# 5
# Toda asymptotics

In this section, we compute the limits of a Toda orbit $S(t)$ for $t \to \pm\infty$. We will be especially interested in an explicit description of the stable and unstable manifolds associated to an equilibrium $\Lambda^\pi$, both in $\mathcal{O}_\Lambda$ and in $\mathcal{T}_\Lambda$. The first statement is common to both manifolds.

**Proposition 14 (Toda Equilibria)** *The only equilibria of the* Toda flow *are the diagonal matrices. Every orbit converges to such an equilibrium for $t \to \pm\infty$.*

**Proof:** This is a standard consequence of the fact that the Toda vector field is the gradient of a height function on a compact manifold, as shown in Theorem 12. The critical points of the function are clearly the equilibria of the vector field. ■

## 5.1
## Stable and unstable manifolds in $\mathcal{T}_\Lambda$

A matrix $T \in \mathcal{T}_\Lambda$ is *unreduced* if all entries $T_{i,i+1} = T_{i+1,i}, i = 1, \ldots, n-1$, (the *off-diagonal* entries) are nonzero. Clearly, zero off-diagonal entries of $T$ allow for a decomposition of the matrix into unreduced blocks $T_i$ of smaller dimension along the diagonal. In particular, the eigenvalues of $\Lambda$ partition into eigenvalues of diagonal matrices $\Lambda_i$, each conjugate to $T_i$.

The Toda flow preserves the partitioning of the initial condition.

**Proposition 15 (Blockwise Toda)** *The sign (positive, zero, negative) of each off-diagonal position of an initial condition $T \in \mathcal{T}_\Lambda$ does not change along a Toda orbit. Moreover, the Toda flow acts on each unreduced block, $\dot{T}_i = [T_i, \Pi_a T_i]$, inducing orbits $T_i(t) \in \mathcal{T}_{\Lambda i}$.*

**Proof:** From the differential equation $\dot{T} = [T, \Pi_a T]$, one obtains the evolution of an off-diagonal entry,

$$\dot{T}_{i+1,i} = T_{i+1,i}(T_{i+1,i+1} - T_{i,i}).$$

Now, if $T_{i+1,i}(t_0) = 0$, the constant function, $T_{i+1,i}(t) = 0$ for all $t$, gives rise to the unique solution of this differential equation. Moreover, off-diagonal entries in one unreduced block do not depend on any entry from another block. The same is true of the diagonal entries from the analogous evolution

$$\dot{T}_{i,i} = 2(T_{i+1,i}^2 - T_{i-1,i}^2).$$

■

Let $\Lambda^e$ (resp. $\Lambda^{\pi_{\max}}$) be the diagonal matrix having the eigenvalues in ascending (resp. descending) order [1].

**Proposition 16** *Let $T \in \mathcal{T}_\Lambda$ be unreduced and consider the Toda orbit $T(t)$. Then*

$$\lim_{t\to-\infty} T(t) = \Lambda^e, \quad \lim_{t\to+\infty} T(t) = \Lambda^{\pi_{\max}}.$$

**Proof:** We consider the case $t \to +\infty$ since the other is similar. From Proposition 14, the limit is a matrix conjugate to $\Lambda$, i.e., some diagonal matrix $\Lambda^\pi = \operatorname{diag}(\lambda_{\pi(1)}, \ldots, \lambda_{\pi(n)})$ for some permutation $\pi$. Say $\delta$ is the spectral gap of $\Lambda$, i.e., the minimal distance between two eigenvalues. Now take $t_0$ so large that, for $t > t_0$, $|T_{i,i} - \lambda_{\pi(i)}| < \delta/2$. Thus, differences of the form $T_{i+1,i+1} - T_{i,i}$ are bounded away from zero. From equation

$$\dot{T}_{i+1,i} = T_{i+1,i}(T_{i+1,i+1} - T_{i,i})$$

it is clear that $T_{i+1,i}$ can only converge to zero if eventually $T_{i+1,i+1} < T_{i,i}$, and hence $\lambda_{\pi(i)} > \lambda_{\pi(i+1)}$ for all $i$, which happens exactly when $\Lambda^\pi = \Lambda^{\pi_{\max}}$. ■

Take a permutation $\pi$ and consider $\Lambda^\pi = \operatorname{diag}(\lambda_{\pi(1)}, \ldots, \lambda_{\pi(n)})$. We now describe the stable manifold $W_s(\Lambda^\pi)$, given by the matrices $T \in \mathcal{T}_\Lambda$ which are initial conditions of orbits $T(t)$ of the Toda flow satisfying $\lim_{t\to\infty} T(t) = \Lambda^\pi$.

The diagonal entries of $\Lambda^\pi$ partition into maximal strings of eigenvalues in descending order. Whenever two consecutive entries $i$ and $i+1$ are in increasing order, a matrix $T \in W_s(\Lambda^\pi)$ ought to satisfy $T_{i,i+1} = T_{i+1,i} = 0$, otherwise the unreduced block containing both positions would converge to a diagonal submatrix with eigenvalues in descending order, by the previous proposition.

**Theorem 17** *The closure of each stable (resp. unstable) manifold of the Toda flow in $\mathcal{T}_\Lambda$ is a product of smaller dimensional manifolds $\mathcal{T}_{\Lambda_i}$, with dimensions adding to $d(\pi)$ (resp. $n - d(\pi) - 1$).*

[1] For terminology related to permutations, see Appendix 8.5

**Proof:** Take a permutation $\pi$ and consider the stable manifold associated to $\Lambda^\pi$. The argument before the statement of the theorem shows that it consists of matrices with off-diagonal entries equal to zero at positions specified by $\pi$. Also, all such matrices do converge to $\Lambda^\pi$ by the previous proposition. Now, it is clear that unreduced matrices form an open dense set on their tridiagonal isospectral manifolds. Thus, taking the closure of each unreduced block separately gives rise to a full $\mathcal{T}_{\Lambda_i}$. Putting them together, one obtains the product in the statement. ■

## 5.2
## Stable and unstable manifolds in $\mathcal{O}_\Lambda$

To describe the asymptotic behavior of Toda orbits in $\mathcal{O}_\Lambda$, we need the notation of the Bruhat decomposition and its variants, as in Appendix 8.4. Recall that $\Lambda^e$ is the diagonal matrix with simple eigenvalues disposed in ascending order. Take $S_0 \in \mathcal{O}_\Lambda$ to be an initial condition for an orbit $S(t)$ and diagonalize $S_0 = Q_0^T \Lambda^e Q_0$, where $Q_0$ is defined up to left multiplication by a sign diagonal matrix $E$.

**Proposition 18** *Let $S_0 \in \mathcal{O}_\Lambda$ be the initial condition of a Toda orbit $S(t)$ with modified Bruhat matrices*[2] $P_{\pi_-} = \mathcal{B}_{NW}(Q_0)$ *and* $P_{\pi_+} = \mathcal{B}_{SW}(Q_0)$. *Then*

$$\lim_{t \to \pm\infty} S(t) = \Lambda^{\pi_\pm}.$$

**Proof:** Consider the case $t \to \infty$ since the other is similar. From Corollary 8, the orbit $S(t)$ starting from $S_0 \in \mathcal{O}_\Lambda$ is of the form

$$S(t) = (\mathbf{Q}(e^{t\Lambda} Q_0))^T \Lambda^e \mathbf{Q}(e^{t\Lambda} Q_0).$$

Thus, when $t$ grows to infinity, the entries of $e^{t\Lambda} Q_0$ in the bottom rows grow *faster*. Consider the first column of $e^{t\Lambda} Q_0$: when normalizing it to obtain the first column of $\mathbf{Q}(e^{t\Lambda} Q_0)$, one obtains a vector which converges to $\pm e_k$, where the vector $e_k$ is the first column of $P_{\pi_+}$. Indeed, $k$ is the lowest row on which an entry on the first column is different from zero, as we learn by interpreting $P_{\pi_+}$ as a table of local ranks of $Q_0$ (and hence of $e^{t\Lambda} Q_0$) counted from the bottom left corner of the matrix.

A similar argument taking into account one more column at a time shows that, indeed, up to signs, $\mathbf{Q}(e^{t\Lambda} Q_0)$ tends to $P_{\pi_+}$ and the result follows. ■

[2]For terminology and notation related to Bruhat decompositions, see Appendix 8.4

Thus, generically, orbits converge to $\Lambda = \Lambda^e$ when $t \to -\infty$ and to $\Lambda^{\pi_{\max}} = P_{\pi_{\max}} \Lambda P_{\pi_{\max}}$, when $t \to \infty$. This is indeed the case for unreduced tridiagonal matrices, from Proposition 16.

It is a general theorem that stable and unstable sets are manifolds isomorphic to Euclidean space. The next proposition provides an explicit coordinatization by triangular coordinates of the stable and unstable manifolds of the Toda flow.

**Proposition 19** *Let $\pi \in S_n$ and consider the stable and unstable manifolds $W_s^\pi(\Lambda^\pi)$ and $W_u^\pi(\Lambda^\pi)$ of the Toda flow. Then $W_s^\pi$ and $W_u^\pi$ belong to a single domain $\mathcal{U}_\mathcal{O}^\pi$. Given the chart $\phi : \mathbb{R}^N \to \mathcal{U}_\mathcal{O}^\pi$, where $N = n(n-1)/2$, the image of $W_s^\pi$ (resp. $W_u^\pi$) under $\phi^{-1}$ consists of the lower triangular matrices $B$ having diagonal equal to $\Lambda^\pi$ and strictly lower entries $(i,j)$ equal to 0 whenever $\lambda_{\pi(i)} > \lambda_{\pi(j)}$ (resp. $\lambda_{\pi(i)} < \lambda_{\pi(j)}$).*

**Proof:** We consider the stable manifolds $W_s^\pi$: the other case is similar. From the previous proposition, $W_s^\pi$ consists of matrices $S = Q^T \Lambda^e Q$ with the same modified Bruhat matrix $P_+$. Set $Q' = P_+^{-1} Q$ and write $S = (Q')^T \Lambda^{\pi_+} Q'$.

To show that $S \in \mathcal{U}_\mathcal{O}^\pi$, we must prove that $Q'$ admits an $LU$-decomposition (which would then imply the existence of an $LU$-positive orthogonal $EQ'$ for some sign diagonal matrix $E$ conjugating $S$ to $\Lambda^{\pi_+}$). Now, by definition,

$$P_+ = P_{\pi_{\max}} \mathcal{B}(P_{\pi_{\max}} Q P_{\pi_{\max}}) P_{\pi_{\max}}.$$

Write $P_{\pi_{\max}} Q P_{\pi_{\max}} = L_1 \mathcal{B}(P_{\pi_{\max}} Q P_{\pi_{\max}}) L_2$ for invertible, lower triangular matrices $L_1$ and $L_2$ to conclude that

$$P_+ = P_{\pi_{\max}} (L_1)^{-1} P_{\pi_{\max}} Q P_{\pi_{\max}} (L_2)^{-1} P_{\pi_{\max}}$$

from which we obtain $Q = U_1 P_+ U_2$ for the invertible upper triangular matrices $U_1 = P_{\pi_{\max}} L_1 P_{\pi_{\max}}$ and $U_2 = P_{\pi_{\max}} L_2 P_{\pi_{\max}}$. Thus $Q' = P_+^{-1} Q = (P_+^{-1} U_1 P_+) U_2$, and it suffices now to show that a conjugation by a permutation of an invertible upper matrix admits an $LU$-decomposition, which is a simple matter of checking that certain simple determinants are not zero.

We are thus entitled to keep track of such an orbit $S(t)$ using triangular coordinates in $\mathcal{U}_\mathcal{O}^\pi$. Since $S(t) \to \Lambda^{\pi_+}$ when $t \to \infty$, we must have that the corresponding $B(t)$ converge to $\Lambda^{\pi_+}$. Now, consider the evolution of $B(t)$ given in Proposition 9: for each subdiagonal entry, $\dot{\beta}_{i,j}^\pi = (\lambda_{\pi(i)} - \lambda_{\pi(j)}) \beta_{i,j}^\pi$, so convergence to zero at $\infty$ happens if and only if $\beta_{i,j}^\pi(0) = 0$ whenever $\lambda_{\pi(i)} > \lambda_{\pi(j)}$. ■

It turns out that, in this case, the closure $\overline{W_s^\pi} \subset \mathcal{O}_\Lambda$ of the stable manifolds are not always manifolds: $\overline{W_s^\pi}$ is the disjoint union of $W_s^{\tilde{\pi}}$ for permutations $\tilde{\pi} \leq \pi$. Here, the order in $S_n$ is the Bruhat order (Appendix 8.5). We will see more of $\overline{W_s^\pi}$ in the next chapter.