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Conclusions

In elasticity, axisymmetric problems can be found in the analysis of several problems, for which analytical solutions are available only for specific geometry, material and boundary conditions. Although useful in engineering practice, even these analytical solutions cannot be generally found in explicit form and are usually given for specific components of displacements and stresses in certain locations of the axisymmetric body.

In this sense, the boundary integral methods are an attractive way of obtaining overall results for axisymmetric problems, since they reduce the analysis of the three-dimensional body to a one-dimensional mesh discretization and the evaluation of linear integrals. After one solves the problems for the boundary, results can be evaluated at any point in the domain .

This work presented the theoretical developments for the implementation of the conventional and simplified-hybrid boundary element for axisymmetric problems in the elastic fullspace and halfspace.

In which concerns the conventional boundary element method, the major contribution of this thesis was the derivation of the axisymmetric fundamental solutions in terms of integrals Lipschitz-Hankel and the implementation of the halfspace fundamental solution by Hasegawa [13, 14] . The former contribution led to explicit expressions for evaluating results at internal points as well as to correctly identifying the singularities that arise in the axisymmetric formulations. The later contribution provided a way to take advantage of the axisymmetric boundary element formulation for halfspace problems without discretizing and truncating the free surface.

For the simplified-hybrid boundary element method, axisymmetric problems had not been addressed up to now. For fullspace problems, the complete formulation of the method was only possible due to new theoretical developments concerning the evaluation of some unknown coefficients of the matrix $\mathbf{U}^*$. This theoretical developments provided a definitive solution to some old issues related to the basis $\mathbf{V}$ for some specific topological configurations, and therefore are also a relevant contribution. Moreover, the simplified-hybrid boundary element method is shown as a particularly attractive method in the case of problems for which the corresponding fundamental solutions are difficult to manipulate, such as axisymmetric elasticity. The ease of post-processing and the need for only integrating the well known influence matrix **H** make this formulation of great simplicity and applicability. For the halfspace, some questions regarding the analytical solutions needed for the evaluation of the submatrices about the main diagonal of $\mathbf{U}^*$ still remain unsolved.

In the following, the conclusions to each aspect treated in the work is presented.

7.1
Fundamental solutions for the axisymmetric elastic fullspace and halfspace

Expressing the fundamental solutions by means of integrals of Lipschitz-Hankel type with products of Bessel functions was shown advantageous for the case of axisymmetric problems.

For both fullspace and halfspace problems, one managed to investigate each integral embedded in the the fundamental solutions, allowing the order of the singularities to be correctly identified and isolated. Moreover, all the expressions manipulated in the boundary element formulations could be written in a more compact manner, providing smaller equations to be implemented computationally and making it easier to find their limiting expressions in the cases of ring loads applied on the axis of axisymmetry. This is evident in the expressions for evaluating displacements and stresses at domain points in terms of Somigliana's identity.

Finally, this more compact representation enabled the explicit expression of the fullspace fundamental solution embedded in the halfspace fundamental solution. As the difference terms between these two fundamental solutions present singularities only on the halfspace surface, the implementation of the halfspace formulation turned out to require only a few modifications in the existing codes for the fullspace.

7.2
The boundary element method for axisymmetric elasticity

For the fullspace problem, the boundary element formulation was first proposed by Cruse et al. [15]. As contributions of this work to this formulation, one may cite the expressions of the generalized inverses of matrix **G** depending on its several cases of singularities; the explicit expressions of the constants c^f_{mn}, for either $\xi > 0$ and $\xi = 0$, referring to the discontinuous part of the singular integral arising in the method; and the explicit expression for evaluating results at internal points.

To the author's best knowledge, the implementation of the boundary element method with the halfspace fundamental solution is original. In this work, one presents all the expressions needed for the complete formulation of the method for halfspace problem. As mentioned before, its implementation requires just a few modifications of existing fullspace codes and can be an advantageous alternative to the use of infinite elements, which may be of cumbersome integration.

7.3
The simplied-hybrid boundary element method for axisymmetric elasticity

As mentioned above, new theoretical developments were presented for the general formulation of the simplified-hybrid boundary element method. By appropriately applying a hybrid contragradient theorem, one managed to derive simple relations that are generally valid and can successfully substitute for the spectral properties in the evaluation of the unknown coefficients of the matrix $\mathbf{U}^*$. With the new developments, once some simple analytical solutions are identified as inherent to a given problem, it is always possible to evaluate the submatrices about the main diagonal of $\mathbf{U}^*$, regardless of topology and spectral properties.

For the fullspace formulation, an orthonormal basis $\mathbf{A}$ was introduced to formally take into account the fact that axisymmetric radial loads applied on the axis of axisymmetry generate no displacements. All the expressions and generalized inverses needed for the implementation of the simplified-hybrid boundary element method for axisymmetric fullspace problems are presented.

For the halfspace, some questions regarding the analytical solutions for evaluating the unknown coefficients of $\mathbf{U}^*$ still remain unsolved, although the completion of the formulation seems just a matter of time.

7.4
Numerical integration

The integration schemes for the evaluation of the integrals arising in the boundary element formulations for axisymmetric problems were presented in detail and can be applied to boundary elements of any order. For axisymmetric boundary element formulations, this topic is usually only superficially discussed in the technical literature and one often finds examples for which the bad accuracy in results is wrongly taken for sparse meshes. Also, simplifications in the geometry to avoid singularities due to the axis of axisymmetry seems to be a common practice.

After the study of the singularities in the integrals of the fullspace formulations, one could identify and treat the singularities of the halfspace formulation with less effort.

7.5
Suggestion for future studies

As suggestions for future works, one may cite:

- The application of the proposed methods to axisymmetric problems of contact, inclusion, indentation and fracture;
- The extension of the proposed methods to problems of torsion, for both fullspace and halfspace;
- The development of a singular element for the axisymmetric halfspace to compare with the halfspace formulations;
- The extension of the new theoretical developments of the simplified-hybrid boundary element methods to two and three-dimensional problems of potential and elasticity, in general.