DEPARTAMENTO DE ENGENHARIA ELÉTRICA

Internal Research Reports

ISSN

Number 11 | August 2010

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PUC RIO

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MAXWELL / LAMBDA-DEE
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Robust Tracking: Relation Between the Numerators, Mode Readability, and Inverse Internal Model

Pedro M. G. Ferreira

***Abstract*—In the robust, multivariable, asymptotic tracking problem with two-output plants, it is shown that if the numerator related to the controlled output is fixed, while the other numerator and the denominator are perturbed, then there is no solution to the problem. If, however, the whole part of the plant related to the controlled output is stable and fixed, while the other part is arbitrarily perturbed, the problem has a solution and the compensator that solves the problem incorporates an "inverse internal model" of the signal to be tracked.**

***Index Terms*—Internal model, robustness, stability, tracking.**

I. Introduction

The linear, multivariable, robust tracking problem has been addressed for more than 25 years, having achieved a degree of maturity in the solution of the main issues. In the mid-1970's, the problem was studied, among others, in the state-space/matrix formulation by [1], in the state-space/geometric approach by [5], in the state-space/Laplace transform by [4], and by [3].

In the early–mid-1980's, the problem in the input–output/Laplace transform was solved by [9] for one-output plant, one-degree-of-freedom compensator, by [8] for one-output plant, two-degrees-of-freedom compensator, and by [7] for the general problem, namely, two-output plant, two-degrees-of-freedom compensator.

Most recently, [6] addressed the issue of the robustness with respect to perturbations of the compensator in the scalar problem with one-output plant, two-degrees-of-freedom compensator. The present note is motivated by a conjecture made by [7], relating the numerators of the two-output plant.

In this note, after the setup of the problem in Section II, we show in Section III that indeed the problem has no solution if the numerators are unrelated, but in Section IV, it is shown that if the part of the plant related to the controlled output is fixed and stable, while the other part is perturbed, then the problem does have a solution: the compensator must incorporate an "inverse internal model" of the exogenous signal.

II. Setting Up the Problem

A. Notation and Abbreviations

The set of proper and stable rational functions, a principle ideal domain, [9], is denoted by S. The set of matrices with elements in S is denoted by $M(S)$. R is the field of real numbers. Left coprime will be abbreviated by *l.c.*, right coprime will be abbreviated by *r.c.*, "such that" will be abbreviated by *s.t.* All the left factors will be denoted with an "l" index, e.g., D_l.

In Fig. 1, $z(s)$ and $r(s)$ are q-valued vectors, $u(s)$ is a m-valued vector and $y(s)$ is a p-valued vector

$$P(s) = \begin{bmatrix} P_1(s) \\ P_2(s) \end{bmatrix}$$

represents the given plant. $C(s) = [C_1(s) \quad -C_2(s)]$ is the compensator to be designed.

Omitting henceforth the argument (s) when convenient, we have:

$$z = P_1 u, \quad y = P_2 u, \quad u = C_1 r - C_2 y.$$

P_1, P_2, C_1, and C_2 are proper rational matrices and have the appropriate dimensions. P_2 is assumed strictly proper for convenience in terms of well-posedness and because this is the case in most practical situations. This assumption might be dropped easily. The exogenous signal r is assumed proper. P_1 and P_2 are assumed to have full rank. All the factorizations in the paper are over S.

Let

$$P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} D^{-1}$$

a *r.c.* factorization.

Let $C = [C_1 \quad -C_2] = D_{lc}^{-1}[N_{lc1} \quad -N_{lc2}]$, a *l.c.* factorization.

We assume that the exogenous signal r has all its poles in the closed right complex plane; those are the relevant poles, since the modes corresponding to stable poles decay asymptotically to zero. This assumption is standard in the literature of the servo problem, but might be easily dropped.

$$r = D_{lr}^{-1} N_{lr} r_0$$

where

D_{lr}	is a known matrix,
N_{lr}	need not be known,
D_{lr} and N_{lr}	are *l.c.*, and
r_0	is an arbitrary vector of real numbers.

ϕ_m will denote the largest invariant factor of D_{lr}. We use the standard definition of closed loop stability. It is known, [2], that if the closed loop is stable, D_{lc}, N_{lc2}, D, and N_2 can be chosen, without loss of generality such that

$$D_{lc} D + N_{lc2} N_2 = I \tag{1}$$

where I is the identity matrix.

Manuscript received March 15, 1999; revised July 28, 1999 and February 14, 2000. Recommended by Associate Editor, X. Zhou.

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Publisher Item Identifier S 0018-9286(00)06083-9.

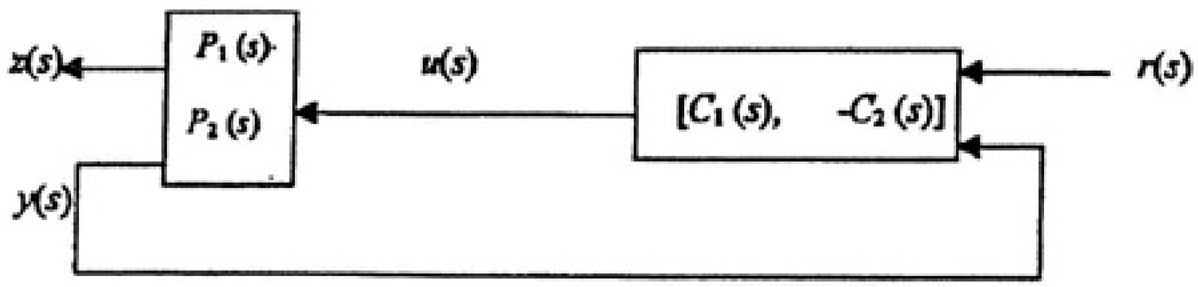

Fig. 1.

Moreover, there exist N_{c2}, D_c, D_l, and N_{l2}, corresponding to a *r.c.* of C_2 and *l.c.* factorization of P_2, respectively, *s.t.*

$$\begin{bmatrix} D_l & -N_{l2} \\ N_{lc2} & D_{lc} \end{bmatrix} \begin{bmatrix} D_c & N_2 \\ -N_{c2} & D \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \tag{1a}$$

Commute the left-hand side (LHS) of (1a) and call it (1b). (1b)

Let H be the transfer function matrix between z and r. Asymptotic tracking is said to take place if and only if the loop is stable and

$$(I - H)r \in \boldsymbol{M}(S). \tag{2}$$

Straightforward calculations give, in view of (1):

$$H = N_1 N_{c1}. \tag{3}$$

Perturb the plant, $P \to P^*$. Let H^* be the resulting transfer matrix between z and r. We say that C is a *robust tracking compensator* if and only if the perturbed closed loop is stable and

$$(I - H^*)r \in \boldsymbol{M}(S) \tag{2*}$$

whatever be the perturbation in a given set.

Remark 1: Recall, [9], that if $\|F\|_\infty < 1$, with $F \in \boldsymbol{M}(S)$, then $I + F$ is unimodular. We have next a technical result:

Lemma: Let Δ_D, $\Delta_N \in \boldsymbol{M}(S)$ be such that $I + D_{lc}\Delta_D + N_{lc2}\Delta_N$ is unimodular. Then there exist ε_1, $\varepsilon_2 \in \boldsymbol{R}$ and Q_1, $Q_2 \in \boldsymbol{M}(S)$ such that

$$(I + D_{lc}\Delta_D + N_{lc2}\Delta_N)^{-1} = I - \varepsilon_1 D_{lc} Q_1 - \varepsilon_2 N_{lc2} Q_2. \tag{4}$$

Conversely, let ε_1, $\varepsilon_2 \in \boldsymbol{R}$ and Q_1, $Q_2 \in \boldsymbol{M}(S)$ be such that $I - \varepsilon_1 D_{lc} Q_1 - \varepsilon_2 N_{lc2} Q_2$ is unimodular. Then there exists Δ_D, $\Delta_N \in \boldsymbol{M}(S)$ such that (4) holds. Moreover,

$$\Delta_D = \varepsilon_1 Q_1 (I - \varepsilon_1 D_{lc} Q_1 - \varepsilon_2 N_{lc2} Q_2)^{-1} \tag{5a}$$

$$\Delta_N = \varepsilon_2 Q_2 (I - \varepsilon_1 D_{lc} Q_1 - \varepsilon_2 N_{lc2} Q_2)^{-1} \tag{5b}$$

$$\varepsilon_1 Q_1 = \Delta_D (I + D_{lc}\Delta_D + N_{lc2}\Delta_N)^{-1} \tag{5c}$$

$$\varepsilon_2 Q_2 = \Delta_N (I + D_{lc}\Delta_D + N_{lc2}\Delta_N)^{-1}. \tag{5d}$$

Proof: From (5a) and (5b), we have:

$$\begin{aligned} I &+ D_{lc}\Delta_D + N_{lc2}\Delta_N \\ &= I + D_{lc}\varepsilon_1 Q_1 (I - \varepsilon_1 D_{lc} Q_1 - \varepsilon_2 N_{lc2} Q_2)^{-1} \\ &\quad + N_{lc2}\varepsilon_2 Q_2 (I - \varepsilon_1 D_{lc} Q_1 - \varepsilon_2 N_{lc2} Q_2)^{-1} \\ &= (I - \varepsilon_1 D_{lc} Q_1 - \varepsilon_2 N_{lc2} Q_2 + \varepsilon_1 D_{lc} Q_1 + \varepsilon_2 N_{lc2} Q_2) \\ &\quad \cdot (I - \varepsilon_1 D_{lc} Q_1 - \varepsilon_2 N_{lc2} Q_2)^{-1} \\ &= (I - \varepsilon_1 D_{lc} Q_1 - \varepsilon_2 N_{lc2} Q_2)^{-1} \end{aligned}$$

which is (4).

By the same token, (4) is obtained from (5c) and (5d) also. ∇

In their important paper [7], Sugie and Vidyasagar assume that N_1 and N_2 are related by

$$N_1(s) = L(s) N_2(s) \tag{6}$$

where the zeros and poles of L are disjoint from those of D_{lr}. Notice that L can be improper and unstable (but of course N_1 is proper and stable, by definition). This relationship between N_1 and N_2 is a rather mild one. The authors call it "mode readability," a weaker condition than "readability," assumed by [1] and [5], in which L is constant. Sugie and Vidyasagar, allow perturbations of L even though not arbitrary. We omit here the class of allowed perturbations of L for the sake of brevity, remitting it to that paper. They make the following *conjecture*: the relationship (6) is necessary for robust tracking. Then they prove that the compensator, which solve the problem must be such that $LD_c\alpha_m^{-1}$ has its poles disjoint from those of r. So, it is generalized the idea that the compensator (with L) must incorporate a replicated—in the multivariable case—internal model of the signal to be tracked.

We show in Section III that if N_1 is fixed, while N_2 and D are perturbed "arbitrarily," the problem has no solution. In Section IV we show that if P_1 is fixed and P_2 is "arbitrarily" perturbed, the problem does have a solution only if P_1 is stable. In this case we will see that the compensator (with P_1) must incorporate an *inverse* internal model of the signal to be tracked.

III. Perturbing N_2 and D and Fixing N_1

The allowed perturbations in this note are those in the sense of the previous Lemma, i.e., they are arbitrary, but sufficiently small *s.t.* $I + D_{lc}\Delta_D + N_{lc2}\Delta_N$ is unimodular.

Theorem 1: Perturb D and N_2 "arbitrarily" (in the sense defined above), while maintaining N_1 fixed. Then the robust tracking problem has no solution.

Proof: Perturb $D \to D + \Delta_D$ and fix N_1 and N_2. Then, it is easy to obtain $z = N_1(I + D_{lc}\Delta_D)^{-1} N_{lc1} r = N_1(I - \varepsilon_1 D_{lc} Q_1) N_{lc1} r$, in view of (4). Hence,

$$\begin{aligned} e &= r - z \\ &= [I - N_1(I - \varepsilon_1 D_{lc} Q_1) N_{lc1}] r \\ &= (I - N_1 N_{lc1}) r + N_1 \varepsilon_1 D_{lc} Q_1 N_{lc1} r. \end{aligned}$$

Now, in view of (2), (2*), and (3), asymptotic tracking implies

$$N_1 D_{lc} Q_1 N_{lc1} r \in \boldsymbol{M}(S), \qquad \forall Q_1 \in \boldsymbol{M}(S).$$

And from the definition of r we get

$$N_1 D_{lc} Q_1 N_{lc1} D_{lr}^{-1} \in \boldsymbol{M}(S), \qquad \forall Q_1 \in \boldsymbol{M}(S). \tag{7}$$

Now, in view of (2) and (3) it is clear that N_{lc1} and D_{lr} are *r.c.*

Let $N_{lc1} D_{lr}^{-1} =: A^{-1}B$, a *l.c.* factorization. It is clear that A and D_{lr} have the same invariant factors. Then, in view of (7), we have

$$N_1 D_{lc} Q_1 A^{-1} \in \boldsymbol{M}(S), \qquad \forall Q_1 \in \boldsymbol{M}(S). \tag{8}$$

Let S_A be the Smith form of A and let U and V be unimodular matrices such that $A = U S_A V$. Define $Q_{11} = Q_1 V^{-1}$. Then from (8)

$$N_1 D_{lc} Q_{11} S_A^{-1} \in \boldsymbol{M}(S), \qquad \forall Q_{11} \in \boldsymbol{M}(S). \tag{9}$$

Let α_j be the invariant factors of A, $j \in \boldsymbol{m}$, where $\boldsymbol{m} = \{1, 2, \cdots, m\}$. Let n_j, $j \in \boldsymbol{p}$, be the columns of $N_1 D_{lc}$. Let q_{ki} be the elements of Q_{11}. Choose Q_{11} such that

$$q_{jm} = 1, \quad q_{ki} = 0 \qquad \forall (k, i) \neq (j, m).$$

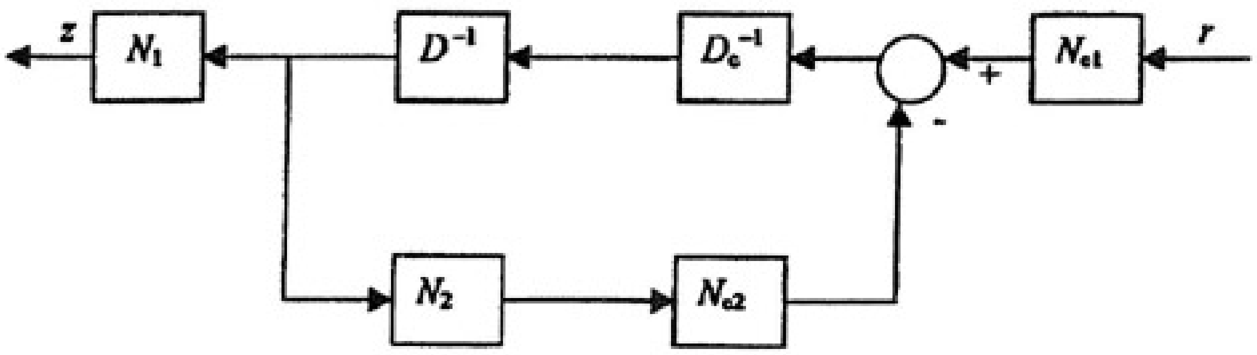

Fig. 2.

Then, straightforward calculations from (9) give $n_j \alpha_m^{-1} \in M(S)$, $\forall j \in p$, or,

$$N_1 D_{lc} \alpha_m^{-1} \in M(S). \tag{10}$$

Now, perturb $N_2 \rightarrow N_2 + \Delta_N$, fixing D and N_1. From the block diagram we obtain

$$z = N_1 (I + N_{lc2} \Delta_N)^{-1} N_{lc1} r = N_1 (I - \varepsilon_2 N_{lc2} Q_2) N_{lc1} r$$

in view of (4). Hence,

$$e = (I - N_1 N_{lc1}) r + N_1 \varepsilon_2 N_{lc2} Q_2 N_{lc1} r.$$

So, robust tracking implies, in view of (2), (2*), and (3),

$$N_1 N_{lc2} Q_2 N_{lc1} D_{lr}^{-1} \in M(S) \qquad \forall Q_2 \in M(S).$$

Defining matrices A, S_A as above [after (8)] and choosing an appropriate matrix in the same way as Q_{11}, we obtain

$$N_1 N_{lc2} \alpha_m^{-1} \in M(S). \tag{11}$$

From (10) and (11), we have

$$\alpha_m^{-1} N_1 [D_{lc}, N_{lc2}] \in M(S).$$

But from (1), D_{lc} and N_{lc2} are *l.c.*, hence, the last implies $\alpha_m^{-1} N_1 \in M(S)$. Hence there exists $N_{11} \in M(S)$ such that $N_1 = \alpha_m N_{11}$. But from (2) and (3), there should exist $W \in M(S)$ such that

$$N_{11} N_{lc1} \alpha_m + W D_r = I.$$

It is clear that there is no solution for this equation in N_{lc1} and W, since $\alpha_m I$ and D_r are not *r.c.*, proving the theorem. ∇

Remark 2: A reviewer of a previous version of this note remarks on the result above: "If P_1 perturbs and y and z have no relation, then the robust tracking has no solution. This result seems to be obvious because we have no way to obtain any information on z in this case."

Example 1: We consider now the following simple scalar example, regarding Theorem 1 and the previous remark $P_1 = P_2 = 1/(s-1)$, $r = A/s$, $A \in R$, $\Delta_D = \delta_1 (s+1)^{-1}$, $\Delta_N = \delta_2 (s+1)^{-1}$, δ_1, $\delta_2 \in R$, arbitrary and sufficiently small.

We redraw the block diagram (Fig. 2) in a more appropriate way for the problem at hand. Let T_1 denote the transfer matrix between the input to N_1 and the output of N_{c1}. In the nominal situation (no perturbation of N_2 and D), we have $T_1 = 1$. Let the perturbed signals and parameters (as a result of perturbation of D and N_2) be denoted with a $(*)$, e.g., T_1^*, e^*, etc. It is clear that $T_1^* = 1/(1 + \delta_1 (s+1)^{-1} D_c + \delta_2 (s+1)^{-1} N_{c2})$. Hence,

$$\begin{aligned} \Delta_{T1} &:= T_1^* - T_1 \\ &= -(\delta_1 (s+1)^{-1} D_c + \delta_2 (s+1)^{-1} N_{c2}) / \\ &\quad (1 + \delta_1 (s+1)^{-1} D_c + \delta_2 (s+1)^{-1} N_{c2}). \end{aligned}$$

Now, $e^* = (1 - N_1 N_{c1}) r - N_1 \Delta_{T1} N_{c1} r$. So, the contribution to the error due to the perturbation in D and N_2 is

$$\begin{aligned} \Delta_e &:= e^* - e \\ &= -N_1 \Delta_{T1} N_{c1} r \\ &= -(s+1)^{-2} N_{c1} (\delta_1 D_c + \delta_2 N_{c2}) A / \\ &\quad s(1 + (s+1)^{-1} (\delta_1 D_c + \delta_2 N_{c2}). \end{aligned}$$

In view of the fact that $s(s+1)^{-1}$ and N_{c1} are coprime, while δ_1 and δ_2 are arbitrary, it is clear that $\Delta_e \notin S$, because D_c and N_{c2} are coprime. Therefore, the problem has no solution.

IV. Solution of the Problem with P_1 Fixed and P_2 Perturbed

Theorem 2: Perturb N_2 and D "arbitrarily" (in the sense above) and fix P_1. Then the asymptotic tracking problem has a solution only if P_1 is stable. If this is the case, C is a robust tracking compensator if and only if it stabilizes the loop and

a) $(I - P_1 D N_{lc1}) D_{lr}^{-1} \in M(S)$;
b) $P_1 N_{c2} \alpha_m^{-1} \in M(S)$.

Proof: We prove first that P_1 has to be stable. Condition a) of the Theorem is necessary for the nominal plant. Perturb $D \rightarrow D + \Delta_D$. Then,

$$\begin{aligned} e &= [I - P_1 (D + \Delta_D)(I + D_{lc} \Delta_D)^{-1} N_{lc1}] r \\ &= [I - (P_1 D + P_1 \Delta_D)(I - \varepsilon_1 D_{lc} Q_1) N_{lc1}] r \\ &= [I - P_1 D N_{lc1} - (-\varepsilon_1 P_1 D D_{lc} Q_1 \\ &\qquad + P_1 \Delta_D (I - \varepsilon_1 D_{lc} Q_1)) N_{lc1}] r \\ &= [I - P_1 D N_{lc1} + (\varepsilon_1 P_1 D D_{lc} Q_1 - \varepsilon_1 P_1 Q_1) N_{lc1}] \\ &\quad \cdot D_{lr}^{-1} N_{lr} r_0. \end{aligned}$$

So, asymptotic tracking implies in view of a) of the Theorem

$$P_1 (I - D D_{lc}) Q_1 N_{lc1} D_{lr}^{-1} \in M(S), \qquad \forall Q_1 \in M(S).$$

And in view of (1b), we have

$$P_1 N_{c2} N_{l2} Q_1 N_{lc1} D_{lr}^{-1} \in M(S), \qquad \forall Q_1 \in M(S).$$

Choosing appropriate Q_1's as in the proof of Theorem 1, the last implies

$$P_1 N_{c2} N_{l2} \alpha_m^{-1} \in M(S). \tag{12}$$

Let $N_{11} D_1^{-1}$ be a *r.c.* factorization of P_1. It is clear that stability of the loop implies the left coprimeness of D_1 and N_{c2}, because D_1 is a left divisor of D. Then perturb N_{l2} at the outset, if necessary, so that (12) is not satisfied. Hence, P_1 has to be stable.

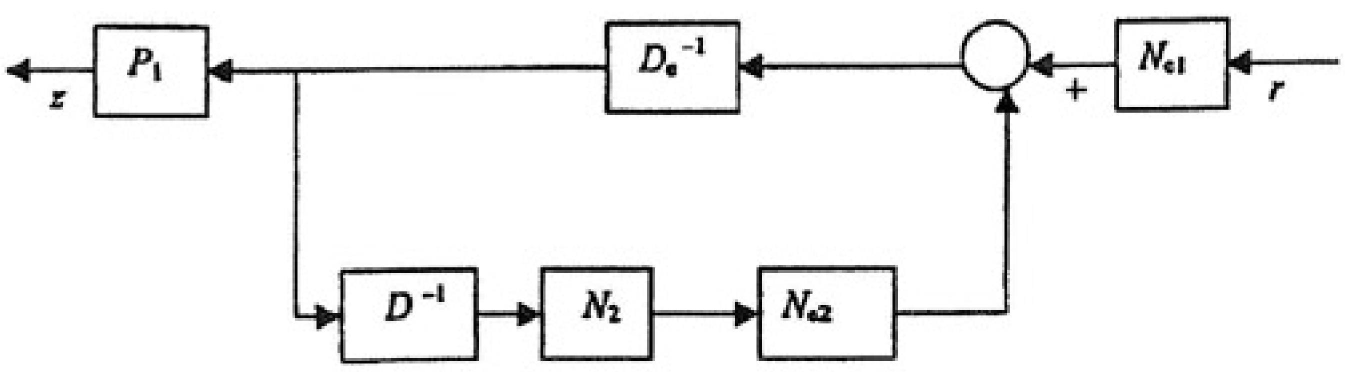

Fig. 3.

We proceed now to the proof of the necessity of condition b) of the theorem, recalling that condition a) is necessary for the nominal loop. Perturb N_2. Then,

$$z = P_1 D(I + N_{lc2}\Delta_N)^{-1} N_{lc1} r = P_1 D(I - \varepsilon_2 N_{lc2} Q_2) N_{lc1} r.$$

Hence, asymptotic tracking implies

$$[I - P_1 D(I - \varepsilon_2 N_{lc2} Q_2) N_{lc1}] D_{lr}^{-1} \in M(S), \qquad \forall Q_2 \in M(S).$$

And this implies in view of a):

$$P_1 D N_{lc2} Q_2 N_{lc1} D_{lr}^{-1} \in M(S), \qquad \forall Q_2 \in M(S).$$

In view of (1b), we have

$$P_1 N_{c2} D_l Q_2 N_{lc1} D_{lr}^{-1} \in M(S), \qquad \forall Q_2 \in M(S).$$

Choosing appropriate Q_2's, it can be shown, as in the proof of Theorem 1, that the last implies

$$P_1 N_{c2} D_l \alpha_m^{-1} \in M(S). \tag{13}$$

From this and (12),

$$P_1 N_{c2} [D_l, N_{l2}] \alpha_m^{-1} \in M(S).$$

And in view of the left coprimeness of D_l and N_{l2}, we have

$$P_1 N_{c2} \alpha_m^{-1} \in M(S). \tag{14}$$

(Sufficiency): Perturbing D and N_2, we have

$$\begin{aligned} z &= P_1 (D + \Delta_D)(I + D_{lc}\Delta_D + N_{lc2}\Delta_N)^{-1} N_{lc1} r \\ e &= [I - P_1(D + \Delta_D)(I - \varepsilon_1 D_{lc} Q_1 - \varepsilon_2 N_{lc2} Q_2) N_{lc1}] r \\ &= (I - P_1 D N_{lc1}) r - [P_1 D(-\varepsilon_1 D_{lc} Q_1 - \varepsilon_2 N_{lc2} Q_{2_2}) \\ &\quad + P_1 \Delta_D (I - \varepsilon_1 D_{lc} Q_1 - \varepsilon_2 N_{lc2} Q_2)] N_{lc1} r. \end{aligned}$$

Call X the first term of the right-hand side (RHS), which according to a), belongs to $M(S)$. Then, in view of (5a),

$$\begin{aligned} e &= X - [P_1(-\varepsilon_1 D D_{lc} Q_1 - \varepsilon_2 D N_{lc2} Q_2) + P_1 \varepsilon_1 Q_1] N_{lc1} r \\ &= X - P_1 \varepsilon_1 (I - D D_{lc}) Q_1 N_{lc1} r + P_1 \varepsilon_2 D N_{lc2} Q_2 N_{lc1} r. \end{aligned}$$

So, in view of (1b), we have

$$e = X - \varepsilon_1 P_1 N_{c2} N_{l2} Q_1 N_{lc1} r + \varepsilon_2 P_1 N_{c2} D_l Q_2 N_{lc1} r.$$

The proof is complete in view of the fact that $\alpha_m r \in M(S)$. ∇

Remark 3: The meaning of this result was given by the reviewer mentioned in the previous remark: "If P_1 is fixed, the information on z can be obtained via u when P_2 perturbs arbitrarily. This is the main reason why the track can be achieved when y contains no information on z. In this case, the feedforward control plays the essential role in tracking. Therefore the feedforward control should be designed in such a way that y does not affect u with respect to the modes of r (i.e., C_2 must contain the blocking zeros with respect to the poles of r)."

Example 2: Let $P_1 = (s+1)^{-1}$, P_2, r, Δ_D, and Δ_N as in the previous example. We redraw the block diagram (Fig. 3).

Let T_2 be the transfer matrix between the input to P_1 and the output of N_{c1}. It is easy to see that $T_2 = D$. And in view of (1a), we have

$$\begin{aligned} \Delta_{T2} &:= T_2^* - T_2 \\ &= (N_2 N_{c2} \Delta_D - D N_{c2} \Delta_N)/(1 + D_c \Delta_D + N_{c2} \Delta_N) \\ &= (s+1)^{-2} N_{c2} (\delta_1 - \delta_2 (s-1))/ \\ &\quad (1 + (s+1)^{-1}(\delta_1 D_c + \delta_2 N_{c2})). \end{aligned}$$

Hence, the contribution to the error, due to the perturbations in the plant is

$$\begin{aligned} \Delta_e = &(s+1)^{-3}(\delta_1 - \delta_2(s-1)) N_{c2} A/ \\ &s(1 + \delta_1 (s+1)^{-1} D_c + \delta_2 (s+1)^{-1} N_{c2}). \end{aligned}$$

It is clear that the problem is solved if $s(s+1)^{-1}$ is a factor of N_{c2}, as pointed out by Remark 3. □

The solvability condition for the problem handled in this section is given next.

Theorem 3: Assume P_1 fixed and stable. Then there exists a two-degrees-of-freedom compensator, which solves the robust asymptotic tracking problem if, and only if, $P_1 D$ and $\alpha_m I$ are left coprime.

Proof: (Only if): From (1b), we have

$$P_1 D D_{lc} + P_1 N_{c2} N_{l2} = P_1.$$

But P_1 has full row rank, otherwise asymptotic tracking would be impossible. Let P_1^R be a right inverse of P_1. Then,

$$P_1 D D_{lc} P_1^R + P_1 N_{c2} N_{l2} P_1^R = I.$$

Let ξ be a mode of r. Then from Theorem 2, b), it is clear in view of the above identity that $P_1 D(\xi)$ has full row rank. This is equivalent to saying that $P_1 D$ and $\alpha_m I$ are left coprime.

(If): If $P_1 D$ and $\alpha_m I$ are left coprime, then $\det(D(\xi)) \neq 0$ for every ξ *s.t.* $\alpha_m(\xi) = 0$. So, we may choose $N_{c2} = \alpha_m X$, satisfying condition b) of Theorem 2, with X and D_c *s.t.* the loop is stable. On the other hand, the left coprimeness of $P_1 D$ and $\alpha_m I$ implies the existence of Y and $W \in M(S)$, *s.t.*

$$P_1 D Y + W \alpha_m I = I.$$

But $\alpha_m I = V D_{lr}$, for some $V \in M(S)$. Hence, we have

$$P_1 D Y + W V D_{lr} = I.$$

Condition a) of Theorem 2 is satisfied choosing $N_{lc1} = Y$, completing the proof. ∇

V. Concluding Remarks

i) The problem handled in Section IV has no solution with one-degree-of-freedom feedback compensator. Indeed, conditions a) and b) of Theorem 1 contradict each other if $N_{lc1} = N_{lc2}$.

ii) The result presented in the last section contrasts evidently with the so called *internal model principle*: see [5] and [1] and, more recently, [7]. According to the internal model principle, in order to obtain robust tracking, the compensator must incorporate a replicated—in the multivariable problem—internal model of the exogenous signal.

Now, in condition b) of our Theorem 2, we have an "inverse internal model" in the sense that the exogenous poles affect the numerator of the feedback channel of the compensator, not the denominator of it. See Remark 3 above, explaining the apparent contradiction. Indeed, in our note we assume that P_1 is fixed, while in the three papers mentioned above, a relationship is assumed between P_1 and P_2, namely, $P_1 = LP_2$. In the first two papers quoted above, L is a fixed matrix of real numbers and then it is said that z is "readable" from y. In the paper by Sugie and Vidyasagar [7], L is a rational matrix, not necessarily proper or stable, and whose zeros and poles are disjoint from the exogenous modes; besides, L is perturbable in a restricted sense and the authors call z "mode readable" from y. It is clear that mode readability is a weaker condition than readability. Sugie and Vidyasagar believe that mode readability is a *necessary* condition for robust tracking. We assumed that P_1 is fixed, while P_2 is arbitrarily perturbed, so in our assumption there is no mode readability and *a fortiori* no readability.

iii) A practical situation of Theorem 2 might occur, say, when P_1 would refer to the digital part, so in most cases virtually unperturbable, of a plant.

Acknowledgment

The author acknowledges the insightful comments of a reviewer after the first draft of this note. The comments are quoted in Remarks 2 and 3. In fact, the note was significantly improved, induced by those remarks.

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H_∞ State Feedback Control for Discrete Singular Systems

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***Abstract*—This note deals with the problem of state feedback H_∞ control for discrete singular systems. It is not assumed that the singular system under consideration is necessarily regular. The problem we address is the design of a state feedback controller, such that the resulting closed-loop system is not only regular, causal, and stable, but also satisfies a prescribed H_∞-norm-bound condition. In terms of certain matrix inequalities, a necessary and sufficient condition for the solution to this problem is obtained, and a suitable state feedback-control law is also given.**

***Index Terms*—Discrete singular systems, H_∞ control, state feedback.**

I. Introduction

The problem of H_∞ control for standard state-space systems has received considerable interest over the last decade. This problem is concerned with constructing a controller such that the closed-loop system is stable and the norm of the closed-loop transfer function is minimized. It has been shown that a solution to this problem for linear time-invariant state-space systems involves solving a set of Riccati equations (see [4], [13], [19], and the references therein). In the context of linear discrete-time state-space systems, the results for continuous systems have been extended, see, e.g., [3], [5], and [17]. Some efforts have also been made to deal with the H_∞ controller design for discrete-time state-space systems subjected to plant parameter perturbations [2], [6], [18].

Recently, much attention has been given to the extensions of the results of H_∞ control theory for state-space systems to singular systems (also known as descriptor systems, implicit systems, generalized state-space systems, differential-algebraic systems, or semistate systems [1], [9]). For example, Takaba *et al.* [14] have considered the H_∞ control problem for singular systems by using the J-spectral factorization approach [(J, J')-spectral factorization for discrete singular systems can be found in [8]]. Masubuchi *et al.* [11] have shown that the solution of the H_∞ control problem for singular systems can be obtained by solving a set of matrix inequalities. Moreover, Wang *et al.* [15] have presented the necessary and sufficient conditions based on two generalized algebraic Riccati equations for the solution to the above problem. It should be pointed out that all of the works mentioned above are concerned with the H_∞ control problem for continuous singular systems. Though there are many publications on the H_∞ controller design for discrete state-space systems, it appears that no effort has been made to extend the available results to the case of discrete singular systems.

In this note, we investigate the problem of H_∞ control for discrete singular systems. The singular system under consideration is not assumed to be necessarily regular. This implies that we study the H_∞ controller design for the general case of discrete singular systems. The motivation for studying these singular systems can be found in [7], [10], and [12]. The purpose of this note is to design a state feedback controller such that the resulting closed-loop system is regular, causal, and stable while satisfying an H_∞ norm condition with a prescribed level. To solve this problem, we first give a necessary and sufficient condition for a discrete singular system to be regular, causal, stable and to

Manuscript received June 16, 1999; revised October 28, 1999. Recommended by Associate Editor, L. Dai.

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Publisher Item Identifier S 0018-9286(00)06091-8.