2 Natural Deduction

2.1 Background and terminology

Natural Deduction is a logical system designed by Gentzen and Jaškowski in the early 30's on an attempt to create a deductive system more compatible with mathematical reasoning. It is also natural in the sense that it allows us to write deductions in a more straightfoward way. We are interested in the fragment $\{\wedge, \vee, \rightarrow, \perp\}$ of intuitionistic logic. We call these logical symbols conjunction, disjunction, implication and falsum respectively.

The properties of each logical operator are given by an elimination and an introduction rule and there is a rule for $\perp$. They are as follows:

$$(\wedge\text{-int}) \quad \frac{A \qquad B}{A \wedge B} \qquad\qquad (\wedge\text{-el}) \quad \frac{A \wedge B}{A} \qquad \frac{A \wedge B}{B}$$

$$(\rightarrow\text{-int}) \quad \frac{\begin{array}{c}[A]\\ \vdots \\ B\end{array}}{A \rightarrow B} \qquad\qquad (\rightarrow\text{-el}) \quad \frac{A \rightarrow B \qquad A}{B}$$

$$(\vee\text{-int}) \quad \frac{A}{A \vee B} \qquad \frac{B}{A \vee B} \qquad\qquad (\vee\text{-el}) \quad \frac{A \vee B \qquad \begin{array}{c}[A]\\ \vdots \\ C\end{array} \qquad \begin{array}{c}[B]\\ \vdots \\ C\end{array}}{C}$$

$$(\perp_i) \quad \frac{\perp}{A}$$

In $\perp_i$ we require A to be different from $\perp$ and we put formulas between brackets when they are *discharged* (see the definition below) and sometimes we indicate with a number the application where it occurs.

We can define negation as a particular case of implication, i.e., $\neg A$ can be defined as $A \rightarrow \perp$ and, putting $B = \perp$ in $\rightarrow$-int and $\rightarrow$-el, we obtain $\neg$-int and ¬-el, respectively:

$$(\neg\text{-int}) \quad \dfrac{\begin{matrix}[A]\\ \vdots \\ \bot\end{matrix}}{\neg A} \qquad\qquad (\neg\text{-el}) \quad \dfrac{A \qquad \neg A}{\bot}$$

We say that a formula A is *atomic* if A has neither of the forms $B \wedge C$, $B \vee C$ nor $B \to C$. If A has any of these forms, then B and C are *subformulas* of A and so are any subformula of B and C and the operator between B and C is called *main connective*. The *major premiss* is the premiss of an elimination rule that has the main connective that is to be eliminated. Every premiss that is not a major premiss is a *minor premiss*. A *thread* is a sequence $A_1, A_2, \ldots, A_n$ of formulas where A_1 is an hypothesis, A_n is the conclusion and A_i, $i \neq n$, stands immediately above A_{i+1}. Let τ be a thread that begins with a formula A. Then A may be *discharged* at B when B is the first formula occurrence in τ that:

1. is a premiss of the application that has $A \to B$ as consequence. For example,

$$\dfrac{\dfrac{[A]^1 \qquad D}{A \wedge D}}{A \to A \wedge D}\,(1)$$

2. is either the minor premiss on the left or the minor premiss on the right on an application of ∨-el that has either $A \vee D$ or $D \vee A$ (for some D) respectively as the major premiss. For example,

$$\dfrac{\dfrac{A \vee D \qquad \dfrac{[A]^1 \qquad D}{A \wedge D}\,(a) \qquad \dfrac{A \qquad [D]^1}{A \wedge D}\,(a)}{A \wedge D}\,(1)}{A}\,(b) \tag{2-1}$$

An assumption that was discharged is called *closed*, otherwise it is called *open*. A *branch* in a deduction is a sequence $A_1, A_2, \ldots, A_n$ of formulas such that A_1 is an assumption not discharged by ∨-el, A_{i+1} occurs immediately below A_i and A_n is either the first occurrence in the thread that is a minor premiss of →-el or the conclusion of the derivation and a *main branch* of a derivation is a branch that is also a thread. A *path* is like a branch but the formula that succeeds the major premiss of an ∨-el rule is one of the hypothesis discharged by the application of this rule (14).

When, in a derivation, an introduction rule α is followed by an elimination rule β, the connective that was introduced by α is immediately eliminated by β, i.e., there was no necessity of introducing it in the first place. Instead of going straight to its goal - the conclusion - the derivation made a *detour*. A formula that is both the conclusion of an introduction rule and a major premiss is called *maximum formula*. A derivation without maximum formulas is said to be *normal*.

In order to find a normal derivation, Prawitz introduced the notion of *reduction*. If a derivation Ψ is achieved from a derivation Π by a sequence of the following steps, then we say that Π reduces to Ψ (denoted $\Pi \rhd \Psi$).

$$(\wedge)\quad \begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\Pi_1\\ A\end{array}\quad \begin{array}{c}\Pi_2\\ B\end{array}}{A\wedge B}}{A}\\ \Pi_3\end{array} \rhd \begin{array}{c}\Pi_1\\ A\\ \Pi_3\end{array} \text{ and } \begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\Pi_1\\ A\end{array}\quad \begin{array}{c}\Pi_2\\ B\end{array}}{A\wedge B}}{B}\\ \Pi_3\end{array} \rhd \begin{array}{c}\Pi_2\\ B\\ \Pi_3\end{array}$$

$$(\rightarrow)\quad \begin{array}{c}\dfrac{\begin{array}{c}\Pi_1\\ A\end{array}\quad \dfrac{\begin{array}{c}[A]\\ \Pi_2\\ B\end{array}}{A\rightarrow B}}{B}\\ \Pi_3\end{array} \rhd \begin{array}{c}\Pi_1\\ [A]\\ \Pi_2\\ B\\ \Pi_3\end{array}$$

$$(\vee)\quad \begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\Pi_1\\ A\end{array}}{A\vee B}\quad \begin{array}{c}[A]\\ \Pi_2\\ C\end{array}\quad \begin{array}{c}[B]\\ \Pi_3\\ C\end{array}}{C}\\ \Pi_4\end{array} \rhd \begin{array}{c}\Pi_1\\ [A]\\ \Pi_2\\ C\\ \Pi_4\end{array} \text{ and } \begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\Pi_1\\ B\end{array}}{A\vee B}\quad \begin{array}{c}[A]\\ \Pi_2\\ C\end{array}\quad \begin{array}{c}[B]\\ \Pi_3\\ C\end{array}}{C}\\ \Pi_4\end{array} \rhd \begin{array}{c}\Pi_1\\ [B]\\ \Pi_3\\ C\\ \Pi_4\end{array}$$

where $\begin{array}{c}\Pi_1\\ [F]\end{array}$, $F = A, B$ means that $\begin{array}{c}\Pi_1\\ F\end{array}$ replaced every occurrence of F that was discharged in the original derivation by the rule in question. Given a derivation Π, we also define $\Pi \rhd \Pi$ as the identity reduction.

Prawitz showed (15) (p.256) that, for every derivation, there exists a finite sequence of reductions leading to a normal derivation which is unique. This result is known as *Normalization Theorem*.

Given reductions α and β, the sequences $\langle\alpha, \beta\rangle$ and $\langle\beta, \alpha\rangle$ are different sequence of reductions and, given reductions α_1 and α_2, if $\alpha_1 \colon \Pi_1 \rhd \Pi_2$ and $\alpha_2 \colon \Pi_1 \rhd \Pi_2'$, then there exist $\beta_1 \colon \Pi_2 \rhd \Pi_3$ and $\beta_2 \colon \Pi_2' \rhd \Pi_3$ such that both $\langle\alpha_1, \beta_1\rangle$ and $\langle\alpha_2, \beta_2\rangle$ go from Π_1 to Π_3. This property is known as the Church-Rosser property.

We do not find, in a branch of a normal derivation, an introduction rule followed by an elimination rule for, if otherwise, it would have maximum formulas and therefore it would not be normal. Thus, we may say that the general structure of a branch of a normal derivation has the shape of an hourglass, with all the elimination rules (if any) on its top and the introduction rules (if any) on its bottom. To the formula that is in between we call *minimum formula*, it is the conclusion of an elimination and the premiss of either an introduction rule or the $\perp_i$ and it is both subformula of an hypothesis and of the end-formula of the branch.

The above reductions are not enough to bring any derivation to its normal form. In a normal derivation, the paths must also have all the elimination rules preceding the introduction rules. In the two paths of (2-1), we have introduction rules (a) preceding an elimination rule (b). Moreover, successive applications of $\vee$-elimination rules form a sequence of formula occurrences of the same shape and we would like to eliminate such a sequence. To deal with situations like this, and with the intention of proving normalization for intuitionistic logic, Prawitz introduced the permutation reduction:

$$\dfrac{\dfrac{\begin{array}{c}\Pi_1\\ A\vee B\end{array} \quad \begin{array}{c}\Pi_2\\ C\end{array} \quad \begin{array}{c}\Pi_3\\ C\end{array}}{C} \qquad \Pi_4}{D} \;\rhd\; \dfrac{\begin{array}{c}\Pi_1\\ A\vee B\end{array} \quad \dfrac{\begin{array}{c}\Pi_2\\ C\end{array} \quad \Pi_4}{D} \quad \dfrac{\begin{array}{c}\Pi_3\\ C\end{array} \quad \Pi_4}{D}}{D}$$

where the lowest occurrence of C is a major premiss, there is at least one occurrence of C in the sequence that is the conclusion of an introduction rule and Π_4 may be empty. Hence, (2-1) can be reduced to $\dfrac{A\vee B \quad \dfrac{\dfrac{[A] \quad D}{A\wedge D}}{A} \quad \dfrac{\dfrac{A \quad [D]}{A\wedge D}}{A}}{A}$ and then to $\dfrac{A\vee B \quad [A] \quad A}{A}$ by $\wedge$-reductions.

There exists a certain symmetry between elimination and introduction rules which is stated by the *inversion principle* (14). We quote Prawitz (15) (p.246):

> the conclusion obtained by an elimination does not state anything more than what must have already been obtained if the major premiss of the elimination was inferred by an introduction.

This principle guarantees that the semantic of a derivation does not changes with its reduction.

Another important principle, the *subformula principle*, states that every formula in a normal derivation is either a subformula of the conclusion or a subformula of an hypothesis. This principle is quite intuitive and guarantees that no formula and no operator different from the expected occur in the normal derivation.

The derivation $\dfrac{\dfrac{\perp}{A \wedge B}}{A}$ is normal according to our definition, i.e, there is no formula that is the conclusion of an introduction rule and a major premiss. However, $A \wedge B$ is of a higher degree than the surrounding formulas and it is neither a subformula of $\perp$ nor a subformula of A. Therefore, in order to preserve the inversion principle, we also say that a formula that is the conclusion of $\perp_i$ and a major premiss is a maximum formula. Thus, in a normal derivation, every rule that occurs below $\perp_i$ is of introduction.

To bring derivations with at least one occurrence of $\perp_i$ to its normal form, we cannot use any of the previous reductions, so we add the following one, where E is an elimination rule and $B \neq \perp$:

$$(\perp\text{-red}) \quad \begin{array}{c}\Pi_1 \\ \dfrac{\dfrac{\perp}{A}}{B}\,(\mathrm{E}) \\ \Pi_2\end{array} \quad \rhd \quad \begin{array}{c}\Pi_1 \\ \dfrac{\perp}{B} \\ \Pi_2\end{array}$$

This reduction is defined as follows:

$$\text{If } E = \wedge\text{-el, then} \quad \begin{array}{c}\Pi_1 \\ \dfrac{\dfrac{\perp}{A_1 \wedge A_2}}{A_i} \\ \Pi_2\end{array} \rhd \begin{array}{c}\Pi_1 \\ \dfrac{\perp}{A_i} \\ \Pi_2\end{array}, \ i = 1, 2$$

$$\text{If } E = \vee\text{-el, then} \quad \begin{array}{c}\dfrac{\begin{array}{c}\Pi_1 \\ \dfrac{\perp}{A \vee B}\end{array} \qquad \begin{array}{c}[A] \\ \Pi_2 \\ C\end{array} \qquad \begin{array}{c}[B] \\ \Pi_3 \\ C\end{array}}{C} \\ \Pi_4\end{array} \rhd \begin{array}{c}\Pi_1 \\ \dfrac{\perp}{C} \\ \Pi_4\end{array}$$

$$\text{If } E = \rightarrow\text{-el, then} \quad \begin{array}{c}\dfrac{\begin{array}{c}\Pi_1 \\ A\end{array} \qquad \begin{array}{c}\Pi_2 \\ \dfrac{\perp}{A \rightarrow B}\end{array}}{B} \\ \Pi_2\end{array} \rhd \begin{array}{c}\Pi_2 \\ \dfrac{\perp}{B} \\ \Pi_3\end{array}$$

To categorically represent the system here presented, we need some more reductions (the reason is shown in later chapters). To begin with, as from $\perp$ we can derive any formula, we can expand ($\perp$-red) to when E is an introduction rule with the restriction that, if it is an $\rightarrow$-int, it does not discharge any formula of the derivation:

$$\text{If } E = \wedge\text{-int, then} \quad \begin{array}{c}\dfrac{\begin{array}{c}\Pi_1 \\ \dfrac{\perp}{A}\end{array} \qquad \begin{array}{c}\Pi_2 \\ B\end{array}}{A \wedge B} \\ \Pi_3\end{array} \rhd \begin{array}{c}\Pi_1 \\ \dfrac{\perp}{A \wedge B} \\ \Pi_3\end{array} \text{ and } \begin{array}{c}\dfrac{\begin{array}{c}\Pi_2 \\ A\end{array} \qquad \begin{array}{c}\Pi_1 \\ \dfrac{\perp}{B}\end{array}}{A \wedge B} \\ \Pi_3\end{array} \rhd \begin{array}{c}\Pi_1 \\ \dfrac{\perp}{A \wedge B} \\ \Pi_3\end{array}$$

If $E = \vee$-int, then $\begin{array}{c}\Pi_1\\ \dfrac{\dfrac{\bot}{A}}{A \vee B}\\ \Pi_2\end{array} \rhd \begin{array}{c}\Pi_1\\ \dfrac{\bot}{A \vee B}\\ \Pi_2\end{array}$ and $\begin{array}{c}\Pi_1\\ \dfrac{\dfrac{\bot}{B}}{A \vee B}\\ \Pi_2\end{array} \rhd \begin{array}{c}\Pi_1\\ \dfrac{\bot}{A \vee B}\\ \Pi_2\end{array}$

If $E = \rightarrow$-int, then $\begin{array}{c}\Pi_1\\ \dfrac{\dfrac{\bot}{B}}{A \rightarrow B}\\ \Pi_2\end{array} \rhd \begin{array}{c}\Pi_1\\ \dfrac{\bot}{A \rightarrow B}\\ \Pi_2\end{array}$

With the introduction of these expansions we loose the unicity of normal form derivations. For example, the derivation $\dfrac{\dfrac{\dfrac{\bot}{A}}{A \vee B} \quad \dfrac{B}{A \vee B} \quad \dfrac{A}{A \vee B}}{A \vee B}$ can be reduced either to $\dfrac{\dfrac{\bot}{A}}{A \vee B}$ by the application of $\vee$-reduction or to $\dfrac{\bot}{A \vee B}$ by $\bot$-reduction applied twice. To deal with this issue we state that an $\bot$-reduction can only be applied whenever $\vee$, $\wedge$ and $\rightarrow$-reduction cannot be applied.

Then we introduce expansions, which have been envisaged by Prawitz (15) to make all minimum formulas atomic. As is the case with reduction, expansion steps form a sequence of derivations and we use the same notation, viz. $\Pi \rhd \Psi$, to signify that Π expands to Ψ ($\rhd$ can also mean a combination of reductions and expansions). We believe that the use of the same notation does not create confusion and it is interesting for practical reasons. We call *rex* either a sequence of reductions, a sequence of expansions or a combination of them. Let C be a minimum formula.

If $C = A \wedge B$, then $\begin{array}{c}\Pi_1\\ A \wedge B\\ \Pi_3\end{array} \rhd \begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\Pi_1\\ A \wedge B\end{array}}{A} \quad \dfrac{\begin{array}{c}\Pi_1\\ A \wedge B\end{array}}{B}}{A \wedge B}\\ \Pi_3\end{array}$

If $C = A \rightarrow B$, then $\begin{array}{c}\Pi_1\\ A \rightarrow B\\ \Pi_2\end{array} \rhd \begin{array}{c}\dfrac{\dfrac{[A]^1 \quad \begin{array}{c}\Pi_1\\ A \rightarrow B\end{array}}{B}}{A \rightarrow B}{\scriptstyle (1)}\\ \Pi_2\end{array}$

If $C = A \vee B$, then $\begin{array}{c}\Pi_1\\ A \vee B\\ \Pi_2\end{array} \rhd \begin{array}{c}\dfrac{\begin{array}{c}\Pi_1\\ A \vee B\end{array} \quad \dfrac{[A]}{A \vee B} \quad \dfrac{[B]}{A \vee B}}{A \vee B}\\ \Pi_2\end{array}$

With the introduction of expansions we introduce the posibility of creating infinite rex sequences. For example, a sequence that begins with the derivation $\dfrac{\dfrac{\dfrac{\begin{matrix}\Pi_1\\ A\wedge B\end{matrix}}{A} \quad \dfrac{\begin{matrix}\Pi_2\\ A\wedge B\end{matrix}}{B}}{A\wedge B}}{A}$ can be as lenghty as we want by successives applications of $\wedge$-reduction followed by $\wedge$-expansion, i.e.,

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\Pi_1\\ A\wedge B\end{matrix}}{A} \quad \dfrac{\begin{matrix}\Pi_2\\ A\wedge B\end{matrix}}{B}}{A\wedge B}}{A} \rhd \dfrac{\begin{matrix}\Pi_1\\ A\wedge B\end{matrix}}{A} \rhd \dfrac{\dfrac{\dfrac{\begin{matrix}\Pi_1\\ A\wedge B\end{matrix}}{A} \quad \dfrac{\begin{matrix}\Pi_2\\ A\wedge B\end{matrix}}{B}}{A\wedge B}}{A} \rhd \dfrac{\begin{matrix}\Pi_1\\ A\wedge B\end{matrix}}{A} \cdots$$

Note that, in this case, both the application of the reduction followed by the application of the expansion and the application of the expansion followed by the application of the reduction yelds the same result as the application of the identity reduction.

We also allow the permutation reduction to work the other way around, i.e.,

$$\dfrac{\begin{matrix}\Pi_1\\ A\vee B\end{matrix} \quad \dfrac{\begin{matrix}\Pi_2\\ C\end{matrix} \quad \Pi_4}{D} \quad \dfrac{\begin{matrix}\Pi_3\\ C\end{matrix} \quad \Pi_4}{D}}{D} \rhd \dfrac{\dfrac{\begin{matrix}\Pi_1\\ A\vee B\end{matrix} \quad \begin{matrix}\Pi_2\\ C\end{matrix} \quad \begin{matrix}\Pi_3\\ C\end{matrix}}{C} \quad \Pi_4}{D}$$

Finally, we add expansions to derivations with at least one application of $\bot_i$:

$$(\bot\text{-exp}) \quad \begin{matrix}\Pi\\ \dfrac{\bot}{B}\\ \Pi'\end{matrix} \quad \rhd \quad \begin{matrix}\Pi\\ \dfrac{\dfrac{\bot}{A}}{B}\,\mathrm{r}\\ \Pi'\end{matrix}$$

which can be expanded so that r is either an introduction or an elimination rule:
If r is an introduction rule and

$$B = A\wedge C, \text{ then } \begin{matrix}\Pi\\ \dfrac{\bot}{A\wedge C}\\ \Pi'\end{matrix} \rhd \begin{matrix}\dfrac{\dfrac{\begin{matrix}\Pi\\ \bot\end{matrix}}{A} \quad \dfrac{\begin{matrix}\Pi\\ \bot\end{matrix}}{C}}{A\wedge C}\\ \Pi'\end{matrix}$$

$$B = A\to C, \text{ then } \begin{matrix}\Pi\\ \dfrac{\bot}{A\to C}\\ \Pi'\end{matrix} \rhd \begin{matrix}\Pi\\ \dfrac{\dfrac{\bot}{C}}{A\to C}\\ \Pi'\end{matrix}$$

$$B = A \vee C, \text{ then } \begin{array}{c}\Pi\\ \dfrac{\perp}{A \vee C}\\ \Pi'\end{array} \rhd \begin{array}{c}\Pi\\ \dfrac{\dfrac{\perp}{A}}{A \vee C}\\ \Pi'\end{array} \text{ or } \begin{array}{c}\Pi\\ \dfrac{\dfrac{\perp}{C}}{A \vee C}\\ \Pi'\end{array}$$

If r is an elimination rule, then

$$(\wedge) \quad \begin{array}{c}\Pi\\ \dfrac{\perp}{B}\\ \Pi'\end{array} \rhd \begin{array}{c}\Pi\\ \dfrac{\dfrac{\perp}{A \wedge B}}{B}\\ \Pi'\end{array} \text{ or } \begin{array}{c}\Pi\\ \dfrac{\dfrac{\perp}{B \wedge A}}{B}\\ \Pi'\end{array}, \text{ for any } A;$$

$$(\rightarrow) \quad \begin{array}{c}\Pi\\ \dfrac{\perp}{B}\\ \Pi'\end{array} \rhd \begin{array}{c}\dfrac{\begin{array}{c}\Pi\\ \dfrac{\perp}{A}\end{array} \qquad \begin{array}{c}\Pi\\ \dfrac{\perp}{A \rightarrow B}\end{array}}{B}\\ \Pi'\end{array}, \text{ for any } A;$$

$$(\vee) \quad \begin{array}{c}\Pi\\ \dfrac{\perp}{B}\\ \Pi'\end{array} \rhd \begin{array}{c}\dfrac{\dfrac{\perp}{A \vee C} \qquad \begin{array}{c}\Pi\\ \dfrac{\perp}{B}\end{array} \qquad \begin{array}{c}\Pi\\ \dfrac{\perp}{B}\end{array}}{B}\\ \Pi'\end{array}, \text{ for any } A \text{ and } C$$

We say that two derivations Π and Ψ are *equivalent* if either $\Pi \rhd \Psi$ or $\Psi \rhd \Pi$. With Normalization Theorem, it is easy to prove that there is not a proof (a derivation where every hypothesis is closed) of $\perp$:

Suppose that there exists a proof of $\perp$. Then, there exists a normal proof of $\perp$ which is the minimum formula of the main branch. As $\perp$ is not inferred by an introduction rule, the only rules in the main branch of the proof are rules of elimination, which do not discharge premisses. Therefore, $\perp$ is subformula of an hypothesis that was not discharged.

We now enounce three properties which our deductive system agrees with. All reduction systems for normalizing natural deduction derivations agree with the properties below. Property (0) means that rex is transitive and properties (1) and (2) mean that we can either reduce a derivation and then apply substitution or apply substitution and then reduce the resulting derivation. Let $\rhd$ be a reduction, an expansion or a combination of both. Then, we have that:

0) If $\Pi \rhd \Pi'$ and $\Pi' \rhd \Pi''$, then $\Pi \rhd \Pi''$;

1) If $\Pi(X) \rhd \Pi'(X)$, then for all Σ, $\Pi(\Sigma) \rhd \Pi'(\Sigma)$;

2) If $\Sigma \rhd \Sigma'$ then $\Pi(\Sigma) \rhd \Pi(\Sigma')$

where $\Pi(X)$ means that X is an hypothesis of Π and $\Pi(\Sigma)$ is the result of substituting every occurrence of the formula X by a derivation Σ whose conclusion is X.

As mathematicians often deal with different proofs for a same theorem, it is natural to try to answer when two proofs (or derivations, as is our case) are equal. We now state a conjecture formulated by Prawitz (15) that we are going to call Prawitz's Conjecture. It is also known as Identity (19) and Normalization (4) Conjecture.

Conjecture: Two derivations represent the same proof (derivation) if, and only if, they are equivalent.

Such a conjecture is plausible due to the inversion principle and, although not proved, is very important in Proof Theory.

2.2 Structural reductions

In contrast to local reductions, that deal with introduction and elimination of logical operators, structural reductions work on a global level, i.e., as the name indicates it, on the structure of the derivation. We show two structural reductions, the first one due to Jan Ekman (5) and the second one due to Pereira and Haeusler (13).

2.2.1 Ekman's reduction

In (5), Ekman worked with a system **N** of Natural Deduction for naïve set theory which comprises the symbols $\{=, \in, \perp, \supset, \&, \forall, \vee, \exists\}$ and their corresponding introduction and elimination rules. He claimed that the rule used to eliminate equality in this system could hide a reduction and he used a derivation that represents the Russel Paradox to give an example of a derivation that does not have a normal form in **N** but has a normal derivation in another system. This another system may be the system **P** of propositional logic which comprises the symbols $\{\perp, \supset, \&, \vee\}$ and their corresponding introduction and elimination rules. It also has $A \Leftrightarrow B$ defined as $(A \supset B) \wedge (B \supset A)$.

With this analysis Ekman reached the following reduction schema where

D is any derivation of $\neg P$:

$$\dfrac{\dfrac{P \Leftrightarrow \neg P}{P \supset \neg P} \qquad \dfrac{\dfrac{P \Leftrightarrow \neg P}{\neg P \supset P} \quad \begin{matrix} D \\ \neg P \end{matrix}}{P}}{\neg P} \rhd \begin{matrix} D \\ \neg P \end{matrix}$$

He generalises this reduction to the following one:

$$\dfrac{\begin{matrix} \Pi_1 \\ B \to A \end{matrix} \qquad \dfrac{\begin{matrix} \Pi_2 \\ A \to B \end{matrix} \quad \begin{matrix} \Pi_3 \\ A \end{matrix}}{B}}{A} \rhd_E \begin{matrix} \Pi_3 \\ A \end{matrix} \tag{2-2}$$

Note that the derivation of the left hand side is normal according to Prawitz's definition but, intuitively, there is too much information in it, for Π_3 is already a derivation of the conclusion A.

Immediately after this reduction, Ekman defined, as follows, a more general reduction:

$$\begin{matrix} \Gamma \\ \Pi_1 \\ A \\ \Pi_2 \\ A \end{matrix} \rhd_E \begin{matrix} \Gamma' \\ \Pi_1 \\ A \end{matrix}$$

where Γ and Γ' are sets of hypothesis and $\Gamma' \subseteq \Gamma$. We use $\rhd_E$ and E-reduction to differentiate Ekman's from Prawitz's reductions. Γ' may have less formulas than Γ because, if there exists a derivation of A from Γ', then there exists a derivation of A from Γ but Γ' cannot have a formula that is not in Γ, for the addition of new hypothesis changes the semantic of the derivation. For example, $\dfrac{\dfrac{A \wedge B}{A} \quad \dfrac{\dfrac{[A]}{A \vee B}}{A \to A \vee B}}{A \vee B} \not\rhd_E \dfrac{A}{A \vee B}$ because the hypothesis o the derivation of the right side is not an hypothesis of the original derivation.

2.2.2 PH's reduction

Pereira and Haeusler defined the following reduction on an attempt to approximate Proof Theory to the categorical semantic (to be discussed in section 3.3.2):

$$\dfrac{\begin{matrix} \Pi_1 \\ \dfrac{\bot}{A} \end{matrix} \qquad \begin{matrix} \Pi_2 \\ A \to B \end{matrix}}{B} \rhd_{P-H} \begin{matrix} \Pi_1 \\ \dfrac{\bot}{B} \end{matrix}$$

we call this reduction *PH's reduction.* We believe that they have reached this reduction by considering what would be missed in (2-2) if the minor premiss of the last rule applied on the derivation of the left hand side had been inferred by $\perp_i$.

With this reduction, we can prove that, if there exist derivations from C to $\perp$, these derivations are equivalent to each other: let Π_1 and Π_2 be two such derivations. Then

$$\dfrac{\dfrac{\begin{matrix} C \\ \Pi_1 \\ \perp \end{matrix}}{A} \qquad \dfrac{\begin{matrix} C \\ \Pi_2 \\ \perp \end{matrix}}{A \to B}}{B} \tag{2-3}$$

reduces to $\dfrac{\begin{matrix} C \\ \Pi_1 \\ \perp \end{matrix}}{B}$ according to PH's reduction and to $\dfrac{\begin{matrix} C \\ \Pi_2 \\ \perp \end{matrix}}{B}$ according to $\perp$-reduction so, according to Prawitz's Conjecture, these derivations are equivalent to (2-3) and, therefore, equivalent to each other and then Π_1 and Π_2 are equivalent derivations.

2.3 Curry-Howard Isomorphism

This section gives a general idea of the history of the Curry-Howard Isomorphism, that is, which path was made to reach its enunciation. We discuss it in more detail in next chapter where we relate typed λ-calculus with Cartesian Closed Category.

There are such basic notions in logic that one take them for granted, as the process of substitution (3). The idea of combinatory logic is the analysis of an adequate foundation for those basic theories. It all seems to have began with an article (16) written by Schönfinkel, where he introduces what is now called *combinators.* Those combinators allow functions and functions values to appear as argument. Schönfinkel introduced the combinators I, C, T, Z and S that represent identity, constancy, interchange, composition and fusion functions respectively (as he called them) defined by the equations

$Ix = x;$

$Cxy = x;$

$T\phi xy = \phi yx;$

$Z\phi\chi x = \phi(\chi x)$;

$S\phi\chi x = (\phi x)(\chi x)$.

where juxtaposition is used to indicate application, and he then showed that I, T and Z can be written in function of S and C only ($I = SCC$, $T = S(ZZS)(CC)$ and $Z = S(CS)C$) and that those combinators can be used to represent any combination of variables.

Without knowing this work, Curry had started to work in this same subject (2). He worked with B, C, W and I before he knew of Schönfinkel's paper. B, C and K represent Schönfinkel's T, Z and C[1], respectively and W is called duplicator and is defined by $Wfx = fxx$, that can also be written in function of S and K (as $SS(SK)$).

To prove that any combination of variables can be written *uniquely* by means of S and K, Curry (1) used the fact that, two combinations of S and K "whose application of a series $x_0x_1x_2\ldots$ yields the same transformation, are equal" (p.383) (e.g., SK and $K(SKK)$ determine the same result).

In (2), Curry shows that those combinators can be written in a notation due to Church: the λ-calculus. He defines $\lambda x.M$ as that function whose value, for any argument a, is the result of substituting a for x in M (3). For multiple arguments, we write $\lambda x_1x_2\ldots x_n.M$ to designate $(\lambda x_1(\lambda x_2\ldots(\lambda x_{n-2}(\lambda x_{n-1}(\lambda x_n.M)))\ldots))$ and the application is indicated by juxtaposition with association to the left.

In this case, we have that:

$S \equiv \lambda xyz.xz(yz)$

$K \equiv \lambda xy.x$

$B \equiv \lambda xyz.x(yz)$

$C \equiv \lambda xyz.xzy$

$W \equiv \lambda xy.xyy$

Kleene and Rosser, in 1935, showed that there was an inconsistency in Church's and Curry's system, they realized the importance of introducing type in their theory. Type symbols are introduced recursively: there exists primitive types and, if α and β are types, then $\alpha\beta$ is a type.

[1]We will adopt Curry's notation from now on

In (3), Curry and Feys pointed out that the types of S and K (viz. $(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$ and $\alpha \to (\beta \to \alpha)$ respectively) are precisely the axioms of intuitionistic implicational logic and what they called Rule P, i.e., the rule that derives β from $\alpha\beta$ and α, can be viewed as the rule of modus ponens.

The type of a combinator (i.e., the type of a λ-term) can be found according to the rules (18)

$$\frac{}{\Gamma, x:\tau \vdash x:\tau}\,{\scriptstyle 1} \qquad \frac{\Gamma, x:\sigma \vdash M:\tau}{\Gamma \vdash \lambda x.M:\sigma \to \tau}\,{\scriptstyle 2} \qquad \frac{\Gamma \vdash M:\sigma \to \tau \qquad \Gamma \vdash N:\sigma}{\Gamma \vdash MN:\tau}\,{\scriptstyle 3}$$

where σ and τ are types, x is a variable and M and N are terms.

Thus, the type of S can be found in the following way where, to save space, we write $\alpha\beta$ instead of $\alpha \to \beta$ and Γ instead of $x:\alpha(\beta\gamma),\ y:\alpha\beta,\ z:\alpha$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash x:\alpha(\beta\gamma)}\,{\scriptstyle (1)} \qquad \dfrac{}{\Gamma \vdash z:\alpha}\,{\scriptstyle (1)}}{\Gamma \vdash xz:\beta\gamma}\,{\scriptstyle (3)} \qquad \dfrac{\dfrac{}{\Gamma \vdash y:\alpha\beta}\,{\scriptstyle (1)} \qquad \dfrac{}{\Gamma \vdash z:\alpha}\,{\scriptstyle (1)}}{\Gamma \vdash yz:\beta}\,{\scriptstyle (3)}}{\Gamma \vdash xz(yz):\gamma}\,{\scriptstyle (3)}}{x:\alpha(\beta\gamma), y:\alpha\beta \vdash \lambda z.xz(yz):\alpha\gamma}\,{\scriptstyle (2)}}{x:\alpha(\beta\gamma) \vdash \lambda yz.xz(yz):\alpha\beta(\alpha\gamma)}\,{\scriptstyle (2)}}{\vdash \lambda xyz.xz(yz):(\alpha(\beta\gamma))(\alpha\beta(\alpha\gamma))}\,{\scriptstyle (2)}$$

Compare this derivation with the proof of $(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$ in intuitionistic implicational calculus:

$$\dfrac{\dfrac{\dfrac{\dfrac{[\alpha \to (\beta \to \gamma)]^3 \qquad [\alpha]^1}{\beta \to \gamma} \qquad \dfrac{[\alpha \to \beta]^2 \qquad [\alpha]^1}{\beta}}{\dfrac{\gamma}{\alpha \to \gamma}\,{\scriptstyle 1}}}{(\alpha \to \beta) \to (\alpha \to \gamma)}\,{\scriptstyle 2}}{(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))}\,{\scriptstyle 3}$$

In (6), Howard stated a correspondence between positive implicational propositional logic ($P(\supset)$) and the combinators, and he introduced what is now known as the Curry-Howard Isomorphism:

> Given any derivation of $\Gamma \to \beta$ in $P(\supset)$ we can find a construction of $\Gamma \to \beta$ and conversely.

where a *construction* of (a term of type) $\Gamma \to \beta$ is "(...) a term F^β of type β such that for every free variable X^α occurring in F^β there is a corresponding occurrence of α in Γ". The correspondence also preserves reductions, that is to say, if a derivation Π reduces to Π', then we can find a construction of Π that reduces to a construction of Π' and conversely. Nowadays we have others correspondences between Proof Theory and Combinatorial Logic, eg., minimal propositional logic corresponds to simply typed λ-calculus, first-order logic corresponds to dependent types, second-order logic corresponds to polymorphic types, etc (18). Howard has also stated a correspondence between a typed λ-calculus and Heyting Arithmetic.