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8
Appendix

8.1
Basic notation

The vector spaces of $n \times n$ real matrices, real diagonal matrices, real symmetric matrices and real skew-symmetric matrices are denoted by $\mathcal{M}_n$, $\mathcal{D}_n$, $\mathcal{S}_n$ and $\mathcal{A}_n$, respectively. The group of orthogonal matrices is $O(n) = \{Q \in \mathcal{M}_n \,|\, Q^T Q = I\}$ and $SO(n)$ is the subgroup of orthogonal matrices with unit determinant. The group of $n \times n$ real upper triangular matrices is denoted by $\mathcal{U}_n$.

8.2
Eigen-Smoothness

In this section we establish the smooth dependence of eigenvalues and eigenvectors on a symmetric simple spectrum matrix S and present a formula for the derivative of the k-th eigenvalue with respect to S. The first step is to show that we are in an open set of $\mathcal{S}_n$, and that, therefore, the question of differentiability make sense.

Proposition 26 *The set of real $n \times n$ symmetric matrices with simple spectrum is open.*

Proof: Let S be a real symmetric matrix with simple spectrum. Then its characteristic polynomial p satisfies the hypotheses of the lemma below, implying that if $\tilde{S}$ is sufficiently close to S, the coefficients of $\tilde{p}$ will be close to those of p and that, therefore, the eigenvalues of $\tilde{S}$ will still be distinct. ■

Lemma 27 (Polynomial roots) *Let $p \colon \mathbb{R} \to \mathbb{R}$ be a polynomial of degree n with real coefficients whose roots are all distinct and real. Then there exists a neighborhood U of p in $\mathcal{P}_n$ - the space of polynomials of degree n - such that all polynomials in U also have distinct real roots.*

Proof: Let

$$f\colon \begin{array}{ccc} \mathcal{P}_n \times \mathbb{R} & \to & \mathbb{R} \\ (q,x) & \mapsto & q(x) \end{array}.$$

Denoting by r_i the roots of p,

$$f(p,x) = p_n \prod_i (x - r_i)$$

and

$$D_2 f(p,x) = p_n \sum_k \prod_{i \neq k} (x - r_i).$$

Therefore, at a root r_j,

$$D_2(p, r_j) = p_n \prod_{i \neq j} (r_j - r_i) \neq 0,$$

and the implicit function theorem applies. We may therefore conclude that, locally around p, each root varies smoothly with the polynomial. Since the roots r_j are all distinct, we may restrict the neighborhood U around p in order to guarantee that they all remain distinct for all $q \in U$. ■

Strictly speaking, these results already show us that each eigenvalue varies smoothly with respect to a symmetric matrix if we restrict ourselves to a sufficiently small neighborhood of a simple spectrum symmetric matrix. However, there is a sense in which the corresponding eigenvectors also vary smoothly with respect to the matrix. This smooth dependence of eigenvalues and eigenvectors is our next result.

Proposition 28 (Eigen-Smoothness) *Let S_0 be a real symmetric $n \times n$ matrix with simple spectrum such that $S_0 v_0 = \lambda_0 v_0$ for $v_0 \in \mathbb{S}^{n-1}$. Then there exist:*

- *A neighborhood $U \subset \mathcal{S}_n$ of S_0.*
- *A neighborhood $V \subset \mathbb{S}^{n-1} \times \mathbb{R}$ of (v_0, λ_0).*
- *A smooth function $G \colon U \to V$.*

such that, in $U \times V$,

$$Sv = \lambda v \Leftrightarrow (v, \lambda) = G(S).$$

Proof: Let us define

$$\begin{array}{rccc} F\colon & \mathcal{S}_n \times (\mathbb{S}^{n-1} \times \mathbb{R}) & \to & \mathbb{R}^n \\ & (S, v, \lambda) & \mapsto & Sv - \lambda v \end{array}.$$

The derivative of F with respect to the variable in $(\mathbb{S}^{n-1} \times \mathbb{R})$ at $p = (S_0, v_0, \lambda_0)$ is

$$\begin{array}{rccc} D_2F(p)\colon & T_{v_0}\mathbb{S}^{n-1} \times \mathbb{R} & \to & \mathbb{R}^n \\ & (w, l) & \mapsto & S_0 w - \lambda w - l v_0 \end{array}.$$

Thus, since $\langle v_0, w\rangle = 0$ and S_0 is symmetric,

$$D_2F(p)(w,l) = 0 \Rightarrow \langle S_0 w - \lambda w - l v_0, v_0\rangle = 0 \Rightarrow l = 0$$

and

$$S_0 w - \lambda w = 0 \Rightarrow w = k v_0 \Rightarrow w = 0,$$

implying that $D_2F(p)$ is injective, and therefore, bijective. The conclusion now follows from the Implicit Function Theorem. ■

Now that we have established the smooth nature of this dependence, we may explicitly compute some derivatives. Since we are primarily interested in eigenvalues, the function we will differentiate is the *ordered spectrum map.*

Definition 29 (Ordered Spectrum Map) *The function*

$$\begin{array}{rccc} \sigma_0\colon & \mathcal{S}_n & \to & \mathbb{R}^n \\ & S & \mapsto & \lambda_1(S) \geq \ldots \geq \lambda_n(S) \end{array}$$

which takes a symmetric matrix S to its ordered eigenvalues is called the ordered spectrum map.

Proposition 30 (Ordered Spectrum Derivative) *Let S_0 be a symmetric matrix with simple spectrum. The derivative of σ_0 at S_0 is*

$$\begin{array}{rccc} D\sigma_0(S_0)\colon & \mathcal{S}_n & \to & \mathbb{R}^n \\ & P & \mapsto & (\langle P v_1, v_1\rangle, \ldots, \langle P v_n, v_n\rangle) \end{array}$$

where v_k is a unit eigenvector of S_0 associated with λ_k.

Proof: First let us restrict ourselves to an adequate neighborhood U of S_0 where σ_0 is smooth. Given an arbitrary differentiable curve $S\colon (-\epsilon, \epsilon) \to U$ with $S(0) = S_0$, the following equation holds:

$$S(t) v_k(t) = \lambda_k(t) v_k(t)$$

where $\lambda_k(t)$ is the k-th eigenvalue of $S(t)$ and $v_k(t)$ the associated unit eigenvector. Differentiating with respect to t we obtain

$$\dot{S} v_k + S \dot{v_k} = \dot{\lambda_k} v_k + \lambda_k \dot{v_k}.$$

Taking the inner product of each side with respect to v_k and recalling that $\langle v_k, v_k\rangle = 1 \Rightarrow \langle v_k, \dot{v_k}\rangle = 0$,

$$\begin{aligned}
\langle \dot{S}v_k, v_k\rangle + \langle S\dot{v_k}, v_k\rangle &= \langle \dot{\lambda_k}v_k, v_k\rangle + \langle \lambda_k\dot{v_k}, v_k\rangle \\
\langle \dot{S}v_k, v_k\rangle + \langle \dot{v_k}, Sv_k\rangle &= \dot{\lambda_k} \\
\langle \dot{S}v_k, v_k\rangle + \langle \dot{v_k}, \lambda_k v_k\rangle &= \dot{\lambda_k}.
\end{aligned}$$

Finally, we are left with:

$$\dot{\lambda_k}(t) = \langle \dot{S}(t)v_k(t), v_k(t)\rangle,$$

an expression for the directional derivative of λ_k along a curve $S(.) \subset U$. Since $\sigma_0|_U$ is smooth, we may compute the total derivative $D\sigma_0(S_0)$ from these directional derivatives. Thus,

$$D\sigma_0(S_0)P = (\langle Pv_1, v_1\rangle, \ldots, \langle Pv_n, v_n\rangle).$$

■

8.3
The manifolds $O(n)$ and $SO(n)$

From the implicit function theorem, given a smooth $f\colon \mathbb{R}^{n+d} \to \mathbb{R}^n$ and a regular value c, the level $f^{-1}(c)$ is a differentiable manifold of dimension d.

Let $\mathcal{M}_n$ be the space of $n \times n$ real matrices with the topology induced by the inner product $\langle A, B\rangle = tr(AB^T)$. Thus, since $\|A\|^2 = \sum_{i,j} {A_{ij}}^2$, $\mathcal{M}_n$ is isometric to $\mathbb{R}^{n^2}$ with the usual Euclidean inner product. We follow the notation in Appendix 8.1.

Proposition 31 *$O(n)$ and $SO(n)$ are compact differentiable manifolds of dimension $N = \frac{n(n-1)}{2}$. Moreover, $O(n)$ has exactly two connected components, $SO(n)$ and $ESO(n)$, both pathwise connected, E being an arbitrary orthogonal matrix of negative determinant.*

Proof:

The function

$$\begin{aligned}
f\colon \quad \mathcal{M}_n &\to \quad \mathcal{S}_n \\
A &\mapsto \quad A^T A
\end{aligned}$$

is clearly smooth and, by definition, $O(n) = f^{-1}(I)$. If the derivative $Df(Q)$ has maximum rank for every point $Q \in O(n)$, I will be a regular value of f

and $O(n)$ will be a differentiable manifold of dimension N. Now,

$$\begin{array}{rccc} Df(Q)\colon & \mathcal{M}_n & \to & \mathcal{S}_n \\ & H & \mapsto & H^TQ+Q^TH \end{array}$$

and its kernel is given by the equation $H^TQ + Q^TH = 0$. In the simple case $Q = I$, $H^T + H = 0$, whose solution set is $\ker(Df(I)) = \mathcal{A}_n$. In the general case we employ the substitution $A = Q^TH$ and the kernel becomes $A^T + A = 0$. Therefore A must be antisymmetric and $H = QA$. Since Q is invertible, $\ker(Df(Q))$ has the same dimension as that of $\mathcal{A}_n$. Thus, whatever the case, the Rank-nullity theorem (7) guarantees us that the dimension of the image of $Df(Q)$ is $n^2 - N = \frac{n(n+1)}{2}$, precisely the dimension of $\mathcal{S}_n$. This proves the surjectivity of $Df(Q)$ and, therefore, that every pre-image of I under f is a regular value.

Since f is continuous, $O(n) = f^{-1}(I)$ is closed. Moreover, an arbitrary matrix in $O(n)$ has norm equal to $\sqrt{n}$, hence $O(n)$ is a compact manifold. However, since it contains matrices with determinants $+1$ and -1, $O(n)$ cannot be connected.

The determinant in $O(n)$ is clearly continuous and only takes the values 1 and -1. Thus a neighborhood U of $Q \in O(n)$ can be restricted to $U' \subseteq U$ in order to ensure that $\det|_{U'}$ is constant. If $Q \in SO(n)$, U' will be, by definition, a neighborhood of Q in the induced topology on $SO(n)$. This means that, locally, a point of $SO(n)$ has a neighborhood common to both topologies in question. Now, if $\phi\colon U \subseteq O(n) \to \mathbb{R}^N$ is the original diffeomorphism, its restriction to U' proves that $SO(n)$ is also a differentiable manifold of dimension N. $SO(n)$ is a closed subset of the compact manifold $O(n)$ and is, therefore, compact.

To prove that $SO(n)$ is pathwise connected is an induction on the dimension. Let us show that given any matrix $Q \in SO(n)$ there is a path in $SO(n)$ connecting Q to I. The case $n = 1$ is trivial for $SO(1) = \{(1)\}$. The inductive step uses the following result:

Lemma 32 *For $n \geq 1$ and $Q \in SO(n)$ there exists an arc*

$$\alpha\colon [0,1] \to SO(n)$$

satisfying $\alpha(0) = I$ and $\alpha(1)(Qe_n) = e_n$.

Proof:

The case $n = 1$ is trivial. Let $n \geq 2$. We know that $v = Qe_n \in \mathbb{S}^{n-1}$. Let us proceed by slicing the proof in cases:

1. $v = e_n$.

Let $\alpha(t) = I$.

2. $v \neq e_n$ and $v \neq -e_n$.

   In this case the vectors v and e_n span a plane Π. Let $w \in \mathbb{S}^{n-1}$ be obtained by removing from e_n its projection onto v and normalizing the resultant vector. Completing the set $\{v, w\}$ to an orthonormal basis $\mathcal{B} = \{v, w, w_3, \cdots, w_n\}$ for $\mathbb{R}^n$, we can construct a family of rotations which act on Π and keep all of the w_i vectors fixed. It is enough to choose an adequate family of rotations $\phi_\theta \colon \Pi \to \Pi$ which gradually take v to e_n. Indeed, in the base $\{v, w\}$,

$$\phi_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in SO(2).$$

   We may now use the function

$$\begin{array}{rccc} R_\theta\colon & \Pi \oplus \Pi^\perp & \to & \Pi \oplus \Pi^\perp \\ & (x, y) & \mapsto & (\phi_\theta(x), y) \end{array}$$

   whose matrix representation in the basis $\mathcal{B}$ is

$$R_\theta = \left(\begin{array}{c|c} \phi_\theta & 0 \\ \hline 0 & I \end{array}\right) \in SO(n).$$

3. $v = -e_n$.

   Since $n \geq 2$, there exists $e_i \in \mathbb{S}^{n-1}$ satisfying $e_i \perp e_n$ and $e_i \perp v$. Using the construction outlined in the previous case we arrive at two arcs, one connecting v to e_i and the other connecting e_i to e_n.

■

Returning to the proposition, let

$$\begin{array}{rccc} \gamma\colon & [0, 1] & \to & SO(n) \\ & t & \mapsto & \alpha(t)Q \end{array}.$$

Then $\gamma(0) = Q$ and $\gamma(1)e_n = e_n$. Thus γ is a curve in $SO(n)$ which starts in Q and undoes, a little at a time, the action of Q over e_n. The matrix representation of $\gamma(1)$ has the form

$$\gamma(1) = \left(\begin{array}{ccc|c} & & & 0 \\ & R & & \vdots \\ & & & 0 \\ \hline 0 & \dots & 0 & 1 \end{array} \right)$$

(the last row starts with $n-1$ zeros because the first $n-1$ columns are orthogonal to e_n). We can easily check that the submatrix R above is in $SO(n-1)$. By the induction hypothesis, there exists a curve $\beta \subseteq SO(n-1)$ connecting R to I_{n-1}. The curve

$$\begin{array}{cccc} \delta\colon & [0,1] & \to & SO(n) \\ & t & \mapsto & \left(\begin{array}{ccc|c} & & & 0 \\ & \beta(t) & & \vdots \\ & & & 0 \\ \hline 0 & \dots & 0 & 1 \end{array} \right) \end{array}$$

connects $\gamma(1)$ to I_n. Concatenating the paths γ and δ we finally join Q and I_n. ■

8.4
QR, LU and Bruhat decompositions

We list some basic facts of linear algebra used in the text.

8.4.1
QR-decomposition

Given a real square matrix M, its *QR-decomposition* is the product $M = QR = \mathbf{Q}(M)\mathbf{R}(M)$, where Q is real orthogonal and R is an upper triangular matrix with strictly positive diagonal. On invertible matrices, the QR-decomposition is unique and the factors $\mathbf{Q}(M)$ and $\mathbf{R}(M)$ vary smoothly with M:

$$GL_n \simeq O(n) \times \mathcal{U}_n^+, \quad GL_n^+ \simeq SO(n) \times \mathcal{U}_n^+.$$

8.4.2
LU-decomposition

The *LU-decomposition* of a real square matrix M is the product $M = LU = \mathbf{L}(M)\mathbf{U}(M)$, where L is lower triangular unipotent (i.e., with diagonal entries equal to 1) and U is upper triangular. Let M_k be the $k \times k$ submatrix consisting of the intersection of the first k rows and columns of M for

$k = 1, \ldots, n$. If $\det M_k \neq 0$ for all k, the LU-decomposition is unique and varies smoothly with M. If all the determinants of M_k are positive, the diagonal of U is positive: such matrices M are called *LU-positive*. Clearly, given a matrix M for which the determinants of M_k are nonzero, there is a (unique) sign diagonal matrix E such that EM is *LU-positive*.

A simple inspection verifies that, for any permutation P, a conjugation $P^T LP$ of an invertible lower triangular matrix still admits an *LU-decomposition*.

8.4.3 Bruhat decomposition

The *Bruhat matrix* $P = \mathcal{B}(M)$ associated to an invertible matrix M is the unique permutation matrix P for which there exist lower triangular matrices L_1 and L_2 with $M = L_1 P L_2$ (the *Bruhat decomposition* of M).

The Bruhat matrix admits an equivalent description in terms of *local ranks*. For an invertible $n \times n$ matrix M, let $r_{NE}(M)$ be the $n \times n$ matrix whose entry $r_{i,j}$ is the rank of the submatrix of M consisting of the intersection of rows $1, \ldots, i$ and columns $j, \ldots, n$. Notice that, if L_1 and L_2 are lower triangular matrices, $r_{NE}(L_1 M L_2) = r_{NE}(M)$: in particular, $r_{NE}(M) = r_{NE}(\mathcal{B}(M))$.

This construction admits rather obvious variants for decompositions $M = U_1 P U_2$, $M = L_1 P U_2$ and $M = U_1 P L_2$, associated in a natural way with local SW, NW and SE ranks, respectively. For instance, $P = \mathcal{B}_{SW}(M)$ is the unique permutation matrix for which there exist upper triangular matrices U_1 and U_2 with $M = U_1 P U_2$. If the r_{ij} entry of $r_{SW}(M)$ is the rank of the submatrix with rows $i, \ldots, n$ and columns $1, \ldots, j$ then $r_{SW}(M) = r_{SW}(\mathcal{B}_{SW}(M))$. Also, for the reversal $\pi_{\max} \in S_n$ given by $\pi_{\max}(i) = n + 1 - i$,

$$\begin{aligned} \mathcal{B}_{SW}(M) &= P_{\pi_{\max}} \mathcal{B}(P_{\pi_{\max}} M P_{\pi_{\max}}) P_{\pi_{\max}}, \\ \mathcal{B}_{NW}(M) &= \mathcal{B}(M P_{\pi_{\max}}) P_{\pi_{\max}}. \end{aligned}$$

Notice that M admits a decomposition $M = PLU$ if $P = \mathcal{B}_{SW}(M)$ or $P = \mathcal{B}_{NW}(M)$.

8.5 Permutations and inversions

The permutations in $\{1, 2, \ldots, n\}$ form the group S_n. The identity permutation is e and the reversal $\pi_{\max} \in S_n$ is given by $\pi_{\max}(i) = n+1-i$. An *inversion* of a permutation π is a pair (i, j) for which $i < j$ but $\pi(i) > \pi(j)$. A *descent* of π is an index i, $1 \leq i \leq n-1$, such that $\pi(i+1) < \pi(i)$ (so that $(i, i+1)$ is an inversion). Let $i(\pi)$ (resp. $d(\pi)$) be the number of inversions (resp. descents) of π. It is well known (11) that

$$\sum_{\pi \in S_n} q^{i(\pi)} = (1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1}).$$

On the other hand, the polynomial

$$A_n(q) = \sum_{\pi \in S_n} q^{1+d(\pi)}$$

is known as the *Eulerian polynomial* and its q^k coefficient is denoted by $A(n,k)$ (an *Eulerian number*).

Local ranks (Appendix 8.4) define the *Bruhat order* in S_n: $\tilde{\pi} \leq \pi$ if and only if $r_{SW}(P_{\tilde{\pi}})$ is entry-wise less than or equal to $r_{SW}(P_\pi)$. Similarly, we write $\tilde{\pi} < \pi$ if $\tilde{\pi} \leq \pi$, $\tilde{\pi} \neq \pi$. Also, $\tilde{\pi}$ is an *immediate predecessor* of π if $\tilde{\pi} < \pi$ and there exists no permutation $\hat{\pi}$ with $\tilde{\pi} < \hat{\pi} < \pi$. It is easy to see that $\tilde{\pi}$ is an immediate predecessor of π if and only if the following conditions hold. First, there exist indices $i_1 < i_2$ and $j_1 < j_2$ such that the matrices P_{π_1} and P_{π_2} differ only in the entries

$$\begin{aligned} &(P_{\tilde{\pi}})_{(i_1,j_1)} = 1, \quad (P_{\tilde{\pi}})_{(i_1,j_2)} = 0, \quad (P_{\tilde{\pi}})_{(i_2,j_1)} = 0, \quad (P_{\tilde{\pi}})_{(i_2,j_2)} = 1, \\ &(P_{\pi})_{(i_1,j_1)} = 0, \quad (P_{\pi})_{(i_1,j_2)} = 1, \quad (P_{\pi})_{(i_2,j_1)} = 1, \quad (P_{\pi})_{(i_2,j_2)} = 0. \end{aligned}$$

Second, these are the only nonzero entries of the submatrices of $P_{\tilde{\pi}}$ and P_π at the intersection of rows $i_1, \ldots, i_2$ and columns $j_1, \ldots, j_2$. In particular, if $\tilde{\pi}$ is an immediate predecessor of π then $i(\pi) = i(\tilde{\pi}) + 1$.

8.6 Stable and unstable manifolds

Let M be a closed smooth manifold of dimension n and X a smooth vector field in M with flow $\phi(u,t), u \in M, t \in \mathbb{R}$. For an equilibrium $p \in M$ (i.e., a point at which $X(p) = 0$), consider the linearization of X, $DX(p)$, which in turn is a vector field on the tangent space TM_p having the origin as an equilibrium. The equilibrium is hyperbolic if no eigenvalue of $DX(p)$ has zero real part. The *stable* and *unstable manifolds* at p are the sets

$$W_s(p) = \{u \in M \mid \lim_{t\to\infty} \phi(u,t) = p\}, \quad W_u(p) = \{u \in M \mid \lim_{t\to-\infty} \phi(u,t) = p\}.$$

From the so called stable manifold theorem, at a hyperbolic equilibrium p, both sets are indeed smooth manifolds and their tangent spaces at p

are naturally identified with the tangent spaces of the stable and unstable manifolds of $DX(p)$ at 0.

8.7 Transition maps

Consider π and its immediate predecessor $\tilde{\pi}$, differing by the inversion $(i, i+1)$, in the sense that $P_{\tilde{\pi}} = P_\pi P$, where P is the matrix associated to the inversion. Let $S \in \mathcal{U}_{\mathcal{O}}^{\pi} \cup \mathcal{U}_{\mathcal{O}}^{\tilde{\pi}}$. We relate the triangular matrices $B_{\tilde{\pi}}$ and B_π associated to S. Write

$$S = Q_\pi^T \Lambda^\pi Q_\pi = Q_{\tilde{\pi}}^T \Lambda^{\tilde{\pi}} Q_{\tilde{\pi}}$$

for LU-positive orthogonal matrices $Q_{\tilde{\pi}}$ and Q_π. Since

$$\Lambda^{\tilde{\pi}} = P_{\tilde{\pi}}^{-1} \Lambda P_{\tilde{\pi}} = P P_\pi^{-1} \Lambda P_\pi P = P \Lambda^\pi P,$$

we learn that $S = Q_{\tilde{\pi}}^T P \Lambda^\pi P Q_{\tilde{\pi}}$ and $Q_\pi = E P Q_{\tilde{\pi}}$ for some sign diagonal matrix E. Now LU-decompose, $Q_\pi = L_\pi U_\pi = E(P L_{\tilde{\pi}}) U_{\tilde{\pi}}$, so that $P L_{\tilde{\pi}}$ also admits an LU-decomposition $P L_{\tilde{\pi}} = \tilde{L}\tilde{U}$. Thus

$$L_\pi U_\pi = E(\tilde{L}\tilde{U}) U_{\tilde{\pi}} = (E\tilde{L}E)(E\tilde{U}U_{\tilde{\pi}})$$

and from uniqueness of the LU-decomposition of an LU-positive matrix and the fact that $E\tilde{L}E$ is lower unipotent, we learn that $L_\pi = E\tilde{L}E$ and $U_\pi = E\tilde{U}U_{\tilde{\pi}}$. Now,

$$B_\pi = L_\pi^{-1} \Lambda^\pi L_\pi = E\tilde{L}^{-1} E \Lambda^\pi E \tilde{L} E$$

$$= E(\tilde{U} L_{\tilde{\pi}}^{-1} P) E \Lambda^\pi E (P L_{\tilde{\pi}} \tilde{U}^{-1}) E = E\tilde{U} B_{\tilde{\pi}} \tilde{U}^{-1} E.$$

Now, P is essentially the identity matrix, up to a 2×2 permutation block in the intersection of two consecutive rows and columns i and $i+1$. Simple computations imply that the decomposition $P L_{\tilde{\pi}} = \tilde{L}\tilde{U}$ is given in block form (for blocks denoted by $[.]_{i,j}$) by

$$\begin{pmatrix} [PL_{\tilde{\pi}}]_{1,1} & 0 & 0 \\ [PL_{\tilde{\pi}}]_{2,1} & [PL_{\tilde{\pi}}]_{2,2} & 0 \\ [PL_{\tilde{\pi}}]_{3,1} & [PL_{\tilde{\pi}}]_{3,2} & [PL_{\tilde{\pi}}]_{3,3} \end{pmatrix} = \begin{pmatrix} * & 0 & 0 \\ * & [\tilde{L}]_{2,2} & 0 \\ * & * & * \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ 0 & [\tilde{U}]_{2,2} & 0 \\ 0 & 0 & I \end{pmatrix}.$$

Explicitly, the LU-decomposition of $[PL_{\tilde\pi}]_{2,2} = [\tilde L]_{2,2}[\tilde U]_{2,2}$ is

$$[PL_{\tilde\pi}]_{2,2} = \begin{pmatrix} \alpha & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{1}{\alpha} & 1 \end{pmatrix} \begin{pmatrix} \alpha & 1 \\ 0 & \frac{-1}{\alpha} \end{pmatrix} = [\tilde L]_{2,2}[\tilde U]_{2,2},$$

where α is the $(i+1,i)$-entry of $L_{\tilde\pi}$. To compute α, recall that $B_\pi = E\tilde U B_{\tilde\pi} \tilde U^{-1} E$: the equality of entries $(i,i+1)$ on both sides gives

$$\alpha = \frac{\beta^{\tilde\pi}_{i+1,i}}{\lambda_{i+1} - \lambda_i}.$$

Since the hypothesis guarantee the existence of the LU-decompositions, we learn in particular that $\alpha \neq 0$.

We now specialize to $\mathcal{T}_\Lambda$. There are three cases regarding the inversion $(i,+1)$: either $i = 1$ or $i = n-1$ or $1 < i < n-1$. We treat the last case first.

The matrix $\tilde U$ is known, from the explicit form for $[\tilde U]_{2,2}$ obtained above. Also, E is necessary in order that the matrix $U_\pi = E\tilde U U_{\tilde\pi}$ has positive diagonal. Since $U_{\tilde\pi}$ has positive diagonal, $E_{j,j} = 1$ for $j \neq i, i+1$, $E_{i,i} = \operatorname{sgn}\alpha$ and $E_{i+1,i+1} = -\operatorname{sgn}\alpha$. Equating entries in $B_\pi = E\tilde U B_{\tilde\pi}\tilde U^{-1}E$,

$$\beta^{\pi}_{j+1,j} = \beta^{\tilde\pi}_{j+1,j}, \quad j \neq i-1, i, i+1,$$

$$\beta^{\pi}_{i,i-1} = (\operatorname{sgn}\alpha)\alpha\beta^{\tilde\pi}_{i,i-1}, \quad \beta^{\pi}_{i+1,i} = \frac{\beta^{\tilde\pi}_{i+1,i}}{\alpha^2}, \quad \beta^{\pi}_{i+2,i+1} = (\operatorname{sgn}\alpha)\alpha\beta^{\tilde\pi}_{i+2,i+1}.$$

Now take $i+1$, the case $i = n-1$ being similar. In this case, only the first two coordinates change,

$$\beta^{\pi}_{2,1} = \frac{\beta^{\tilde\pi}_{2,1}}{\alpha^2}, \quad \beta^{\pi}_{3,2} = (\operatorname{sgn}\alpha)\alpha\beta^{\tilde\pi}_{3,2}.$$

8.8
Orientability of $\mathcal{T}_\Lambda$

We need a preliminary fact of independent interest. Recall that a tridiagonal matrix T is unreduced if its off-diagonal entries $T_{i+1,i}, T_{i,i+1}$ are different from zero and that, for any chart $\mathcal{U}^\pi_T$, the triangular matrix B_π is lower bidiagonal with off-diagonal entries $\beta^\pi_{i+1,i}$.

Proposition 33 *Unreduced matrices belong to all charts $\mathcal{U}^\pi_T$.*

Proof: The proof requires two facts. The first one follows from the change of charts in Appendix 8.7: if $\beta^\pi_{i+1,i} \neq 0$, then $\beta^{\tilde\pi}_{i+1,i} \neq 0$, for a permutation π and its immediate predecessor $\tilde\pi$. Thus the property holds when changing between arbitrary charts, and bidiagonal matrices whose triangular coordinates at entries $(i+1,i)$ are nonzero then belong to any chart domain $\mathcal{U}_{\mathcal{T}}^{\pi}$.

The second fact is that $\beta_{i+1,i}^{\pi} = 0$ for some chart π if and only if, for the associated matrix T, one has $T_{i+1,i} = 0$. This follows directly from the fact that T is a conjugation of B_π by an upper triangular matrix. Thus the unreduced tridiagonal matrices are exactly those for which the nontrivial triangular coordinates at entries $(i+1,i)$ are nonzero. ■

We return to the proof of orientability of $\mathcal{T}_\Lambda$. Since every chart domain $\mathcal{U}_{\mathcal{T}}^{\pi}$ is dense in $\mathcal{T}_\Lambda$, we do not have to check the compatibility between arbitrary pairs of charts. Instead, we only consider the change of charts between $\phi_{\tilde{\pi}}$ and ϕ_π, for permutations differing by an inversion of consecutive elements, so that the matrices P_π and P_π induced by the permutations are related by $P_\pi = P_\pi P$ where $P = P^{-1}$ is associated to the permutation $(i, i+1)$. This is exactly the context of Appendix 8.7.

Each $\mathcal{U}_{\mathcal{T}}^{\pi}$ gets an orientation by pushing forward one of the two orientations of $\mathbb{R}^{n-1}$ by ϕ_π: take the standard orientation or the other, according to the evenness of the permutation π.

Thus, it suffices to check that the map

$$((\beta_{2,1}^{\pi}, \ldots, \beta_{n,n-1}^{\pi}), +) \mapsto ((\beta_{2,1}^{\pi}, \ldots, \beta_{n,n-1}^{\pi}), -)$$

is orientation preserving, where the signs are reminders of the opposite orientations chosen for $\mathbb{R}^{n-1}$ in the domain and image.

Again, there are three cases $i = 1$ or $i = n-1$ or $1 < i < n-1$ for the inversion $(i, i+1)$ and we start with the last.

Simply compute the Jacobian of the map which shows how entries $\beta_{i,i-1}^{\pi}$, $\beta_{i+1,i}^{\pi}$ and $\beta_{i+2,i+1}^{\pi}$ vary, keeping in mind the expression of α in terms of $\beta_{i+1,i}^{\pi}$:

$$\begin{pmatrix} \frac{(\operatorname{sgn}\alpha)\beta_{i+1,i}^{\pi}}{\lambda_{i+1}-\lambda_i} & \frac{(\operatorname{sgn}\alpha)}{\lambda_{i+1}-\lambda_i}\beta_{i,i-1}^{\pi} & 0 \\ 0 & \frac{-(\lambda_{i+1}-\lambda_i)^2}{(\beta_{i+1,i}^{\pi})^2} & 0 \\ 0 & \frac{(\operatorname{sgn}\alpha)\beta_{i+2,i+1}^{\pi}}{\lambda_{i+1}-\lambda_i} & \frac{(\operatorname{sgn}\alpha)\beta_{i+1,i}^{\pi}}{\lambda_{i+1}-\lambda_i} \end{pmatrix},$$

whose determinant is always negative, showing that the orientations prescribed for $\mathcal{U}_{\mathcal{T}}^{\pi}$ and $\mathcal{U}_{\mathcal{T}}^{\rho}$ are indeed compatible.

Now take $i+1$: $i = n-1$ is similar. Take derivatives of the first two coordinates,

$$\begin{pmatrix} \frac{-(\lambda_{i+1}-\lambda_i)^2}{(\beta_{i+1,i}^{\pi})^2} & 0 \\ \frac{(\operatorname{sgn}\alpha)\beta_{i+2,i+1}^{\pi}}{\lambda_{i+1}-\lambda_i} & \frac{(\operatorname{sgn}\alpha)\beta_{i+1,i}^{\pi}}{\lambda_{i+1}-\lambda_i} \end{pmatrix},$$

and the determinant is negative again, finally yielding the orientability of $\mathcal{T}_\Lambda$.