3
Category Theory

A category $\mathcal{C}$ is composed of $\mathcal{C}$-objects or objects (A, B, $C\ldots$) and $\mathcal{C}$-arrows, arrows or even morphisms (f, g, $h\ldots$) between pairs of these objects. For every $f\colon A \to B$ and $g\colon B \to C$ there exists $h\colon A \to C$ that is the composition of f and g, i.e., $h = g \circ f$, which is associative. Every object A defines an identity arrow $id_A\colon A \to A$ and for every $f\colon A \to B$, $id_B \circ f = f \circ id_A = f$. We define:

Isomorphism: An arrow $f\colon A \to B$ is an isomorphism if there exist $g\colon B \to A$ such that $g \circ f = id_A$ and $f \circ g = id_B$. Then, the objects A and B are isomorphic ($A \cong B$).

Product: A product of two $\mathcal{C}$-objects A and B is a $\mathcal{C}$-object $A \times B$ together with projection arrows $\pi_1\colon A \times B \to A$ and $\pi_2\colon A \times B \to B$ such that for every $\mathcal{C}$-object C and arrows $f\colon C \to A$ and $g\colon C \to B$ there exists a unique arrow $\langle f, g\rangle\colon C \to A \times B$ such that $\pi_1 \circ \langle f, g\rangle = f$ and $\pi_2 \circ \langle f, g\rangle = g$. This definition can be represented by the following diagram:

$$
\begin{tikzcd}
 & C \arrow[dl, "f"'] \arrow[d, dotted, "{\langle f,g \rangle}"] \arrow[dr, "g"] & \\
A & A \times B \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & B
\end{tikzcd}
$$

Note that we say *a* instead of *the* product. That is because this notion, as many in Category Theory, is defined up to isomorphism, for example, $B \times A$ is also a product of A and B. We use a dotted arrow to represent unique arrows.

Co-product: A co-product (or sum) of two $\mathcal{C}$-objects A and B is a $\mathcal{C}$-object $A + B$ together with inclusion arrows $i_1\colon A \to A + B$ and $i_2\colon B \to A + B$ such that for every $\mathcal{C}$-object C and arrows $f\colon A \to C$ and $g\colon B \to C$ there exists a unique arrow $[f,g]\colon A+B \to C$ such that $[f,g]\circ i_1 = f$ and $[f,g]\circ i_2 = g$

$$\begin{tikzcd} & C & \\ A \arrow[ur, "f"] \arrow[r, "i_1"'] & A+B \arrow[u, dotted, "{[f,g]}"] & B \arrow[ul, "g"'] \arrow[l, "i_2"] \end{tikzcd}$$

Exponentiation: An exponential of two $\mathcal{C}$-objects A and B is a $\mathcal{C}$-object B^A together with an evaluation arrow $eval\colon A \times B^A \to B$ such that, for every $f\colon A \times C \to B$, there exists a unique arrow $\hat{f}\colon C \to B^A$ that makes the following diagram to commute:

$$\begin{tikzcd} A \times C \arrow[d, "id_A"'] \arrow[d, dotted, shift left=4, "\hat{f}"] \arrow[dr, "f"] & \\ A \times B^A \arrow[r, "eval"'] & B \end{tikzcd}$$

The exponential establishes a natural bijection between $Hom(A\times C, B)$ and $Hom(C, B^A)$, where $Hom\ (X,\ Y)$ is the set of all morphisms that goes from X to Y.

Initial object: A $\mathcal{C}$-object 0 is an initial object if, for every $\mathcal{C}$-object A, there exists only one arrow from 0 to A.

Factorization: An arrow f is said to *factor through* g if there exists h such that $g \circ h = f$. For example, in the product diagram above, f factors through π_1 uniquely.

Functor: Given two categories $\mathcal{A}$ and $\mathcal{B}$, a functor F from $\mathcal{A}$ to $\mathcal{B}$ is a function that takes to $\mathcal{B}$ the categorical structure of $\mathcal{A}$, i.e.,
(i) if A is an $\mathcal{A}$-object, then $F(A)$ is a $\mathcal{B}$-object;
(ii) if $f\colon A \to B$, then $F(f)\colon F(A) \to F(B)$;
(iii) for every $\mathcal{A}$-object A, $F(id_A) = id_{F(A)}$ and;
(iv) if $h = f \circ g$, then $F(h) = F(f) \circ F(g)$.

Cartesian Closed Category: A category with finite objects and arrows and where there exist product and exponentiation for every pair of objects is called a Cartesian Closed Category.

3.1 The Curry-Howard Isomorphism

In this section, which is based on (10), we define a functor **L** from the category **Cart** whose objects are cartesian closed categories and whose arrows are functors to the category λ-**Calc** of typed λ-calculi, which is to be defined, in order to show one side of their relation. The other side is the functor **K** from λ-**Calc** to **Cart** which is usually called the categorical semantics of λ-calculus. Given a Cartesian Closed Category $\mathcal{A}$, the types of $\mathbf{L}(\mathcal{A})$ are the objects of $\mathcal{A}$ and its terms are formed by applying one of the following operations:

$$\frac{}{x\colon A[x\colon A]} \qquad \frac{M\colon A[\Delta]}{M\colon A[\Delta, x\colon B]}$$

$$\frac{N\colon B[x\colon A] \qquad M\colon C[y\colon B]}{M[y \leftarrow N]\colon C[x\colon A]} \qquad \frac{M\colon A[\Delta] \qquad N\colon B[\Delta]}{\langle M, N\rangle\colon A\times B[\Delta]}$$

$$\frac{M\colon A\times B[\Delta]}{\pi_1(M)\colon A[\Delta]} \qquad \frac{M\colon A\times B[\Delta]}{\pi_2(M)\colon B[\Delta]}$$

$$\frac{M\colon B[\Delta, x\colon A]}{\lambda x.M\colon B^A[\Delta]} \qquad \frac{M\colon B^A[\Delta] \qquad N\colon A[\Delta']}{App(M, N)\colon B[\Delta, \Delta']}$$

where M and N are metavariables for terms and Δ is a set of variables. When there is no doubt which term is to be substituted, we can write $M[N]$ instead of $M[y \leftarrow N]$.

The cartesian closed categorical semantic for the typed λ-calculus is given by the function $[\![\cdot]\!]$ that takes as arguments types and terms and yields values in $\mathcal{A}$ (either objects or morphisms):

Types

If A is a basic type, then $[\![A]\!]$ is an object of $\mathcal{A}$, else

$[\![A\times B]\!]= [\![A]\!]\times[\![B]\!]$

$[\![B^A]\!]= [\![B]\!]^{[\![A]\!]}$

Terms

$[\![x\colon A[x\colon A]]\!]= I_{[\![A]\!]}$

$[\![M\colon A[\Delta, x\colon B]]\!]= f\colon [\![\Delta]\!]\times[\![B]\!]\rightarrow[\![A]\!]$

$[\![M[N]\colon C[x\colon A]]\!]= [\![N\colon B[x\colon A]]\!]\circ[\![M\colon C[x\colon B]]\!]$

$[\![\langle M, N\rangle\colon A\times B[\Delta]]\!]= \langle[\![M\colon A[\Delta], N\colon B[\Delta]]\!]\rangle$

$[\![\pi_1(M)\colon A[\Delta]]\!]= \pi_1([\![M\colon A\times B[\Delta]]\!])$

$[\![\pi_2(M)\colon B[\Delta]]\!]= \pi_2([\![M\colon A \times B[\Delta]]\!])$

$[\![\lambda x.M\colon B^A[\Delta]]\!]= \widehat{f}$, that is the only arrow that makes the diagram to commute.

$$\begin{tikzcd} B^A \times A \arrow[dr, "eval"] & \\ \Delta \times A \arrow[u, dotted, "\langle \widehat{f}, {}_A \rangle"] \arrow[r, "f"'] & B \end{tikzcd}$$

$[\![App(M,N)\colon B[\Delta, x\colon A]]\!]= eval([\![M\colon B^A[\Delta]]\!], [\![N\colon A[\Delta]]\!])$

During the process of term formation, it can happen that we apply more operations then it is necessary, i.e., some combinations of operations can work as a kind of identity operation, making its application redundant. We say that a term M *reduces* to a term M' ($M \triangleright M'$) if M' is obtained from M by the application of any of the following operations:

$\lambda x^A.M[x \leftarrow N] \triangleright M[x \leftarrow N]$

$\pi_1\langle M, N\rangle \triangleright M$

$\pi_2\langle M, N\rangle \triangleright N$

$M \triangleright M'$ implies $M[N] \triangleright M'[N]$

$M \triangleright M'$ implies $N[M] \triangleright N[M']$

$M \triangleright M'$ implies $\lambda x^A.M \triangleright \lambda x^A.M'$

$M \triangleright M'$ implies $\pi_1(M) \triangleright \pi_1(M')$

$M \triangleright M'$ implies $\pi_2(M) \triangleright \pi_2(M')$

$\lambda x^A.App(M, x) \triangleright M$

$\langle \pi_1(M), \pi_2(M)\rangle \triangleright M$

Theorem 3.1.1. *Let $M\colon A[\Delta]$ be a λ-term such that $M \triangleright M'$ and $M'\colon A[\Delta]$. Then we have that $[\![M\colon A[\Delta]]\!]= [\![M'\colon A[\Delta]]\!]$.*

Given a λ-term $M\colon B$, the set $\mathrm{FV}(M)$ of the free variables of M, is defined as follows:

if $M \equiv x$, then $\mathrm{FV}(M) = \{x\}$;

if $M \equiv \lambda x : A.N$, then $\mathrm{FV}(M) = \mathrm{FV}(N)\backslash\{x\}$;

if $M \equiv N[P]$, then $\mathrm{FV}(M) = \mathrm{FV}(N)\cup\mathrm{FV}(P)$;

if $M \equiv \langle N, P\rangle$, then $\mathrm{FV}(M) = \mathrm{FV}(N) \cup \mathrm{FV}(P)$;

if $M \equiv \pi_1(N)$, then $\mathrm{FV}(M) = \mathrm{FV}(N)$;

if $M \equiv \pi_2(N)$, then $\mathrm{FV}(M) = \mathrm{FV}(N)$;

A variable that is not free is said to be closed.

Let $\mathcal{A}$ and $\mathcal{A}'$ be two objects of **Cart**. A morphism $\Phi : \mathbf{L}(\mathcal{A}) \to \mathbf{L}(\mathcal{A}')$ is to be called *translation* and is defined as:

1. if a has type A, then $\Phi(a)$ has type $\Phi(A)$, if a is closed (respectively free), so is $\Phi(a)$ and the i^{th} variable of type A is sent to the i^{th} variable of type $\Phi(A)$.

2. Φ preserves type and term forming operations, e.g.:

   $\Phi(A \times B) = \Phi(A) \times \Phi(B)$, $\Phi(B^A) = \Phi(B)^{\Phi(A)} \ldots$

   $\Phi(\pi_1(c)) = \pi_1(\Phi(c))$, where $c \colon A \times B$; $\Phi(\lambda x.M) = \lambda\Phi(x).\Phi(M) \ldots$

Proposition 3.1.2. **L** *is a functor.*

Proof. **(i)** to each **Cart**-object $\mathcal{A}$, $\mathbf{L}(\mathcal{A})$ is a λ-**Calc**-object

That comes from the construction of $\mathbf{L}(\mathcal{A})$

(ii) to each **Cart**-arrow $F \colon \mathcal{A} \to \mathcal{A}'$, $\mathbf{L}F \colon \mathbf{L}(\mathcal{A}) \to \mathbf{L}(\mathcal{A}')$ is a λ-**Calc**-arrow

It is the same as to show that $\mathbf{L}F$ is a translation:

1. As F is a functor, for every object A in $\mathcal{A}$, $F(A)$ is an object in $\mathcal{A}'$ and a type in $\mathbf{L}(\mathcal{A}')$, i.e., $\mathbf{L}F$ takes a type A in $\mathbf{L}(\mathcal{A})$ to a type $\mathbf{L}FA = FA$ in $\mathbf{L}(\mathcal{A}')$

2. If $A \times B$ is a type in $\mathbf{L}(\mathcal{A})$, then $\mathbf{L}F(A \times B) = \mathbf{L}(FA \times FB)$ (for F preserves product) and then $\mathbf{L}FA \times \mathbf{L}FB$ is a type in $\mathbf{L}(\mathcal{A}')$; Analogous for exponentiation.

   We show that $\mathbf{L}F$ preserves terms by induction on the size of the term:

   As functors preserve arrows, we have that $\mathbf{L}F$ sends $x \colon B[y \colon A]$ to $x \colon FB[y \colon FA]$. Suppose it works for terms M and N, then

$$\frac{\mathbf{L}F(N) \colon \mathbf{L}F(B)[x \colon \mathbf{L}F(A)] \qquad \mathbf{L}F(M) \colon \mathbf{L}F(C)[y \colon \mathbf{L}F(B)]}{\mathbf{L}F(M)[\mathbf{L}F(N)] \colon \mathbf{L}F(C)[x \colon \mathbf{L}F(A)]}$$

$$\frac{\mathbf{L}F(M) \colon \mathbf{L}F(A)[\Delta] \qquad \mathbf{L}F(N) \colon \mathbf{L}F(B)[\Delta]}{\langle \mathbf{L}F(M), \mathbf{L}F(N)\rangle \colon \mathbf{L}F(A) \times \mathbf{L}F(B)[\Delta]}$$

$$\frac{\mathbf{L}F(M)\colon \mathbf{L}F(A)\times \mathbf{L}F(B)[\Delta]}{\pi_1(\mathbf{L}F(M))\colon \mathbf{L}F(A)[\Delta]}$$

$$\frac{\mathbf{L}F(M)\colon \mathbf{L}F(A)\times \mathbf{L}F(B)[\Delta]}{\pi_2(\mathbf{L}F(M))\colon \mathbf{L}F(B)[\Delta]}$$

$$\frac{\mathbf{L}F(M)\colon \mathbf{L}F(B)[\Delta, x\colon \mathbf{L}F(A)]}{\lambda x.\mathbf{L}F(M)\colon \mathbf{L}F(B)^{\mathbf{L}F(A)}[\Delta]}$$

$$\frac{\mathbf{L}F(M)\colon \mathbf{L}F(B)^{\mathbf{L}F(A)}[\Delta] \qquad \mathbf{L}F(N)\colon \mathbf{L}F(A)[\Delta]}{App(\mathbf{L}F(M), \mathbf{L}F(N))\colon \mathbf{L}F(B)[\Delta]}$$

(iii) $\mathbf{L}(id_{\mathcal{A}}) = id_{\mathbf{L}(\mathcal{A})}$

We show that $\mathbf{L}(id_{\mathcal{A}})$ is a translation:

1. For every object A in $\mathcal{A}$, $id_{\mathcal{A}}(A)$ is an object in $\mathcal{A}$ and a type in $\mathbf{L}(\mathcal{A})$, i.e., $\mathbf{L}(id_{\mathcal{A}})$ takes a type A in $\mathbf{L}(\mathcal{A})$ to a type $\mathbf{L}(id_{\mathcal{A}}(A)) = id_{\mathcal{A}}(A) = A$ in $\mathbf{L}(\mathcal{A})$
2. If $A \times B$ is a type in $\mathbf{L}(\mathcal{A})$, then $\mathbf{L}(id_{\mathcal{A}}(A \times B)) = \mathbf{L}(A \times B)$ and then $A \times B$ is a type in $\mathbf{L}(\mathcal{A})$; Analogous for exponentiation.

   We show that $\mathbf{L}(id_{\mathcal{A}})$ preserves terms by induction on the size of the term:

   As functors preserve arrows, we have that $\mathbf{L}(id_{\mathcal{A}})$ sends a term $x\colon B[y\colon A]$ to a term $x\colon B[y\colon A]$. Suppose it works for terms M and N, then

$$\frac{\mathbf{L}(N)\colon \mathbf{L}(B)[x\colon \mathbf{L}(A)] \qquad \mathbf{L}(M)\colon \mathbf{L}(C)[y\colon \mathbf{L}(B)]}{\mathbf{L}(M)[\mathbf{L}(N)]\colon \mathbf{L}(C)[x\colon \mathbf{L}(A)]}$$

$$\frac{\mathbf{L}(M)\colon \mathbf{L}(A)[\Delta] \qquad \mathbf{L}(N)\colon \mathbf{L}(B)[\Delta]}{\langle \mathbf{L}(M), \mathbf{L}(N)\rangle\colon \mathbf{L}(A)\times \mathbf{L}(B)[\Delta]}$$

$$\frac{\mathbf{L}(M)\colon \mathbf{L}(A)\times \mathbf{L}(B)[\Delta]}{\pi_1(\mathbf{L}(M))\colon \mathbf{L}(A)[\Delta]}$$

$$\frac{\mathbf{L}(M)\colon \mathbf{L}(A)\times \mathbf{L}(B)[\Delta]}{\pi_2(\mathbf{L}(M))\colon \mathbf{L}(B)[\Delta]}$$

$$\frac{\mathbf{L}(M)\colon \mathbf{L}(B)[\Delta, x\colon \mathbf{L}(A)]}{\lambda x.\mathbf{L}(M)\colon \mathbf{L}(B)^{\mathbf{L}(A)}[\Delta]}$$

$$\frac{\mathbf{L}(M)\colon \mathbf{L}(B)^{\mathbf{L}(A)}[\Delta] \qquad \mathbf{L}(N)\colon \mathbf{L}(A)[\Delta]}{App(\mathbf{L}(M), \mathbf{L}(N))\colon \mathbf{L}(B)[\Delta]}$$

(iv) $\mathbf{L}(G \circ F) = \mathbf{L}(G) \circ \mathbf{L}(F)$, whenever $G \circ F$ is defined

Let A be a formula in $\mathcal{A}$. As $\mathbf{L}F$ and $\mathbf{L}G$ are translations (from (ii)), we have that $\mathbf{L}F(A)$ is a type in a λ-Calculus $\mathbf{L}(\mathcal{B})$ and $\mathbf{L}G(\mathbf{L}F(A))$ is a type in a λ-Calculus $\mathbf{L}(\mathcal{C})$. As $G \circ F$ is a functor, then $\mathbf{L}(G \circ F)$ is a translation and $\mathbf{L}(G \circ F)(A)$ is a type in $\mathbf{L}(\mathcal{C})$. The same works for arrow instead of formula. Hence, the equality holds.

□

We call attention to the obvious correspondence between Proof Theory and λ-Calculus: some of the term formating rules can be seen as $\wedge$ and $\rightarrow$ elimination and introduction rules; the reduction operations have a strong resemblance with the three properties presented in the ending of section 2.1 and it is impossible not to think about Prawitz's Conjecture after reading Theorem 1.

3.2 Categorical view of Proof Theory

Natural Deduction, from the point of view of Proof Theory, forms a category by thinking of formulas as objects and derivations as morphisms. A derivation f of A from C can be seen as an arrow f whose source is C and whose target is A. For typographical reasons, sometimes we use the λ-Calculus notation $\Pi\colon B[x\colon A]$ instead of $\begin{array}{c} A \\ \Pi \\ B \end{array}$ and if $\Psi\colon A[y\colon B]$, then $\Pi(\Psi)$ is represented by $\Psi\colon A[y\colon B](y \leftarrow \Pi\colon B[x\colon A])$. Formally we have:

Let $[\![\cdot]\!]$ be a functor that takes elements of our deduction system to elements of Category Theory. If A, B and C are formulas and Π and Ψ are derivations, we have:

$[\![A]\!]$ is an object

$[\![\Pi\colon A[x\colon B]]\!] = f$, where f is an arrow from B to A

$[\![\Pi\colon C[y\colon B](y \leftarrow \Psi\colon B[x\colon A])]\!] = g \circ f$, where g is an arrow from B to C and f is an arrow from A to B.

In Category Theory, two equivalent derivations are represented by the same arrow. That is not a problem for it is in accordance with Prawitz's Conjecture.

We are interested in relating conjunction, disjunction, implication and falsum to elements in Category Theory. Mann (12), based on Lambek's works (8), (7) and (9), explains the relation between product and conjunction and co-product and disjunction. He uses the definitions of these categorical operators to achieve their respective reduction and expansion step and concludes that $\land$-reduction comes from the commutativity of the product diagram, $\to$-reduction comes from the commutativity of the exponent diagram and that $\land$ and $\to$ expansions come from the uniqueness of the factorization defined by product and exponential respectively (this is one of the reasons we have added expansions to our system).

3.3 Problems with the Curry-Howard Isomorphism

3.3.1 Disjunction

For disjunction to be seen as co-product, one more reduction must to be added:

In (11), Mann shows that $\lor$-reduction comes from the commutativity of the co-product diagram, but for $\lor$-expansion, from the uniqueness of the factorization, he concludes that

$$\begin{array}{c} \dfrac{A \lor B \quad \dfrac{[A]}{A \lor B} \quad \dfrac{[B]}{A \lor B}}{A \lor B} \\ \Pi \\ C \end{array} \quad = \quad \dfrac{A \lor B \quad \begin{array}{c}\dfrac{[A]}{A \lor B} \\ \Pi \\ C\end{array} \quad \begin{array}{c}\dfrac{[B]}{A \lor B} \\ \Pi \\ C\end{array}}{C}$$

but, if there is an application of $\to$-int in Π that discharges the major premiss of the $\lor$-el (i.e., the formula $A \lor B$), then there is no derivation of the form of the right hand side. One way of solving this issue is by adding the reduction:

$$\begin{array}{c}\dfrac{\begin{array}{c}\Pi_1 \\ A \lor B\end{array} \quad \begin{array}{c}[A] \\ \Pi_2 \\ C\end{array} \quad \begin{array}{c}[B] \\ \Pi_3 \\ C\end{array}}{C} \\ \Pi_4 \\ D\end{array} \quad \rhd \quad \dfrac{\begin{array}{c}\Pi_1 \\ A \lor B\end{array} \quad \begin{array}{c}[A] \\ \Pi_2 \\ C \\ \Pi_4 \\ D\end{array} \quad \begin{array}{c}[B] \\ \Pi_3 \\ C \\ \Pi_4 \\ D\end{array}}{D}$$

where no rule in Π_4 discharges any hypothesis of Π_1. Note that this reduction is more general than the permutation reduction defined by Prawitz. For instance, the lowest occurrence of C does not need to be a major premiss. We call this reduction *Mann's $\lor$-permutation* (MDP).

3.3.2
Initial object

In a Cartesian Closed Category $\mathcal{C}$ with initial object $\perp$ and an object A, there exists only one arrow from A to $\perp$: as $Hom(\perp \times \perp, \perp) \cong Hom(\perp, \perp^{\perp})$, we have that $|Hom(\perp \times \perp, \perp)| = |Hom(\perp, \perp^{\perp})| = 1$. Suppose that there exists two arrows f and g from A to $\perp$. From the product diagram

$$\begin{tikzcd} & A \arrow[dl, bend right, "f"'] \arrow[d, dotted, "!"] \arrow[dr, bend left, "g"] & \\ \perp & \perp \times \perp \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & \perp \end{tikzcd}$$

$f = \pi_1 \circ !$ and $g = \pi_2 \circ !$, but as there is only one arrow from $\perp \times \perp$ to $\perp$, it follows that $\pi_1 = \pi_2$ and consequently $f = g$. This observation is due to Joyal (10)(p.116) and such a result, if we only consider Prawitz's reductions, is not supported by proof theoretical means: if we follow the same line of idea of the above proof representing the arrows of the product diagram as derivations, we would form the derivation $\dfrac{\dfrac{\begin{matrix} A \\ f \\ \perp \end{matrix} \quad \begin{matrix} A \\ g \\ \perp \end{matrix}}{\perp \wedge \perp}\,h}{\perp}\,\pi$ that reduces to $\begin{matrix} A \\ f \\ \perp \end{matrix}$ if $\pi = \pi_1$ and to $\begin{matrix} A \\ g \\ \perp \end{matrix}$ if $\pi = \pi_2$, in other words, we do not know if π is the rule $\dfrac{A \wedge B}{A}$ or if it is the rule $\dfrac{A \wedge B}{B}$ and thus we would come to no conclusion about a relation between f and g. The two derivations below are an example of two different derivations from a same formula to $\perp$:

$$\dfrac{\dfrac{\perp \wedge (A \wedge \perp)}{A \wedge \perp}}{\perp} \qquad\qquad \dfrac{\perp \wedge (A \wedge \perp)}{\perp}$$

Note that there is no rex between them. According to PH's reduction, every derivation from a formula A to $\perp$ is equivalent to each other and that is the reason we said that this reduction approximates Proof Theory to the categorical semantic. The derivation

$$\dfrac{\dfrac{A}{A \to A} \qquad \dfrac{\dfrac{A \qquad \neg A}{\perp}}{\neg(A \to A)}\,\perp_i}{\perp}$$

is not normal and it cannot be normalized by any of the reductions presented so far. For cases like this, we introduce the following reduction:

$$\begin{array}{c}\Pi_1\\ \dfrac{\dfrac{\perp}{A}\,\perp_i}{\perp}\\ \Pi_2\end{array} \rhd \begin{array}{c}\Pi_1\\ \perp\\ \Pi_2\end{array} \tag{3-1}$$

which is defined as follows:

If $E = \wedge$-el, then $\begin{array}{c}\Pi_1\\ \dfrac{\dfrac{\perp}{A_1 \wedge \perp}}{\perp}\\ \Pi_2\end{array} \rhd \begin{array}{c}\Pi_1\\ \perp\\ \Pi_2\end{array}$ and $\begin{array}{c}\Pi_1\\ \dfrac{\dfrac{\perp}{\perp \wedge A_2}}{\perp}\\ \Pi_2\end{array} \rhd \begin{array}{c}\Pi_1\\ \perp\\ \Pi_2\end{array}$

If $E = \vee$-el, then $\begin{array}{c}\dfrac{\begin{array}{c}\Pi_1\\ \dfrac{\perp}{A \vee B}\end{array} \quad \begin{array}{c}[A]\\ \Pi_2\\ \perp\end{array} \quad \begin{array}{c}[B]\\ \Pi_3\\ \perp\end{array}}{\perp}\\ \Pi_4\end{array} \rhd \begin{array}{c}\Pi_1\\ \perp\\ \Pi_4\end{array}$

If $E = \to$-el, then $\begin{array}{c}\dfrac{\begin{array}{c}\Pi_1\\ A\end{array} \quad \begin{array}{c}\Pi_2\\ \dfrac{\perp}{A \to \perp}\end{array}}{\perp}\\ \Pi_2\end{array} \rhd \begin{array}{c}\Pi_2\\ \perp\\ \Pi_3\end{array}$

With this reduction, the above derivation reduces to $\dfrac{A \quad \neg A}{\perp}$

Besides this ad hoc situation, we can prove that every derivation from $\perp$ to A can be reduced to $\dfrac{\perp}{A}$, i.e., every derivation from $\perp$ to A is equivalent to $\dfrac{\perp}{A}$ and, therefore, every derivation from $\perp$ to A is categorically represented by the same arrow, i.e, by the only arrow from the initial object to A.

Proposition 1. *Let Σ be a derivation from $\perp$ to A. Then there exists at least one rex τ from Σ to* $\dfrac{\perp}{A}$.

Proof. If Σ reduces to a derivation Π, and Π reduces to $\dfrac{\perp}{A}$ then, by property (0), Σ reduces to $\dfrac{\perp}{A}$. Let ρ be the rex from Σ to Π. If $\Pi = \dfrac{\perp}{A}$, then $\tau = \rho$.

Suppose $\Pi = \begin{array}{c}\dfrac{\dfrac{\perp}{B}}{C}\\ \Psi\\ A\end{array} \neq \dfrac{\perp}{A}$ and that there exists a rex α such that $\begin{array}{c}\dfrac{\perp}{C}\\ \Psi\\ A\end{array} \rhd_\alpha$ $\dfrac{\perp}{A}$. Then,

– if $C = C_1 \wedge C_2$, then $\Pi = \begin{array}{c}\dfrac{\dfrac{\perp}{C_1} \quad \dfrac{\perp}{C_2}}{C_1 \wedge C_2}\\ \Psi\\ A\end{array} \rhd \begin{array}{c}\dfrac{\perp}{C_1 \wedge C_2}\\ \Psi\\ A\end{array} \rhd_\alpha \dfrac{\perp}{A}$ and τ can be the sequence $\langle \rho, \perp\text{-red}, \alpha \rangle$.

– if $C = C_1 \vee C_2$, then $\Pi = \dfrac{\dfrac{\perp}{C_i}}{\dfrac{C_1 \vee C_2}{\begin{array}{c}\Psi\\ A\end{array}}} \rhd \dfrac{\perp}{\begin{array}{c}C_1 \vee C_2\\ \Psi\\ A\end{array}} \rhd_\alpha \dfrac{\perp}{A}$ and τ can be the sequence $\langle \rho, \perp\text{-red}, \alpha \rangle$.

– if $C = C_1 \rightarrow C_2$, then $\Pi = \dfrac{\dfrac{\perp}{C_2}}{\dfrac{C_1 \rightarrow C_2}{\begin{array}{c}\Psi\\ A\end{array}}} \rhd \dfrac{\perp}{\begin{array}{c}C_1 \rightarrow C_2\\ \Psi\\ A\end{array}} \rhd_\alpha \dfrac{\perp}{A}$ and τ can be the sequence $\langle \rho, \perp\text{-red}, \alpha \rangle$.

$\square$

3.3.3 Ekman's reduction

Considering Prawitz's conjecture, when a derivation Π E-reduces to a derivation Π', we cannot, in contrast to Prawitz's Conjecture, say that they are equivalent. For instance, the derivation $\dfrac{\dfrac{A}{A \vee B} \quad \dfrac{A \vee B}{A \vee B \rightarrow A \vee B}}{A \vee B}$ E-reduces to both $\dfrac{A}{A \vee B}$ and $A \vee B$ but they are not equivalent. Categorically, we would not have the same arrow representing equivalent derivations. Thus, Ekman's reduction does not seems to be completely adequate to a categorical interpretation yet.