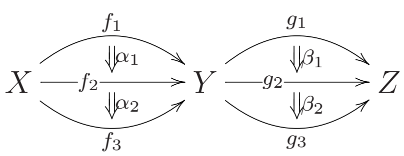
4
2-Category Theory

Considering the category whose objects are small categories [1] and whose arrows are functors, it would be interesting that the maps between them were structure preserving. For example, in theorem 3.1.1 of section 3.1 (p. 27), we would like a realtion less strict than equality. As objects in a category are defined "up to isomorphism", one way of achieving it is by seeing the arrows as objects. This is the idea of 2-Category Theory.

In a 2-category, the set of morphisms with same source and target is seen as a category, that is, the morphisms are seen as objects and, at another level, the morphisms between them as morphisms. That means that, as 1-categorical objects are defined up to isomorphism, so are 2-categorical arrows. To prevent confusion, the objects are called 0-cells, the morphisms 1-cells and the morphisms between morphisms 2-cells, which are represented by double arrows, and we write $Hom_{Cat}(X, Y)$ when $Hom(X, Y)$ is seen as a category. The 2-cells have the properties of the categorical arrows with the addition of a new way of composing them. This new composition is called vertical and is represented by "$\cdot$" and the other is called horizontal and is represented by ";": given 2-cells $\alpha_1 \colon f_1 \Rightarrow f_2$ and $\alpha_2 \colon f_2 \Rightarrow f_3$, there exists a 2-cell $\alpha_2 \cdot \alpha_1 \colon f_1 \Rightarrow f_3$ that is their *vertical* composition and given 2-cells $\alpha_1 \colon f_1 \Rightarrow f_2$ and $\beta_1 \colon g_1 \Rightarrow g_2$ such that $g_1 \circ f_1$ and $g_2 \circ f_2$ are defined, there exists a 2-cell $\beta_1; \alpha_1 \colon g_1 \circ f_1 \Rightarrow g_2 \circ f_2$ that is their *horizontal* composition.

The vertical and horizontal composition have the following property, which we call 2-composition property: given a 2-categorical diagram



we can either first compose vertically and then horizontally or first horizontally

[1]categories whose colection of objects and morphisms are sets

and then vertically, that is,

$$(\beta_2 \cdot \beta_1); (\alpha_2 \cdot \alpha_1) = (\beta_2; \alpha_2) \cdot (\beta_1; \alpha_1)$$

2-product: The definition of categorical product is inspired in its 1-categorical similar. A 1-categorical product is determined by a natural isomorphism between the sets of arrows $Hom(X, A \times B)$ and $Hom(X, A) \otimes Hom(X, B)$, so it is natural to define 2-categorical product as coming from the natural isomorphism between the categories $Hom_{Cat}(X, A \times B)$ and $Hom_{Cat}(X, A) \otimes Hom_{Cat}(X, B)$. Hence, a *2-categorical product*, or simply 2-product, of two 0-cells A and B is a 0-cell $A \times B$ together with two projection arrows $\pi_1 \colon A \times B \to A$ and $\pi_2 \colon A \times B \to B$ such that for any $f \colon X \to A$ and $g \colon X \to B$, there exist $h \colon X \to A \times B$ and isomorphisms $\pi_1 \circ h \cong f$ and $\pi_2 \circ h \cong g$ and such that for all $k \colon X \to A \times B$ and 2-cells $\alpha \colon \pi_1 \circ h \Rightarrow \pi_1 \circ k$ and $\beta \colon \pi_2 \circ h \Rightarrow \pi_2 \circ k$ there exist a unique $\gamma \colon h \Rightarrow k$ such that $id_{\pi_1}; \gamma = \alpha$ and $id_{\pi_2}; \gamma = \beta$.

2-initial object: Given a 0-cell A, a 0-cell 0 is initial if $Hom_{Cat}(0, A)$ is equivalent to **1**, the terminal category [2]. In other words, 0 is initial if there exists an arrow from 0 to A and, for every pair of arrows $f, g \colon 0 \to A$, there exists a unique 2-cell from f to g which is an isomorphism.

2-functor: Given two 2-categories $\mathcal{A}$ and $\mathcal{B}$, a 2-functor F from $\mathcal{A}$ to $\mathcal{B}$ is a function that takes to $\mathcal{B}$ the 2-categorical structure of $\mathcal{A}$, i.e., besides (i) to (iv) in the definition of categorical functor (see p. 25) with equalities replaced by isomorphisms, we also have:
(v) if $\alpha \colon f \Rightarrow g$, then $F(\alpha) \colon F(f) \Rightarrow F(g)$;
(vi) for every 1-cell f, $F(id_f) = id_{F(f)}$;
(vii) if $\gamma = \alpha \cdot \beta$, then $F(\gamma) = F(\alpha) \cdot F(\beta)$ and;
(viii) if $\gamma = \alpha; \beta$, then $F(\gamma) = F(\alpha); F(\beta)$.

Natural Transformation: Given two categories $\mathcal{A}$ and $\mathcal{B}$ and functors F, $G \colon \mathcal{A} \to \mathcal{B}$, a natural transformation $\sigma \colon F \to G$ is such that, for every $h \colon A \to B$ in $\mathcal{A}$, the following diagram commutes:

$$\begin{array}{ccc} F(A) & \xrightarrow{\sigma_A} & G(A) \\ {\scriptstyle F(h)}\downarrow & & \downarrow{\scriptstyle G(h)} \\ F(B) & \xrightarrow{\sigma_B} & G(B) \end{array}$$

[2] a category with only one object and only one arrow - the identity arrow

Although a 1-categorical definition, it plays an important role in some of the proofs in this section. Note that one can define a 2-category whose 0-cells are small categories, 1-cells are functors and 2-cells are natural transformations.

The word *lax* preceding a notion means that the given notion is "weakened" - the equivalences in the definition can be replaced by arrows, e.g., if F is a 2-functor and f and g are 1-cells, a lax 2-functor would, instead of $F(f) \circ F(g) = F(f \circ g)$, have, for example, $F(f) \circ F(g) \Rightarrow F(f \circ g)$, weakening the definition of 2-functor.

2-adjunction: Given two 2-categories $\mathcal{A}$ and $\mathcal{B}$ and 2-functors $F\colon \mathcal{A} \to \mathcal{B}$ and $G\colon \mathcal{B} \to \mathcal{A}$, a 2-adjunction occurs when, for every 0-cell A in $\mathcal{A}$ and B in $\mathcal{B}$, there exists a natural isomorphism between the categories $Hom_{Cat}(F(A), B)$ and $Hom_{Cat}(A, G(B))$. A lax 2-adjunction would have lax 2-functors instead of 2-functors and arrows instead of isomorphism between the categories. Depending on the direction of the arrows, we have different lax 2-adjunction.

For every $f\colon A' \to A$ in $\mathcal{A}$ and $g\colon B \to B'$ in $\mathcal{B}$ we define: for every h in $Hom(A, B)$,

- $Hom(f, B) = h \circ f$ is the functor that goes from $Hom(A, B)$ to $Hom(A', B)$;
- $Hom(A, g) = g \circ h$ is the functor that goes from $Hom(A, B)$ to $Hom(A, B')$.

4.1 A proof-theoretical 2-category

As reductions and expansions are relations between derivations - that are represented by arrows - it is natural to try to represent them explicitly as 2-cells instead of having equality between arrows (1-cells). More precisely, consider derivations $\Pi_i\colon Y[x : X]$ and $\Psi_i\colon Z[x : Y]$, $i = 1, 2, 3$. Consider also the reductions α_1 and α_2 from Π_1 to Π_2 and from Π_2 to Π_3 respectively and the reductions β_1 and β_2 from Ψ_1 to Ψ_2 and from Ψ_2 to Ψ_3 respectively. We have then the following diagram:

$$
\begin{array}{ccccc}
X & & X & & X \\
\Pi_1 & \rhd_{\alpha_1} & \Pi_2 & \rhd_{\alpha_2} & \Pi_3 \\
Y & & Y & & Y \\
\Psi_1 & \rhd_{\beta_1} & \Psi_2 & \rhd_{\beta_2} & \Psi_3 \\
Z & & Z & & Z
\end{array}
$$

According to the kind of reductions usually defined in Proof Theory in order to achieve normal proofs we have some possibilities when applying the reductions above: we can, for instance, apply α_1 followed by α_2 followed by β_1 followed by β_2 or α_1 followed by β_1, followed by α_2 followed by β_2. Any of the choices should produce the same result, namely, the proof on the right side. Of course the particular meaning of "followed by" is not always the same, so to say, α_1 followed by α_2 can be seen as a sequencing of reductions while α_1 followed by β_1 can be seen as a reduction integrated to a substitution process.

If we represent the formulas of the above schema by 0-cells, the derivations by 1-cells and the reductions by 2-cells, we achieve a 2-categorical diagram like the one presented in the last section and then sequencing of reductions can be viewed as vertical composition and reduction integrated to substitution can be viewed as horizontal composition. Note that the 2-composition property corresponds to the three properties presented in section 2.1, meaning that there might exist a direct correspondence between 2-categorical and proof-theoretical composition.

Let us define now a rex that eliminates simultaneously maximum formulas of different branches. Given a derivation of the form $\dfrac{\begin{matrix}\Pi_1 & \Pi_2 \\ A & B\end{matrix}}{C}$, if $\Pi_1 \rhd_\alpha \Pi'_1$ and $\Pi_2 \rhd_\beta \Pi'_2$, then $\alpha \mid \beta$ is the rex that takes $\dfrac{\begin{matrix}\Pi_1 & \Pi_2 \\ A & B\end{matrix}}{C}$ to $\dfrac{\begin{matrix}\Pi'_1 & \Pi'_2 \\ A & B\end{matrix}}{C}$ by first reducing Π_1 and then Π_2. Note that the application of β followed by α should produce the same result.

Proposition 4.1.1. *For every pair of reductions* (α_1, α_2) *and* (β_1, β_2), $(\alpha_1 \mid \alpha_2) \cdot (\beta_1 \mid \beta_2) = (\alpha_1 \cdot \beta_1) \mid (\alpha_2 \cdot \beta_2)$.

Proof. Given reductions $\Pi_i \rhd_{\alpha_i} \Pi'_i$ and $\Pi'_i \rhd_{\beta_i} \Pi''_i$, $i = 1, 2$, as the reductions α_1 and β_1 occur in a branch and the reductions α_2 and β_2 occur in another branch, applying α_1 together with α_2 followed by β_1 together with β_2 produce the same result as applying α_1 followed by β_1 together with α_2 followed by β_2. □

4.2 Conjunction

As categorical product is related to conjunction, it would be natural to try to achieve a similar result to Mann's (11) in 2-Category Theory, namely, a relation between conjunction and 2-product: given two 0-cells A and B, if we represent the formula $A \wedge B$ by the product $A \times B$ and the derivation rules $\dfrac{A \wedge B}{A}$ and $\dfrac{A \wedge B}{B}$ by the projection arrows $\pi_1\colon A \times B \to A$ and $\pi_2\colon A \times B \to B$ respectively, h would be represented by $\dfrac{\begin{array}{cc} C & C \\ f & g \\ A & B \end{array}}{A \wedge B}$. In order to $A \wedge B$ be a 2-categorical product, we would have $\pi_1 \circ h$ and f isomorphic and hence

$$\dfrac{\dfrac{\begin{array}{cc} C & C \\ f & g \\ A & B \end{array}}{A \wedge B}{\scriptstyle h}}{A}{\scriptstyle \pi_1} \quad \begin{array}{c} \rhd \\ \lhd \end{array} \quad \begin{array}{c} C \\ f \\ A \end{array} . \tag{4-1}$$

However, there is no meaningful proof-theoretical relation from $\begin{array}{c} C \\ f \\ A \end{array}$ to $\dfrac{\dfrac{\begin{array}{cc} C & C \\ f & g \\ A & B \end{array}}{A \wedge B}}{A}$. Such a reason is enough to make us realise that conjunction does not have the same structure of 2-categorical product and for a similar reason it is not possible to relate co-product with disjunction and exponential with implication using such a strategy.

As we have only one of the relations in (4-1), the one that represents $\wedge$-reduction, we can try to relate conjunction to lax 2-product. Let us keep on this path: let k be an arrow from C to $A \times B$. Then the 2-cells α and β from $\pi_1 \circ h$ to $\pi_1 \circ k$ and from $\pi_2 \circ h$ to $\pi_2 \circ k$ respectively can be represented by the two rex

$$\dfrac{\begin{array}{c} C \\ h \\ A \wedge B \end{array}}{A} \rhd_\alpha \dfrac{\begin{array}{c} C \\ k \\ A \wedge B \end{array}}{A} \qquad \dfrac{\begin{array}{c} C \\ h \\ A \wedge B \end{array}}{B} \rhd_\beta \dfrac{\begin{array}{c} C \\ k \\ A \wedge B \end{array}}{B}$$

To achieve the wanted relation, we would have to show that there exists only one $\gamma\colon h \Rightarrow k$ such that $id_{\pi_1};\gamma = \alpha$ and $id_{\pi_2};\gamma = \beta$. The unicity of γ is discussed in the conclusion.

Seely (17) states that conjunction is lax right 2-adjoint to the diagonal lax 2-functor and that disjunction is lax left 2-adjoint to the diagonal lax 2-functor. Seely does not use this nomenclature, but defines lax and rax 2-adjunction:

lax 2-adjunction: Given two 2-categories $\mathcal{A}$ and $\mathcal{B}$ and lax 2-functors $F\colon \mathcal{A} \to \mathcal{B}$ and $G\colon \mathcal{B} \to \mathcal{A}$, a lax 2-adjunction occurs when, for every 0-cell A in $\mathcal{A}$ and B in $\mathcal{B}$, there exist arrows K_{AB} from $Hom_{Cat}(F(A), B)$ to $Hom_{Cat}(A, G(B))$

and L_{AB} from $Hom_{Cat}(A, G(B))$ to $Hom_{Cat}(F(A), B)$ such that those arrows define the following natural transformations, as defined by Seely (17): Let $f\colon A' \to A$ and $g\colon B \to B'$. Then,

1. $L_{A'B} \circ Hom(f, G(B)) \Rightarrow Hom(F(f), B) \circ L_{AB}$;
2. $L_{A'B'} \circ Hom(A', G(g)) \Rightarrow Hom(F(A'), g) \circ L_{A'B}$;
3. $Hom(f, G(B)) \circ K_{AB} \Rightarrow K_{A'B} \circ Hom(F(f), B)$;
4. $Hom(A', G(g)) \circ K_{A'B} \Rightarrow K_{A'B'} \circ Hom(F(A'), g)$;
5. $\alpha\colon L_{AB} \circ K_{AB} \Rightarrow id_{Hom_{Cat}(F(A),B)}$;
6. $\beta\colon id_{Hom_{Cat}(A,G(B))} \Rightarrow K_{AB} \circ L_{AB}$

We also have that the following equalities hold:

$(id_{K_{AB}}; \alpha) \cdot (\beta; id_{K_{AB}}) = id_{K_{AB}}$ and
$(\alpha; id_{L_{AB}}) \cdot (id_{L_{AB}}; \beta) = id_{L_{AB}}$

If we invert the direction of the natural transformations, we still have a lax 2-adjunction but, to differentiate these two notions, Seely called this new one *rax* 2-adjunction. In a rax 2-adjunction, instead of the above equalities, we have the following: $(\beta; id_{K_{AB}}) \cdot (id_{K_{AB}}; \alpha) = id_{K_{AB}}$ and $(id_{L_{AB}}; \beta) \cdot (\alpha; id_{L_{AB}}) = id_{L_{AB}}$. Seely only gives an idea of the proof. Let us prove it in detail.

We present the definition of lax 2-adjunction in four parts on an attempt to make the proof easier to read:

first part given the 2-categories $\mathcal{PT}$ and $\mathcal{PT} \times \mathcal{PT}$, where $\mathcal{PT}$ is the 2-category whose 0-cells are formulas, 1-cells are derivations and 2-cells are reductions, we show that both $\Delta\colon \mathcal{PT} \to \mathcal{PT} \times \mathcal{PT}$ and $\wedge\colon \mathcal{PT} \times \mathcal{PT} \to \mathcal{PT}$ are lax 2-functors;

second part we show that there exists an arrow K_{AB} from $Hom_{Cat}(\Delta A, B)$ to $Hom_{Cat}(A, \wedge B)$ and an arrow L_{AB} from $Hom_{Cat}(A, \wedge B)$ to $Hom_{Cat}(\Delta A, B)$, for every 0-cell A in $\mathcal{PT}$ and B in $\mathcal{PT} \times \mathcal{PT}$;

third part we show that the above arrows define the natural transformations correspondent to this definition;

fourth part finally, we show that $(id_{K_{AB}}; \alpha) \cdot (\beta; id_{K_{AB}}) = id_{K_{AB}}$ and $(\alpha; id_{L_{AB}}) \cdot (id_{L_{AB}}; \beta) = id_{L_{AB}}$.

first part

Lemma 4.2.1. *$\Delta\colon \mathcal{PT} \to \mathcal{PT} \times \mathcal{PT}$ such that $\Delta(A) = (A, A)$, $\Delta(f) = (f, f)$ and $\Delta(\alpha) = (\alpha, \alpha)$ for every formula A, derivation f and reduction α, is a lax 2-functor.*

Proof. The items (i), (ii) and (v) of the lax 2-functor definition come from the definition of Δ.

(iii) $\Delta(id_A) = id_{\Delta A}$, for every formula A

Given a formula A, both $\Delta(id_A)$ and $id_{\Delta A}$ represent the pair of formulas (A, A).

(iv) $\Delta f \circ \Delta g = \Delta(f \circ g)$ for every $f\colon A \to B$ and $g\colon B \to C$

Given derivations f and g, $\Delta f \circ \Delta g = (f, f) \circ (g, g) = (f \circ g, f \circ g) = \Delta(f \circ g)$

(vi) $\Delta(id_f) = id_{\Delta f}$ for every $f\colon A \to B$

Given a derivation f, $\Delta(id_f) = (id_f, id_f)$ which is the identity reduction that goes from Δf to Δf.

(vii) $\Delta(\alpha \cdot \beta) = \Delta\alpha \cdot \Delta\beta$ for every $\alpha\colon f \Rightarrow g$ and $\beta\colon g \Rightarrow h$

Given reductions $\alpha\colon f \Rightarrow g$ and $\beta\colon g \Rightarrow h$, $\Delta(\alpha \cdot \beta) = (\alpha \cdot \beta, \alpha \cdot \beta) = (\alpha, \alpha) \cdot (\beta, \beta) = \Delta\alpha \cdot \Delta\beta$

(viii) $\Delta(\alpha; \beta) = \Delta\alpha; \Delta\beta$ for every $\alpha\colon f \Rightarrow f'$ and $\beta\colon g \Rightarrow g'$ such that $g \circ f$ and $g' \circ f'$ are defined

Given reductions $\alpha\colon f \Rightarrow f'$ and $\beta\colon g \Rightarrow g'$, $\Delta(\alpha; \beta) = (\alpha; \beta, \alpha; \beta) = (\alpha, \alpha); (\beta, \beta) = \Delta\alpha; \Delta\beta$

□

Lemma 4.2.2. *$\wedge\colon \mathcal{PT} \times \mathcal{PT} \to \mathcal{PT}$ such that $\wedge(A) = A_1 \wedge A_2$, $\wedge(f) =$*

$$f_1 \wedge f_2 = \dfrac{\begin{array}{cc} \dfrac{A_1 \wedge A_2}{A_1} & \dfrac{A_1 \wedge A_2}{A_2} \\ f_1 & f_2 \\ A'_1 & A'_2 \end{array}}{A'_1 \wedge A'_2} = \langle f_1 \circ \pi_1, f_2 \circ \pi_2 \rangle$$

[3] *and $\wedge(\alpha) = (\alpha_1; id_{\pi_1}) \mid (\alpha_2; id_{\pi_2})$ for every formula $A = (A_1, A_2)$, derivation $f = (f_1, f_2)$, $f_i\colon A_i \to A'_i$ and rex $\alpha = (\alpha_1, \alpha_2)$, is a lax 2-functor.*

Proof. The items (i), (ii) and (v) of the lax 2-functor definition (p. 36) come from the definition of $\wedge$.

[3]like in the definition of product (p. 24)

(iii) $\wedge(id_A) = id_{\wedge A}$ for every 0-cell $A = (A_1, A_2)$

Given a formula A, both $id_{A_1} \wedge id_{A_2}$ and $id_{A_1 \wedge A_2}$ are the derivation $A_1 \wedge A_2$;

(iv) $\wedge f \circ \wedge g \Rightarrow \wedge(f \circ g)$ for every $f = (f_1, f_2)\colon (A_1, A_1) \to (B_1, B_2)$ and $g = (g_1, g_2)\colon (B_1, B_1) \to (C_1, C_2)$

Given derivations $f\colon A \to B$ and $g\colon B \to C$, $\wedge f \circ \wedge g$ is the derivation

$$\dfrac{\begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\dfrac{A_1 \wedge A_2}{A_1}\\ f_1\\ B_1\end{array} \qquad \begin{array}{c}\dfrac{A_1 \wedge A_2}{A_2}\\ f_2\\ B_2\end{array}}{B_1 \wedge B_2}}{B_1}\\ g_1\\ C_1\end{array} \qquad \begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\dfrac{A_1 \wedge A_2}{A_1}\\ f_1\\ B_1\end{array} \qquad \begin{array}{c}\dfrac{A_1 \wedge A_2}{A_2}\\ f_2\\ B_2\end{array}}{B_1 \wedge B_2}}{B_2}\\ g_2\\ C_2\end{array}}{C_1 \wedge C_2}$$

which reduces to

$$\dfrac{\begin{array}{c}\dfrac{A_1 \wedge A_2}{A_1}\\ f_1\\ B_1\\ g_1\\ C_1\end{array} \qquad \begin{array}{c}\dfrac{A_1 \wedge A_2}{A_2}\\ f_2\\ B_2\\ g_2\\ C_2\end{array}}{C_1 \wedge C_2} = \wedge(f \circ g).$$

(vi) $\wedge(id_f) = id_{\wedge f}$ for every $f = (f_1, f_2)\colon (A_1, A_1) \to (B_1, B_2)$

Given a derivation $f\colon A \to B$, $\wedge(id_f) = (id_{f_1}; id_{\pi_1}) \mid (id_{f_2}; id_{\pi_2})$ is the identity reduction that goes from $\wedge f$ to $\wedge f$.

(vii) $\wedge(\alpha \cdot \beta) = \wedge\alpha \cdot \wedge\beta$ for every $\alpha\colon f \Rightarrow g$ and $\beta\colon g \Rightarrow h$

Given reductions $\alpha\colon f \Rightarrow g$ and $\beta\colon g \Rightarrow h$,

$$\begin{array}{rcl}
\wedge\alpha \cdot \wedge\beta & = & \\
((\alpha_1; id_{\pi_1}) \mid (\alpha_2; id_{\pi_2})) \cdot ((\beta_1; id_{\pi_1}) \mid (\beta_2; id_{\pi_2})) & = & \text{(Lemma 4.1.1)}\\
((\alpha_1; id_{\pi_1}) \cdot (\beta_1; id_{\pi_1})) \mid ((\alpha_2; id_{\pi_2}) \cdot (\beta_2; id_{\pi_2})) & = & \text{(2-comp. property)}\\
((\alpha_1 \cdot \beta_1); (id_{\pi_1} \cdot id_{\pi_1})) \mid ((\alpha_2 \cdot \beta_2); (id_{\pi_2} \cdot id_{\pi_2})) & = & \text{(identities comp.)}\\
((\alpha_1 \cdot \beta_1); id_{\pi_1}) \mid ((\alpha_2 \cdot \beta_2); id_{\pi_2}) & = & \wedge(\alpha \cdot \beta)
\end{array}$$

(viii) $\wedge\beta; \wedge\alpha \Rightarrow \wedge(\beta; \alpha)$ for every $\alpha\colon f \Rightarrow f'$ and $\beta\colon g \Rightarrow g'$ such that $g \circ f$ and $g' \circ f'$ are defined

We only remark that applying $\wedge\alpha$ and $\wedge\beta$ followed by $\wedge$-reduction is the same as applying $\wedge$-reduction followed by $\wedge\alpha$ and $\wedge\beta$.

□

<u>second part</u>

Given the 2-categories $\mathcal{PT}$ and $\mathcal{PT} \times \mathcal{PT}$, a 0-cell A in $\mathcal{PT}$ and a 0-cell $B = (B_1, B_2)$ in $\mathcal{PT} \times \mathcal{PT}$ define the categories $Hom_{Cat}(\Delta A, B)$ and $Hom_{Cat}(A, \wedge B)$.

Lemma 4.2.3. *There exists a functor K_{AB} from $Hom_{Cat}(\Delta A, B)$ to $Hom_{Cat}(A, \wedge B)$.*

Let us define K_{AB} as a function that takes a pair of derivations from $Hom_{Cat}(\Delta A, B)$ and applies $\wedge$-int, i.e., for every $\Pi = (\Pi_1, \Pi_2) \in Hom_{Cat}(\Delta A, B)$, $K_{AB}(\Pi) = \dfrac{\begin{matrix} A \\ \Pi_1 \\ B_1 \end{matrix} \quad \begin{matrix} A \\ \Pi_2 \\ B_2 \end{matrix}}{B_1 \wedge B_2} \in Hom_{Cat}(A, \wedge B)$. Given a rex $\alpha = (\alpha_1, \alpha_2) \colon \Pi \rhd \Pi'$ in $Hom_{Cat}(\Delta A, B)$, $K_{AB}(\alpha) = \alpha_1 \mid \alpha_2$. Let us prove that K_{AB} thus defined is a functor (p. 36):

Proof. (i) Comes from the definition of K_{AB};
(ii) If $\Pi_1 \rhd_{\alpha_1} \Pi'_1$ and $\Pi_2 \rhd_{\alpha_2} \Pi'_2$, then

$$K_{AB}(\Pi) = \dfrac{\begin{matrix} A \\ \Pi_1 \\ B_1 \end{matrix} \quad \begin{matrix} A \\ \Pi_2 \\ B_2 \end{matrix}}{B_1 \wedge B_2} \rhd_{\alpha_1|\alpha_2} \dfrac{\begin{matrix} A \\ \Pi'_1 \\ B_1 \end{matrix} \quad \begin{matrix} A \\ \Pi'_2 \\ B_2 \end{matrix}}{B_1 \wedge B_2} = K_{AB}(\Pi')$$

i.e., $K_{AB}(\alpha) \colon K_{AB}(\Pi) \rhd K_{AB}(\Pi')$
(iii) Given a derivation Π,

$$K_{AB}(\Pi) = \dfrac{\begin{matrix} A \\ \Pi_1 \\ B_1 \end{matrix} \quad \begin{matrix} A \\ \Pi_2 \\ B_2 \end{matrix}}{B_1 \wedge B_2} \rhd_{id_{\Pi_1}|id_{\Pi_1}} \dfrac{\begin{matrix} A \\ \Pi_1 \\ B_1 \end{matrix} \quad \begin{matrix} A \\ \Pi_2 \\ B_2 \end{matrix}}{B_1 \wedge B_2} = K_{AB}(\Pi)$$

an thus $id_{K_{AB}(\Pi)} = K_{AB}(id_\Pi) \colon K_{AB}(\Pi) \rhd K_{AB}(\Pi)$
(iv) Comes from proposition 4.1.1. □

Lemma 4.2.4. *There exists a functor L_{AB} from $Hom_{Cat}(A, \wedge B)$ to $Hom_{Cat}(\Delta A, B)$.*

For every derivation Ψ in $Hom_{Cat}(A, \wedge B)$, put $L_{AB}(\Psi)$ as being the pair $\left(\dfrac{\begin{matrix} A \\ \Psi \\ B_1 \wedge B_2 \end{matrix}}{B_1}, \dfrac{\begin{matrix} A \\ \Psi \\ B_1 \wedge B_2 \end{matrix}}{B_2} \right)$ and, for every rex $\alpha \colon \Psi \rhd \Psi'$ in $Hom_{Cat}(A, \wedge B)$, $L_{AB}(\alpha) = (id_{\pi_1}; \alpha, id_{\pi_2}; \alpha)$.

Let us prove that L_{AB} thus defined is a functor:

Proof. (i) and (ii) come from the definition of L_{AB};
(iii) Given a derivation Ψ, $L_{AB}(id_\Psi) = (id_{\pi_1}; id_\Psi, id_{\pi_2}; id_\Psi)$, which is the identity reduction that goes from $L_{AB}(\Psi)$ to $L_{AB}(\Psi)$.
(iv) Given rex α and β,

$$\begin{array}{rcl}
L_{AB}(\alpha)\cdot L_{AB}(\beta) & = & \\
(id_{\pi_1};\alpha, id_{\pi_2};\alpha)\cdot(id_{\pi_1};\beta, id_{\pi_2};\beta) & = & \\
((id_{\pi_1};\alpha)\cdot(id_{\pi_1};\beta),(id_{\pi_2};\alpha)\cdot(id_{\pi_2};\beta)) & = & \text{(2-comp. property)}\\
((id_{\pi_1}\cdot id_{\pi_1});(\alpha\cdot\beta),(id_{\pi_2}\cdot id_{\pi_2});(\alpha\cdot\beta)) & = & \text{(comp. of identities)}\\
(id_{\pi_1};(\alpha\cdot\beta), id_{\pi_2};(\alpha\cdot\beta)) & = & L_{AB}(\alpha\cdot\beta)
\end{array}$$

□

<u>third part</u>

Given derivations $g = (g_1, g_2)$ in $\mathcal{A}$, $g_1\colon B_1'[x\colon B_1]$ and $g_2\colon B_2'[x\colon B_2]$, and $f\colon A[x\colon A']$ in $\mathcal{B}$, the following diagram defines the following natural transformations:

$$\begin{array}{ccc}
Hom_{Cat}(\Delta A, B) & \underset{\longleftarrow L_{AB} \longrightarrow}{\overset{\longrightarrow K_{AB} \longrightarrow}{\rightleftarrows}} & Hom_{Cat}(A, \wedge B)\\
\downarrow{\scriptstyle Hom(\Delta f, B)} & & \downarrow{\scriptstyle Hom(f, \wedge B)}\\
Hom_{Cat}(\Delta A', B) & \underset{L_{A'B}}{\overset{K_{A'B}}{\rightleftarrows}} & Hom_{Cat}(A', \wedge B)\\
\downarrow{\scriptstyle Hom(\Delta A', g)} & & \downarrow{\scriptstyle Hom(A', \wedge g)}\\
Hom_{Cat}(\Delta A', B') & \underset{L_{A'B'}}{\overset{K_{A'B'}}{\rightleftarrows}} & Hom_{Cat}(A', \wedge B')
\end{array}$$

Note that we can define a category whose objects are functors and whose arrows are natural transformations. Therefore, to show that the following natural transformations hold, we only need to show that they correspond to a rex.

1. $L_{A'B} \circ Hom(f, \wedge B) \Rightarrow Hom(\Delta f, B) \circ L_{AB}$

Given a derivation $\Sigma\colon B_1 \wedge B_2[x\colon A]$,

$$\begin{array}{c} A\\ \Sigma\\ B_1\wedge B_2 \end{array} \xrightarrow{Hom(f,\wedge B)} \begin{array}{c} A'\\ f\\ A\\ \Sigma\\ B_1\wedge B_2 \end{array} \xrightarrow{L_{A'B}} \left(\begin{array}{c} A'\\ f\\ A\\ \Sigma\\ \dfrac{B_1\wedge B_2}{B_1} \end{array}, \begin{array}{c} A'\\ f\\ A\\ \Sigma\\ \dfrac{B_1\wedge B_2}{B_2} \end{array} \right)$$

$$\begin{array}{c} A \\ \Sigma \\ B_1 \wedge B_2 \end{array} \xrightarrow{L_{AB}} \left(\begin{array}{c} A \\ \Sigma \\ \dfrac{B_1 \wedge B_2}{B_1} \end{array}, \begin{array}{c} A \\ \Sigma \\ \dfrac{B_1 \wedge B_2}{B_2} \end{array} \right) \xrightarrow{Hom(\Delta f, B)} \left(\begin{array}{c} A' \\ f \\ A \\ \Sigma \\ \dfrac{B_1 \wedge B_2}{B_1} \end{array}, \begin{array}{c} A' \\ f \\ A \\ \Sigma \\ \dfrac{B_1 \wedge B_2}{B_2} \end{array} \right)$$

As $L_{A'B} \circ Hom(f, \wedge B)(\Sigma) = Hom(\Delta f, B) \circ L_{AB}(\Sigma)$, this natural transformation corresponds to the identity reduction.

2. $L_{A'B'} \circ Hom(A', \wedge g) \Rightarrow Hom(\Delta A', g) \circ L_{A'B}$

Given a derivation $\Sigma \colon B_1 \wedge B_2[x \colon A']$,

$$\begin{array}{c} A' \\ \Sigma \\ B_1 \wedge B_2 \end{array} \xrightarrow{Hom(A', \wedge g)} \begin{array}{c} A' \\ \Sigma \\ B_1 \wedge B_2 \\ g_1 \wedge g_2 \\ B_1' \wedge B_2' \end{array} \xrightarrow{L_{A'B'}} \left(\begin{array}{c} A' \\ \Sigma \\ B_1 \wedge B_2 \\ g_1 \wedge g_2 \\ \dfrac{B_1' \wedge B_2'}{B_1'} \end{array}, \begin{array}{c} A' \\ \Sigma \\ B_1 \wedge B_2 \\ g_1 \wedge g_2 \\ \dfrac{B_1' \wedge B_2'}{B_2'} \end{array} \right)$$

$$\begin{array}{c} A' \\ \Sigma \\ B_1 \wedge B_2 \end{array} \xrightarrow{L_{A'B}} \left(\begin{array}{c} A' \\ \Sigma \\ \dfrac{B_1 \wedge B_2}{B_1} \end{array}, \begin{array}{c} A' \\ \Sigma \\ \dfrac{B_1 \wedge B_2}{B_2} \end{array} \right) \xrightarrow{Hom(\Delta A', g)} \left(\begin{array}{c} A' \\ \Sigma \\ \dfrac{B_1 \wedge B_2}{B_1} \\ g_1 \\ B_1' \end{array}, \begin{array}{c} A' \\ \Sigma \\ \dfrac{B_1 \wedge B_2}{B_2} \\ g_2 \\ B_2' \end{array} \right)$$

where $g_1 \wedge g_2$ is as in Lemma 4.2.2. We have that $L_{A'B'} \circ Hom(A', \wedge g)(\Sigma) =$

$$\left(\begin{array}{c} \dfrac{\dfrac{\begin{array}{c} A' \\ \Sigma \\ \dfrac{B_1 \wedge B_2}{B_1} \\ g_1 \\ B_1' \end{array} \quad \begin{array}{c} A' \\ \Sigma \\ \dfrac{B_1 \wedge B_2}{B_2} \\ g_2 \\ B_2' \end{array}}{B_1' \wedge B_2'}}{B_1'} \end{array}, \begin{array}{c} \dfrac{\dfrac{\begin{array}{c} A' \\ \Sigma \\ \dfrac{B_1 \wedge B_2}{B_1} \\ g_1 \\ B_1' \end{array} \quad \begin{array}{c} A' \\ \Sigma \\ \dfrac{B_1 \wedge B_2}{B_2} \\ g_2 \\ B_2' \end{array}}{B_1' \wedge B_2'}}{B_2'} \end{array} \right)$$

reduces to $Hom(\Delta A', g) \circ L_{A'B}(\Sigma)$ and this natural transformation corresponds to a pair of $\wedge$-reductions.

3. $Hom(f, \wedge B) \circ K_{AB} \Rightarrow K_{A'B} \circ Hom(\Delta f, B)$

Given a pair of derivations $\Sigma = (\Sigma_1 \colon B_1[x \colon A], \Sigma_2 \colon B_2[x \colon A])$,

$$\left(\begin{array}{c} A \\ \Sigma_1 \\ B_1 \end{array}, \begin{array}{c} A \\ \Sigma_2 \\ B_2 \end{array} \right) \xrightarrow{K_{AB}} \dfrac{\begin{array}{c} A \\ \Sigma_1 \\ B_1 \end{array} \quad \begin{array}{c} A \\ \Sigma_2 \\ B_2 \end{array}}{B_1 \wedge B_2} \xrightarrow{Hom(f, \wedge B)} \dfrac{\begin{array}{c} A' \\ f \\ A \\ \Sigma_1 \\ B_1 \end{array} \quad \begin{array}{c} A' \\ f \\ A \\ \Sigma_2 \\ B_2 \end{array}}{B_1 \wedge B_2}$$

$$\begin{pmatrix} A & & A \\ \Sigma_1 & , & \Sigma_2 \\ B_1 & & B_2 \end{pmatrix} \xrightarrow{K_{AB}} \begin{pmatrix} A' & & A' \\ f & & f \\ A & , & A \\ \Sigma_1 & & \Sigma_2 \\ B_1 & & B_2 \end{pmatrix} \xrightarrow{Hom(f,\wedge B)} \dfrac{\begin{array}{cc} A' & A' \\ f & f \\ A & A \\ \Sigma_1 & \Sigma_2 \\ B_1 & B_2 \end{array}}{B_1 \wedge B_2}$$

As $Hom(f, \wedge B) \circ K_{AB}(\Sigma) = K_{A'B} \circ Hom(\Delta f, B)(\Sigma)$, this natural transformation corresponds to the identity reduction.

4.$Hom(A', \wedge g) \circ K_{A'B} \Rightarrow K_{A'B'} \circ Hom(\Delta A', g)$

Given a pair of derivations $\Sigma = (\Sigma_1 \colon B_1[x \colon A'], \Sigma_2 \colon B_2[x \colon A'])$,

$$\begin{pmatrix} A' & & A' \\ \Sigma_1 & , & \Sigma_2 \\ B_1 & & B_2 \end{pmatrix} \xrightarrow{K_{A'B}} \dfrac{\begin{array}{cc} A' & A' \\ \Sigma_1 & \Sigma_2 \\ B_1 & B_2 \end{array}}{B_1 \wedge B_2} \xrightarrow{Hom(A',\wedge g)} \begin{array}{c} \dfrac{\begin{array}{cc} A' & A' \\ \Sigma_1 & \Sigma_2 \\ B_1 & B_2 \end{array}}{B_1 \wedge B_2} \\ g_1 \wedge g_2 \\ B_1' \wedge B_2' \end{array}$$

$$\begin{pmatrix} A' & & A' \\ \Sigma_1 & , & \Sigma_2 \\ B_1 & & B_2 \end{pmatrix} \xrightarrow{Hom(\Delta A', g)} \begin{pmatrix} A' & & A' \\ \Sigma_1 & & \Sigma_2 \\ B_1 & , & B_2 \\ g_1 & & g_2 \\ B_1' & & B_2' \end{pmatrix} \xrightarrow{K_{A'B'}} \dfrac{\begin{array}{cc} A' & A' \\ \Sigma_1 & \Sigma_2 \\ B_1 & B_2 \\ g_1 & g_2 \\ B_1' & B_2' \end{array}}{B_1 \wedge B_2}$$

We have that $Hom(A', \wedge g)K_{A'B}(\Sigma) = \dfrac{\begin{array}{cc} \dfrac{\dfrac{\begin{array}{cc} A' & A' \\ \Sigma_1 & \Sigma_2 \\ B_1 & B_2 \end{array}}{B_1 \wedge B_2}}{\begin{array}{c} B_1 \\ g_1 \\ B_1' \end{array}} & \dfrac{\dfrac{\begin{array}{cc} A' & A' \\ \Sigma_1 & \Sigma_2 \\ B_1 & B_2 \end{array}}{B_1 \wedge B_2}}{\begin{array}{c} B_2 \\ g_2 \\ B_2' \end{array}} \end{array}}{B_1' \wedge B_2'}$ $\wedge$-reduces to $K_{A'B'}Hom(\Delta A', g)(\Sigma)$ and this natural transformation corresponds to $\wedge$-red| $\wedge$-red.

5. $\alpha \colon L_{AB} \circ K_{AB} \Rightarrow id_{Hom_{Cat}(\Delta A, B)}$

Given a pair of derivations $\Sigma = (\Sigma_1 \colon B_1[x \colon A], \Sigma_2 \colon B_2[x \colon A])$,

$$\begin{pmatrix} A & & A \\ \Sigma_1 & , & \Sigma_2 \\ B_1 & & B_2 \end{pmatrix} \xrightarrow{K_{AB}} \dfrac{\begin{array}{cc} A & A \\ \Sigma_1 & \Sigma_2 \\ B_1 & B_2 \end{array}}{B_1 \wedge B_2} \xrightarrow{L_{AB}} \begin{pmatrix} \dfrac{\dfrac{\begin{array}{cc} A & A \\ \Sigma_1 & \Sigma_2 \\ B_1 & B_2 \end{array}}{B_1 \wedge B_2}}{B_1} , & \dfrac{\dfrac{\begin{array}{cc} A & A \\ \Sigma_1 & \Sigma_2 \\ B_1 & B_2 \end{array}}{B_1 \wedge B_2}}{B_2} \end{pmatrix}$$

As $L_{AB}K_{AB}(\Sigma)$ $\wedge$-reduces to $id_{Hom_{Cat}(A, \wedge B)}(\Sigma)$ this natural transformation corresponds to a pair of $\wedge$-reductions.

6. $\beta\colon id_{Hom_{Cat}(A,\wedge B)} \Rightarrow K_{AB} \circ L_{AB}$

Given a derivation $\Sigma\colon B_1 \wedge B_2[x\colon A]$,

$$\begin{array}{c} A \\ \Sigma \\ B_1 \wedge B_2 \end{array} \xrightarrow{L_{AB}} \left(\dfrac{\begin{array}{c} A \\ \Sigma \\ B_1 \wedge B_2 \end{array}}{B_1}, \dfrac{\begin{array}{c} A \\ \Sigma \\ B_1 \wedge B_2 \end{array}}{B_2} \right) \xrightarrow{K_{AB}} \dfrac{\dfrac{\begin{array}{c} A \\ \Sigma \\ B_1 \wedge B_2 \end{array}}{B_1} \quad \dfrac{\begin{array}{c} A \\ \Sigma \\ B_1 \wedge B_2 \end{array}}{B_2}}{B_1 \wedge B_2}$$

As $id_{Hom_{Cat}(A,\wedge B)}(\Sigma)$ $\wedge$-expands to $K_{AB}L_{AB}(\Sigma)$ this natural transformation corresponds to $\wedge$-expansion.

<u>fourth part</u>

As $\alpha\colon L_{AB} \circ K_{AB} \Rightarrow id_{Hom_{Cat}(\Delta A,B)}$ and $\beta\colon id_{Hom_{Cat}(A,\wedge B)} \Rightarrow K_{AB} \circ L_{AB}$, we have that:

1. $(id_{K_{AB}}; \alpha) \cdot (\beta; id_{K_{AB}}) = id_{K_{AB}}$

$\beta; id_{K_{AB}}$ goes from $id_{Hom_{Cat}(\Delta A,B)} \circ K_{AB} = K_{AB}$ to $(K_{AB} \circ L_{AB}) \circ K_{AB}$ and $id_{K_{AB}}; \alpha$ goes from $K_{AB} \circ (L_{AB} \circ K_{AB})$ to $K_{AB} \circ id_{Hom_{Cat}(\Delta A,B)} = K_{AB}$ therefore their composition goes from K_{AB} to K_{AB}.

2. $(\alpha; id_{L_{AB}}) \cdot (id_{L_{AB}}; \beta) = id_{L_{AB}}$

$id_{L_{AB}}; \beta$ goes from $L_{AB} \circ id_{Hom_{Cat}(A,\wedge B)} = L_{AB}$ to $L_{AB} \circ (K_{AB} \circ L_{AB})$ and $\alpha; id_{L_{AB}}$ goes from $(L_{AB} \circ K_{AB}) \circ L_{AB}$ to $id_{Hom_{Cat}(\Delta A,B)} \circ L_{AB} = L_{AB}$ therefore their composition goes from L_{AB} to L_{AB}.

4.3 Disjunction

Similarly to conjunction, we provide a more detailed proof that disjunction is rax right 2-adjoint to the diagonal lax 2-functor as stated by Seely (17). We also present the definition of rax 2-adjunction, presented in the preceding section, in four parts on an attempt to make the proof easier to read:

first part given the 2-categories $\mathcal{PT}$ and $\mathcal{PT} \times \mathcal{PT}$, we show that $\vee\colon \mathcal{PT} \times \mathcal{PT} \to \mathcal{PT}$ is a lax 2-functor (that $\Delta\colon \mathcal{PT} \to \mathcal{PT} \times \mathcal{PT}$ is a lax 2-functor is proved in the preceding section);

second part we show that there exists an arrow L_{AB} from $Hom_{Cat}(A, \Delta B)$ to $Hom_{Cat}(\vee A, B)$ and an arrow K_{AB} from $Hom_{Cat}(\vee A, B)$ to $Hom_{Cat}(A, \Delta B)$ for every 0-cell A in $\mathcal{PT}$ and B in $\mathcal{PT} \times \mathcal{PT}$;

third part we show that the above arrows define the natural transformations that correspond to this definition

fourth part finally, we show that $(\beta; id_{K_{AB}}) \cdot (id_{K_{AB}}; \alpha) = id_{K_{AB}}$ and $(id_{L_{AB}}; \beta) \cdot (\alpha; id_{L_{AB}}) = id_{L_{AB}}$ hold

As Mann in (11), Seely worked with MDP.

<u>first part</u>

Lemma 4.3.1. *$\vee\colon \mathcal{PT} \times \mathcal{PT} \to \mathcal{PT}$ such that $\vee(A) = A_1 \vee A_2$, $\vee(f) =$*

$$f_1 \vee f_2 = \cfrac{A'_1 \vee A'_2 \qquad \cfrac{\begin{matrix}[A'_1]\\ f_1\\ A_1\end{matrix}}{A_1 \vee A_2} \qquad \cfrac{\begin{matrix}[A'_2]\\ f_2\\ A_2\end{matrix}}{A_1 \vee A_2}}{A_1 \vee A_2} = [\iota_1 \circ f_1, \iota_2 \circ f_2]$$ [4] *and* $\vee(\alpha) =$

$id_{\vee A'} \mid ((id_{\iota_1}; \alpha_1) \mid (id_{\iota_2}; \alpha_2))$, for every formula $A = (A_1, A_2)$, derivation $f = (f_1, f_2), f_i\colon A'_i \to A_i$ and reduction $\alpha = (\alpha_1, \alpha_2)$, is a lax 2-functor.

Proof. The items (i), (ii) and (v) of the lax 2-functor definition (p. 36) come from the definition of $\vee$.

(iii) $\vee(id_A) = id_{\vee A}$ for every formula $A = (A_1, A_2)$

Given a formula A, both $id_{A_1} \vee id_{A_2}$ and $id_{A_1 \vee A_2}$ are the derivation $A_1 \vee A_2$;

(iv) $\vee f \circ \vee g \Rightarrow \vee(f \circ g)$ for every $f = (f_1, f_2)\colon (A_1, A_1) \to (B_1, B_2)$ and $g = (g_1, g_2)\colon (B_1, B_1) \to (C_1, C_2)$

Given derivations $f\colon A \to B$ and $g\colon B \to C$, $\vee f \circ \vee g$ is the derivation

$$\cfrac{\cfrac{A_1 \vee A_2 \qquad \cfrac{\begin{matrix}[A_1]\\ f_1\\ B_1\end{matrix}}{B_1 \vee B_2} \qquad \cfrac{\begin{matrix}[A_2]\\ f_2\\ B_2\end{matrix}}{B_1 \vee B_2}}{B_1 \vee B_2} \qquad \cfrac{\begin{matrix}[B_1]\\ g_1\\ C_1\end{matrix}}{C_1 \vee C_2} \qquad \cfrac{\begin{matrix}[B_2]\\ g_2\\ C_2\end{matrix}}{C_1 \vee C_2}}{C_1 \vee C_2}$$

which reduces to

$$\cfrac{A_1 \vee A_2 \qquad \cfrac{\begin{matrix}[A_1]\\ f_1\\ B_1\\ g_1\\ C_1\end{matrix}}{C_1 \vee C_2} \qquad \cfrac{\begin{matrix}[A_2]\\ f_2\\ B_2\\ g_2\\ C_2\end{matrix}}{C_1 \vee C_2}}{C_1 \vee C_2} = \vee(f \circ g).$$

(vi) $\vee(id_f) = id_{\vee f}$ for every $f = (f_1, f_2)\colon (A_1, A_1) \to (B_1, B_2)$

Given a derivation f, $\vee(id_f) = id_{\vee A} \mid ((id_{\iota_1}; id_{f_1}) \mid (id_{\iota_2}; id_{f_2}))$ which is the identity reduction that goes from $\vee f$ to $\vee f$.

[4]as in the definition of co-product (p. 24)

(vii) $\vee(\alpha \cdot \beta) = \vee\alpha \cdot \vee\beta$ for every $\alpha\colon f \Rightarrow g$ and $\beta\colon g \Rightarrow h$

Given reductions $\alpha\colon f \Rightarrow g$ and $\beta\colon g \Rightarrow h$,

$$\begin{aligned}
\vee\alpha \cdot \vee\beta &= \\
\{id_{\vee A} \mid ((id_{\iota_1};\alpha_1) \mid (id_{\iota_2};\alpha_2))\} \cdot \{id_{\vee A} \mid ((id_{\iota_1};\beta_1) \mid (id_{\iota_2};\beta_2))\} &= \\
(id_{\vee A} \cdot id_{\vee A}) \mid \{[(id_{\iota_1};\alpha_1) \mid (id_{\iota_2};\alpha_2)] \cdot [(id_{\iota_1};\beta_1) \mid (id_{\iota_2};\beta_2)]\} &= \\
(id_{\vee A} \cdot id_{\vee A}) \mid \{[(id_{\iota_1};\alpha_1) \cdot (id_{\iota_1};\beta_1)] \mid [(id_{\iota_2};\alpha_2) \cdot (id_{\iota_2};\beta_2)]\} &= \\
(id_{\vee A} \cdot id_{\vee A}) \mid \{[(id_{\iota_1} \cdot id_{\iota_1});(\alpha_1 \cdot \beta_1)] \mid [(id_{\iota_2} \cdot id_{\iota_2});(\alpha_2 \cdot \beta_2)]\} &= \\
id_{\vee A} \mid [id_{\iota_1};(\alpha_1 \cdot \beta_1)] \mid [id_{\iota_2};(\alpha_2 \cdot \beta_2)] &= \\
\vee(\alpha \cdot \beta) &
\end{aligned}$$

(viii) $\vee\alpha;\vee\beta \Rightarrow \vee(\alpha;\beta)$ for every $\alpha\colon f \Rightarrow f'$ and $\beta\colon g \Rightarrow g'$ such that $g \circ f$ and $g' \circ f'$ are defined

We only remark that applying $\vee\alpha$ and $\vee\beta$ followed by $\vee$-reduction is the same as applying $\vee$-reduction followed by $\vee\alpha$ and $\vee\beta$.

□

<u>second part</u>

Given 2-categories $\mathcal{A}$ and $\mathcal{B}$, a 0-cell $A = (A_1, A_2)$ in $\mathcal{A}$ and a 0-cell B in $\mathcal{B}$ define the categories $Hom_{Cat}(A, \Delta B)$ and $Hom_{Cat}(\vee A, B)$.

Lemma 4.3.2. *There exists a functor* $K_{AB}\colon Hom_{Cat}(\vee A, B) \to Hom_{Cat}(A, \Delta B)$

Let us define K_{AB} as the function that composes every derivation $\Pi \in Hom_{Cat}(\vee A, B)$ with $\vee$-int, i.e., $K_{AB}(\Pi) = \left(\begin{array}{c} \dfrac{A_1}{A_1 \vee A_2} \\ \Pi \\ B \end{array}, \begin{array}{c} \dfrac{A_2}{A_1 \vee A_2} \\ \Pi \\ B \end{array} \right)$ and for every $\alpha \in Hom_{Cat}(\vee A, B)$, $K_{AB}(\alpha) = (\alpha; id_{\iota_1}, \alpha; id_{\iota_2})$.

Proof. (i) e (ii) of the definition of functor come from the definition of K_{AB}; (iii) Given a derivation Π,$K_{AB}(id_\Pi) = (id_\Pi; id_{\iota_1}, id_\Pi; id_{\iota_2})$ which is the identity reduction that goes from $K_{AB}(\Pi)$ to $K_{AB}(\Pi)$.

(iv) Given reductions α and β,

$$\begin{aligned}
K_{AB}(\alpha)\cdot K_{AB}(\beta) &= \\
(\alpha; id_{\iota_1}, \alpha; id_{\iota_2})\cdot(\beta; id_{\iota_1}, \beta; id_{\iota_2}) &= \\
((\alpha; id_{\iota_1})\cdot(\beta; id_{\iota_1}), (\alpha; id_{\iota_2})\cdot(\beta; id_{\iota_2})) &= \text{(2-comp. property)}\\
((\alpha\cdot\beta); (id_{\iota_1}\cdot id_{\iota_1}), (\alpha\cdot\beta); (id_{\iota_2}\cdot id_{\iota_2})) &= \text{(comp. of identities)}\\
((\alpha\cdot\beta); id_{\iota_1}, (\alpha\cdot\beta); id_{\iota_2}) &= K_{AB}(\alpha\cdot\beta)
\end{aligned}$$

□

Lemma 4.3.3. *There exists a functor* $L_{AB}\colon Hom_{Cat}(A, \Delta B) \rightarrow Hom_{Cat}(\vee A, B)$

Let us define L_{AB} as the function that takes every $\Pi = (\Pi_1, \Pi_2) \in Hom_{Cat}(A, \Delta B)$ to $L_{AB}(\Pi) = \dfrac{A_1 \vee A_2 \quad \begin{matrix}[A_1]\\ \Pi_1\\ B\end{matrix} \quad \begin{matrix}[A_2]\\ \Pi_2\\ B\end{matrix}}{B}$ and for every $\alpha = (\alpha_1, \alpha_2)\colon \Pi \rhd \Pi'$, $L_{AB}(\alpha) = id_{A_1\vee A_2} \mid (\alpha_1 \mid \alpha_2)$.

Proof. (i) and (ii) of the definition of functor come from the definition of L_{AB};
(iii) Given a derivation Π,

$L_{AB}(id_\Pi) = id_{A_1\vee A_2} \mid (id_{\Pi_1} \mid id_{\Pi_2})$ which is the identity reduction that goes from $L_{AB}(\Pi)$ to $L_{AB}(\Pi)$.
(iv) Given reductions α and β,

$$\begin{aligned}
L_{AB}(\alpha)\cdot L_{AB}(\beta) &= \\
(id_{A_1\vee A_2} \mid (\alpha_1 \mid \alpha_2))\cdot(id_{A_1\vee A_2} \mid (\beta_1 \mid \beta_2)) &= \text{(Lemma 4.1.1)}\\
(id_{A_1\vee A_2}\cdot id_{A_1\vee A_2}) \mid ((\alpha_1 \mid \alpha_2)\cdot(\beta_1 \mid \beta_2)) &= \text{(Lemma 4.1.1)}\\
(id_{A_1\vee A_2}\cdot id_{A_1\vee A_2}) \mid ((\alpha_1\cdot\beta_1) \mid (\alpha_2\cdot\beta_2)) &= \text{(comp. of identities)}\\
id_{A_1\vee A_2} \mid ((\alpha_1\cdot\beta_1) \mid (\alpha_2\cdot\beta_2)) &= L_{AB}(\alpha\cdot\beta)
\end{aligned}$$

□

<u>third part</u>

Given derivations $f = (f_1, f_2)$ in $\mathcal{A}$, $f_1\colon A_1'[x\colon A_1]$ and $f_2\colon A_2'[x\colon A_2]$, and $g\colon B[x\colon B']$ in $\mathcal{B}$, the following diagram defines the following natural transformations:

$$\begin{array}{ccc}
Hom_{Cat}(\vee A, B) & \underset{\longleftarrow L_{AB} \longrightarrow}{\overset{K_{AB}}{\longrightarrow}} & Hom_{Cat}(A, \Delta B) \\
\downarrow {\scriptstyle Hom(\vee f, B)} & & \downarrow {\scriptstyle Hom(f, \Delta B)} \\
Hom_{Cat}(\vee A', B) & \underset{\longleftarrow L_{A'B}}{\overset{K_{A'B}}{\longrightarrow}} & Hom_{Cat}(A', \Delta B) \\
\downarrow {\scriptstyle Hom(id_{\vee A'}, g)} & & \downarrow {\scriptstyle Hom(A', \Delta g)} \\
Hom_{Cat}(\vee A', B') & \underset{\longleftarrow L_{A'B'}}{\overset{K_{A'B'}}{\longrightarrow}} & Hom_{Cat}(A', \Delta B')
\end{array}$$

1. $Hom(\vee f, B) \circ L_{AB} \Rightarrow L_{A'B} \circ Hom(f, \Delta B)$

   Given a pair of derivations $\Sigma = (\Sigma_1\colon B[x\colon A_1], \Sigma_2\colon B[x\colon A_2])$,

$$\begin{pmatrix} A_1 & & A_2 \\ \Sigma_1 & , & \Sigma_2 \\ B & & B \end{pmatrix} \xrightarrow{L_{AB}} \frac{A_1 \vee A_2 \quad \begin{matrix}[A_1]\\ \Sigma_1 \\ B\end{matrix} \quad \begin{matrix}[A_2]\\ \Sigma_2 \\ B\end{matrix}}{B} \xrightarrow{Hom(\vee f, B)}$$

$$\cfrac{\cfrac{A_1' \vee A_2' \quad \cfrac{\begin{matrix}[A_1']\\ f_1 \\ A_1\end{matrix}}{A_1 \vee A_2} \quad \cfrac{\begin{matrix}[A_2']\\ f_2 \\ A_2\end{matrix}}{A_1 \vee A_2}}{A_1 \vee A_2} \quad \begin{matrix}[A_1]\\ \Sigma_1 \\ B\end{matrix} \quad \begin{matrix}[A_2]\\ \Sigma_2 \\ B\end{matrix}}{B}$$

$$\begin{pmatrix} A_1 & & A_2 \\ \Sigma_1 & , & \Sigma_2 \\ B & & B \end{pmatrix} \xrightarrow{Hom(f, \Delta B)} \begin{pmatrix} [A_1'] & & [A_2'] \\ f_1 & & f_2 \\ A_1 & , & A_2 \\ \Sigma_1 & & \Sigma_2 \\ B & & B \end{pmatrix} \xrightarrow{L_{A'B}} \frac{A_1' \vee A_2' \quad \begin{matrix}[A_1']\\ f_1 \\ A_1 \\ \Sigma_1 \\ B\end{matrix} \quad \begin{matrix}[A_2']\\ f_2 \\ A_2 \\ \Sigma_2 \\ B\end{matrix}}{B}$$

and this natural transformation corresponds to the sequence $\langle$MDP, $\vee$-reduction$\rangle$.

2. $Hom(\vee A', g) \circ L_{A'B} \Rightarrow L_{A'B'} \circ Hom(A', \Delta g)$

   Given a pair of derivations $\Sigma = (\Sigma_1\colon B[x\colon A_1'], \Sigma_2\colon B[x\colon A_2'])$,

$$\begin{pmatrix} A_1' & & A_2' \\ \Sigma_1 & , & \Sigma_2 \\ B & & B \end{pmatrix} \xrightarrow{L_{A'B}} \frac{A_1' \vee A_2' \quad \begin{matrix}[A_1']\\ \Sigma_1 \\ B\end{matrix} \quad \begin{matrix}[A_2']\\ \Sigma_2 \\ B\end{matrix}}{B} \xrightarrow{Hom(\vee A', g)}$$

$$\cfrac{A_1' \vee A_2' \qquad \begin{matrix}[A_1']\\ \Sigma_1\\ B\end{matrix} \qquad \begin{matrix}[A_2']\\ \Sigma_2\\ B\end{matrix}}{\begin{matrix}B\\ g\\ B'\end{matrix}}$$

$$\begin{pmatrix}A_1 & & A_2\\ \Sigma_1 & , & \Sigma_2\\ B & & B\end{pmatrix} \xrightarrow{Hom(A',\Delta g)} \begin{pmatrix}[A_1'] & & [A_1']\\ \Sigma_1 & & \Sigma_2\\ B & , & B\\ g & & g\\ B' & & B'\end{pmatrix} \xrightarrow{L_{A'B'}} \cfrac{A_1' \vee A_2' \qquad \begin{matrix}[A_1']\\ f_1\\ A_1\\ \Sigma_1\\ B\\ g\\ B'\end{matrix} \qquad \begin{matrix}[A_1']\\ f_1\\ A_2\\ \Sigma_2\\ B\\ g\\ B'\end{matrix}}{B'}$$

$Hom(\vee A', g) \circ L_{A'B}(\Sigma)$ reduces to $L_{A'B'} \circ Hom(A', \wedge g)(\Sigma)$ and this natural transformation corresponds to MDP.

3. $K_{A'B} \circ Hom(\vee f, B) \Rightarrow Hom(f, \Delta B) \circ K_{AB}$

Given a derivation $\Sigma \colon B[x \colon A_1 \vee A_2]$,

$$\begin{matrix}A_1 \vee A_2\\ \Sigma\\ B\end{matrix} \xrightarrow{K_{AB}} \begin{pmatrix}\cfrac{A_1}{A_1 \vee A_2} & & \cfrac{A_1}{A_1 \vee A_2}\\ \Sigma & , & \Sigma\\ B & & B\end{pmatrix} \xrightarrow{Hom(f,\Delta B)} \begin{pmatrix}A_1' & & A_2'\\ f_1 & & f_2\\ \cfrac{A_1}{A_1 \vee A_2} & , & \cfrac{A_2}{A_1 \vee A_2}\\ \Sigma & & \Sigma\\ B & & B\end{pmatrix}$$

$$\begin{matrix}A_1 \vee A_2\\ \Sigma\\ B\end{matrix} \xrightarrow{Hom(\vee f, B)} \begin{matrix}A_1' \vee A_2'\\ f_1 \vee f_2\\ A_1 \vee A_2\\ \Sigma\\ B\end{matrix} \xrightarrow{K_{A'B}} \begin{pmatrix}\cfrac{A_1'}{A_1' \vee A_2'} & & \cfrac{A_2'}{A_1' \vee A_2'}\\ f_1 \vee f_2 & , & f_1 \vee f_2\\ A_1 \vee A_2 & & A_1 \vee A_2\\ \Sigma & & \Sigma\\ B & & B\end{pmatrix}$$

As $Hom(f, \Delta B) \circ K_{AB}(\Sigma) =$

$$\left(\begin{matrix}\cfrac{\cfrac{A_1'}{A_1' \vee A_2'} \qquad \cfrac{\begin{matrix}[A_1']\\ f_1\\ A_1\end{matrix}}{A_1 \vee A_2} \qquad \cfrac{\begin{matrix}[A_2']\\ f_2\\ A_2\end{matrix}}{A_1 \vee A_2}}{\begin{matrix}A_1 \vee A_2\\ \Sigma\\ B\end{matrix}}\end{matrix} , \begin{matrix}\cfrac{\cfrac{A_2'}{A_1' \vee A_2'} \qquad \cfrac{\begin{matrix}[A_1']\\ f_1\\ A_1\end{matrix}}{A_1 \vee A_2} \qquad \cfrac{\begin{matrix}[A_2']\\ f_2\\ A_2\end{matrix}}{A_1 \vee A_2}}{\begin{matrix}A_1 \vee A_2\\ \Sigma\\ B\end{matrix}}\end{matrix}\right)$$

$\vee$-reduces to $K_{A'B} \circ Hom(\vee f, B)$, this natural transformation corresponds to a pair of $\vee$-reductions.

4. $K_{A'B'} \circ Hom(\vee A', g) \Rightarrow Hom(A', \Delta g) \circ K_{A'B}$

Given a derivation $\Sigma\colon B[x\colon A_1' \vee A_2']$,

$$\begin{array}{c} A_1' \vee A_2' \\ \Sigma \\ B \end{array} \xrightarrow{Hom(\vee A', g)} \left(\begin{array}{c} A_1' \vee A_2' \\ \Sigma \\ B \\ g \\ B' \end{array} , \begin{array}{c} A_1' \vee A_2' \\ \Sigma \\ B \\ g \\ B' \end{array} \right) \xrightarrow{K_{A'B'}} \left(\begin{array}{c} \dfrac{A_1'}{A_1' \vee A_2'} \\ \Sigma \\ B \\ g \\ B' \end{array} , \begin{array}{c} \dfrac{A_2'}{A_1' \vee A_2'} \\ \Sigma \\ B \\ g \\ B' \end{array} \right)$$

$$\begin{array}{c} A_1' \vee A_2' \\ \Sigma \\ B \end{array} \xrightarrow{K_{A'B}} \left(\begin{array}{c} \dfrac{A_1'}{A_1' \vee A_2'} \\ \Sigma \\ B \end{array} , \begin{array}{c} \dfrac{A_2'}{A_1' \vee A_2'} \\ \Sigma \\ B \end{array} \right) \xrightarrow{Hom(A', \Delta g)} \left(\begin{array}{c} \dfrac{A_1'}{A_1' \vee A_2'} \\ \Sigma \\ B \\ g \\ B' \end{array} , \begin{array}{c} \dfrac{A_2'}{A_1' \vee A_2'} \\ \Sigma \\ B \\ g \\ B' \end{array} \right)$$

As $K_{A'B'} \circ Hom(\vee A', g)(\Sigma) = Hom(A', \Delta g) \circ K_{A'B}$, this natural transformation corresponds to the identity reduction.

5. $\alpha\colon id_{Hom_{Cat}(\vee A, B)} \Rightarrow L_{AB} \circ K_{AB}$

Given a derivation $\Sigma\colon B[x\colon A_1 \vee A_2]$,

$$\begin{array}{c} A_1 \vee A_2 \\ \Sigma \\ B \end{array} \xrightarrow{K_{AB}} \left(\begin{array}{c} \dfrac{A_1}{A_1 \vee A_2} \\ \Sigma \\ B \end{array} , \begin{array}{c} \dfrac{A_2}{A_1 \vee A_2} \\ \Sigma \\ B \end{array} \right) \xrightarrow{L_{AB}} \dfrac{A_1 \vee A_2 \quad \begin{array}{c} \dfrac{[A_1]}{A_1 \vee A_2} \\ \Sigma \\ B \end{array} \quad \begin{array}{c} \dfrac{[A_2]}{A_1 \vee A_2} \\ \Sigma \\ B \end{array}}{B}$$

$id_{Hom_{Cat}(\vee A, B)}(\Sigma)$ reduces to $L_{AB}K_{AB}(\Sigma)$ and this natural transformation corresponds to the sequence $\langle \vee\text{-expansion}, \mathrm{MDP} \rangle$

6. $\beta\colon K_{AB} \circ L_{AB} \Rightarrow id_{Hom_{Cat}(A, \Delta B)}$

Given a pair of derivations $\Sigma = (\Sigma_1\colon B[x\colon A_1], \Sigma_2\colon B[x\colon A_2])$,

$$\left(\begin{array}{c} A_1 \\ \Sigma_1 \\ B \end{array} , \begin{array}{c} A_2 \\ \Sigma_2 \\ B \end{array} \right) \xrightarrow{L_{AB}} \dfrac{A_1 \vee A_2 \quad \begin{array}{c} [A_1] \\ \Sigma_1 \\ B \end{array} \quad \begin{array}{c} [A_2] \\ \Sigma_2 \\ B \end{array}}{B} \xrightarrow{K_{AB}}$$

$$\left(\dfrac{\dfrac{A_1}{A_1 \vee A_2} \quad \begin{array}{c} [A_1] \\ \Sigma_1 \\ B \end{array} \quad \begin{array}{c} [A_2] \\ \Sigma_2 \\ B \end{array}}{B} , \dfrac{\dfrac{A_2}{A_1 \vee A_2} \quad \begin{array}{c} [A_1] \\ \Sigma_1 \\ B \end{array} \quad \begin{array}{c} [A_2] \\ \Sigma_2 \\ B \end{array}}{B} \right)$$

$K_{AB}L_{AB}(\Sigma)$ reduces to $id_{Hom_{Cat}(\vee A, B)}(\Sigma)$ and this natural transformation corresponds to a pair of $\vee$-reductions.

fourth part

As $\alpha\colon id_{Hom_{Cat}(\vee A,B)} \Rightarrow L_{AB} \circ K_{AB}$ and $\beta\colon K_{AB} \circ L_{AB} \Rightarrow id_{Hom_{Cat}(A,\Delta B)}$, we have that:

1. $(\beta; id_{K_{AB}}) \cdot (id_{K_{AB}}; \alpha) = id_{K_{AB}}$
   $id_{K_{AB}}; \alpha$ goes from $K_{AB} \circ id_{Hom_{Cat}(\vee A,B)} = K_{AB}$ to $K_{AB} \circ (L_{AB} \circ K_{AB})$ and $\beta; id_{K_{AB}}$ goes from $(K_{AB} \circ L_{AB}) \circ K_{AB}$ to $id_{Hom_{Cat}(A,\Delta B)} \circ K_{AB} = K_{AB}$ and therefore their composition goes from K_{AB} to K_{AB}.

2. $(id_{L_{AB}}; \beta) \cdot (\alpha; id_{L_{AB}}) = id_{L_{AB}}$
   $\alpha; id_{L_{AB}}$ goes from $id_{Hom_{Cat}(A,\Delta B)} \circ L_{AB} = L_{AB}$ to $(L_{AB} \circ K_{AB}) \circ L_{AB}$ and $id_{L_{AB}}; \beta$ goes from $L_{AB} \circ (K_{AB} \circ L_{AB})$ to $L_{AB} \circ id_{Hom_{Cat}(\vee A,B)} = L_{AB}$ therefore their composition goes from L_{AB} to L_{AB}.

4.4 Implication

Implication is neither lax right 2-adjoint nor rax right 2-adjoint to conjunction but it has some of these two properties and we thought it would be interesting to show and discuss it here. We show what can be proved and point out the reasons why it is not a lax 2-ajunction. We also separate it in four parts:

first part given the 2-category $\mathcal{PT}$, we show that $\rightarrow\colon \mathcal{PT} \rightarrow \mathcal{PT}$ and that $\wedge\colon \mathcal{PT} \rightarrow \mathcal{PT}$ are lax 2-functors;

second part we show that there exists an arrow K_{BC} from $Hom_{Cat}(C \wedge X, B)$ to $Hom_{Cat}(C, X \rightarrow B)$ and an arrow L_{BC} from $Hom_{Cat}(C, X \rightarrow B)$ to $Hom_{Cat}(C \wedge X, B)$;

third part we show that the above arrows define natural transformations that correspond either to the lax 2-adjunction definition or to the rax 2-adjunction definition, i.e., we show that the natural transformations correspond to the items 1, 2 and 3 are in accordance to the rax 2-adjunction definition, the item 4 is in accordance to the lax 2-adjunction definition and that 5 and 6 is neither lax nor rax 2-adjunction;

fourth part we show that neither of the equalities of the definitions of lax and rax 2-adjunction hold, i.e., we show that neither $(\beta; id_{K_{CB}}) \cdot (id_{K_{CB}}; \alpha) = id_{K_{CB}}$, $(id_{L_{CB}}; \beta) \cdot (\alpha; id_{L_{CB}}) = id_{L_{CB}}$, $(id_{K_{CB}}; \alpha) \cdot (\beta; id_{K_{CB}}) = id_{K_{CB}}$ and $(\alpha; id_{L_{CB}}) \cdot (id_{L_{CB}}; \beta) = id_{L_{CB}}$ hold.

first part

Proposition 4.4.1. *$\wedge\colon \mathcal{PT} \rightarrow \mathcal{PT}$ such that $\wedge(A) = A \wedge B$, $\wedge(f) = f \wedge B = \langle f \circ \pi_1, \pi_2\rangle$ and $\wedge(\alpha) = \langle \alpha; id_{\pi_1}, id_{\pi_2}\rangle$ for every formula A, derivation f and rex α, is a lax 2-functor.*

Proof. This proposition is a corollary of proposition 4.2.2 ☐

Lemma 4.4.2. *$\rightarrow\colon \mathcal{PT} \rightarrow \mathcal{PT}$ such that $\rightarrow(B) = X \rightarrow B$, $\rightarrow(f) = X \rightarrow f =$*

$$\cfrac{\begin{array}{c}\dfrac{X \rightarrow B \qquad [X]}{B}\\ f\\ B'\end{array}}{X \rightarrow B'} = f \circ \widehat{eval\langle \rightarrow B, X\rangle}$$

[5] and $\rightarrow(\alpha) = id_r; (\alpha; id_s)$, where $r\colon B \rightarrow (X \rightarrow B)$ and $s\colon (X \rightarrow B, X) \rightarrow B$, for every formula B, derivation $f\colon B \rightarrow B'$ and rex $\alpha\colon f \Rightarrow f'$, is a lax 2-functor.

Proof. The items (i), (ii) and (v) of the functor definition come from the definition of $\rightarrow$.

(iii) $\rightarrow(id_A) = id_{\rightarrow(A)}$, for every formula A

Given a formula A, both $\rightarrow(id_A)$ and $id_{\rightarrow(A)}$ are the derivation $X \rightarrow A$;

(iv) $(\rightarrow(f) \circ \rightarrow(g)) \Rightarrow (\rightarrow(f \circ g))$ for every $f\colon A \rightarrow B$ and $g\colon B \rightarrow C$

Given derivations $f\colon A \rightarrow B$ and $g\colon B \rightarrow C$, $\rightarrow(f) \circ \rightarrow(g)$ is the derivation

$$\cfrac{\begin{array}{c}\dfrac{\cfrac{\begin{array}{c}\dfrac{X \rightarrow A \qquad [X]}{A}\\ f\\ B\end{array}}{X \rightarrow B} \qquad [X]}{B}\\ g\\ C\end{array}}{X \rightarrow C}$$

which reduces to

$$\cfrac{\begin{array}{c}\dfrac{X \rightarrow A \qquad [X]}{A}\\ f\\ B\\ g\\ C\end{array}}{X \rightarrow C}$$

$= \rightarrow(f \circ g)$.

(vi) $\rightarrow(id_f) = id_{\rightarrow(f)}$ for every $f\colon A \rightarrow B$

Given a derivation f, $\rightarrow(id_f) = id_r; id_f; id_s$ which is the identity reduction that goes from $\rightarrow(f)$ to $\rightarrow(f)$.

(vii) $\rightarrow(\alpha \cdot \beta) = \rightarrow(\alpha) \cdot \rightarrow(\beta)$ for every $\alpha\colon f \Rightarrow g$ and $\beta\colon g \Rightarrow h$

[5] as in the definition of exponentiation (p. 25)

Given reductions $\alpha\colon f \Rightarrow g$ and $\beta\colon g \Rightarrow h$,

$$\begin{aligned} \rightarrow(\alpha)\cdot\rightarrow(\beta) &= \\ (id_r;(\alpha;id_s))\cdot(id_r;(\beta;id_s)) &= \text{(2-comp. property)} \\ (id_r\cdot id_r);((\alpha;id_s)\cdot(\beta;id_s)) &= \text{(2-comp. property)} \\ (id_r\cdot id_r);((\alpha\cdot\beta);(id_s\cdot id_s)) &= \text{(comp. of identities)} \\ id_r;((\alpha\cdot\beta);id_s) &= \rightarrow(\alpha\cdot\beta) \end{aligned}$$

(viii) $\rightarrow(\beta);\rightarrow(\alpha)\Rightarrow\rightarrow(\beta;\alpha)$ for every $\alpha\colon f\Rightarrow f'$ and $\beta\colon g\Rightarrow g'$ such that $g\circ f$ and $g'\circ f'$ are defined

We only remark that applying $\rightarrow\alpha$ and $\rightarrow\beta$ followed by $\rightarrow$-reduction is the same as applying $\rightarrow$-reduction followed by $\rightarrow\alpha$ and $\rightarrow\beta$.

□

second part

Given the 2-category $\mathcal{PT}$, 0-cells A and B in $\mathcal{PT}$ define the categories $Hom_{Cat}(C\wedge A,B)$ and $Hom_{Cat}(C,A\rightarrow B)$.

Lemma 4.4.3. *There exists a functor* $K_{BC}\colon Hom_{Cat}(C\wedge X,B)\rightarrow Hom_{Cat}(C,X\rightarrow B)$.

Define $K_{BC}(\Pi) = \dfrac{\dfrac{C \quad [X]}{C\wedge X}\\ \Pi \\ B}{X\rightarrow B}$ for every Π in $Hom_{Cat}(C\wedge X,B)$ and $K_{BC}(\alpha)=id_r;\alpha;id_p$, where $p\colon(C,X)\rightarrow C\wedge X$, for every $\alpha\in Hom_{Cat}(C\wedge X,B)$.

Proof. (i) and (ii) of the definition of functor come from the definition of K_{BC};
(iii) Given a derivation Π, $K_{BC}(id_(\Pi)=id_r;id_\Pi;id_p$ which is the identity reduction that goes from $K_{BC}(\Pi)$ to $K_{BC}(\Pi)$;
(iv) Given reductions α and β,

$$\begin{aligned} K_{BC}(\alpha)\cdot K_{BC}(\beta) &= \\ (id_r;(\alpha;id_p))\cdot(id_r;(\beta;id_p)) &= \text{(2-comp. property)} \\ (id_r\cdot id_r);((\alpha;id_p)\cdot(\beta;id_p)) &= \text{(2-comp. property)} \\ (id_r\cdot id_r);((\alpha\cdot\beta);(id_p\cdot id_p)) &= \text{(comp. of identities)} \\ id_r;((\alpha\cdot\beta);id_p) &= K_{BC}(\alpha\cdot\beta) \end{aligned}$$

□

Lemma 4.4.4. *There exists a functor* $L_{BC}\colon Hom_{Cat}(C, X \to B) \to Hom_{Cat}(C \wedge X, B)$.

Put $L_{BC}(\Psi) = \dfrac{\begin{array}{c}\dfrac{C \wedge X}{C}\\ \Psi \\ X \to B\end{array} \qquad \dfrac{C \wedge X}{X}}{B}$ for every $\Psi \in Hom_{Cat}(C, X \to B)$

and $L_{BC}(\alpha) = (\alpha; id_{\pi_1}) \mid id_{\pi_2}$ for every $\alpha\colon \Pi \Rightarrow \Pi' \in Hom_{Cat}(C, X \to B)$.

Proof. (i) and (ii) of the definition of functor come from the definition of L_{BC};
(iii) Given a derivation Π, $L_{BC}(id_\Pi) = (id_\Pi; id_{\pi_1}) \mid id_{\pi_2}$ which is the identity reduction that goes from $L_{BC}(\Pi)$ to $L_{BC}(\Pi)$;
(iv) Given reductions α and β,

$$\begin{array}{rcl}
L_{BC}(\alpha) \cdot L_{BC}(\beta) & = & \\
((\alpha; id_{\pi_1}) \mid id_{\pi_2}) \cdot ((\beta; id_{\pi_1}) \mid id_{\pi_2}) & = & \text{(Lemma 4.1.1)} \\
((\alpha; id_{\pi_1}) \cdot (\beta; id_{\pi_1})) \mid (id_{\pi_2} \cdot id_{\pi_2}) & = & \text{(2-comp. property)} \\
((\alpha \cdot \beta); (id_{\pi_1} \cdot id_{\pi_1})) \mid (id_{\pi_2} \cdot id_{\pi_2}) & = & \text{(comp. of identities)} \\
((\alpha \cdot \beta); id_{\pi_1}) \mid id_{\pi_2} & = & L_{BC}(\alpha \cdot \beta)
\end{array}$$

□

third part

Given derivations $g\colon B'[x\colon B]$ and $f\colon C'[x\colon C]$ in $\mathcal{PT}$, we have the following natural transformations:

1. $Hom(\wedge(f), B) \circ L_{CB} \Rightarrow L_{C'B} \circ Hom(f, \to (B))$ in accordance to rax 2-adjunction

Given a derivation $\Sigma\colon A \to B[x\colon C]$,

$$\begin{array}{c} C \\ \Sigma \\ A \to B \end{array} \xrightarrow{L_{CB}} \dfrac{\begin{array}{c}\dfrac{C \wedge A}{C} \\ \Sigma \\ A \to B\end{array} \quad \dfrac{C \wedge A}{A}}{B} \xrightarrow{Hom(\wedge(f), B)}$$

$$\dfrac{\begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\dfrac{C' \wedge A}{C'} \\ f \\ C\end{array} \quad \dfrac{C' \wedge A}{A}}{C \wedge A}}{C} \\ \Sigma \\ A \to B\end{array} \quad \dfrac{\dfrac{\begin{array}{c}\dfrac{C' \wedge A}{C'} \\ f \\ C\end{array} \quad \dfrac{C' \wedge A}{A}}{C \wedge A}}{A}}{B}$$

$$\begin{array}{c} C \\ \Sigma \\ A \to B \end{array} \xrightarrow{Hom(f, \to (B))} \begin{array}{c} C' \\ f \\ C \\ \Sigma \\ A \to B \end{array} \xrightarrow{L_{C'B}} \dfrac{\begin{array}{c}\dfrac{C' \wedge A}{C'} \\ f \\ C \\ \Sigma \\ A \to B\end{array} \quad \dfrac{C' \wedge A}{A}}{B}$$

As $Hom(\wedge(f), B) \circ L_{CB}(\Sigma)$ $\wedge$-reduces to $L_{C'B} \circ Hom(f, \to (B))(\Sigma)$, this natural transformation corresponds to $\wedge$-red| $\wedge$-red.

2. $Hom(\wedge(C'), g) \circ L_{C'B} \Rightarrow L_{C'B'} \circ Hom(C', \to (g))$ in accordance to rax 2-adjunction.

Given a derivation $\Sigma\colon A \to B[x\colon C']$,

$$\begin{array}{c} C' \\ \Sigma \\ A \to B \end{array} \xrightarrow{L_{C'B}} \dfrac{\begin{array}{c}\dfrac{\begin{array}{c}C' \\ \Sigma \\ A \to B\end{array} \quad [A]}{B} \\ g \\ B'\end{array}}{A \to B'} \xrightarrow{Hom(\wedge(C'), g)} \dfrac{\dfrac{\begin{array}{c}\dfrac{\begin{array}{c}\dfrac{C' \wedge A}{C'} \\ \Sigma \\ A \to B\end{array} \quad [A]}{B} \\ g \\ B'\end{array}}{A \to B'} \quad \dfrac{C' \wedge A}{A}}{B'}$$

$$\begin{array}{c} C' \\ \Sigma \\ A\to B\end{array} \xrightarrow{Hom(C',\to(g))} \dfrac{\begin{array}{c}\dfrac{C'\wedge A}{C'} \\ \Sigma \\ A\to B\end{array} \quad \dfrac{C'\wedge A}{A}}{B} \xrightarrow{L_{C'B'}} \begin{array}{c}\dfrac{\begin{array}{c}\dfrac{C'\wedge A}{C'}\\ \Sigma\\ A\to B\end{array}\quad \dfrac{C'\wedge A}{A}}{B}\\ g \\ B'\end{array}$$

As $Hom(\wedge(C'), g) \circ L_{C'B}(\Sigma)$ $\to$-reduces to $L_{C'B'} \circ Hom(C', \to (g))(\Sigma)$, this natural transformation corresponds to $\to$-reduction.

3. $K_{C'B} \circ Hom(\wedge(f), B) \Rightarrow Hom(f, \to B) \circ K_{CB}$ in accordance to rax 2-adjunction.

Given a derivation $\Sigma\colon B[x\colon C \wedge A]$,

$$\begin{array}{c} C\wedge A \\ \Sigma \\ B\end{array} \xrightarrow{Hom(\wedge(f),B)} \begin{array}{c}\dfrac{\begin{array}{c}\dfrac{C'\wedge A}{C'} \\ f \\ C\end{array} \quad \dfrac{C'\wedge A}{A}}{C\wedge A}\\ \Sigma \\ B\end{array} \xrightarrow{K_{C'B}} \begin{array}{c}\dfrac{\begin{array}{c}\dfrac{\dfrac{C' \quad [A]}{C'\wedge A}}{C'} \\ f \\ C\end{array} \quad \dfrac{\dfrac{C' \quad [A]}{C'\wedge A}}{A}}{C\wedge A}\\ \Sigma \\ \dfrac{B}{A\to B}\end{array}$$

$$\begin{array}{c} C\wedge A \\ \Sigma \\ B\end{array} \xrightarrow{K_{CB}} \begin{array}{c}\dfrac{C \quad [A]}{C\wedge A}\\ \Sigma \\ \dfrac{B}{A\to B}\end{array} \xrightarrow{Hom(f,\to B)} \begin{array}{c}\dfrac{\begin{array}{c}C'\\ f\\ C\end{array} \quad [A]}{C\wedge A}\\ \Sigma \\ \dfrac{B}{A\to B}\end{array}$$

As $Hom(f, \to B) \circ K_{CB}(\Sigma)$ $\wedge$-reduces to $Hom(f, \to B) \circ K_{CB}(\Sigma)$, this natural transformation corresponds to ($\wedge$-red| $\wedge$-red); $id_{\Sigma}; id_r$.

4. $Hom(C', \to (g)) \circ K_{C'B} \Rightarrow K_{C'B'} \circ Hom(\wedge(C'), g)$ in accordance to lax 2-adjunction.

Given a derivation $\Sigma\colon B[x\colon C' \wedge A]$,

$$\begin{array}{c} C'\wedge A \\ \Sigma \\ B\end{array} \xrightarrow{K_{C'B}} \begin{array}{c}\dfrac{C' \quad [A]}{C'\wedge A}\\ \Sigma \\ \dfrac{B}{A\to B}\end{array} \xrightarrow{Hom(C',\to(g))} \begin{array}{c}\dfrac{\begin{array}{c}\dfrac{C' \quad [A]}{C'\wedge A}\\ \Sigma \\ \dfrac{B}{A\to B}\end{array} \quad [A]}{B}\\ g \\ \dfrac{B'}{A\to B'}\end{array}$$

$$\begin{array}{c} C'\wedge A\\ \Sigma\\ B\end{array} \xrightarrow{Hom(\wedge(C'),g)} \begin{array}{c} C'\wedge A\\ \Sigma\\ B\\ g\\ B'\end{array} \xrightarrow{K_{C'B'}} \dfrac{\begin{array}{c}\dfrac{C' \quad [A]}{C'\wedge A}\\ \Sigma\\ B\\ g\\ B'\end{array}}{A\to B'}$$

As $Hom(C', \to (g)) \circ K_{C'B}(\Sigma)$ $\to$-reduces to $K_{C'B'} \circ Hom(\wedge(C'), g)(\Sigma)$, this natural transformation corresponds to $\to$-reduction.

5. Neither $id_{Hom_{Cat}(C\wedge A,B)} \Rightarrow L_{CB} \circ K_{CB}$ nor $L_{CB} \circ K_{CB} \Rightarrow id_{Hom_{Cat}(C\wedge A,B)}$

Given a derivation $\Sigma\colon B[x\colon C\wedge A]$,

$$\begin{array}{c} C\wedge A\\ \Sigma\\ B\end{array} \xrightarrow{Hom(\wedge(f),B)} \dfrac{\begin{array}{c}\dfrac{C\quad [A]}{C\wedge A}\\ \Sigma\\ B\end{array}}{A\to B} \xrightarrow{K_{C'B}} \dfrac{\dfrac{\begin{array}{c}\dfrac{\dfrac{C\wedge A}{C}\quad [A]}{C\wedge A}\\ \Sigma\\ B\end{array}}{A\to B} \quad \dfrac{C\wedge A}{A}}{B}$$

and there is neither a rex from $id_{Hom_{Cat}(C\wedge A,B)}$ to $L_{CB}\circ K_{CB}$ nor from $L_{CB}\circ K_{CB}$ to $id_{Hom_{Cat}(C\wedge A,B)}$.

If we use the reduction that corresponds to $\langle \pi_1(M), \pi_2(M)\rangle \rhd M$ in section 3.1, that is, the reduction $\dfrac{\dfrac{A\wedge B}{A}\quad \dfrac{A\wedge B}{B}}{A\wedge B} \rhd A\wedge B$, then there exists a rex from $L_{CB}\circ K_{CB}(\Sigma)$ to $id_{Hom_{Cat}(C\wedge A,B)}(\Sigma)$, which is the $\to$-red followed by this λ-calculus reduction and then $L_{CB}\circ K_{CB} \Rightarrow id_{Hom_{Cat}(C\wedge A,B)}$ is in accordance to rax 2-adjunction.

6. Neither $K_{CB} \circ L_{CB} \Rightarrow id_{Hom_{Cat}(C,A\to B)}$ nor $id_{Hom_{Cat}(C,A\to B)} \Rightarrow K_{CB} \circ L_{CB}$

Given a derivation $\Sigma\colon A\to B[x\colon C]$,

$$\begin{array}{c}C\\ \Sigma\\ A\to B\end{array} \xrightarrow{L_{CB}} \dfrac{\begin{array}{c}\dfrac{C\wedge A}{C}\\ \Sigma\\ A\to B\end{array}\quad \dfrac{C\wedge A}{A}}{B} \xrightarrow{Hom(\wedge(f),B)} \dfrac{\dfrac{\begin{array}{c}\dfrac{\dfrac{C\quad [A]}{C\wedge A}}{C}\\ \Sigma\\ A\to B\end{array}\quad \dfrac{\dfrac{C\quad [A]}{C\wedge A}}{A}}{B}}{A\to B}$$

and there is neither a rex from $id_{Hom_{Cat}(C,A\to B)}$ to $K_{CB}\circ L_{CB}$ nor from $K_{CB}\circ L_{CB}$ to $id_{Hom_{Cat}(C,A\to B)}$.

If we use the reduction that corresponds to $\lambda x.App(M,x) \rhd M$ in section 3.1, that is, the reduction $\dfrac{\dfrac{A\to B\quad A}{B}}{A\to B} \rhd A\to B$, then there exists a rex from $K_{CB}\circ L_{CB}(\Sigma)$ to $id_{Hom_{Cat}(C,A\to B)}(\Sigma)$, which is the $\wedge$-red followed by this λ-calculus reduction and then $K_{CB}\circ L_{CB} \Rightarrow id_{Hom_{Cat}(C\wedge A,B)}$ is in accordance to lax 2-adjunction.

fourth part

If we included the λ-calculus reduction in the system presented in chapter 1, we would have the natural transformation α as in the definition of lax 2-adjunction and β as in the definition of rax 2-adjunction. Even though, we would not have neither the equalities as in the definition of lax 2-adjunction or the equalities as in the definition of rax 2-adjunction:

As $\alpha\colon L_{CB} \circ K_{CB} \Rightarrow id_{Hom_{Cat}(C \wedge A, B)}$ and $\beta\colon K_{CB} \circ L_{CB} \Rightarrow id_{Hom_{Cat}(C, A \to B)}$, we have that:

1. $\beta; id_{K_{CB}}$ goes from $(K_{CB} \circ L_{CB}) \circ K_{CB}$ to $id_{Hom_{Cat}(C, A \to B)} \circ K_{CB} = K_{CB}$ and $id_{K_{CB}}; \alpha$ goes from $K_{CB} \circ (L_{CB} \circ K_{CB})$ to $K_{CB} \circ id_{Hom_{Cat}(C \wedge A, B)} = K_{CB}$, therefore there is not a composition of $\beta; id_{K_{CB}}$ with $id_{K_{CB}}; \alpha$.

2. $\alpha; id_{L_{CB}}$ goes from $(L_{CB} \circ K_{CB}) \circ L_{CB}$ to $id_{Hom_{Cat}(C \wedge A, B)} \circ L_{CB} = L_{CB}$ and $id_{L_{CB}}; \beta$ goes from $L_{CB} \circ (K_{CB} \circ L_{CB})$ to $L_{CB} \circ id_{Hom_{Cat}(C, A \to B)} = L_{CB}$, therefore there is not a composition of $\alpha; id_{L_{CB}}$ with $id_{L_{CB}}; \beta$.

4.5 2-initial object

$\bot$ cannot be seen as a 2-initial object: given derivations Π and Ψ from $\bot$ to a formula A, it is not the case that there always exists a rex from Π to Ψ that is an isomorphism because, for example, the derivation $\dfrac{\dfrac{\bot}{\bot \wedge A}}{\bot}$ reduces to $\bot$ but $\bot$ does not expands to $\dfrac{\dfrac{\bot}{\bot \wedge A}}{\bot}$.

For a similar reason, $\bot$ cannot be seen as a lax 2-initial object. For example, there is no rex between $\dfrac{\dfrac{\bot}{\bot \wedge A}}{\bot}$ and $\dfrac{\dfrac{\bot}{A} \quad \dfrac{\bot}{A \to \bot}}{\bot}$. Note that both these derivations reduce to $\bot$ and therefore, they are 1-categorically represented by the same arrow, viz., the identity arrow from $\bot$ to $\bot$.

4.6 Ekman's reduction

Ekman's reduction cannot be 2-categorically interpreted. When we begin examining the 2-categorical commutativity, the property (1) in section (2.1) does not hold if we add Ekman's reduction to Prawitz's, e.g.,

$$\dfrac{A \qquad \dfrac{\begin{matrix}[A]\\ \Pi\\ B\end{matrix}}{A \to B}}{B} \rhd_E \begin{matrix}A\\ \Pi\\ B\end{matrix} \qquad \text{but} \qquad \dfrac{\dfrac{\begin{matrix}A\\ \Pi'\\ C\end{matrix}}{A} \qquad \dfrac{\begin{matrix}[A]\\ \Pi\\ B\end{matrix}}{A \to B}}{B} \not\rhd_E \begin{matrix}A\\ \Pi'\\ \dfrac{C}{A}\\ \Pi\\ B\end{matrix}$$

meaning that, when Ekman's reduction is involved, there is no correspondence between 2-categorical and proof-theoretical composition. In other words this means that there exists 1-cells $f, g\colon A \to B$ and $f', g'\colon A \to A$ and 2-cells $\alpha\colon f \Rightarrow g$ and $\beta\colon f' \Rightarrow g'$ such that there is no 2-cell $\alpha;\beta$ from $f \circ f'$ to $g \circ g'$.