa\sum\nolimits_{k=0}^{j-1}[1-r+rC_k]y_{t-k} - (1-r)\theta_t - r\sum\nolimits_{m=0}^{j} c_m\theta_{t-m}.
\end{aligned}
$$

Note that $1-r+rC_j = \Omega_j$ and

$$
\begin{aligned}
&-(1-r)\theta_t - r\sum\nolimits_{m=0}^{j} c_m\theta_{t-m}\\
&= -[1-r+rc_0]\theta_t - r\sum\nolimits_{m=1}^{j} c_m\theta_{t-m}\\
&= -\Omega_0\theta_t - \sum\nolimits_{m=1}^{j}[\Omega_m - \Omega_{m-1}]\theta_{t-m}\\
&= -\sum\nolimits_{m=0}^{j-1}\Omega_m(\theta_{t-m} - \theta_{t-m-1}) - \Omega_j\theta_{t-j}\\
&= -\sum\nolimits_{m=0}^{j-1}\Omega_m\varepsilon_{t-m} - \Omega_j(\theta_{t-j-1} + \varepsilon_{t-j})\\
&= -\sum\nolimits_{m=0}^{j}\Omega_m\varepsilon_{t-m} - \Omega_j\theta_{t-j-1}
\end{aligned}
$$

where

$$
\Omega_j = \left[\frac{1-r}{1-r\big[1-\rho(1-\lambda)^j\big]}\right].
$$

These observations allows us to write $u(x_{t-j},\Theta_{t-j-1},Y_t)$ as a function of independent shocks

$$
\begin{aligned}
u(x_{t-j},\Theta_{t-j-1},Y_t) &= \Omega_j\{(1-\delta)[\theta_{t-j-1}+\varepsilon_{t-j}+\xi_{t-j}(z)]+\delta\theta_{t-j-1}+\delta\kappa(\varepsilon_{t-j}+\eta_{t-j})\} \\
&\quad + \kappa\sum\nolimits_{k=0}^{j-1}\Omega_k(\varepsilon_{t-k}+\eta_{t-k}) - \sum\nolimits_{m=0}^{j}\Omega_m\varepsilon_{t-m} - \Omega_j\theta_{t-j-1} \\
&= \Omega_j\{-\delta(1-\kappa)\varepsilon_{t-j}+(1-\delta)\xi_{t-j}(z)+\delta\kappa\eta_{t-j}\} \\
&\quad - (1-\kappa)\sum\nolimits_{k=0}^{j-1}\Omega_k\varepsilon_{t-k} + \kappa\sum\nolimits_{k=0}^{j-1}\Omega_k\eta_{t-k}
\end{aligned}
$$

As $\delta = \dfrac{\alpha+\gamma}{\alpha+\beta+\gamma}$, and $\kappa = \left(\dfrac{\gamma}{\alpha+\gamma}\right)$,

we have that

$$
\begin{aligned}
u(x_{t-j},\Theta_{t-j-1},Y_t) &= \frac{\Omega_j}{(\alpha+\beta+\gamma)}\{-\alpha\varepsilon_{t-j}+\beta\xi_{t-j}(z)+\gamma\eta_{t-j}\} \\
&= \sum\nolimits_{k=0}^{j-1}\frac{\Omega_k}{(\alpha+\gamma)}[-\alpha\varepsilon_{t-k}+\gamma\eta_{t-k}]
\end{aligned}
$$

We use this expression to obtain the efficience criterion $E\Pi$ as

$$
\begin{aligned}
&-\lambda\int_{(\Theta_t,Y_t)}\left[\sum\nolimits_{j=0}^{\infty}(1-\lambda)^j\int_{x_{t-j}}u(x_{t-j},\Theta_{t-j-1},Y_t)^2 dF(x_{t-j}\mid\Theta_t,Y_t)\right]dF(\Theta_t,Y_t) \\
&= -\lambda\sum\nolimits_{j=0}^{\infty}(1-\lambda)^j\left[\begin{array}{c}\frac{\Omega_j^2}{(\alpha+\beta+\gamma)^2}\{\alpha^2\alpha^{-1}+\beta^2\beta^{-1}+\gamma^2\gamma^{-1}\} \\ +\sum\nolimits_{k=0}^{j-1}\frac{\Omega_k^2}{(\alpha+\gamma)^2}\{\alpha^2\alpha^{-1}+\gamma^2\gamma^{-1}\}\end{array}\right] \\
&= -\lambda\sum\nolimits_{j=0}^{\infty}(1-\lambda)^j\left[\frac{\Omega_j^2}{(\alpha+\beta+\gamma)}+\sum\nolimits_{k=0}^{j-1}\frac{\Omega_k^2}{(\alpha+\gamma)}\right] \\
&= -\lambda\sum\nolimits_{j=0}^{\infty}(1-\lambda)^j\left(\frac{1}{\alpha+\beta+\gamma}\right)\Omega_j^2 + \left(\frac{1}{\alpha+\gamma}\right) - \lambda\sum\nolimits_{k=0}^{\infty}\Omega_k^2\sum\nolimits_{j=k}^{\infty}(1-\lambda)^j \\
&= -\lambda\sum\nolimits_{j=0}^{\infty}(1-\lambda)^j\left(\frac{1}{\alpha+\beta+\gamma}\right)\Omega_j^2 + -\lambda\left(\frac{1}{\alpha+\gamma}\right)\sum\nolimits_{k=0}^{\infty}\Omega_k^2\sum\nolimits_{j=k}^{\infty}(1-\lambda)^j \\
&= -\left(\frac{\lambda}{\alpha+\beta+\gamma}\right)\sum\nolimits_{j=0}^{\infty}(1-\lambda)^j\Omega_j^2 - \left(\frac{1}{\alpha+\gamma}\right)\sum\nolimits_{k=0}^{\infty}(1-\lambda)^{k+1}\Omega_k^2 \\
&= -\left(\frac{\lambda}{\alpha+\beta+\gamma}+\frac{1-\lambda}{\alpha+\gamma}\right)\sum\nolimits_{j=0}^{\infty}(1-\lambda)^j\Omega_j^2
\end{aligned}
$$

5.3. Appendix of Chapter 3

5.3.1. Expectation

In this appendix, we derive equation (3.10). In order to compute $E[\theta_{t-m} \mid \Im_{t-j}(z)]$ when $m < j$, we need to obtain $E[u_{t-i} \mid w_{t-i}]$ and

$E\left[u_{t-j} \mid w_{t-j}, t_{t-j}(z)\right]$. First, we are going to obtain the distribution of $e_{t-i} \mid u_{t-i}$. From the Bayes theorem, we know that

$$f(e_{t-i} \mid u_{t-i}) = \frac{f(e_{t-i}, u_{t-i})}{f(u_{t-i})} = \frac{f(u_{t-i} \mid e_{t-i}) f(e_{t-i})}{\int f(e_{t-i}, u_{t-i}) de_{t-i}}.$$

But, using (3.8), we have that

$$\begin{aligned}
&f(e_{t-i}, u_{t-i}) \\
&= f(u_{t-i} \mid e_{t-i}) f(e_{t-i}) \\
&= k_1 \exp -\frac{1}{2}\left\{ \frac{\left(u_{t-i} + \left(\frac{\sigma\phi}{1+\sigma\phi}\right)e_{t-i}\right)^2}{\left((1+\sigma\phi)^2\alpha\right)^{-1}} + \frac{e_{t-i}^2}{\omega^{-1}} \right\} \\
&= k_1 \exp -\frac{1}{2}\left\{ \left((1+\sigma\phi)^2\alpha\right)u_{t-i}^2 + \psi\left(e_{t-i}^2 + 2\frac{(1+\sigma\phi)\alpha\sigma\phi}{\psi}e_{t-i}u_{t-i}\right)\right\} \\
&= k_1 k_2 \sqrt{\frac{\psi}{2\pi}} \exp -\frac{1}{2}\left\{ \frac{\left(e_{t-i} + \frac{(1+\sigma\phi)\alpha\sigma\phi}{\psi}u_{t-i}\right)^2}{\psi^{-1}} \right\}
\end{aligned}$$

where

$$\begin{aligned}
k_1 &= \frac{\sqrt{\left((1+\sigma\phi)^2\alpha\right)\omega}}{2\pi} \\
k_2 &= \sqrt{\frac{2\pi}{\psi}}\left[(1+\sigma\phi)^2\alpha - \frac{((1+\sigma\phi)\alpha\sigma\phi)^2}{\psi} \right] u_{t-i}^2 \\
\psi &= \alpha(\sigma\phi)^2 + \omega
\end{aligned}$$

Therefore,

$$f(e_{t-i} \mid u_{t-i}) = \frac{f(e_{t-i}, u_{t-i})}{\int f(e_{t-i}, u_{t-i}) de_{t-i}} = N\left(-\frac{(1+\sigma\phi)\alpha\sigma\phi}{\psi}u_{t-i}, \psi^{-1}\right)$$

With this result, it is easy to see that

$$f(w_{t-i} \mid u_{t-i}) = u_{t-i} + f(e_{t-i} \mid u_{t-i}) = N\left(\frac{\omega - \alpha\sigma\phi}{\psi}u_{t-i}, \psi^{-1}\right)$$

We use this result to compute $E[u_{t-i} \mid w_{t-i}]$. Since

$$\begin{aligned}
f(w_{t-i}, u_{t-i}) &= f(w_{t-i} \mid u_{t-i}) f(u_{t-i}) \\
&= k \exp -\frac{1}{2}\left\{ \psi w_{t-i}^2 - 2(\omega - \alpha\sigma\phi) w_{t-i} u_{t-i} + \left(\frac{(\omega - \alpha\sigma\phi)^2}{\psi} + \varphi\right) u_{t-i}^2 \right\}
\end{aligned}$$

$$= k\exp -\frac{1}{2}\left\{\psi w_{t-i}^2 - 2(\omega-\alpha\sigma\phi)w_{t-i}u_{t-i} + \left(\frac{\omega^2+(\alpha\sigma\phi)^2+\alpha\omega\left(1+(\sigma\phi)^2\right)}{\psi}\right)u_{t-i}^2\right\}$$

$$= k\exp -\frac{1}{2}\left\{\psi w_{t-i}^2 + (\alpha+\omega)\left(u_{t-i}^2 - 2\left(\frac{\omega-\alpha\sigma\phi}{\alpha+\omega}\right)w_{t-i}u_{t-i}\right)\right\}$$

$$= k\exp\left\{-\frac{1}{2}\left[\psi w_{t-i}^2 - \frac{(\omega-\alpha\sigma\phi)^2}{\alpha+\omega}w_{t-i}^2\right]\right\}\exp\left\{-\frac{1}{2}\left[\frac{\left(u_{t-i}-\left(\frac{\omega-\alpha\sigma\phi}{\alpha+\omega}\right)w_{t-i}\right)^2}{(\alpha+\omega)^{-1}}\right]\right\},$$

using Bayes theorem we obtain

$$f(u_{t-i} \mid w_{t-i}) = \frac{f(w_{t-i},u_{t-i})}{\int f(w_{t-i},u_{t-i})du_{t-i}} = N\left(\left(\frac{\omega-\alpha\sigma\phi}{\alpha+\omega}\right)w_{t-i},(\alpha+\omega)^{-1}\right).$$

This means that

$$E[u_{t-i} \mid w_{t-i}] = \left(\frac{\omega-\alpha\sigma\phi}{\alpha+\omega}\right)w_{t-i}.$$

Alternatively, we can obtain this result from

$$E[u_{t-i} \mid w_{t-i}] = \left[\frac{cov(u_{t-i},w_{t-i})}{var(w_{t-i})}\right]w_{t-i}$$

since

$$\begin{aligned} cov(u_{t-i},w_{t-i}) &= cov(u_{t-i},u_{t-i}+e_{t-i}) \\ &= \left(\frac{1}{1+\sigma\phi}\right)^2 cov(\varepsilon_{t-i}-\sigma\phi e_{t-i},\varepsilon_{t-i}+e_{t-i}) \\ &= \left(\frac{1}{1+\sigma\phi}\right)^2 [var(\varepsilon_{t-i})-\sigma\phi var(e_{t-i})] \\ &= \left(\frac{1}{1+\sigma\phi}\right)^2 [\alpha^{-1}-\sigma\phi\omega^{-1}] = \frac{\omega-\sigma\phi\alpha}{\alpha\omega(1+\sigma\phi)^2} \end{aligned}$$

and

$$\begin{aligned} var(w_{t-i}) &= var(u_{t-i}+e_{t-i}) \\ &= \left(\frac{1}{1+\sigma\phi}\right)^2 var(\varepsilon_{t-i}+e_{t-i}) \\ &= \frac{\alpha^{-1}+\omega^{-1}}{(1+\sigma\phi)^2} = \frac{\alpha+\omega}{\alpha\omega(1+\sigma\phi)^2}. \end{aligned}$$

Nevertheless, computing $f(w_{t-j} \mid u_{t-j})$ is useful to assess $E[u_{t-j} \mid w_{t-j},t_{t-j}(z)]$. As before, we use Bayes Theorem to compute $f(u_{t-j} \mid w_{t-j},t_{t-j})$. That is,

$$f(u_{t-j} \mid w_{t-j}, t_{t-j}) = \frac{f(u_{t-j}, w_{t-j}, t_{t-j})}{f(w_{t-j}, t_{t-j})} = \frac{f(t_{t-j}, w_{t-j} \mid u_{t-j}) f(u_{t-j})}{\int f(u_{t-j}, w_{t-j}, t_{t-j}) du_{t-j}}.$$

Since

$$\begin{aligned} t_{t-j}(z) &\equiv x_{t-j} - \theta_{t-j-1} = u_{t-j} + \xi_{t-j}(w), \\ w_{t-j} &\equiv \phi^{-1} i_t = u_{t-j} + e_{t-j}, \end{aligned}$$

and e_{t-j} is independent of $\xi_{t-j}(z)$, we have

$$f(u_{t-j} \mid w_{t-j}, t_{t-j}) = \frac{f(t_{t-j} \mid u_{t-j}) f(w_{t-j} \mid u_{t-j}) f(u_{t-j})}{\int f(t_{t-j} \mid u_{t-j}) f(w_{t-j} \mid u_{t-j}) f(u_{t-j}) du_{t-j}}.$$

As $f(t_{t-j} \mid u_{t-j}) = N(u_{t-j}, \beta^{-1})$, $f(w_{t-j}) = N(u_{t-j}, \omega^{-1})$, and $f(u_{t-j}) = N(0, \varphi^{-1})$, we have that

$$\begin{aligned}
&f(t_{t-j} \mid u_{t-j}) f(w_{t-j} \mid u_{t-j}) f(u_{t-j}) \\
&= \left(\frac{\beta\psi\varphi}{(2\pi)^3}\right)^{1/2} \exp -\frac{1}{2}\left\{\frac{(t_{t-j} - u_{t-j})^2}{\beta^{-1}} + \frac{\left(w_{t-j} - \frac{\omega - \alpha\sigma\phi}{\psi} u_{t-j}\right)^2}{\psi^{-1}} + \frac{u_{t-j}^2}{\varphi^{-1}}\right\} \\
&= \left(\frac{\beta\omega\varphi}{(2\pi)^3}\right)^{1/2} \exp -\frac{1}{2}\left\{\beta t_{t-j}^2 - 2\beta u_{t-j} t_{t-j} + \beta u_{t-j}^2\right\} \\
&\times \exp -\frac{1}{2}\left\{\psi w_{t-j}^2 - 2(\omega - \alpha\sigma\phi) u_{t-j} w_{t-j} + \left(\frac{(\omega - \alpha\sigma\phi)^2}{\psi}\right) u_{t-j}^2 + \varphi u_{t-j}^2\right\} \\
&= \left(\frac{\beta\omega\varphi}{(2\pi)^3}\right)^{1/2} \exp -\frac{1}{2}\left\{\beta t_{t-j}^2 + \psi w_{t-j}^2 + \left[\beta + \left(\frac{(\omega - \alpha\sigma\phi)^2}{\psi}\right) + \varphi\right] u_{t-j}^2\right\} \\
&\times \exp -\frac{1}{2}\{-2(\beta t_{t-j} + (\omega - \alpha\sigma\phi) w_{t-j}) u_{t-j}\} \\
&= \left(\frac{\beta\omega\varphi}{(2\pi)^3}\right)^{1/2} \exp\left\{-\frac{1}{2}\left[\beta t_{t-j}^2 + \psi w_{t-j}^2 - \frac{(\beta t_{t-j} + (\omega - \alpha\sigma\phi) w_{t-j})^2}{\beta + \omega + \alpha}\right]\right\} \\
&\times \exp\left\{-\frac{1}{2}\left(\frac{u_{t-j} - \left(\frac{\beta t_{t-j} + (\omega - \alpha\sigma\phi) w_{t-j}}{\beta + \omega + \alpha}\right)}{(\beta + \omega + \alpha)^{-1}}\right)^2\right\}
\end{aligned}$$

where the last equality holds because

$$\varphi \equiv \frac{\alpha\omega(1 + \sigma\phi)^2}{\omega + (\sigma\phi)^2\alpha}.$$

From this expression we finally obtain

$$f(u_{t-j} \mid w_{t-j}, t_{t-j}) = N\left(\frac{\beta t_{t-j} + (\omega - \alpha\sigma\phi) w_{t-j}}{\beta + \omega + \alpha}, (\beta + \omega + \alpha)^{-1}\right),$$

and consequently,

$$E[u_{t-j} \mid w_{t-j}, t_{t-j}(z)] = \frac{\beta t_{t-j}(z) + (\omega - \alpha\sigma\phi) w_{t-j}}{\beta + \omega + \alpha}.$$

5.3.2. *Ex-ante* total profit

In this appendix we derive (3.21). First we are going to compute the equilibrium price of each firm z, $p_t(z)$.Substituting (3.1) in (3.5) and using the fact that in equilibrium the price index is guiven by (3.13), we get

$$\begin{aligned}
p_t(z) &= p_t(x_{t-j}, \Theta_{t-j-1}, I_t) \\
&= E[(1-r)\theta_t + rP_t \mid \Im_{t-j}(z)] \\
&= (1-r)E[\theta_t \mid \Im_{t-j}(z)] + r\sum_{m=0}^{\infty} c_m E[\theta_{t-m} \mid \Im_{t-j}(z)] + r\sum_{m=0}^{\infty} d_m i_{t-m} \\
&= (1-r)\left[(1-\delta)x_{t-j}(z) + \delta\theta_{t-j-1} + \delta\kappa i_{t-j} + \kappa\sum_{k=0}^{j-1} i_{t-k}\right] \\
&+ r\sum_{m=0}^{j} c_m \left[(1-\delta)x_{t-j}(z) + \delta\theta_{t-j-1} + \delta\kappa i_{t-j} + \kappa\sum_{k=m}^{j-1} i_{t-k}\right] \\
&+ r\sum_{m=j+1}^{\infty} c_m \theta_{t-m} + r\sum_{m=0}^{\infty} d_m i_{t-m} \\
&= (1-r)\left[(1-\delta)x_{t-j}(z) + \delta\theta_{t-j-1} + \delta\kappa i_{t-j} + \kappa\sum_{k=0}^{j-1} i_{t-k}\right] \\
&+ [r(1-\delta)x_{t-j}(z) + r\delta\theta_{t-j-1} + r\delta\kappa i_{t-j}]C_j + r\kappa\sum_{k=0}^{j-1} C_k i_{t-k} \\
&+ r\sum_{m=j+1}^{\infty} c_m \theta_{t-m} + r\sum_{m=0}^{\infty} d_m i_{t-m} \\
&= [1-r+rC_j]\{(1-\delta)x_{t-j}(z) + \delta\theta_{t-j-1} + \delta\kappa i_{t-j}\} + \kappa\sum_{k=0}^{j-1}[1-r+rC_k]i_{t-k} \\
&+ r\sum_{m=j+1}^{\infty} c_m \theta_{t-m} + r\sum_{m=0}^{\infty} d_m i_{t-m}
\end{aligned}$$

where

$$C_j \equiv \sum_{m=0}^{j} c_m.$$

This expression shows that the price set by each firm z is a function of the signals present on the information set $\Im_{t-j}(z)$, i.e. $p_t(z) = p_t(x_{t-j}(z), \Theta_{t-j-1}, I_t)$. As a result,

$$\begin{aligned}
& p_t(z) - [(1-r)\theta_t + rP_t] \\
&= [1-r+rC_j]\{(1-\delta)x_{t-j}(z) + \delta\theta_{t-j-1} + \delta\kappa i_{t-j}\} \\
&+ \kappa\sum_{k=0}^{j-1}[1-r+rC_k]i_{t-k} - (1-r)\theta_t - r\sum_{m=0}^{j} c_m \theta_{t-m}.
\end{aligned}$$

is a function of (x_{t-j}, Θ_t, I_t). To simplify this expression, it is important to obtain C_k and compute $-(1-r)\theta_t - r\sum_{m=0}^{j} c_m \theta_{t-m}$. We calculate C_k as

$$\begin{aligned} C_j &\equiv \sum_{m=0}^{j} c_m = c_0 + \sum_{m=1}^{j} c_m \\ &= \frac{(1-r)(1-\rho)}{1-r(1-\rho)} + \left(\frac{1-r}{r}\right)\left\{\frac{1}{1-r\left[1-\rho(1-\lambda)^j\right]} - \frac{1}{1-r(1-\rho)}\right\} \\ &= \left(\frac{1-r}{r}\right)\left[\frac{1}{1-r+r\rho(1-\lambda)^j} - 1\right]. \end{aligned} \tag{4.6}$$

Although this derivation assumes $j > 0$, it also holds for $j = 0$. Furthermore,

$$\begin{aligned} &-(1-r)\theta_t - r\sum_{m=0}^{j} c_m \theta_{t-m} \\ &= -[1-r+rc_0]\theta_t - r\sum_{m=1}^{j} c_m \theta_{t-m} \\ &= -\Omega_0 \theta_t - \sum_{m=1}^{j} (\Omega_m - \Omega_{m-1})\theta_{t-m} \\ &= -\sum_{m=0}^{j-1} \Omega_m (\theta_{t-m} - \theta_{t-m-1}) - \Omega_j \theta_{t-j} \\ &= -\sum_{m=0}^{j-1} \Omega_m u_{t-m} - \Omega_j(\theta_{t-j-1} + u_{t-j}) \\ &= -\sum_{m=0}^{j} \Omega_m u_{t-m} - \Omega_j \theta_{t-j-1} \end{aligned}$$

where

$$\Omega_j(\rho) = \left[\frac{1-r}{1-r\left[1-\rho(1-\lambda)^j\right]}\right].$$

Thus,

$$\begin{aligned} &p_t(z) - [(1-r)\theta_t + rP_t] \\ &= [1-r+rC_j]\{(1-\delta)x_{t-j}(z) + \delta\theta_{t-j-1} + \delta\kappa i_{t-j}\} \\ &\quad + \kappa\sum_{k=0}^{j-1}[1-r+rC_k]i_{t-k} - \sum_{m=0}^{j} \Omega_m u_{t-m} - \Omega_j \theta_{t-j-1} \\ &= \Omega_j\{(1-\delta)x_{t-j}(z) + \delta\theta_{t-j-1} + \delta\kappa i_{t-j}\} + \kappa\sum_{k=0}^{j-1} \Omega_k i_{t-k} - \sum_{m=0}^{j} \Omega_m u_{t-m} - \Omega_j \theta_{t-j-1} \\ &= \Omega_j\{(1-\delta)[\theta_{t-j-1} + u_{t-j} + \xi_{t-j}(z)] + \delta\theta_{t-j-1} + \delta\kappa[\phi u_{t-j} + \phi e_{t-j}]\} + \\ &\quad \kappa\sum_{k=0}^{j-1} \Omega_k[\phi u_{t-k} + \phi e_{t-k}] - \sum_{m=0}^{j} \Omega_m u_{t-m} - \Omega_j \theta_{t-j-1} \\ &= \Omega_j\{(1-\delta)[u_{t-j} + \xi_{t-j}(z)] + \phi\delta\kappa[u_{t-j} + e_{t-j}]\} \\ &\quad + \phi\kappa\sum_{k=0}^{j-1} \Omega_k[u_{t-k} + e_{t-k}] - \sum_{m=0}^{j} \Omega_m u_{t-m} \end{aligned}$$

$$= \Omega_j\left\{\left(\frac{1}{1+\sigma\phi}\right)(1-\delta)[\varepsilon_{t-j}-(\sigma\phi)e_{t-j}]+(1-\delta)\xi_{t-j}(z)+\left(\frac{1}{1+\sigma\phi}\right)\phi\delta\kappa[\varepsilon_{t-j}+e_{t-j}]\right\}$$
$$+\phi\kappa\left(\frac{1}{1+\sigma\phi}\right)\sum_{k=0}^{j-1}\Omega_k[\varepsilon_{t-k}+e_{t-k}]-\left(\frac{1}{1+\sigma\phi}\right)\sum_{m=0}^{j}\Omega_m[\varepsilon_{t-m}-(\sigma\phi)e_{t-m}]$$
$$= \Omega_j\left\{\left(\frac{\phi\kappa-1}{1+\sigma\phi}\right)\delta\varepsilon_{t-j}+(1-\delta)\xi_{t-j}(z)+\left(\frac{\phi(\sigma+\kappa)}{1+\sigma\phi}\right)\delta e_{t-j}\right\}$$
$$+\left(\frac{\phi\kappa-1}{1+\sigma\phi}\right)\sum_{k=0}^{j-1}\Omega_k\varepsilon_{t-k}+\left(\frac{\phi(\sigma+\kappa)}{1+\sigma\phi}\right)\sum_{k=0}^{j-1}\Omega_k e_{t-k}$$

Using this expression we write the criterion $E\Pi$ as a function of the parameters (κ,δ). That is,

$$E\Pi(\kappa,\delta) = -\lambda\sum_{j=0}^{\infty}(1-\lambda)^j\Omega_j^2\left[\left(\frac{\phi\kappa-1}{1+\sigma\phi}\right)^2\delta^2\alpha^{-1}+(1-\delta)^2\beta^{-1}+\left(\frac{\phi(\sigma+\kappa)}{1+\sigma\phi}\right)^2\delta^2\omega^{-1}\right]$$
$$-\lambda\sum_{j=0}^{\infty}(1-\lambda)^j\left[\left(\frac{\phi\kappa-1}{1+\sigma\phi}\right)^2\sum_{k=0}^{j-1}\Omega_k^2\alpha^{-1}+\left(\frac{\phi(\sigma+\kappa)}{1+\sigma\phi}\right)^2\omega^{-1}\sum_{k=0}^{j-1}\Omega_k^2\right]$$
$$= -\left[\left(\frac{(\phi\kappa-1)^2}{\alpha}+\frac{\phi^2(\sigma+\kappa)^2}{\omega}\right)\left(\frac{\lambda\delta^2+(1-\lambda)}{(1+\sigma\phi)^2}\right)+\frac{\lambda(1-\delta)^2}{\beta}\right]\sum_{j=0}^{\infty}(1-\lambda)^j\Omega_j^2$$

From this expression, we compute $E\Pi(\hat{\kappa},\hat{\delta})$ and $E\Pi(\tilde{\kappa},\tilde{\delta})$ using respectively (3.11) and (3.16). We obtain

$$E\Pi(\hat{\kappa},\hat{\delta}) = -\left[\frac{\lambda}{(\beta+\omega+\alpha)}+\frac{(1-\lambda)}{(\alpha+\omega)}\right]\sum_{j=0}^{\infty}(1-\lambda)^j\hat{\Omega}_j^2$$

and

$$E\Pi(\tilde{\kappa},\tilde{\delta}) = -\left[\frac{\lambda}{(\alpha+\beta)}+\frac{(1-\lambda)}{\alpha}\right]\sum_{j=0}^{\infty}(1-\lambda)^j\tilde{\Omega}_j^2.$$

where $\hat{\Omega}_j=\Omega_j(\hat{\rho})$ and $\tilde{\Omega}_j=\Omega_j(\tilde{\rho})$.

5.3.3. Cross-sectional dispersion

In this appendix, we derive (3.22) to show that the cross-sectional dispersion can be writen as function of $E\Pi$. due to

$$
\begin{aligned}
&EV \\
&= -\lambda \int_{(\Theta_t, I_t)} \left[\sum_{j=0}^{\infty} (1-\lambda)^j \int_{x_{t-j}} \left[(p_t(z) - P_t)^2 \right] dF(x_{t-j} \mid \Theta_t, I_t) \right] dF(\Theta_t, I_t) \\
&= -\lambda \int_{(\Theta_t, I_t)} \left[\sum_{j=0}^{\infty} (1-\lambda)^j \int_{x_{t-j}} \left[((p_t(z) - p_t^*) + (p_t^* - P_t))^2 \right] dF(x_{t-j} \mid \Theta_t, I_t) \right] dF(\Theta_t, I_t) \\
&= -\lambda \int_{(\Theta_t, I_t)} \left[\sum_{j=0}^{\infty} (1-\lambda)^j \int_{x_{t-j}} (p_t(z) - p_t^*)^2 dF(x_{t-j} \mid \Theta_t, I_t) \right] dF(\Theta_t, I_t) \\
&- \int_{(\Theta_t, I_t)} 2(p_t^* - P_t) \left[\lambda \sum_{j=0}^{\infty} (1-\lambda)^j \int_{x_{t-j}} p_t(z) dF(x_{t-j} \mid \Theta_t, I_t) - p_t^* \right] dF(\Theta_t, I_t) \\
&- \int_{(\Theta_t, I_t)} (p_t^* - P_t)^2 dF(\Theta_t, I_t) \\
&= E\Pi - 2 \int_{(\Theta_t, I_t)} (p_t^* - P_t)[P_t - p_t^*] dF(\Theta_t, I_t) - \int_{(\Theta_t, I_t)} (p_t^* - P_t)^2 dF(\Theta_t, I_t) \\
&= E\Pi + 2E\left[(p_t^* - P_t)^2 \right] - E\left[(p_t^* - P_t)^2 \right] \\
&= E\Pi + E\left[(p_t^* - P_t)^2 \right] \\
&= E\Pi + (1-r)^2 E\left[(\theta_t - P_t)^2 \right]
\end{aligned}
$$

Considering the equilibrium expression for P_t, equation (3.13),and the fact that, according to (4.6), $C_\infty = \lim_{j \to \infty} C_j = \sum_{m=0}^{\infty} c_m = 1$, we have

$$
\begin{aligned}
\theta_t - P_t &= \theta_t - \sum_{m=0}^{\infty} c_m \theta_{t-m} - \sum_{m=0}^{\infty} d_m y_{t-m} \\
&= \sum_{m=0}^{\infty} c_m \theta_t - \sum_{m=0}^{\infty} c_m \theta_{t-m} - \sum_{m=0}^{\infty} d_m y_{t-m} \\
&= \sum_{m=0}^{\infty} c_m (\theta_t - \theta_{t-m}) - \sum_{m=0}^{\infty} d_m y_{t-m} \\
&= \sum_{m=0}^{\infty} c_m \sum_{k=0}^{j-1} u_{t-k} - \phi \sum_{m=0}^{\infty} d_m (u_{t-m} + e_{t-m}) \\
&= \sum_{k=0}^{\infty} u_{t-k} \left(\sum_{m=k+1}^{\infty} c_m \right) - \phi \sum_{m=0}^{\infty} d_m (u_{t-m} + e_{t-m}) \\
&= \frac{1}{\kappa} \sum_{k=0}^{\infty} d_k u_{t-k} - \phi \sum_{m=0}^{\infty} d_m (u_{t-m} + e_{t-m}).
\end{aligned}
$$

The last equality holds because c_m is given by (3.14) for $m > 0$. Using the expression for u_{t-k}, equation (3.8),we obtain $\theta_t - P_t$ as a function of independent shocks.

$$
\theta_t - P_t = \left(\frac{1}{1 + \sigma\phi} \right) \left[\frac{1 - \kappa\phi}{\kappa} \sum_{k=0}^{\infty} d_k \varepsilon_{t-m} - \frac{\phi(\sigma + \kappa)}{\kappa} \sum_{k=0}^{\infty} d_k e_{t-m} \right]
$$

Therefore, denoting $EV_1(\kappa, \delta) \equiv (1-r)^2 E\left[(\theta_t - P_t)^2 \right]$, we have

$$EV_1(\kappa,\delta)$$
$$=\left(\frac{\rho}{1+\sigma\phi}\right)^2\left[\frac{(1-\kappa\phi)^2}{\alpha}+\frac{[\phi(\sigma+\kappa)]^2}{\omega}\right]\sum_{k=0}^{\infty}(1-\lambda)^{2j}\Omega_k^2$$

where Ω_k is defined as in (3.20). Therefore, using the expressions for $(\hat{\kappa},\hat{\delta})$ and $(\tilde{\kappa},\tilde{\delta})$, we get

$$EV_1\left(\hat{\kappa},\hat{\delta}\right)=\left(\frac{\rho}{\alpha+\omega}\right)^2\sum_{k=0}^{\infty}(1-\lambda)^{2j}\hat{\Omega}_k^2$$
$$EV_1\left(\tilde{\kappa},\tilde{\delta}\right)=\left(\frac{\rho}{\alpha}\right)^2\sum_{k=0}^{\infty}(1-\lambda)^{2j}\tilde{\Omega}_k^2$$

This expression shows that EV_1 is in fact a function of ω.