



**Marcos de Oliveira Lage Ferreira**

**Simulation of flows with suspended and  
floating particles**

**TESE DE DOUTORADO**

Thesis presented to the Postgraduate Program in  
Mathematics of the Departamento de Matemática, PUC–Rio  
as partial fulfillment of the requirements for the degree of  
Doutor em Matemática

Advisor : Prof. Hélio Côrtes Vieira Lopes  
Co–Advisor: Prof. Marcio da Silveira Carvalho

Rio de Janeiro  
December 2009



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**Prof. Hélio Côrtes Vieira Lopes**

Advisor

Departamento de Matemática — PUC–Rio

**Prof. Marcio da Silveira Carvalho**

Co–Advisor

Departamento de Engenharia Mecânica — PUC–Rio

**Prof. Geovan Tavares dos Santos**

Departamento de Matemática — PUC–Rio

**Prof. Eurípedes do Amaral Vargas Junior**

Departamento de Engenharia Civil — PUC–Rio

**Prof. Francisco Duarte Moura Neto**

Instituto Politécnico do Rio de Janeiro — IPRJ

**Prof. Marcio Arab Murad**

Laboratório Nacional de Computação Científica — LNCC

**Prof. Luiz Carlos Pacheco Rodrigues Velho**

Instituto Nacional de Matemática Pura e Aplicada — IMPA

**Prof. José Eugenio Leal**

Coordinator of the Centro Técnico Científico da PUC–Rio

Rio de Janeiro, — December 21, 2009

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### Marcos de Oliveira Lage Ferreira

Graduated in Mathematics from Univesidade do Estado do Rio de Janeiro in 2004, he obtained the degree of Mestre em Matemática at PUC–Rio in 2006.

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## Abstract

Ferreira, Marcos de Oliveira Lage; Lopes, Hélio Côrtes Vieira; Carvalho, Marcio da Silveira. **Simulation of flows with suspended and floating particles**. Rio de Janeiro, 2009. 94p. D.Sc. Thesis — Departamento de Matemática, Pontifícia Universidade Católica do Rio de Janeiro.

Flows with particles in suspension is still a challenging task important in many applications such as sedimentation, rheology and fluidized suspensions. The coupling between the suspending liquid flow and the particles' motion is the central point in the complete understanding of these phenomena. Moreover, the study of the evolution of the configuration of particles at an interface between two immiscible fluid phases is also a very important research area, since it occurs in many engineering and industrial processes like slurries transport and drying processes of micro and nano suspension coating. This work proposes a new fictitious domain formulation based on Lagrange multipliers that solves the Navier–Stokes and rigid body equations to perform the simulation of the flow and of the flotation of particles embedded on one or more immiscible fluid phases that we numerically discretize using a fully implicit and coupled finite element approach. The method is validated using different test problems. The results obtained are compared with previous works, and the agreement is excellent.

## Keywords

Fluid Simulation; Capilarity Force; Finite Elements; Fictitious Domain; Lagrange Multipliers;

## Resumo

Ferreira, Marcos de Oliveira Lage; Lopes, Hélio Côrtes Vieira; Carvalho, Marcio da Silveira. **Simulação de fluxos com partículas suspensas e flutuantes.** Rio de Janeiro, 2009. 94p. Tese de Doutorado — Departamento de Matemática, Pontifícia Universidade Católica do Rio de Janeiro.

Fluxos de partículas em suspensão ainda são um desafio importante em muitas aplicações, tais como sedimentação, reologia e suspensões em leitos fluidizados. O acoplamento entre o fluxo da fase líquida e o movimento das partículas é o ponto central para a compreensão completa deste fenômeno. Além disso, o estudo da evolução da disposição das partículas na interface entre duas fases de fluido imiscíveis é também uma área de pesquisa muito importante, pois tal fenômeno ocorre em engenharia em muitos processos industriais, tais como transporte de pastas e a secagem de micro e nano coberturas. Este trabalho propõe uma nova formulação baseada em domínios fictícios e multiplicadores de Lagrange que resolve as equações de Navier–Stokes e de corpo rígido para realizar a simulação do fluxo e da flutuação de partículas submersas em uma ou mais fases de fluidos imiscíveis. Para obtermos a solução discreta das equações utilizamos o método dos elementos finitos e uma abordagem totalmente implícita e acoplada. Esta formulação foi validada usando diferentes problemas de teste. Os resultados obtidos foram comparados com trabalhos anteriores e a concordância foi excelente.

## Palavras-chave

Simulação de Fluidos; Força de Capilaridade; Elementos Finitos; Domínio Fictício; Multiplicadores de Lagrange;

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*“Só sabemos com exatidão quando sabemos pouco; à medida que vamos adquirindo conhecimentos, instala-se a dúvida.”*

**Johan Wolfgang Von Goethe**, escritor alemão.