4
Height functions

As usual, let Λ have simple spectrum. Let D be a real diagonal matrix of simple spectrum and consider $\mathcal{P}_{\mathcal{D}}$, the hyperplane in $\mathcal{S}_n$ of matrices orthogonal to D. We are interested in the *height function*

$$\begin{array}{rccc} h_D\colon & \mathcal{O}_\Lambda & \to & \mathbb{R} \\ & S & \mapsto & \operatorname{tr}(DS) = \langle D, S\rangle \end{array},$$

which measures the signed distance of a given matrix $S \in \mathcal{O}_\Lambda$ to $\mathcal{P}_{\mathcal{D}}$. We also consider its restriction

$$\begin{array}{rccc} \tilde{h}_D\colon & \mathcal{T}_\Lambda & \to & \mathbb{R} \\ & T & \mapsto & \operatorname{tr}(DT) \end{array}.$$

4.1
The Toda vector field and ∇h_D

The following theorem allows us to combine, in the computation of the homology of $\mathcal{T}_\Lambda$ and $\mathcal{O}_\Lambda$, the standard techniques of Morse theory applied on height functions to those in dynamical systems related to gradient vector fields, stable and unstable manifolds.

Theorem 12 *Let D be a diagonal matrix with entries in strictly descending order. There exists a Riemannian metric on $\mathcal{O}_\Lambda$ for which the gradient of the height function $h_D(S) = \operatorname{tr}(DS)$ is the Toda vector field $[S, \Pi_a S]$. The analogous statement holds for $\mathcal{T}_\Lambda$ and $\tilde{h}_D$.*

Proof: We consider $\mathcal{O}_\Lambda$: the other case follows since $\mathcal{T}_\Lambda$ is a submanifold of $\mathcal{O}_\Lambda$. The Toda flow is invariant on both manifolds.

Recall that the tangent space of $\mathcal{O}_\Lambda$ at S consists of the symmetric matrices of the form $\{[S, A]\}$ for A skew-symmetric. Also, the map $i : A \in \mathcal{A}_n \mapsto [S, A] \in T_S\mathcal{O}_\Lambda$ is a linear isomorphism, since a matrix in the kernel is both skew-symmetric and symmetric. Thus, we may prescribe a Riemannian structure on $\mathcal{O}_\Lambda$ by pushing forward by i an inner product on $\mathcal{A}_n$.

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We need to prescribe an inner product $\langle\langle .\,,.\rangle\rangle_{T_S\mathcal{O}_\Lambda}$ at each point $S \in \mathcal{O}_\Lambda$ so that the derivative of the height function h_D along a tangent vector $[S, A]$

$$Dh_D(S)[S,A] = \mathrm{tr} D[S,A] = \mathrm{tr}[D,S]A = \langle -[D,S], A\rangle$$

equals

$$\langle\langle [S,\Pi_a S], [S,A]\rangle\rangle_{T_S\mathcal{O}_\Lambda} = \langle\langle \Pi_a S, A\rangle\rangle_{\mathcal{A}_n}.$$

The inner products in $\mathcal{A}_n$ are of the form

$$\langle\langle A_1, A_2\rangle\rangle_{\mathcal{A}_n} = \langle \mathcal{P}A_1, A_2\rangle$$

for an appropriate positive definite operator $\mathcal{P}$ from $\mathcal{A}_n$ to itself. Thus

$$\langle\langle \Pi_a S, A\rangle\rangle_{\mathcal{A}_n} = \langle \mathcal{P}\Pi_a S, A\rangle$$

must equal

$$\langle -[D,S], A\rangle$$

for all A in $\mathcal{A}_n$, and we only need to find $\mathcal{P}$ such that $\mathcal{P}\Pi_a S = -[D,S]$. Now notice that for all pairs i,j such that $i > j$, the entries of $-[D,S]$ (resp. $\Pi_a S$) are $(d_j - d_i)S_{i,j}$ (resp. $S_{i,j}$). Let $\mathcal{P}$ be the linear map from $\mathcal{A}_n$ to itself satisfying

$$\mathcal{P}E_{i,j} = (d_j - d_i)E_{i,j},$$

where $E_{i,j}$ is the matrix whose only nonzero entries are a one in position (i,j) and a minus one in position (j,i). The equation $\mathcal{P}\Pi_a S = -[D,S]$ is satisfied and $\mathcal{P}$ is clearly symmetric and positive definite. Thus the Riemannian structure

$$\langle\langle [S,A_1], [S,A_2]\rangle\rangle_{T_S\mathcal{O}_\Lambda} = \langle \mathcal{P}A_1, A_2\rangle$$

realizes the required equality. ■

Corollary 13 *The height functions h_D and $\tilde{h}_D$ are Morse, with critical points given by the diagonal matrices Λ^π. In $\mathcal{O}_\Lambda$, the index of Λ^π is $i(\pi)$. In $\mathcal{T}_\Lambda$, the index is $d(\pi)$.*

Proof: The critical points of h_D are the matrices Λ^π, from 12. The nondegeneracy and the expression for the index follow from 10. ■

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