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Tightness and Tautness

A *linear height function* is the signed distance of points in a manifold $M^n \subset \mathbb{R}^m$ to a hyperplane in $\mathbb{R}^m$. An immersion $f\colon M^n \to \mathbb{R}^m$ is *tight* if every non-degenerate linear height function has precisely the minimum possible number of critical points on M (1). More precisely, by making use of the Morse inequalities (10), the number of critical points of index k has to be equal to the dimension of $H_k(M;\mathbb{R})$.

A *distance function* is the square of the Euclidean distance between points in $M^n \subset \mathbb{R}^m$ and a given point $p \in \mathbb{R}^m$. An immersion $f\colon M^n \to \mathbb{R}^m$ is *taut* if every non-degenerate distance function has precisely the minimum possible number of critical points on M (2).

It turns out that, in an isospectral scenario, the two concepts are equivalent.

Proposition 22 *If M is an isospectral manifold, tightness and tautness are equivalent properties.*

Proof: We compare the height and distance functions $h_N\colon S \to \langle S, N\rangle$ and $L_N\colon S \to \|S-N\|^2$. Since $L_N(S) = \langle S-N, S-N\rangle = \langle S,S\rangle - 2\langle S,N\rangle + \langle N,N\rangle$, and $\langle S,S\rangle$ is constant throughout M,

$$DL_N(S) = -2Dh_N(S).$$

Thus the critical sets of h_N and L_N are equal. ■

A subset M of $\mathbb{R}^n$ has the two-piece property (<u>TPP</u>) if, for every hyperplane $\mathcal{P}$ in $\mathbb{R}^n$, the complement $M \setminus \mathcal{P}$ contains at most two connected components. As shown in (2), an equivalent formulation of the *TPP* for a compact connected smoothly immersed manifold M is the following: every non-degenerate linear height function on M has precisely one local minimum and one local maximum. Said differently, for such functions, a local extremum is necessarily global.

Proposition 23 *In a compact, connected, oriented manifold, tightness implies TPP.*

Proof: The hypotheses imply that $H_0 \simeq \mathbb{Z} \simeq H_n$. The Morse inequalities (10) and tightness are sufficient to conclude that an arbitrary non-degenerate linear height function has exactly one local maximum and one local minimum. ∎

Proposition 24 *The natural immersion of $\mathcal{O}_\Lambda$ into $\mathcal{S}_n$ satisfies tightness, tautness and the TPP.*

Proof: We compute the critical set $\mathcal{C}_N$ of a height function $h_N(S) = \mathrm{tr} NS$. Diagonalize $N = Q^T DQ$, with Q orthogonal and D a diagonal matrix with entries in descending order. Let $h_D(S) = \mathrm{tr} DS$ with critical set $\mathcal{C}_D$. Clearly the map $M \to QMQ^T$ is a bijection from $\mathcal{C}_N$ to $\mathcal{C}_D$ respecting indices.

From Corollary 13, h_D is a Morse function when the diagonal entries of D are distinct and, moreover, the number of critical points of index k is $m(k)$. From the Morse inequalities applied to homology coefficients in the field $\mathbb{Z}/2\mathbb{Z}$, the number of critical points of index k of a Morse function are bounded below by the $\mathbb{Z}/2\mathbb{Z}$-Betti numbers, which are also given by $m(k)$, by Theorem 21. Equality of both numbers implies tightness. When the entries of D are not distinct, a simple computation shows that h_D is not a morse function.

Tautness and TPP follow from the propositions above. ∎

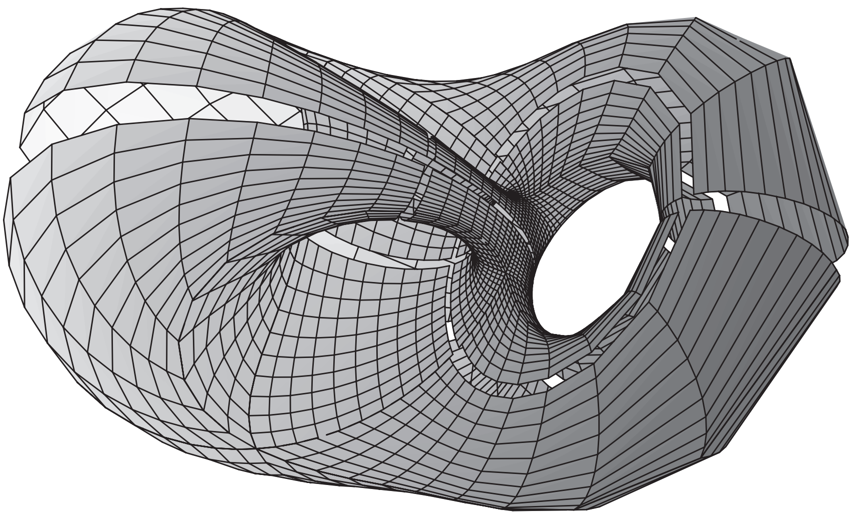

Figure 7.1: A 3d rendition of $\mathcal{T}_\Lambda$ for $\Lambda = \mathrm{diag}(4, 5, 7)$

Proposition 25 *The natural immersion of $\mathcal{T}_\Lambda$ into $\mathcal{T}_n \cap \mathcal{S}_n$ is <u>not</u> tight. Still, the height function h_D, for $D = \mathrm{diag}(d_1, \ldots, d_n)$ with $d_1 > \ldots > d_n$ is perfect, in the sense that it has the minimal number of critical points of each index.*

Proof: We consider the 3×3 case. Within the 9-dimensional vector space of 3×3 matrices endowed with the usual inner product $\langle A, B\rangle = \mathrm{tr} A^T B$, consider the 5-dimensional subspace $\mathcal{V}$ of real, symmetric, tridiagonal matrices.

Set $\Lambda = \mathrm{diag}(7,5,4)$: the manifold $\mathcal{T}_\Lambda$ lies in the 4-dimensional subspace $\mathcal{V}_{16}$ of matrices with trace equal to 16. Since the sum of squares of entries of matrices in $\mathcal{T}_\Lambda$ is $7^2 + 5^2 + 4^2$, $\mathcal{T}_\Lambda$ actually lies in a 3-sphere of radius $3\sqrt{10}$. The stereographic projection now takes this sphere to standard $\mathbb{R}^3$ (with a point at infinity) and $\mathcal{T}_\Lambda$ may be represented as in figure (7).

Clearly, there are (generic) 2-planes tangent to the bitorus at more than one point. Pulling back such planes by the stereographic projection, one finds 2-spheres in $\mathcal{V}$ which are sections of 4-spheres, which in turn can be interpreted as levels of a distance function with two local maxima in $\mathcal{T}_\Lambda$.

To show that h_D is perfect under the hypothesis above for D, proceed as in the proposition above: the number of critical points of index k equals the k-th Betti number. ■

The picture appears in (8), as an application of triangular coordinates, used to provide charts on $\mathcal{T}_\Lambda$. The gaps indicate the position of matrices which are not unreduced.

There is an interesting connection between the above proposition and some problems in numerical spectral theory: height/distance functions have unique local extrema. In a sense, such problems behave as well as optimization problems under convexity hypothesis — a steepest descent/ascent method always leads to the solution.

Say, for example, that one searches for a matrix $S \in \mathcal{S}_n$ with fixed given spectrum and smallest $(1,n)$- coordinate. It is not hard to see that this is equivalent to searching for a matrix in $\mathcal{O}_\Lambda$ which minimizes the height function to the hyperplane of matrices perpendicular to $E_{1,n} + E_{n,1}$ ($E_{i,j}$ is the matrix whose only nonzero entry, equal to one, is in position (i,j)). This height function turns out *not* to be Morse, but its set of minima is connected (not necessarily a singleton! Indeed, it is not a single matrix in this case) and may be reached by a steepest descent type of algorithm.

Another natural question is the following. Given two matrices $S, S' \in \mathcal{O}_\Lambda$ with corner entries $S_{1,n}, S'_{1,n} \leq C$, is there a path joining them in $\mathcal{O}_\Lambda$ so that the corner entries are always bounded by C? The answer is yes, as implied by the TPP.