2
The manifolds $\mathcal{O}_\Lambda$ and $\mathcal{T}_\Lambda$

Throughout this chapter $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$ will be a real $n \times n$ diagonal matrix with simple spectrum. Let $\mathcal{O}_\Lambda$ be the set of real symmetric matrices conjugate to Λ and $\mathcal{T}_\Lambda$ be the subset of $\mathcal{O}_\Lambda$ consisting of tridiagonal matrices.

We first prove that both sets are indeed manifolds. For $\mathcal{O}_\Lambda$, this follows from a simple application of the implicit function theorem, but the first proof for $\mathcal{T}_\Lambda$ was more delicate (12). Here, instead, we provide an atlas for each space, following (8).

2.1
Triangular coordinates

The *triangular coordinates* were originally intended for $\mathcal{T}_\Lambda$ in (8), where they are called bidiagonal coordinates, but they turn out to be a much more general construct.

Diagonalize $S \in \mathcal{O}_\Lambda$ as $S = Q^T \Lambda Q$ with Q orthogonal. The matrix Q is defined only up to left multiplication by a sign diagonal matrix E. Let $\mathcal{U}_\mathcal{O} \subset \mathcal{O}_\Lambda$ be the set of matrices S for which the related Q (or any EQ) admits an LU-decomposition. Thus, for $S \in \mathcal{U}_\mathcal{O}$, there is indeed a (unique) decomposition $S = Q^T \Lambda Q$ such that Q is *LU-positive* (see Appendix 8.4). More generally, given a permutation $\pi \in S_n$, let $\Lambda^\pi = \mathrm{diag}(\lambda_{\pi(1)}, \dots, \lambda_{\pi(n)})$ and $\mathcal{U}_\mathcal{O}^\pi \subset \mathcal{O}_\Lambda$ be the set of matrices S admitting a spectral decomposition $S = Q_\pi^T \Lambda^\pi Q_\pi$ for an orthogonal *LU-positive* matrix Q_π. Clearly each $\mathcal{U}_\mathcal{O}^\pi$ is an open, dense subset of $\mathcal{O}_\Lambda$ and their union covers $\mathcal{O}_\Lambda$.

Now, for $S \in \mathcal{U}_\mathcal{O}^\pi$, write the (unique) diagonalization $S = Q_\pi^T \Lambda^\pi Q_\pi$ where Q_π is an orthogonal *LU-positive* matrix and LU-decompose $Q_\pi = L_\pi U_\pi$ so that L_π is lower triangular unipotent and the upper triangular matrix U_π has positive diagonal. Set $B_\pi = L_\pi^{-1}\Lambda^\pi L_\pi = U_\pi S U_\pi^{-1}$, so B_π is lower triangular:

$$B_\pi = \begin{pmatrix} \lambda_{\pi(1)} & & & & \\ \beta^\pi_{2,1} & \lambda_{\pi(2)} & & & \\ \beta^\pi_{3,1} & \beta^\pi_{3,2} & \lambda_{\pi(3)} & & \\ \vdots & & \ddots & \ddots & \\ \beta^\pi_{n,1} & \beta^\pi_{n,2} & \dots & \beta^\pi_{n,n-1} & \lambda_{\pi(n)} \end{pmatrix}.$$

Proposition 1 *There is a diffeomorphism $\phi_\pi\colon \mathbb{R}^N \to \mathcal{U}^\pi_\mathcal{O}$ taking the entries $\beta^\pi_{i,j}, i = 2, \dots, n, j = 1, \dots, i-1$, to $S \in \mathcal{U}^\pi_\mathcal{O}$. Here $N = \frac{n(n-1)}{2}$.*

Proof: With the numbers $\beta^\pi_{i,j}$, construct B_π as above. Now diagonalize $B_\pi = L_\pi^{-1}\Lambda^\pi L_\pi$, which can be done in a unique fashion such that L_π is unipotent. Now consider the (unique) *QR-decomposition* $L_\pi = Q_\pi R_\pi$, where Q_π is orthogonal and R_π is upper triangular with positive diagonal. Since $Q_\pi = L_\pi R_\pi^{-1}$, we see that Q_π is *LU-positive*. Finally set $S = Q_\pi^T \Lambda^\pi Q_\pi$, clearly a matrix in $\mathcal{U}^\pi_\mathcal{O}$. To go from S to B_π, proceed as described previously. By construction, both maps are inverse to each other. ■

We now specialize the above construction for tridiagonal matrices $T \in \mathcal{T}_\Lambda$ — which values of the triangular coordinates $\beta_{i,j}$ give rise to such a T? Given $\pi \in S_n$, let $\mathcal{U}^\pi_\mathcal{T} \subset \mathcal{T}_\Lambda$ be the set of matrices T admitting a spectral decomposition $T = Q_\pi^T \Lambda^\pi Q_\pi$, for some orthogonal, *LU-positive* matrix Q_π. As before, each $\mathcal{U}^\pi_\mathcal{T}$ is an open, dense subset of $\mathcal{T}_\Lambda$ (and each contains all matrices in $\mathcal{T}_\Lambda$ with nonzero off-diagonal entries) and their union covers $\mathcal{T}_\Lambda$. The only real difference is that since

$$B_\pi = L_\pi^{-1}\Lambda^\pi L_\pi = U_\pi T U_\pi^{-1},$$

from the first equality B_π is lower triangular and, from the second, it is upper Hessenberg (a real square matrix H is upper Hessenberg if $H_{ij} = 0$ whenever $i > j+1$): B_π must be lower bidiagonal! The following proposition is immediate.

Proposition 2 *There is a diffeomorphism $\phi_\pi\colon \mathbb{R}^{n-1} \to \mathcal{U}^\pi_\mathcal{T}$ taking the entries $\beta^\pi_{i,i-1}, i = 2, \dots, n$, to $T \in \mathcal{U}^\pi_\mathcal{T}$.*

Proposition 3 *The sets $\mathcal{O}_\Lambda$ and $\mathcal{T}_\Lambda$ are compact, connected, orientable manifolds of dimension $N = \frac{n(n-1)}{2}$ and $n-1$, respectively.*

Proof: Charts were provided for both spaces. Compactness follows from the fact that both spaces are closed in $\mathcal{S}_n$ and lie in a sphere of radius $\|\Lambda\| = \sqrt{\mathrm{tr}\Lambda^2}$

centered at the origin. Connectivity of $\mathcal{O}_\Lambda$ follows from the connectivity of $SO(n)$, proved in Appendix 8.3, since $\mathcal{O}_\Lambda$ is the image of the continuous function

$$\begin{array}{rccc} F\colon & SO(n) & \to & \mathcal{O}_\Lambda \\ & Q & \mapsto & Q^T \Lambda Q \end{array}.$$

To compute the tangent space of $\mathcal{O}_\Lambda$ at a matrix S, take curves of the form $S(t) = e^{-tA} S e^{tA}$ for skew-symmetric matrices A, which clearly stay in $\mathcal{O}_\Lambda$ and satisfy $S(0) = S$. Differentiating, we learn that the matrices $[S, A]$ are tangent vectors. Once we show that such vectors are independent, this has to be the full tangent space, since the vector space of skew-symmetric matrices also has dimension N. Now, suppose $[S, A] = 0$ for some skew-symmetric matrix A. This means that A commutes with S, which in turn has simple spectrum. From Lemma 4 below, A must be a function of S, and thus, symmetric. But the only matrix which is simultaneously symmetric and skew-symmetric is 0. This also provides orientability, by identifying N independent vector fields along $\mathcal{O}_\Lambda$: just take a basis A_i of skew-symmetric matrices and consider the vector field $[S, A_i]$.

We now consider the connectivity of $\mathcal{T}_\Lambda$. Take $T \in \mathcal{T}_\Lambda$, hence, in some $\mathcal{U}^\pi_T$. In triangular coordinates, join T to the diagonal matrix Λ^π. We only need to construct paths joining diagonal matrices Λ^{π_1} and Λ^{π_2}. This is easy to accomplish if π_2 differs from π_1 by an inversion of consecutive diagonal entries i and $i+1$. The path in this case is simple: it consists of a conjugation of Λ^{π_1} by a rotation in the $(i, i+1)$-plane, which clearly stays within $\mathcal{T}_\Lambda$. Arbitrary permutations π_2 differ from π_1 by a product of inversions and connectivity follows.

Finally, the proof that $\mathcal{T}_\Lambda$ is orientable is given in Appendix 8.8. ■

In the proof above, and later in the text, we use the following fact from linear algebra.

Lemma 4 *Let $A \in \mathcal{M}_n$ have simple spectrum. Then A and B commute if and only if B is a polynomial of A, $B = p(A)$.*

Proof: Write $A = PDP^{-1}$ where D is a diagonal matrix with the eigenvalues of A. Then,

$$[A, B] = 0 \Leftrightarrow (PDP^{-1})B = B(PDP^{-1}) \Leftrightarrow D(P^{-1}BP) = (P^{-1}BP)D.$$

But D is a diagonal matrix with simple spectrum, so

$$[D, P^{-1}BP] = 0 \Leftrightarrow P^{-1}BP = \tilde{D}$$

with $\tilde{D}$ also a diagonal matrix. Finally,

$$P^{-1}BP = \tilde{D} \Leftrightarrow B = P\tilde{D}P^{-1} = p(A)$$

for some polynomial p. ■

The manifold $\mathcal{O}_\Lambda$ is special in many senses. Thus, for example, it is an *adjoint orbit* — a matter not considered in this text — and this allowed the computation of its homology by Faybusovich (4) following general ideas of Bott.

Proposition 5 *If the spectrum of Λ is simple, $SO(n)$ is a covering space for $\mathcal{O}_\Lambda$.*

Proof: The function

$$\begin{array}{rccc} F: & SO(n) & \to & \mathcal{O}_\Lambda \\ & Q & \mapsto & Q^T\Lambda Q \end{array}$$

fails to be injective: $F(Q) = F(W)$ whenever $(WQ^T)\Lambda = \Lambda(WQ^T)$. This means that WQ^T is a function of Λ (hence diagonal, from Lemma 4) and orthogonal, so $Q = WE$ for some sign diagonal matrix E with unit determinant.

Let us check that F is a covering map for $\mathcal{O}_\Lambda$ with fibers consisting of 2^{n-1} elements. The continuity and surjectivity of F are obvious. From Appendix 8.3, $T_Q SO(n) = Q\mathcal{A}_n$[1], and the derivative at $Q \in SO(n)$ along QA is given by

$$DF(Q)(QA) = -AQ^T\Lambda Q + Q^T\Lambda QA = [F(Q), A].$$

It is easy to see then that $DF(Q)$ is an isomorphism between tangent spaces. From the inverse function theorem, F is a local diffeomorphism and we are done. ■

We have not yet computed the tangent spaces of $\mathcal{T}_\Lambda$: this will be easier once we have introduced the Toda flows. However, the tangent space $T_D\mathcal{T}_\Lambda$ at a diagonal matrix $D \in \mathcal{T}_\Lambda$ is easily seen to be the $(n-1)$-dimensional vector space of tridiagonal symmetric matrices with a null main diagonal.

[1]For notation used throughout this text, see Appendix 8.