6
Computing homologies

In this section we compute the homologies $H_*(\mathcal{T}_\Lambda;\mathbb{Z})$ and $H_*(\mathcal{O}_\Lambda;\mathbb{Z}/2\mathbb{Z})$ via height functions $\tilde{h}_D$ and h_D, the *Toda Flow* and *Morse Theory*. The approach is similar for both cases. Choose a real, diagonal matrix D with diagonal entries in (strictly) decreasing order for which the associated height function $h(S) = \operatorname{tr}(DS)$ (either $\tilde{h}_D$ or h_D) has distinct critical values $c_\pi = \operatorname{tr}(D\Lambda^\pi)$ indexed by permutations $\pi \in S_n$ [1]. In particular, c_e and $c_{\pi_{\max}}$ are respectively (global) minimum and maximum of h, where e is the identity permutation and $\pi_{\max}(j) = n+1-j$ is the reversal permutation. Take $\epsilon > 0$ such that, for all $\pi_1 \neq \pi_2$, $|\, c_{\pi_1} - c_{\pi_2}| > 2\epsilon$. As usual, we compute the homology of the nested manifolds with boundary $M_\pi^\pm = h^{-1}((-\infty, c_\pi \pm \epsilon])$. Clearly, $M_e^- = \emptyset$ and $M = M_{\pi_{\max}}^+$ is either $\mathcal{T}_\Lambda$ or $\mathcal{O}_\Lambda$. Also, for consecutive values $c_{\pi_1} < c_{\pi_2}$, the manifolds $M_{\pi_1}^+$ and $M_{\pi_2}^-$ are diffeomorphic (10). The homologies of M_π^- and M_π^+ are related by the long exact sequence (9) of the pair (M_π^+, M_π^-):

$$\begin{array}{c} \vdots \\ H_{k+1}(M_\pi^-;A) \longrightarrow H_{k+1}(M_\pi^+;A) \longrightarrow H_{k+1}(M_\pi^+, M_\pi^-;A) \\ \xrightarrow{\ \partial\ } \\ H_k(M_\pi^-;A) \longrightarrow H_k(M_\pi^+;A) \longrightarrow H_k(M_\pi^+, M_\pi^-;A) \\ \longrightarrow \\ \vdots \end{array}$$

Here $A = \mathbb{Z}$ (resp. $\mathbb{Z}/2\mathbb{Z}$) for $M = \mathcal{T}_\Lambda$ (resp. $\mathcal{O}_\Lambda$). If the critical point Λ^π has index i, M_π^+ is homotopically equivalent to a space obtained from M_π^- by attaching an i-cell (10). This allows us to compute the relative homology of the pair (M_π^+, M_π^-):

$$H_k(M_\pi^+, M_\pi^-;A) \simeq H_k(\mathbb{B}^i, \mathbb{S}^{i-1};A) \simeq \begin{cases} A & \text{if } k = i, \\ 0 & \text{if } k \neq i. \end{cases}$$

[1]Notations and facts about permutations are given in Appendix 8.5

Lemma 20 *Let i be the index of the critical point Λ^π. Then the connecting homomorphism $\partial : H_i(M_\pi^+, M_\pi^-; A) \to H_{i-1}(M_\pi^-; A)$ is zero.*

Proof: We first recall the topological meaning of the connecting homomorphism ∂. Start with an element a of $H_i(M_\pi^+, M_\pi^-; A)$: a representative of a is given by a linear combination $\tilde{a}$ of i-simplices in M_π^+ with boundary contained in M_π^-. The boundary $\tilde{b}$ of $\tilde{a}$ (a linear combination of $i-1$ simplices, the faces) is clearly closed in $C_{i-1}(M_\pi^-)$ and therefore represents an element $b \in H_{i-1}(M_\pi^-; A)$: b is ∂a. In our context, we may take $\tilde{a}$ to be the closure of $W_s(\Lambda^\pi) \smallsetminus M_\pi^-$ and therefore $\tilde{b}$ is the intersection of $W_s(\Lambda^\pi)$ with the boundary of M_π^-.

In the case $\mathcal{T}_\Lambda$, Theorem 17 implies that the closure of $W_s(\Lambda^\pi)$ is a compact orientable manifold N_π: $\tilde{b}$ is the boundary of $N_\pi \cap M_\pi^-$ and therefore exact in $C_{i-1}(M_\pi^i)$. Thus, in this case, $\partial = 0$.

From Morse theory, $H_*(\mathcal{O}_\Lambda; \mathbb{Z}/2\mathbb{Z})$ is spanned by the stable manifolds $W_s(\pi)$ associated to the critical points Λ^π of the Toda flow. The closures of $W_s(\pi)$ are *not* manifolds with boundary, but they are still homotopic to polyhedra.

The Bruhat ordering $\tilde{\pi} \leq \pi$ is equivalent to the inclusion $W_s(\tilde{\pi}) \subset \overline{W_s}(\pi)$. Also, by Proposition 19, if $W_s(\tilde{\pi}) \subset \overline{W_s}(\pi)$ and the dimensions of $W_s(\tilde{\pi})$ and $W_s(\pi)$ differ by one then $\tilde{\pi}$ is an immediate predecessor of π. Said differently, in order to study the connecting homomorphism $\partial : H_k(M_\pi^+, M_\pi^-) \to H_{k-1}(M_\pi^-)$ it suffices to study $W_s(\pi)$ in a neighborhood of each $W_s(\tilde{\pi})$, $\tilde{\pi}$ an immediate predecessor of π.

We show the triviality of the connecting homomorphism ∂ in a sufficiently rich example. Let $\tilde{\pi}$ and π be given by

$$P_{\tilde{\pi}} = \begin{pmatrix} & & & 1 & & & & \\ & & \underline{1} & & & \underline{0} & & \\ 1 & & & & & & & \\ & 1 & & & & & & \\ & & & & & & 1 & \\ & & & & & & & 1 \\ & & \underline{0} & & & \underline{1} & & \\ & & & & 1 & & & \end{pmatrix}, \quad P_\pi = \begin{pmatrix} & & & 1 & & & & \\ & & \underline{0} & & & \underline{1} & & \\ 1 & & & & & & & \\ & 1 & & & & & & \\ & & & & & & 1 & \\ & & & & & & & 1 \\ & & \underline{1} & & & \underline{0} & & \\ & & & & 1 & & & \end{pmatrix}.$$

The fact that the interior of the rectangle with underlined vertices contains no nonzero entries indicates that $\tilde{\pi}$ is an immediate predecessor of π. The set $\mathcal{U}_\mathcal{O}^{\tilde{\pi}}$ is an open neighborhood of $W_s(\tilde{\pi})$: we shall study the sets $W_s(\tilde{\pi})$ and $W_s(\pi) \cap \mathcal{U}_\mathcal{O}^{\tilde{\pi}}$.

For $S = Q^T \Lambda Q \in \mathcal{U}^{\tilde{\pi}}_{\mathcal{O}}$, write $Q = P_{\tilde{\pi}} L_{\tilde{\pi}} U_{\tilde{\pi}}$; conversely, given

$$\tilde{L} = P_{\tilde{\pi}} L_{\tilde{\pi}} = \begin{pmatrix} x_{11} & x_{12} & x_{13} & 1 & & & & \\ x_{21} & x_{22} & \underline{\mathbf{1}} & & & & & \\ 1 & & & & & & & \\ x_{41} & 1 & & & & & & \\ x_{51} & x_{52} & x_{53} & x_{54} & x_{55} & x_{56} & 1 & \\ x_{61} & x_{62} & x_{63} & x_{64} & x_{65} & x_{66} & x_{67} & 1 \\ x_{71} & x_{72} & x_{73} & x_{74} & x_{75} & \underline{\mathbf{1}} & & \\ x_{81} & x_{82} & x_{83} & x_{84} & 1 & & & \end{pmatrix},$$

let $Q = \mathbf{Q}(\tilde{L})$ and $S = Q^T \Lambda Q$ thus defining a diffeomorphism ξ from $\mathbb{R}^{n(n-1)/2}$ to $\mathcal{U}^{\tilde{\pi}}_{\mathcal{O}}$. Recall that $S = Q^T \Lambda Q$ belongs to $W_s(\tilde{\pi})$ (resp. $W_s(\pi)$) if and only if $\mathcal{B}_{SW}(Q) = \mathcal{B}_{SW}(P_{\tilde{\pi}} L_{\tilde{\pi}}) = P_{\tilde{\pi}}$ (resp. $\mathcal{B}_{SW}(Q) = \mathcal{B}_{SW}(P_{\tilde{\pi}} L_{\tilde{\pi}}) = P_{\pi}$). Thus, $W^{\tilde{\pi}}_s$ is the image under ξ of the subspace

$$\tilde{L} = \begin{pmatrix} x_{11} & x_{12} & x_{13} & 1 & & & & \\ x_{21} & x_{22} & \underline{\mathbf{1}} & & & & & \\ 1 & & & & & & & \\ 0 & 1 & & & & & & \\ 0 & 0 & 0 & 0 & x_{55} & x_{56} & 1 & \\ 0 & 0 & 0 & 0 & x_{65} & x_{66} & 0 & 1 \\ 0 & 0 & 0 & 0 & x_{75} & \underline{\mathbf{1}} & & \\ 0 & 0 & 0 & 0 & 1 & & & \end{pmatrix},$$

where x_{ij} is arbitrary if its row (resp. column) contains a 1 to its right (resp. below), or, equivalently, if $(j, \tilde{\pi}^{-1}(i))$ is an inversion of $\tilde{\pi}$. Similarly, $W^{\pi}_s \cap \mathcal{U}^{\tilde{\pi}}_{\mathcal{O}}$ is the image under ξ of

$$\tilde{L} = \begin{pmatrix} x_{11} & x_{12} & x_{13} & 1 & & & & \\ x_{21} & x_{22} & \underline{\mathbf{1}} & & & & & \\ 1 & & & & & & & \\ 0 & 1 & & & & & & \\ 0 & 0 & x_{73}x_{56} & 0 & x_{55} & x_{56} & 1 & \\ 0 & 0 & x_{73}x_{66} & 0 & x_{65} & x_{66} & 0 & 1 \\ 0 & 0 & \underline{x_{73}} & 0 & x_{75} & \underline{\mathbf{1}} & & \\ 0 & 0 & 0 & 0 & 1 & & & \end{pmatrix}, \quad x_{73} \neq 0.$$

Indeed, for all such matrices $\tilde{L}$, we have $\mathcal{B}_{SW}(\tilde{L}) = P_{\pi}$, proving that the image is contained in $W^{\pi}_s \cap \mathcal{U}^{\tilde{\pi}}_{\mathcal{O}}$. Conversely, given $\tilde{L}$ with $\mathcal{B}_{SW}(\tilde{L}) = P_{\pi}$ (so that $\xi(\tilde{L}) \in W^{\pi}_s$), we must show that $\tilde{L}$ has the form above. For instance,

$(r_{SW}(\tilde{L}))_{41} = (r_{SW}(\tilde{L}))_{52} = (r_{SW}(\tilde{L}))_{84} = 0$ imply that $(\tilde{L})_{ij} = 0$ for entries $41, 51, 61, 71, 81, 52, 62, 72, 82, 83, 84$. On the other hand, $(r_{SW}(\tilde{L}))_{73} = 1$ implies $(\tilde{L})_{73} = x_{73} \neq 0$. From $(r_{SW}(\tilde{L}))_{24} = 3$ and the fact that the first three entries of rows $2, 3, 4$ are linearly independent it follows that $(\tilde{L})_{ij} = 0$ for entries $54, 64, 74$. Finally, $(r_{SW}(\tilde{L}))_{56} = 2$ implies the indicated values at entries $53, 63$.

In general, free entries for $\xi(\tilde{L}) \in W_s(\tilde{\pi})$ remain free for $\xi(\tilde{L}) \in W_s(\pi)$. Also, the south-west corner of the underlined rectangle becomes free and nonzero. Some entries which were zero for $\xi(\tilde{L}) \in W_s(\tilde{\pi})$ become smooth functions of the free entries.

Thus, the triple $W_s(\tilde{\pi}) \subset \overline{W_s}(\pi) \cap \mathcal{U}_{\mathcal{O}}^{\tilde{\pi}} \subset \mathcal{U}_{\mathcal{O}}^{\tilde{\pi}}$ is diffeomorphic to $\mathbb{R}^{i(\tilde{\pi})} \subset \mathbb{R}^{i(\pi)} \subset \mathbb{R}^{n(n-1)/2}$. Thus, Bruhat cells of codimension 1 always come up in pairs and since we are working with coefficients in $\mathbb{Z}/2\mathbb{Z}$, $\partial = 0$. ■

Theorem 21 *Let m_k (resp. n_k) be the number of permutations $\pi \in S_n$ with k inversions (resp. descents). Then*

$$H_k(\mathcal{O}_\Lambda; \mathbb{Z}/2\mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z})^{m_k}, \quad H_k(\mathcal{T}_\Lambda; \mathbb{Z}) = \mathbb{Z}^{n_k}.$$

Proof: Again, say Λ^π is a critical point of index i. Suppose first that $k, k+1 \neq i$: from the long exact sequence of the pair, we obtain

$$0 \longrightarrow H_k(M_\pi^-; A) \longrightarrow H_k(M_\pi^+; A) \longrightarrow 0\ ,$$

which implies that $H_k(M_\pi^-; A) \simeq H_k(M_\pi^+; A)$. Now, if $k + 1 = i$, from the triviality of ∂ proved in Lemma 20, the sequence

$$H_i(M_\pi^+, M_\pi^-; A) \simeq A \xrightarrow{\partial} H_{i-1}(M_\pi^-; A) \longrightarrow H_{i-1}(M_\pi^+; A) \longrightarrow 0.$$

yields $H_{i-1}(M_\pi^-; A) \simeq H_{i-1}(M_\pi^+; A)$. Finally, if $k = i$, again from Lemma 20, we obtain an exact sequence at the three intermediate stages,

$$0 \longrightarrow H_i(M_\pi^-; A) \longrightarrow H_i(M_\pi^+; A) \longrightarrow A \xrightarrow{\partial} 0.$$

Since A is a free A-modulus, the sequence splits and

$$H_i(M_\pi^+; A) \simeq H_i(M_\pi^-; A) \oplus A.$$

Thus, a critical point of index i contributes (freely) with a generator of $H_i(M; A)$ and the theorem is proved. ■