1 Introduction

Let $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ be a real, $n \times n$ diagonal matrix of simple spectrum, i.e., with n distinct eigenvalues, say $\lambda_1 < \ldots < \lambda_n$. In this work we consider two manifolds: the space $\mathcal{O}_\Lambda$ of real, symmetric matrices conjugate to Λ and its subset $\mathcal{T}_\Lambda$ consisting of the tridiagonal matrices in $\mathcal{O}_\Lambda$.

The fact that $\mathcal{O}_\Lambda$ is a compact, orientable manifold naturally embedded in $\mathcal{S}_n$, the vector space of real, symmetric matrices of order n, is essentially an application of the implicit function theorem[1]. The same properties hold for $\mathcal{T}_\Lambda$, for more elaborate reasons. Our main purpose is to compute the homologies $H_*(\mathcal{T}_\Lambda; \mathbb{Z})$ and $H_*(\mathcal{O}_\Lambda; \mathbb{Z}/2\mathbb{Z})$. These computations suffice to obtain some interesting geometric information about the natural embedding $\mathcal{O}_\Lambda \hookrightarrow \mathcal{S}_n$ — it is *tight* and *taut* — a result with unexpected consequences to matrix spectral theory. This is not the case for $\mathcal{T}_\Lambda$, as will be shown, but we still present some *perfect* Morse functions on it.

The basic ingredient is the *height function*

$$h(S) = \mathrm{tr} DS,$$

where $D = \mathrm{diag}(d_1, \ldots, d_n)$ for distinct numbers d_k. Height functions are Morse on $\mathcal{O}_\Lambda$ and $\mathcal{T}_\Lambda$ and one may expect to compute homology by the standard procedure from Morse theory (10).

In a nutshell, one computes sequentially the homology of subsets of the form $M_c = h^{-1}(-\infty, c])$ for increasing regular values c. The topological type of M_c changes at critical values of h in a controlled fashion. A simple computation shows that the critical points of h are the diagonal matrices $\Lambda^\pi = \mathrm{diag}(\lambda_{\pi(1)}, \ldots, \lambda_{\pi(n)})$, where $\pi \in S_n$ is a permutation. In $\mathcal{T}_\Lambda$, Λ^π is a critical point of index i if and only if the permutation π has i *descents*, i.e., $d(\pi) = i$ (for terminology and notation related to permutations, see Appendix 8.5). In $\mathcal{O}_\Lambda$, the index of the same critical point is the number of *inversions* $i(\pi)$.

The difficulty, as usual, lies in the understanding of the connecting

[1] Notation used throughout this text is listed in Appendix 8.1.

homomorphisms $\partial : H_{k+1}(M_{c+\epsilon}, M_{c-\epsilon}; A) \to H_k(M_{c-\epsilon}; A)$ associated to the attachment of a new cell at the transition of a critical value $c = h(\Lambda^\pi)$. It turns out that all connecting homomorphisms are trivial, both for and $\mathcal{T}_\Lambda$ and $\mathcal{O}_\Lambda$, for the appropriate choice of coefficients. Thus, the homology essentially counts the critical points of a given index.

To verify the triviality of these maps, one uses the *Toda flow*,

$$\dot{S} = [S, \Pi_a S],$$

a very special differential equation which keeps both manifolds invariant. Here, $\Pi_a S$ is the real, skew symmetric matrix whose lower triangular part equals that of S. The argument simplifies considerably because, on both manifolds, there is a Riemannian structure for which the gradient of the height function is the Toda vector field.

Indeed, a generic gradient of a height function h may be hard to track, but the stable and unstable manifolds of the Toda flow may be computed explicitly, and this is how they become handy in the study of the connecting homomorphisms.

For $\mathcal{T}_\Lambda$, the generator of the relative homology $H_{k+1}(M_{c+\epsilon}, M_{c-\epsilon}; A)$ at a critical point Λ^π is homotopic to a cycle γ contained in the stable manifold $W_s(\Lambda^\pi)$. This cycle, as will be shown, is trivialized by the closure of $W_s(\Lambda^\pi)$, which turns out to be an orientable manifold. The upshot is that $H_*(\mathcal{T}_\Lambda; \mathbb{Z})$ is freely generated by such closures.

The argument on $\mathcal{O}_\Lambda$ has to be refined. In this case, the closure of $W_s(\Lambda^\pi)$ is not a manifold, but it still yields a trivialization for the generator of $H_{k+1}(M_{c+\epsilon}, M_{c-\epsilon}; A)$ when computations are performed in $\mathbb{Z}/2\mathbb{Z}$.

The universal covering space of $\mathcal{T}_\Lambda$ was shown to be $\mathbb{R}^{n-1}$ in (12). Soon after, Fried (5) computed its cohomology ring using techniques related to those employed in this text. The manifold $\mathcal{O}_\Lambda$ is, in a sense, simpler, being an adjoint orbit of an appropriate group action (for complex matrices), for which Lie group techniques may be applied. Indeed, using ideas of Bott, Faybusovich and Kocherlakota computed $H_*(\mathcal{O}_\Lambda)$ in (4) and (6), respectively.

The height function h defined on each manifold is *perfect*: for diagonal matrices D with simple spectrum, h is a Morse function whose number of critical points of index k equals the k-th Betti number. From the Morse inequalities, this is the minimal number of critical points for a Morse function on a compact manifold.

A mild amplification of this fact has an interesting geometric interpretation. Consider the embedding $\iota : \mathcal{O}_\Lambda \hookrightarrow \mathcal{S}_n$. We will show that ι is *tight* and *taut*, in the sense that, for any hyperplane $\mathcal{P} \subset \mathcal{S}_n$ (resp. point $S \in \mathcal{S}_n$), the height function $h_{\mathcal{P}} : \mathcal{O}_\Lambda \to \mathbb{R}$ giving the signed distance of a point $p \in \mathcal{O}_\Lambda$ to $\mathcal{P}$ (resp. $\ell_S : \mathcal{O}_\Lambda \to \mathbb{R}$, the square of the distance of $p \in \mathcal{O}_\Lambda$ to S), if Morse, is perfect. This is not the case for the analogous embedding of $\mathcal{T}_\Lambda$ into the space of real, tridiagonal, symmetric matrices: we show a counterexample for 3×3 matrices.

In general, tightness implies the *two-piece property (TPP)*: an embedding $M \subset \mathbb{R}^n$ satisfies the TPP if, for every hyperplane $\mathcal{P} \subset \mathbb{R}^n$, the complement $M \setminus \mathcal{P}$ contains at most two connected components. As shown in (2), an equivalent formulation of the TPP for a compact smoothly immersed manifold M is the following: every Morse height function admits a single local minimum and maximum. Said differently, for such functions, a local extremum is necessarily global. We finish the text with some natural applications of these ideas to problems in numerical spectral theory.