8
References

1 SELVADURAI, A. P. S. The analytical method in geomechanics. *Applied Mechanics Reviews*, v. 60, p. 87–106, 2007.

2 SELVADURAI, A. P. S.; RAJAPAKSE, R. K. N. D. On the load transfer from a rigid cylindrical inclusion into an elastic half space. *International Journal of Solids and Structures*, v. 21, n. 12, p. 1213–1229, 1985.

3 HASEGAWA, H.; LEE, V. G.; MURA, T. Green-functions for axisymmetrical problems of dissimilar elastic solids. *Journal of Applied Mechanics*, v. 59, n. 2, Part 1, p. 312–320, 1992.

4 SELVADURAI, A. P. S.; DUMONT, N. A.; OLIVEIRA, M. F. F. Mindlin's problem for a halfspace indented by a flexible circular plate. In: *Proceedings of PACAM XI - 11th Pan-American Congress of Applied Mechanics*. Foz do Iguaçu, Brazil: Submitted, 2010. p. 1–6.

5 TAN, C. L.; SELVADURAI, A. P. S. Internally indented penny-shaped cracks: A comparison of analytical and boundary integral equation estimates. *Theoretical and Applied Fracture Mechanics*, v. 5, n. 3, p. 181–188, 1986.

6 GAO, Y. L.; TAN, C. L.; SELVADURAI, A. P. S. Stress intensity factors for cracks around or penetrating an elliptic inclusion using the boundary element method. *Engineering Analysis with Boundary Elements*, v. 10, p. 59–68, 1992.

7 SELVADURAI, A. P. S. Fracture evolution during indentation of a brittle elastic solid. *Mechanics of Cohesive-Frictional Materials*, v. 5, n. 4, p. 325–339, 2000.

8 LIAO, J. J.; WANG, C. D. Elastic solutions for a transversely isotropic half-space subjected to a point load. *International Journal for Numerical and Analytical Methods in Geomechanics*, v. 22, p. 425–447, 1998.

9 WANG, C. D.; LIAO, J. J. Elastic solutions for a transversely isotropic half-space subjected to buried axisymmetric-loads. *International Journal for Numerical and Analytical Methods in Geomechanics*, v. 23, p. 115–139, 1999.

10 WANG, C. D. et al. Displacements and stresses due to a vertical point load in an inhomogeneous transversely isotropic half-space. *International Journal of Rock Mechanics & Mining Sciences*, v. 40, p. 667–685, 2003.

PUC-Rio - Certificação Digital Nº 0410745/CA

11 WIDEBERG, J.; BENITEZ, F. G. Elastic stress and displacement distribution in an orthotropic layer due to a concentrated load. *Engineering Analysis with Boundary Elements*, v. 17, n. 205–221, 1996.

12 KERMANIDIS, T. A numerical solution for axially symmetrical elasticity problems. *International Journal of Solids and Structures*, v. 11, p. 493–500, 1975.

13 HASEGAWA, H. Investigation on axisymmetric concentrated load problems of elasticity - Applications of solutions with singularities. *Bulletin of the Japan Society of Mechanical Engineers*, v. 21, n. 160, p. 1462–1468, 1978.

14 HASEGAWA, H. Greens-functions for axisymmetric body force problems of an elastic half space and their application. *Bulletin of the Japan Society of Mechanical Engineers*, v. 27, n. 231, p. 1829–1835, 1984.

15 CRUSE, T. A.; SNOW, D. W.; WILSON, R. B. Numerical solutions in axisymmetric elasticity. *Computers and Structures*, v. 7, n. 3, p. 445–451, 1977.

16 MAYR, M. On the numerical solution of axisymmetric elasticity problems using an integral approach. *Mechanics Research Communications*, v. 3, n. 5, p. 393–398, 1976.

17 RIZZO, F. J.; SHIPPY, D. J. A boundary integral approach to potential and elasticity problems for axisymmetric bodies with arbitrary boundary conditions. *Mechanics Research Communications*, v. 6, n. 2, p. 99–193, 1979.

18 RIZZO, F. J.; SHIPPY, D. J. A boundary element method for axisymmetric elastic bodies. In: BANERJEE, P. K.; WATSON, J. O. (Ed.). *Developments in Boundary Element Methods*. London, New York: Elsevier, 1986. (Developments Series, v. 4), p. 67–90.

19 PARK, K. H. A BEM formulation for axisymmetric elasticity with arbitrary body force using particular integrals. *Computers and Structures*, v. 80, n. 32, p. 2507–2514, 2002.

20 ISHIDA, R.; OCHIAI, Y. On boundary integral equation formulation for axisymmetric problems of transversely isotropic media. *Archive of Applied Mechanics*, v. 61, p. 414–421, 1991.

21 BAKR, A. A.; FENNER, R. T. Boundary integral equation analysis of axisymmetric thermoelastic problems. *Journal of Strain Analysis*, v. 18, n. 4, p. 239–251, 1983.

22 CATHIE, D. N.; BANERJEE, P. K. Numerical solutions in axisymmetric elastoplasticity. In: AL., R. S. et (Ed.). *Innovative Numerical Analysis for the Engineering Sciences*. Charlottesville, VA: University of Virginia Press, 1980. p. 331.

23 SARIHAN, V.; MUKHERJEE, S. Axisymmetric viscoplastic deformation by the boundary element method. *International Journal of Solids and Structures*, v. 18, n. 12, p. 1113–1128, 1982.

PUC-Rio - Certificação Digital Nº 0410745/CA

24 WANG, H. C.; BANERJEE, P. K. Generalized axisymmetric elastodynamic analysis by boundary element method. *International Journal for Numerical Methods in Engineering*, v. 30, p. 115–131, 1990.

25 WANG, H. C.; BANERJEE, P. K. Axisymmetric transient elastodynamic analysis by boundray element method. *International Journal of Solids and Structures*, v. 26, p. 401–415, 1990.

26 TSINOPOULOS, S. V. et al. An advanced boundary element method for axisymmetric elastodynamic analysis. *Computational Methods in Applied Mechanics and Engineering*, v. 175, n. 1–2, p. 53–57, 1999.

27 YANG, Z.; ZHOU, H. A boundary element method in elastodynamics for geometrically axisymmetric problems. *Applied Mathematical Modelling*, v. 23, p. 501–508, 1999.

28 BAKR, A. A.; FENNER, R. T. Axisymmetric fracture mechanics analysis by the boundary integral equation method. *International Journal of Pressure Vessels and Piping*, v. 18, n. 1, p. 55–75, 1985.

29 ABDUL-MIHSEIN, M. J.; BAKR, A. A.; PARKER, A. P. A boundary integral equation method for axisymmetric elastic contact problems. *Computers and Structures*, v. 33, p. 787–793, 1986.

30 DOMÍNGUEZ, J. Twenty five years of boundary elements for dynamic soil-structure interaction. In: HALL, W. S.; OLIVETO, G. (Ed.). *Boundary Element Methods for Soil-Structure Interaction*. Dordrecht: Kluwer Academic Publishers, 2003. cap. 1.

31 WATSON, J. O. Advanced implementation of the boundary element method for two- and three-dimensional elastostatics. In: BANERJEE, P. K.; BUTTERFIELD, R. (Ed.). *Developments in Boundary Element Methods*. London: Applied Science, 1979. v. 1, p. 31–63.

32 BEER, G.; WATSON, J. O. Infinite boundary elements. *International Journal for Numerical Methods in Engineering*, v. 28, n. 6, p. 1233–1247, 1989.

33 LIU, M.; FARRIS, T. N. Three-dimensional infinite boundary elements for contact problems. *International Journal for Numerical Methods in Engineering*, v. 36, p. 3381–3398, 1993.

34 DAVIES, T. G.; BU, S. Infinite boundary elements for the analysis of halfspace problems. *Computers and Geotechniques*, v. 19, n. 2, p. 137–151, 1996.

35 BU, S. 3D boundary element analysis of axisymmetric halfspace problems. *Engineering Analysis with Boundary Elements*, v. 17, p. 75–84, 1996.

36 GAO, X.; DAVIES, T. G. 3-D infinite boundary elements for half-space problems. *Engineering Analysis with Boundary Elements*, v. 21, p. 207–213, 1998.

PUC-Rio - Certificação Digital Nº 0410745/CA

37 LIANG, J.; LIEW, M. Boundary elements for half-space problems via fundamental solutions: A three-dimensional analysis. *International Journal for Numerical Methods in Engineering*, v. 52, p. 1189–1202, 2001.

38 TELLES, J. C. F.; BREBBIA, C. A. Boundary element solution for half-plane problems. *International Journal of Solids and Structures*, v. 17, n. 12, p. 1149–1158, 1981.

39 DUMIR, P. C.; MEHTA, A. K. Boundary element solution for elastic orthotropic half-plane problems. *Computers and Structures*, v. 26, n. 3, p. 431–438, 1987.

40 OLIVEIRA, M. F. F. de; SELVADURAI, A. P. S.; DUMONT, N. A. Implementação da solução fundamental de elasticidade para problemas axissimétricos no semi-espaço no método dos elementos de contorno. In: *Proceedings of CMNE/CILAMCE 2007 — Congresso de Métodos Numéricos em Engenharia and XXVIII Iberian Latin-American Congress on Computational Methods in Engineering*. Porto: APMTAC/FEUP, 2007. p. 1–20.

41 DUMONT, N. A. The variational formulation of the boundary element method. In: BREBBVIA, C. A.; VENTURINI, W. A. (Ed.). *Boundary Element Techniques: Applications in Fluid Flow and Computational Aspects*. Southampton: Adlard and Son Ltd., 1987. p. 225–239.

42 DUMONT, N. A. The hybrid boundary element method: An alliance between mechanical consistency and simplicity. *Applied Mechanics Reviews*, v. 42, n. 11, part 2, p. S54–S63, 1989.

43 DUMONT, N. A. Variationally-based, hybrid boundary element methods. *Computer Assisted Mechanics and Engineering Sciences*, v. 10, p. 407–430, 2003.

44 PIAN, T. H. H. Element stiffness matrices for boundary compatibility and for prescribed boundary stresses. In: *Proceedings of the Conference on Matrix Methods in Structural Mechanics*. Ohio: Wright Patterson Air Force Base, 1966. (AFFDL-TR-66-80), p. 457–477.

45 DUMONT, N. A.; OLIVEIRA, R. From frequency-depencent mass and stiffness matrices to the dynamic response of elastic systems. *International Journal of Solids and Structures*, v. 38, p. 1813–1830, 2001.

46 DUMONT, N. A.; COSSIO, M. U. de L. Q. Sensitivity analysis with the hybrid boundary element method. *Building Research Journal*, v. 49, p. 35–58, 2001.

47 DUMONT, N. A.; CHAVES, R. A. General time-dependent analysis with the frequency-domain hybrid boundary element method. *Computer Assisted Mechanics and Engineering Sciences*, n. 10, p. 431–452, 2003.

48 DUMONT, N. A.; LOPES, A. A. O. On the explicit evaluation of stress intensity factor in the hybrid boundary element method. *Fatigue & Fracture of Engineering Materials & Structures*, v. 26, p. 161–165, 2003.

PUC-Rio - Certificação Digital Nº 0410745/CA

49 DUMONT, N. A.; CHAVES, R. A. P.; PAULINO, G. H. The hybrid boundary element method applied to problems of potential theory in nonhomogeneous materials. *International Journal of Computational Engineering Science*, v. 5, n. 4, p. 863–891, 2004.

50 DUMONT, N. A.; CHAVES, R. A. P. The simplified hybrid boundary element method. In: *Proceedings of XX CILAMCE — XX Iberian Latin-American Congress on Computational Methods in Engineering*. São Paulo: EPUSP, 1999. p. 1–20.

51 CHAVES, R. A. P. *Estudo do Método Híbrido dos Elementos de Contorno e Proposta de uma Formulação Simplificada*. Dissertação (Master Thesis) — Pontifical Catholic University of Rio de Janeiro, Rio de Janeiro, 1999. Supervisor: N. A. Dumont.

52 CHAVES, R. A. P. *O Método Híbrido Simplificado dos Elementos de Contorno aplicado a Problemas Dependentes do Tempo*. Dissertação (PhD Thesis) — Pontifical Catholic University of Rio de Janeiro, Rio de Janeiro, 2003. Supervisor: N. A. Dumont.

53 DUMONT, N. A.; CHAVES, R. A. P. The simplified hybrid boundary element method applied to general transient problems. In: *Proceedings of XXI CILAMCE — XXI Iberian Latin-American Congress on Computational Methods in Engineering*. Rio de Janeiro: UFPA, 2000. p. 1–20.

54 DUMONT, N. A.; CHAVES, R. A. P. Simplified hybrid boundary element method applied to general time-dependent problems. In: *Computational Mechanics — New Frontiers for the New Millenium*. Sydney: Elsevier, 2001. (Proceedings of the First Asian Pacific Congress on Computational Mechanics, v. 1), p. 1009–1018.

55 DUMONT, N. A.; CHAVES, R. A. P. Simplified hybrid boundary element method for the transient heat conduction analysis of functionally graded materials. In: *Proceedings of XXIV CILAMCE — XXIV Iberian Latin Amercian Congress on Computational Methods in Engineering*. Ouro Preto: UFOP, 2003. p. 1–15.

56 DUMONT, N. A.; OLIVEIRA, M. F. F.; LOPES, A. A. O. Sensitivity analysis with the simplified hybrid boundary element method. In: *Proceedings of WCSMO6 - 6th World Congress on Structural and Multidisciplinary Optimization*. Rio de Janeiro: COPPE/UFRJ, 2005. p. 1–10.

57 DUMONT, N. A.; OLIVEIRA, M. F. F.; CHAVES, R. A. P. Conventional, hybrid and simplified boundary element methods. In: *26th World Conference on Boundary Elements and other Mesh Reduction Methods*. Bologna, Italy: WIT Press, 2004. (International Series on Advances in Boundary Elements, v. 19), p. 23–33.

58 OLIVEIRA, M. F. F. *Métodos de Elementos de Contorno Convencional, Híbridos e Simplificados*. Dissertação (Master Thesis) — Pontifical Catholic University of Rio de Janeiro, Brazil, 2004. In Portuguese.

PUC-Rio - Certificação Digital Nº 0410745/CA

59 DUMONT, N. A. Linear algebra aspects in the equilibrium-based implementaton of finite/boundary element methods for FGMs. In: *Proceedings of FGM2006 — Multiscale and Functionally Graded Materials Conference 2006*. Honolulu: AIP Conference Proceedings, 2008. v. 973, p. 658–663.

60 OLIVEIRA, M. F. F. de; DUMONT, N. A. Conceptual completion of the simplified hybrid boundary element method. In: *Proceedings of BETEQ 2009 - International Conference on Boundary Element Techniques*. Athens: Elsevier, 2009. p. 1–6.

61 SOUTAS-LITTLE, R. W. *Elasticity*. Mineola, New York: Dover Pulbications, Inc., 1999.

62 SHIPPY, D. J.; RIZZO, F. J.; NIGAM, R. K. A boundary integral equation method for axisymmetric elastostatic bodies under arbitrary surface loads. In: *Proceedings of the 2nd International Symposium on Innovative Numerical Analysis in Applied Engineering Sciences*. Montreal: Unversity Press of Virginia, 1980. p. 423–432.

63 HASEGAWA, H. Axisymmetric body force problems of an elastic half-space. *Bulletin of the Japan Society of Mechanical Engineers*, v. 19, n. 137, p. 1262–1269, 1976.

64 MUKI, R. Asymmetric problems of the theory of elasticity ofr a semi-Infinite solid and a thick plate. In: SNEDDON, N.; HILL, R. (Ed.). *Progress in Solid Mechanics*. 1. ed. Amsterdam: North-Holland Publ. Co., 1960. v. 1, cap. 8.

65 SNEDDON, I. N. On Muki's solution of the equations of linear elasticity. *International Journal of Engineering Science*, v. 30, n. 10, p. 1237–1246, 1992.

66 HANSON, M. T.; WANG, Y. Concentrated ring loadings in a full space or half space: Solutions for transverse isotropy and isotropy. *International Journal of Solids and Structures*, v. 34, n. 11, p. 1379–1418, 1997.

67 EASON, G.; NOBLE, B.; SNEDDON, I. N. On certain integrals of Lipschitz-Hankel type involving products of Bessel functions. *Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences*, v. 247, n. 935, p. 529–551, 1955.

68 ABRAMOWITZ, M.; STEGUN, I. A. Applied mathematics series. In: *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*. Washington: National Bureau of Standards of the United States of America, 1972. v. 55.

69 FRIEDMAN, A. *Generalized Functions and Partial Differential Equations*. Englewood Cliffs: Prentice-Hall, 1963.

70 BRACEWELL, R. N. *The Fourier Transform and Its Applications*. New York: McGraw-Hill, 1986.

PUC-Rio - Certificação Digital Nº 0410745/CA

71 DUMONT, N. A. An assessment of the spectral properties of the matrix G used in the boundary element methods. *Computational Mechanics*, v. 22, n. 1, p. 32–41, 1998.

72 CRUSE, T. A. An improved boundary-integral equation method for three-dimensional elastic stress analysis. *Computers and Structures*, v. 4, n. 4, p. 741–754, 1974.

73 BREBBIA, C. A.; TELLES, J. C. F.; WROBEL, L. C. *Boundary Element Techniques: Theory and Applications in Engineering*. Berlin-Heidelberg-New York-Tokyo: Springer-Verlag, 1984.

74 ALIABADI, M. H. *The Boundary Element Method - Applications in Solids and Structures*. London: John Wiley & Sons, LTD, 2002.

75 BEN-ISRAEL, A.; GREVILLE, T. N. E. *Generalized Inverses: Theory and Applications*. Huntington: R. E. Krieger, 1980.

76 BOTT, R.; DUFFIN, R. J. On the algebra of networks. *Transactions of the American Mathematical Society*, v. 74, p. 99–109, 1953.

77 OLIVEIRA, M. F. F.; DUMONT, N. A. On a mesh-reduced boundary element formulation. In: *Proceedings of XXV CILAMCE — XXV Iberian Latin American Congress on Computational Methods in Engineering*. Recife: UFPE, 2004. p. 1–19.

78 HARTMANN, F. Computing the C-matrix in non-smooth boundary points. In: BREBBIA, C. A. (Ed.). *New Developments in Boundary Element Methods*. [S.l.]: CML Southampton, 1983. p. 367–379.

79 BAKR, A. A.; FENNER, R. T. Treatment of singular integrals in the boundary integral equation method for axisymmetric elastostatics problems. *Communications in Applied Numerical Methods*, v. 1, n. 2, p. 81–84, 1985.

80 TAN, C. T. Interior point solutions of boundary integral equation axisymmetric stress analysis. *Applied Mathematical Modelling*, v. 8, n. 1, p. 57–60, 1984.

81 TAN, C. L. Letter to the editor. *Applied Mathematical Modelling*, v. 8, n. 4, p. 293, 1984.

82 LACERDA, L. A. de; WROBEL, L. C. Hypersingular boundary integral equation for axisymmetric elasticity. *International Journal for Numerical Methods in Engineering*, v. 52, p. 1337–1354, 2001.

83 MUKHERJEE, S. Regularization of hypersingular boundary integral equations: A new approach for axisymmetric elasticity. *Engineering Analysis with Boundary Elements*, v. 26, p. 839–844, 2002.

84 DUMONT, N. A. On the spectral properties of the double layer potential matrix H of the boundary element methods. In: *Proceedings of PACAM XI - 11th Pan-American Congress of Applied Mechanics*. Foz do Iguaçu, Brazil: Submitted, 2010. p. 1–6.

PUC-Rio - Certificação Digital Nº 0410745/CA

85 GAUL, L.; WAGNER, M.; WENZEL, W. Efficient field point evaluation by combined direct and hybrid boundary element methods. *Engineering Analysis with Boundary Elements*, v. 21, p. 215–222, 1998.

86 STROUD, A. H.; SECREST, D. *Gaussian Quadrature Formulas*. New York: Prentice-Hall, 1966.

87 BIALECK, R. A. et al. Weakly singular 2D quadratures for some fundamental solutions. *Engineering Analysis with Boundary Elements*, v. 18, n. 4, p. 333–336, 1996. Technical Note.

88 DUMONT, N. A.; SOUZA, R. M. A simple, unified technique for the evaluation of quasi-singular, singular and strongly singular integrals. In: BREBBIA, C. A.; DOMINGUEZ, J.; PARIS, F. (Ed.). *Boundary Elements XIV – Field Problems and Applications*. Southampton: Computational Mechanics Publications, 1992. v. 1, p. 619–632.

89 TIMOSHENKO, S. P.; GOODIER, J. N. *Theory of Elasticity*. Third edition. New York: McGraw-Hill, 1970.

90 DUMONT, N. A.; NORONHA, M. A simple, accurate scheme for the numerical evaluation of integrals with complex singularity poles. *Computational Mechanics*, v. 22, p. 42–49, 1998.

91 SELVADURAI, A. P. S. *Elastic Analysis of Soil-Foundation Interaction*. Amsterdam: Elsevier, 1979. (Developments in Geotechnical Engineering, v. 17).

92 MILOVIC, D. *Stresses and Displacements for Shallow Foundations*. Amsterdam: Elsevier, 1992. (Developments in Geotechnical Engineering, v. 70).

93 CARLSON, B. C. Computing elliptic integrals by duplication. *Numerical Mathematics*, v. 33, n. 1, p. 1–16, 1979.

94 CARLSON, B. C. Three improvements in reduction and computation of elliptic integrals. *Journal of Research of the National Institute of Standards and Technology*, v. 107, n. 5, p. 413–418, 2002.

95 KUTT, H. R. *On the Numerical Evaluation of Finite Part Integrals Involving an Algebraic Singularity*. Pretoria, 1975.

PUC-Rio - Certificação Digital Nº 0410745/CA

A
Integrals of Lipschitz-Hankel type involving products of Bessel functions

The integrals of Lipschitz-Hankel type involving products of Bessel functions can be represented by

$$I_{pq\lambda}(\xi, r; c) = \int_0^\infty J_p(\xi t)\, J_q(rt)\, e^{-ct}\, t^\lambda\, \mathrm{d}t \tag{A-1}$$

where p, q and λ are integers; and $J_p(\xi t)$ and $J_q(rt)$ are Bessel functions of the first kind of order p and q, respectively. The convergent integrals of this type were tabulated by Eason et al. [67]. The expressions used in the previous chapters are

$$I_{000} = \frac{2k}{\pi A_1} K(m) \tag{A-2}$$

$$I_{110} = -\frac{2(k^2-2)}{\pi k A_1} K(m) - \frac{4}{\pi k A_1} E(m) \tag{A-3}$$

$$I_{100} = \begin{cases} -\frac{k c}{2\pi\xi\sqrt{\xi r}} K(m) - \frac{\Lambda_0(n,m)}{2\xi} + \frac{1}{\xi} & \text{if} \quad \xi > r \\ -\frac{k c}{2\pi\xi^2} K(m) + \frac{1}{2\xi} & \text{if} \quad \xi = r \\ -\frac{k c}{2\pi\xi\sqrt{\xi r}} K(m) + \frac{\Lambda_0(n,m)}{2\xi} & \text{if} \quad \xi < r \end{cases} \tag{A-4}$$

$$I_{001} = \frac{2\, c\, k^3}{\pi \bar{k}^2 A_1^3} E(m) \tag{A-5}$$

$$I_{111} = -\frac{4\, c\, k}{\pi A_1^3} K(m) - \frac{2\, c\, k\,(k^2-2)}{\pi \bar{k}^2 A_1^3} E(m) \tag{A-6}$$

$$I_{101} = \frac{k}{\pi \xi\, A_1} K(m) + \frac{k^3 A_2}{\pi \xi \bar{k}^2 A_1^3} E(m) \tag{A-7}$$

$$I_{002} = -\frac{2\, c^2 k^5}{\pi \bar{k}^2 A_1^5} K(m) - \frac{2\, k^3}{\pi \bar{k}^2 A_1^3}\left[1 + \frac{2\, c^2 k^2 (k^2-2)}{\bar{k}^2 A_1^2}\right] E(m) \tag{A-8}$$

$$I_{112} = \frac{2\, k}{\pi A_1^3}\left[2 + \frac{c^2 k^2 (k^2-2)}{\bar{k}^2 A_1^2}\right] K(m) + \frac{2\, k}{\pi \bar{k}^2 A_1^3}\left[k^2 - 2 + \frac{2\, c^2 k^2 (k^4 + \bar{k}^2)}{\bar{k}^2 A_1^2}\right] E(m) \tag{A-9}$$

$$I_{102} = -\frac{c\, k^5 A_2}{\pi \xi \bar{k}^2 A_1^5} K(m) + \frac{c\, k^3}{\pi \xi \bar{k}^2 A_1^3}\left[3 - \frac{2\, k^2 A_2 (k^2-2)}{\bar{k}^2 A_1^2}\right] E(m) \tag{A-10}$$

PUC-Rio - Certificação Digital Nº 0410745/CA

$$I_{003} = \frac{2\,c\,k^5}{\pi\,\bar{k}^2\,A_1^5}\left[3 + \frac{4\,c^2\,k^2\,(k^2-2)}{\bar{k}^2\,A_1^2}\right]K(m) - \frac{2\,c\,k^5}{\pi\,\bar{k}^4\,A_1^5}\left[-6(k^2-2) - \frac{c^2\,k^2}{A_1^2}\left(\frac{8\,k^4}{\bar{k}^2} + 23\right)\right]E(m) \tag{A-11}$$

$$I_{113} = -\frac{2\,c\,k^3}{\pi\,\bar{k}^2\,A_1^5}\left[3(k^2-2) + \frac{2c^2\,k^2}{A_1^2}\left(\frac{2k^4}{\bar{k}^2} + 3\right)\right]K(m) - \frac{2c\,k^3}{\pi\,\bar{k}^4\,A_1^5}\left[6\,(k^4+\bar{k}^2) + \frac{c^2\,k^2\,(k^2-2)\,(3\bar{k}^2+8k^4)}{\bar{k}^2\,A_1^2}\right]E(m) \tag{A-12}$$

$$I_{103} = \frac{k^5}{\pi\,\xi\,\bar{k}^2\,A_1^5}\left[A_2 - 5c^2 + \frac{4\,k^2\,(k^2-2)\,c^2\,A_2}{\bar{k}^2\,A_1^2}\right]K(m) + \frac{k^3}{\pi\,\xi\,\bar{k}^2\,A_1^3}\left[-3 + \frac{2\,k^2\,(k^2-2)\,(A_2-5c^2)}{\bar{k}^2\,A_1^2} + \frac{c^2\,k^4\,A_2}{\bar{k}^2\,A_1^4}\left(\frac{8\,k^4}{\bar{k}^2} + 23\right)\right]E(m) \tag{A-13}$$

$$I_{004} = -\frac{2\,k^5}{\pi\,\bar{k}^2\,A_1^5}\left[3 + \frac{24\,c^2\,k^2\,(k^2-2)}{\bar{k}^2\,A_1^2} + \frac{c^4\,k^4\,(24k^4+41\bar{k}^2)}{\bar{k}^4\,A_1^4}\right]K(m) - \frac{4k^5}{\pi\,\bar{k}^4\,A_1^5}\left[3(k^2-2) + \frac{3\,c^2\,k^2\,(8k^4+23\bar{k}^2)}{\bar{k}^2\,A_1^2} + \frac{4\,c^4\,k^4\,(k^2-2)\,(6\,k^4+11\bar{k}^2)}{\bar{k}^4\,A_1^4}\right]E(m) \tag{A-14}$$

$$I_{114} = \frac{6\,k^3}{\pi\,\bar{k}^2\,A_1^5}\left[k^2 - 2 + \frac{4c^2\,k^2\,(2k^4+3\bar{k}^2)}{\bar{k}^2\,A_1^2} + \frac{c^4\,k^4\,(k^2-2)\,(8k^4+5\bar{k}^2)}{\bar{k}^4\,A_1^4}\right]K(m) + \frac{12k^3}{\pi\,\bar{k}^4\,A_1^5}\left[k^2 + \bar{k}^4 + \frac{c^2\,k^2\,(k^2-2)\,(3\bar{k}^2+8k^4)}{\bar{k}^2\,A_1^2} + \frac{c^4\,k^4\,(8-4\bar{k}^2-3\bar{k}^4-4\bar{k}^6+8\bar{k}^8)}{\bar{k}^4\,A_1^4}\right]E(m) \tag{A-15}$$

$$I_{104} = -\frac{c\,k^5}{\pi\,\xi\,\bar{k}^2\,A_1^5}\left[-15 + \frac{4\,k^2\,(k^2-2)\,(3A_2-7c^2)}{\bar{k}^2\,A_1^2} + \frac{c^2\,k^4\,A_2\,(24k^4+71\bar{k}^2)}{\bar{k}^4\,A_1^4}\right]K(m) - \frac{c\,k^5}{\pi\,\xi\,\bar{k}^4\,A_1^5}\left[-30(k^2-2) + \frac{k^2\,(8k^4+23\bar{k}^2)\,(3A_2-7c^2)}{\bar{k}^2\,A_1^2} + \frac{8c^3\,k^4\,A_2\,(k^2-2)\,(6k^4+11\bar{k}^2)}{\bar{k}^4\,A_1^4}\right]E(m) \tag{A-16}$$

in which $I_{pq\lambda}(\xi, r; c) = I_{qp\lambda}(r, \xi; c)$ and

$$A_1 = 2\sqrt{\xi\,r}, \qquad A_2 = \xi^2 - r^2 - c^2, \qquad A_3 = -\xi^2 + r^2 - c^2 \tag{A-17}$$

PUC-Rio - Certificação Digital Nº 0410745/CA

In the above expressions, $K(m)$ and $E(m)$ are the complete elliptic integrals of the first and second kinds, respectively,

$$K(m) = \int_0^{\pi/2} (1 - m\,\sin\theta^2)^{-1/2}\mathrm{d}\theta \tag{A-18}$$

$$E(m) = \int_0^{\pi/2} (1 - m\,\sin\theta^2)^{1/2}\mathrm{d}\theta \tag{A-19}$$

the modulus k, the complementary modulus $\bar{k}$ and the parameter m are given by

$$k = \frac{2\sqrt{\xi\, r}}{\sqrt{(\xi + r)^2 + c^2}}, \qquad \bar{k} = \sqrt{1 - k^2} \qquad \text{and} \qquad m = k^2 \tag{A-20}$$

In Eq. (A-4), $\Lambda_0(n, m)$ is the Heuman complete elliptic integral expressed as

$$\Lambda_0(n, m) = \frac{2}{\pi}\left[\sqrt{1 - n}\,\sqrt{1 - \frac{m}{n}}\,\Pi(n, m)\right] \tag{A-21}$$

where $\Pi(n, m)$ is the complete elliptic integral of the third kind defined as

$$\Pi(n, m) = \int_0^{\pi/2} (1 - n\,\sin\theta^2)^{-1}\,(1 - m\,\sin\theta^2)^{-1/2}\mathrm{d}\theta \tag{A-22}$$

and n is the characteristic number

$$n = \frac{A_1^2}{(\xi + r)^2} \tag{A-23}$$

Notice that all Lipschitz-Hankel integrals $I_{pq\lambda}(\xi, r; c)$ listed above are written in terms of $K(m)$, $E(m)$ and $\Pi(m)$, which can be numerically evaluated by duplication as proposed by Carlson [93, 94].

In the following, some useful limits are given, for $\rho_0 = \sqrt{r^2 + c^2}$:

$$\lim_{\xi\to 0} I_{000} = \frac{1}{\rho_0} \tag{A-24}$$

$$\lim_{\xi\to 0} I_{110}/\xi = \frac{r}{2\rho^3} \tag{A-25}$$

$$\lim_{\xi\to 0} I_{100}/\xi = \frac{c}{2\rho_0^3} \tag{A-26}$$

$$\lim_{\xi\to 0} I_{010} = \frac{c - \rho_0}{r\,\rho_0} \tag{A-27}$$

$$\lim_{\xi\to 0} I_{001} = \frac{c}{\rho_0^3} \tag{A-28}$$

$$\lim_{\xi\to 0} I_{111}/\xi = \frac{3rc}{2\rho_0^5} \tag{A-29}$$

$$\lim_{\xi\to 0} I_{101}/\xi = -\frac{r^2 - 2c^2}{2\rho_0^5} \tag{A-30}$$

$$\lim_{\xi\to 0} I_{011} = \frac{r}{\rho_0^3} \tag{A-31}$$

$$\lim_{\xi\to 0} I_{002} = -\frac{r^2 - 2c^2}{\rho_0^5} \tag{A-32}$$

$$\lim_{\xi\to 0} I_{112}/\xi = -\frac{3r\,(r^2 - 4c^2)}{2\rho_0^7} \tag{A-33}$$

$$\lim_{\xi\to 0} I_{102}/\xi = -\frac{3c\,(3r^2 - 2c^2)}{2\rho_0^7} \tag{A-34}$$

$$\lim_{\xi\to 0} I_{012} = \frac{3rc}{\rho_0^5} \tag{A-35}$$

$$\lim_{\xi\to 0} I_{003} = -\frac{3c\,(3r^2 - 2c^2)}{\rho_0^7} \tag{A-36}$$

$$\lim_{\xi\to 0} I_{113}/\xi = -\frac{15rc\,(3r^2 - 4c^2)}{2\rho_0^9} \tag{A-37}$$

PUC-Rio - Certificação Digital Nº 0410745/CA

$$\lim_{\xi\to 0} I_{103}/\xi = \frac{3\,(3r^4 - 24r^2c^2 + 8c^4)}{2\rho_0^9} \quad \text{(A-38)}$$

$$\lim_{\xi\to 0} I_{013} = -\frac{3r\,(r^2 - 4c^2)}{\rho_0^7} \quad \text{(A-39)}$$

$$\lim_{\xi\to 0} I_{004} = \frac{3(3r^4 - 24r^2c^2 + 8c^4)}{\rho_0^9} \quad \text{(A-40)}$$

$$\lim_{\xi\to 0} I_{114}/\xi = \frac{45r\,(r^4 - 12r^2c^2 + 8c^4)}{2\rho_0^{11}} \quad \text{(A-41)}$$

$$\lim_{\xi\to 0} I_{104}/\xi = \frac{15c\,(15r^4 - 40r^2c^2 + 8c^4)}{2\rho_0^{11}} \quad \text{(A-42)}$$

$$\lim_{\xi\to 0} I_{014} = -\frac{15cr\,(3r^2 - 4c^2)}{\rho_0^9} \quad \text{(A-43)}$$

The expressions listed in Eqs. (A-2) to (A-16) and Eqs. (A-24) to (A-43) are part of the integrand of the boundary integrals arising in the formulations presented in this work. Depending on the value of m and ρ_0, some of these integrals become weakly and others strongly singular, as dealt with in Appendix B.

PUC-Rio - Certificação Digital Nº 0410745/CA

B
Numerical integration

This appendix presents the numerical schemes used to evaluate the integrals arising in the boundary element formulations for axisymmetric problems. As only the meridian of the axisymmetric boundary needs to be discretized, these integrals are calculated along the boundary $\Gamma(r,z)$, for each portion between consecutive nodes of an element.

B.1
Regular integral

Let $f(r,z)$ be a regular function on Γ, in the sense that it can be approximated by a polynomial of a not too high degree in the domain of interest. Then, its integral can be expressed in a natural coordinate system η in the interval $[-1,1]$ and approximated by the Gauss-Legendre quadrature rule [86], arriving at

$$\int_\Gamma f(r,z)\,\mathrm{d}\Gamma = \int_{-1}^{1} f(\eta)J(\eta)\,\mathrm{d}\eta \cong \sum_{m=1}^{n_g} \left[f(\eta)\,J(\eta)\right]_{\eta=\eta_m^g}\, w_m^g \qquad \text{(B-1)}$$

where

$$J(\eta) = \sqrt{\left(\frac{\mathrm{d}r}{\mathrm{d}\eta}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}\eta}\right)^2} \qquad \text{(B-2)}$$

is the Jacobian transformation between the global and natural coordinate systems. The coefficients η_m^g and w_m^g are the abscissas and weights of the Gauss-Legendre quadrature rule for n_g points within the interval $(-1,1)$, which suffice to exactly evaluate the integral of a polynomial of order $2n_g - 1$.

B.2
Weakly singular integral of logarithmic terms

Let $f(r,z)$ be a regular function and $\rho(r,z)$ the distance between the points $\mathrm{P}(\xi,z')$ and $\mathrm{Q}(r,z)$ on the boundary $\Gamma(r,z)$. One is concerned with the evaluation of the following weakly singular integral

$$\int_\Gamma f(r,z)\,\ln\rho(r,z)\,\mathrm{d}\Gamma = \int_{-1}^{1} f(\eta)\,\ln\rho(\eta)\,J(\eta)\,\mathrm{d}\eta \qquad \text{(B-3)}$$

PUC-Rio - Certificação Digital Nº 0410745/CA

for the case $\rho(-1) = 0$ or $\rho(1) = 0$. A unified treatment of both cases may be obtained by expressing

$$\rho(\eta) = \bar{\rho}(\eta, \eta')\,(1 - \eta'\,\eta) \tag{B-4}$$

where η' is equal to either -1 or 1 and $\bar{\rho}(\eta, \eta')$ is the non-vanishing part of $\rho(\eta)$ for $\eta \in (-1, 1)$. Then, the integral of Eq. (B-3) may be decomposed as [74]

$$\int_\Gamma f(r, z)\, \ln \rho(r, z)\, \mathrm{d}\Gamma = \int_{-1}^{1} f(\eta)\, \ln[2\bar{\rho}(\eta)]\, J(\eta)\, \mathrm{d}\eta + 2 \int_0^1 f(\tilde{\eta})\, \ln \tilde{\eta}\, J(\tilde{\eta})\, \mathrm{d}\tilde{\eta} \tag{B-5}$$

in which the transformation to the natural coordinate system $\tilde{\eta} \in [0, 1]$ is given by

$$\tilde{\eta} = \frac{1}{2}(1 - \eta'\,\eta) \tag{B-6}$$

The resulting integrals can be approximated by the Gauss-Legendre and logarithmic weighted Gauss quadratures rules [86], leading to

$$\int_\Gamma f(r, z)\, \ln \rho(r, z)\, \mathrm{d}\Gamma \cong \sum_{m=1}^{n_g} [f(\eta)\, \ln[2\bar{\rho}(\eta)]\, J(\eta)]_{\eta=\eta^g_m}\, w^g_m + \sum_{m=1}^{n_l} [f(\tilde{\eta})\, \ln \tilde{\eta}\, J(\tilde{\eta})]_{\tilde{\eta}=\eta^l_m}\, w^l_m \tag{B-7}$$

The coefficients η^l_m and w^l_m are the abscissas and weights of the logarithmic weighted Gauss quadrature rule for n_l points within the interval (0, 1), which suffice to exactly evaluate the integral of a polynomial of order $n_l - 1$.

The above integration scheme is obtained from a transformation of variables and the use of Gauss-Legendre and logarithmic weighted Gauss quadratures rules. However, other approaches can also be employed, such as performing a transformation of variables so that the singular terms vanish, or regularizing the kernels to be evaluated in the numerical scheme with analytical integration of the regular part [74].

PUC-Rio - Certificação Digital Nº 0410745/CA

B.2.1
Weakly singular integral of terms with the complete elliptic integral of the first order

Let $f(r, z)$ be a regular function and $K(m)$ the complete elliptic integral of the first order given by Eq. (A-18), of modulus

$$m = \frac{4\xi r}{(\xi + r)^2 + (z' - z)^2} \tag{B-8}$$

given in terms of the coordinates of points $\mathrm{P}(\xi, z')$ and $\mathrm{Q}(r, z)$ on the boundary $\Gamma(r, z)$. One is concerned with the evaluation of the following weakly singular integral

$$\int_\Gamma f(r, z)\, K(m)\, \mathrm{d}\Gamma = \int_{-1}^{1} f(\eta)\, K(m)\, J(\eta)\, \mathrm{d}\eta \tag{B-9}$$

which actually encompasses to singularity in the case of $K(m) = \infty$ since $m = 1$ for either $\eta = -1$ or $\eta = 1$. The integration scheme to be presented was proposed by Bialecki et al. [87], by approximating the complete elliptic integral $K(m)$ and isolating its singular term.

The complete elliptic integral $K(m)$ can be approximated, for $0 \leq m < 1$ and within an error $\epsilon < 2 \cdot 10^{-8}$, by the expression [68]

$$K(m) = K_1(\bar{m}) - K_2(\bar{m}) \ln \bar{m} \tag{B-10}$$

where

$$\bar{m} = \frac{\rho^2}{(\xi + r)^2 + (z' - z)^2} \tag{B-11}$$

is the complementary modulus of the complete elliptic integral and

$$\begin{aligned} K_1(\bar{m}) &= a_0 + a_1 \bar{m} + \ldots + a_4 \bar{m}^4 \\ K_2(\bar{m}) &= b_0 + b_1 \bar{m} + \ldots + b_4 \bar{m}^4 \end{aligned} \tag{B-12}$$

are polynomials whose coefficients are given by

$$\begin{aligned} a_0 &= 1.38629436112 & b_0 &= 0.5 \\ a_1 &= 0.09666344259 & b_1 &= 0.12498593597 \\ a_2 &= 0.03590092383 & b_2 &= 0.06880248576 \\ a_3 &= 0.03742563713 & b_3 &= 0.03328355346 \\ a_4 &= 0.01451196212 & b_4 &= 0.00441787012 \end{aligned} \tag{B-13}$$

Substituting the approximation given by Eq. (B-10) into the weakly singular integral in Eq. (B-9), one may isolate the singular term to obtain

$$\begin{aligned} \int_\Gamma f(r,z)\, K(m)\, \mathrm{d}\Gamma = \int_\Gamma f(r,z) \left[K_1(\bar{m}) + K_2(\bar{m}) \ln \frac{\rho(r,z)^2}{\bar{m}} \right] \mathrm{d}\Gamma - \\ 2 \int_\Gamma f(r,z)\, K_2(\bar{m}) \ln \rho(r,z)\, \mathrm{d}\Gamma \end{aligned} \tag{B-14}$$

Applying the scheme for the regular integral presented in Section B.1 to the first integral and the scheme for the weakly singular integral in Section B.2 to the second integral, one obtains

$$\begin{aligned} \int_\Gamma f(r,z)\, K(m)\, \mathrm{d}\Gamma \cong \sum_{m=1}^{n_g} \left\{ f(\eta) \left[K_1(\bar{m}) + 2K_2(\bar{m}) \ln \frac{1 - \eta' \eta}{2\sqrt{\bar{m}}} \right] J(\eta) \right\}_{\eta = \eta_m^g} w_m^g - \\ 4 \sum_{m=1}^{n_l} \left[f(\tilde{\eta})\, K_2(\bar{m}) \ln \tilde{\eta}\, J(\tilde{\eta}) \right]_{\tilde{\eta} = \eta_m^l} w_m^l \end{aligned} \tag{B-15}$$

for $\tilde{\eta}$ according to the development that has led to Eq. (B-6).

PUC-Rio - Certificação Digital Nº 0410745/CA

B.2.2
Weakly singular integral of terms with the complete elliptic integral of the second order

Let $f(r,z)$ be a regular function and $E(m)$ the complete elliptic integral of the second order, of modulus m, given by Eq. (A-19) in terms of the coordinates of points $\mathrm{P}(\xi, z')$ and $\mathrm{Q}(r,z)$ on the boundary $\Gamma(r,z)$. One needs to evaluate the following weakly singular integral

$$\int_\Gamma f(r,z)\, E(m)\, \mathrm{d}\Gamma = \int_{-1}^{1} f(\eta)\, E(m)\, J(\eta)\, \mathrm{d}\eta \tag{B-16}$$

for the case of $m = 1$ for either $\eta = -1$ or $\eta = 1$. Although $E(m) \neq \infty$ for this case, one may isolate the quasi-singular terms to enhance the convergence of the numerical integration.

The complete elliptic integral $E(m)$ can be approximated, for $0 \leq m < 1$ and within an error $\epsilon < 2 \cdot 10^{-8}$, by the expression [68]

$$E(m) = E_1(\bar{m}) - E_2(\bar{m}) \ln \bar{m} \tag{B-17}$$

where $\bar{m}$ is given by Eq. (B-8) and

$$\begin{aligned} E_1(\bar{m}) &= 1 + a_1\, \bar{m} + \ldots + a_4\, \bar{m}^4 \\ E_2(\bar{m}) &= b_1\, \bar{m} + \ldots + b_4\, \bar{m}^4 \end{aligned} \tag{B-18}$$

are the polynomials whose coefficients are given by

$$\begin{array}{ll} a_1 = 0.44325141463 & b_1 = 0.24998368310 \\ a_2 = 0.06260601220 & b_2 = 0.09200180037 \\ a_3 = 0.04757383546 & b_3 = 0.04069697526 \\ a_4 = 0.01736506451 & b_4 = 0.00526449639 \end{array} \tag{B-19}$$

The polynomial approximation of $E(m)$ presents no singularity, since $E_2(\bar{m})$ has no free coefficients, according to Eq. (B-17). However, the presence of $\ln \bar{m}$ causes the integrand of Eq. (B-16) to be non-analytical, which requires a special numerical treatment.

In a manner similar to that used in the previous section, one arrives at the following expression for the numerical evaluation of the weakly singular integral given by Eq. (B-16)

$$\begin{aligned} \int_\Gamma f(r,z)\, E(m)\, \mathrm{d}\Gamma \cong & \sum_{m=1}^{n_g} \left\{ f(\eta) \left[E_1(\bar{m}) + 2E_2(\bar{m}) \ln \frac{1 - \eta'\, \eta}{2\sqrt{\bar{m}}} \right] J(\eta) \right\}_{\eta=\eta_m^g} w_m^g - \\ & 4 \sum_{m=1}^{n_l} \left[f(\tilde{\eta})\, E_2(\bar{m}) \ln \tilde{\eta}\, J(\tilde{\eta}) \right]_{\tilde{\eta}=\eta_m^l} w_m^l \end{aligned} \tag{B-20}$$

for $\tilde{\eta}$ given by Eq. (B-6).

PUC-Rio - Certificação Digital Nº 0410745/CA

B.3 Cauchy principal value of the singular integral of order $1/\rho$

Let $f(r,z)$ be a regular function and $\rho(r,z)$ the distance beteween the points $\mathrm{P}(\xi, z')$ and $\mathrm{Q}(r,z)$ on the boundary $\Gamma(r,z)$. One needs to evaluate the strongly singular integral

$$\int_\Gamma \frac{f(r,z)}{\rho(r,z)}\,\mathrm{d}\Gamma \tag{B-21}$$

for the case $\rho(-1) = 0$ or $\rho(1) = 0$. This integral may be obtained as a sum of a Cauchy principal value and a discontinuous term as

$$\int_\Gamma \frac{f(r,z)}{\rho(r,z)}\,\mathrm{d}\Gamma = \mathrm{PV}\int_\Gamma \frac{\mathrm{f(r,z)}}{\rho(\mathrm{r,z})}\,\mathrm{d}\Gamma + \mathrm{c} \tag{B-22}$$

The evaluation of the discontinuous term c of the strongly singular integrals appearing in the boundary element formulations is addressed in Section 3.1.5.

The Cauchy principal value is best evaluated in terms of two finite-part integrals, denoted by $⨍$, for the boundary segments adjacent to the singularity point $\rho(r,z) = 0$.

In what follows, one makes use of the integration scheme proposed by Dumont & Souza [88]. Using the same notation of Eq. (B-6), one may expand the regular function by Taylor and obtain the following normalized integral of Eq. (B-21) over the curved boundary Γ

$$⨍_\Gamma \frac{f(r,z)}{\rho(r,z)}\,\mathrm{d}\Gamma = -\eta' \left[f(\eta)\ln|\bar{\rho}|\right]_{\eta=\eta'} + ⨍_{-1}^{1} \frac{f(\eta)}{\rho(\eta)}\,J(\eta)\,\mathrm{d}\eta \tag{B-23}$$

The resulting quadrature rule for evaluating Cauchy's principal value of the strongly singular integral of (B-21) is given by

$$\begin{aligned} ⨍_\Gamma \frac{f(r,z)}{\rho(r,z)}\,\mathrm{d}\Gamma \cong \sum_{m=1}^{n_g}\left[\frac{f(\eta)}{\rho(\eta)}\,J(\eta)\right]_{\eta=\eta_m^g} w_m^g - \\ \eta'\,[f(\eta)]_{\eta=\eta'}\left\{[\ln|2\bar{\rho}|]_{\eta=\eta'} - \sum_{m=1}^{n_g}\frac{w_m^g}{1-\eta_m^g}\right\} \end{aligned} \tag{B-24}$$

where

$$[\bar{\rho}(\eta)]_{\eta=\eta'} = [J(\eta)]_{\eta=\eta'} \tag{B-25}$$

The above scheme, that employs the Gauss-Legendre quadrature rule and an additional correction term, evaluates exactly this integral for a polynomial function of order $2n_g - 2$. Other possible numerical integration scheme for the strongly singular integral of Eq. (B-21) makes use of Kutt's weighted quadrature rule [95] or the technique of subtraction of the singularity with further numerical integration [74]. The last is another way of expressing the procedure outlined above.

PUC-Rio - Certificação Digital Nº 0410745/CA