3
The boundary element method for axisymmetric elasticity

The boundary element method for axisymmetric elasticiy was first formulated by Cruse et al. [15], using the fullspace fundamental solution derived by Kermanidis [12], which was presented in Section 2.2. One may cite several contributions regarding the axisymmetric formulation, as the expansion of asymmetric boundary conditions by Fourier series suggested by Mayr [16] and Rizzo & Shippy [17, 18]; and the assessment of body forces by means of particular integrals incorporated by Park [19]. Also, axisymmetric formulations have been developed for transversely isotropy by Ishida & Ochiai [20], thermoelasticity by Bakr & Fenner [21], elastoplasticity by Cathie & Banerjee [22] and viscoplasticity by Sarihan & Mukherjee [23]. In elastodynamics, one may cite the works by Wang & Banerjee [24, 25], Tsinopoulos et al. [26] and Yang & Zhou [27] in the frequency domain.

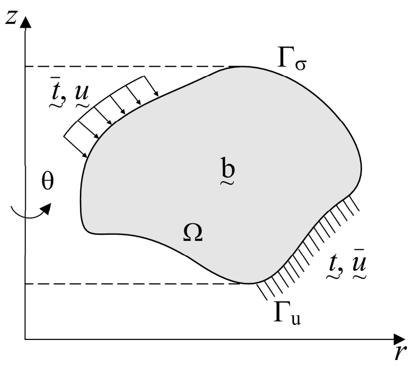
For problems in the halfspace, boundary element formulations that make use of fullspace fundamental solutions require the discretization of the free infinite surface, with truncation at a reasonable distance from the axis z. The errors caused by this approximation can be attenuated by using infinite elements [31, 32], which simulates the decay of the displacement and stress fields as $r \to \infty$. Alternatively, on may implement fundamental solutions that satisfy in advance the traction free boundary condition on the free surface, thus dispensing with its discretization. In elasticity, this approach was used by Telles & Brebbia [38] and Dumir & Mehta [39] to deal with problems in the isotropic and orthotropic halfplane, respectively.

This chapter presents the axisymmetric boundary element formulation using the fundamental solutions presented in Chapter 2. In the case of the fullspace, some relevance of the developments outlined may be claimed in relation tothe numerical integration schemes as well to the systematic evaluation of stresses at internal points, presenting explicit expressions in terms of integrals of Lipschitz-Hankel type. The axisymmetric halfspace formulation has not been addressed in the literature up to now. In this case, all developments in theory and implementation, which seem to be original, are presented in more detail.

The next sections address problems involving radial and axial loads. The formulation for torsional loads can be derived in a similar manner. In the following, the superscript f refers to fullspace and h to halfspace.

3.1
Formulation for the axisymmetric fullspace problem

3.1.1
Boundary integral equation



(a)

Figure 3.1: Meridian plane of an elastic axisymmetric body submitted to body forces, traction forces and prescribed displacements

Consider an axisymmetric body submitted to body forces b_i in Ω, traction forces $\bar{t}_i$ on Γ_σ and prescribed displacements $\bar{u}_i$ on Γ_u, as shown in Fig. 3.1. One is looking for displacements u_i and tractions t_i along the boundary Γ that best satisfy the equilibrium equation in the domain Ω, as well the boundary conditions.

The solution of this problem can be derived by using Betti's reciprocity theorem, which relates the self-equilibrate state (u_i^*, t_i^*, b_i^*) and the approximately self-equilibrated state (u_i, t_i, b_i) by the expression

$$\int_\Gamma t_i u_i^* \mathrm{d}\Gamma + \int_\Omega b_i u_i^* \mathrm{d}\Omega = \int_\Gamma t_i^* u_i \mathrm{d}\Gamma + \int_\Omega b_i^* u_i \mathrm{d}\Omega \qquad (3\text{-}1)$$

where $u_i = \bar{u}_i$ on Γ_u and $t_i = \bar{t}_i$ on Γ_σ; and the displacements and traction forces of the remaining part of Γ are to be obtained. In fact. Eq. (3-1) is valid only within some approximation error.

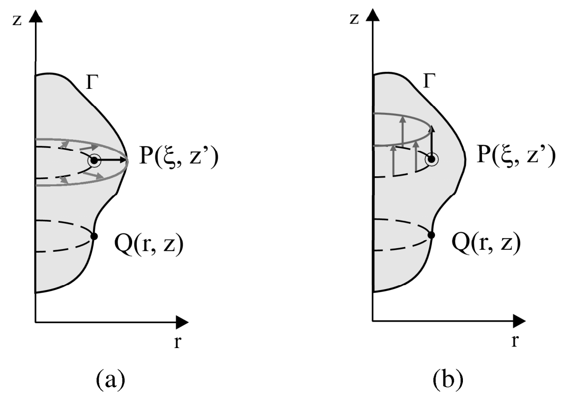


Figure 3.2: Axisymmetric body submitted to: a) radial ring load; b) axial ring load.

The auxiliary state (u_i^*, t_i^*, b_i^*) results from the application of ring loads of intensity p_m^* at a point $\mathrm{P}(\xi, z')$ in Ω, weith $b_i^* = \Delta_{im}^* p_m^*$, as depicted in Fig. 3.2. Hence, displacements and stresses at any point $\mathrm{Q}(r, z)$ can be evaluated by using the fundamental solution, so that

$$u_i^*(\mathrm{Q}) = u_{im}^{*\mathrm{f}}(\mathrm{Q}, \mathrm{P})\, p_m^*(\mathrm{P}) + c_i^r \tag{3-2}$$

$$t_i^*(\mathrm{Q}) = t_{im}^{*\mathrm{f}}(\mathrm{Q}, \mathrm{P})\, p_m^*(\mathrm{P}) \tag{3-3}$$

The fundamental solutions $u_{im}^{*\mathrm{f}}$ and $t_{im}^{*\mathrm{f}} = \sigma_{ijm}^{*\mathrm{f}} \eta_j$ correspond to displacements and traction forces in a fullspace submitted to radial and axial loads, presented in Chapter 2. In the above equation, c_i^r are arbitrary rigid body constants that are intrinsic to the fundamental solution. As presented, the index m refers to both the direction and the location $\mathrm{P}(\xi, z')$ of the applied load. The index j refers to the direction of the displacements and traction forces measured at $\mathrm{Q}(r, z)$.

Substituting the above relations into Betti's reciprocity theorem leads to

$$\int_\Gamma t_i(\mathrm{Q})\, u_{im}^*(\mathrm{Q}, \mathrm{P})\, \mathrm{d}\Gamma + \int_\Omega b_i(\mathrm{Q})\, u_{im}^*(\mathrm{Q}, \mathrm{P})\, \mathrm{d}\Omega = \int_\Gamma t_{im}^*(\mathrm{Q}, \mathrm{P})\, u_i(\mathrm{Q})\, \mathrm{d}\Gamma + \int_\Omega \Delta_{im}^*\, u_i(\mathrm{Q})\, \mathrm{d}\Omega \tag{3-4}$$

in which the error in estimating the work of tractions $t_i(\mathrm{Q})$ and body forces $b_i(\mathrm{Q})$ over rigid body displacements c_i^r is ommited. This error tends to zero as $t_i(\mathrm{Q})$ and $b_i(\mathrm{Q})$ tend to represent a self-equilibrated stress field [71, 53].

For an axisymmetric body, the surface corresponding to the three-dimensional boundary is given by

$$\mathrm{d}\Gamma(r, z, \theta) = r\, \mathrm{d}\theta\, \mathrm{d}\Gamma(r, z) \tag{3-5}$$

where $\Gamma(r, z)$ is the boundary of the meridian plane, as indicated in Fig. 3.2. Then, one may integrate Eq. (3-4) over θ and apply the property of Δ_{im}^* expressed in

Eq. (2-75) to arrive at

$$u_m(\mathrm{P}) = -2\pi \int_\Gamma t^{*\mathrm{f}}_{im}(\mathrm{P},\mathrm{Q})\, u_i(\mathrm{Q})\, r\, \mathrm{d}\Gamma(r,z) + 2\pi \int_\Gamma u^{*\mathrm{f}}_{im}(\mathrm{P},\mathrm{Q})\, t_i(\mathrm{Q})\, r\, \mathrm{d}\Gamma(r,z) \quad (3\text{-}6)$$

The above equation is the Somigliana's identity for axisymmetric problems [15] in the absence of body forces, which relates displacements at a point $\mathrm{P}(\xi, z')$ in the domain to displacements and stresses along the boundary. From this relation, one may obtain a system of integral equations on the boundary and the matrix governing equation of the problem, as presented in Section 3.1.3.

3.1.2
Approximation of displacements and tractions

Let the boundary $\Gamma(r,z)$ of the meridian plane be subdivided into n_e elements of boundary Γ_e with a total of n_n nodes. The displacements and tractions can be approximated along the boundary by

$$\begin{aligned} u_i &= u_{in}\, u_n \qquad \text{where} \quad n = 1, n_u \\ t_i &= t_{il}\, t_l \qquad \text{where} \quad l = 1, n_t \end{aligned} \quad (3\text{-}7)$$

where u_n and t_l are nodal values; u_{in} and t_{il} are the respective interpolation functions. The index n refers to each displacement degree of freedom of a total $n_u = 2n_n$; and the index l refers to each traction force degree of freedom of a total $n_t \geq n_u$. Note that the numbers of degrees of freedom for traction forces and displacements are not necessarily the same, since one may need additional traction parameters to represent load discontinuities along the boundary [72].

3.1.3
Matrix governing equation

Let the point $\mathrm{P}(\xi, z')$ in the Somigliana's identity given by Eq. (3-6) approach the boundary Γ. In the limit, the first integral becomes singular and the second term can be rearranged to arrive at the following integral equation

$$2\pi \int_\Gamma t^{*\mathrm{f}}_{im}(\mathrm{P},\mathrm{Q})\, u_i(\mathrm{Q})\, r\, \mathrm{d}\Gamma(r,z) + \delta^*_{im}\, u_i(\mathrm{Q}) = 2\pi \int_\Gamma u^{*\mathrm{f}}_{im}(\mathrm{P},\mathrm{Q})\, t_i(\mathrm{Q})\, r\, \mathrm{d}\Gamma(r,z) \quad (3\text{-}8)$$

Substituting the approximations expressed in Eq. (3-7) for displacements u_i and tractions t_i, one arrives at

$$H_{mn}\, u_n = G_{ml}\, t_l \quad \text{or} \quad \mathbf{H\,u} = \mathbf{G\,t} \quad (3\text{-}9)$$

where

$$H_{mn} = 2\pi \int_{\Gamma} t^{*\mathrm{f}}_{im}(\mathrm{P},\mathrm{Q})\, u_{in}(\mathrm{Q})\, r\, \mathrm{d}\Gamma(r,z) + \delta^{*}_{mn} = \widehat{H}_{mn} + c^{\mathrm{f}}_{mn} \quad (3\text{-}10)$$

$$G_{ml} = 2\pi \int_{\Gamma} u^{*\mathrm{f}}_{im}(\mathrm{P},\mathrm{Q})\, t_{il}(\mathrm{Q})\, r\, \mathrm{d}\Gamma(r,z) \quad (3\text{-}11)$$

in terms of the generalized Kronecker delta

$$\delta^{*}_{mn} = \begin{cases} 1 & \text{if } m \text{ and } n \text{ refer to both the same node and coordinate direction} \\ 0 & \text{otherwise} \end{cases} \quad (3\text{-}12)$$

and the Cauchy principal value of the singular integral

$$\widehat{H}_{mn} = 2\pi \,⨍_{\Gamma} t^{*\mathrm{f}}_{im}(\mathrm{P},\mathrm{Q})\, u_{in}(\mathrm{Q})\, r\, \mathrm{d}\Gamma(r,z) \quad (3\text{-}13)$$

The constants c^{f}_{mn} refer from the inclusion of the discontinuous part of the singular integral and their evaluation is detailed in Section 3.1.5.

In the governing equation given by Eq. (3-9), $\mathbf{H} = [H_{mn}] \in \mathbb{R}^{n_u \times n_u}$ and $\mathbf{G} = [G_{ml}] \in \mathbb{R}^{n_u \times n_t}$ are influence matrices, $\mathbf{u} = [u_n] \in \mathbb{R}^{n_u}$ and $\mathbf{t} = [t_l] \in \mathbb{R}^{n_t}$ are displacements and tractions in each coordinate direction at each nodal point along the boundary. As mentioned before, the number of nodal traction forces can be greater than the number of nodal displacements and, in this case, $\mathbf{G}$ becomes a rectangular matrix.

By rearranging each term of the above equation according to the prescribed boundary conditions for displacements and traction forces, $\bar{u}_i$ and $\bar{t}_i$ respectively, a linear system of equations $\mathbf{A}\,\mathbf{Y} = \mathbf{B}$ is obtained, in which $\mathbf{Y}$ contains the unknown parameters of the problem [73].

3.1.4
Stiffness matrix

The solution can also be found by means of a stiffness matrix $\mathbf{K} = [K_{mn}] \in \mathbb{R}^{2n_u \times 2n_u}$ for the following linear system of equations

$$\mathbf{K}\,\mathbf{u} = \mathbf{p} \quad (3\text{-}14)$$

where $\mathbf{u} = [u_n] \in \mathbb{R}^{n_u}$ and $\mathbf{p} = [p_m] \in \mathbb{R}^{n_u}$ contain nodal displacements and equivalent nodal forces, respectively. This approach is usually applied to couple the finite element method and the boundary element method. In this case, the portion of the body modeled by the boundary element method is considered as a superelement whose stiffness contributes to the global stiffness matrix of the problem [73, 74].

The equivalent nodal forces p_m can be obtained from the traction forces t_i in terms of virtual work

$$p_m \, \delta u_m = 2\,\pi\,\delta u_m \int_\Gamma u_i m \, t_i \, r \, \mathrm{d}\Gamma(r,z) \tag{3-15}$$

which leads to

$$p_m = 2\pi \int_\Gamma u_{im} \, t_i \, r \, \mathrm{d}\Gamma(r,z) \tag{3-16}$$

Moreover, the traction forces can be approximated by Eq. (3-7) and the above expression becomes

$$\mathbf{p} = \mathbf{L}^{\mathrm{T}} \, \mathbf{t} \tag{3-17}$$

where $\mathbf{L} = [L_{lm}] \in \mathbb{R}^{n_t \times n_u}$ is given by

$$L_{lm} = 2\pi \int_\Gamma t_{il} \, u_{im} \, r \, \mathrm{d}\Gamma(r,z) \tag{3-18}$$

Then, isolating the vector of traction forces $\mathbf{t}$ in Eq. (3-9) and substituting into Eq. (3-15) one arrives at the following expression for the stiffness matrix

$$\mathbf{K} = \mathbf{L}^{\mathrm{T}} \, \mathbf{G}^{(-1)} \, \mathbf{H} \tag{3-19}$$

Notice that matrix $\mathbf{G}$ may be rectangular and non-singular, in which cases its inverse should be obtained in the frame of generalized inverses [75], denoted by the superscript (-1), according to the following developments [58].

3.1.4.1 Generalized inverses of G

If the number of nodal traction forces n_t is greater then the number of nodal displacements n_u, then matrix $\mathbf{G}$ is rectangular. Also, if the boundary Γ contains any node on the axis of axisymmetry, all the coefficients of the rows of $\mathbf{G}$ corresponding to the r-direction are void, since $u^{*\mathrm{f}}_{i(r)} = 0$ for $\xi = 0$. Thus, one must distinguish the following cases concerning the form and rank of $\mathbf{G}$:

Case 1: $n_u = n_t$ and $\Gamma(r,z)$ does no intercept the axis of axisymmetry

The simplest case is a toroid. The matrix $\mathbf{G}$ is square and non-singular, thus can be inverted directly.

Case 2: $n_u = n_t$ and $\Gamma(r, z)$ intercepts the axis of axisymmetry

The simplest case is a sphere. Let n_a be the points at which $\Gamma(r, z)$ intersects the axis of axisymmetry. The matrix $\mathbf{G}$ is square of order n_u and singular with $\text{rank}(\mathbf{G}) = n_u - n_a$. To deal with the linear algebra problem, one resorts to a matrix $\mathbf{A} = [A_{il}] \in \mathbb{R}^{n_a \times n_t}$ whose columns are defined as following.

For each node i of intersection, for which $\xi = 0$, A_{il} is written as

$$A_{i_l} = \begin{cases} 1 & \text{if } l \text{ refers to both node } i \text{ and } r\text{-direction} \\ 0 & \text{otherwise} \end{cases} \tag{3-20}$$

The matrix $\mathbf{A}$ is by definition orthonormal, with columns spanning the null space of $\mathbf{G}$. This null space means that a radial load applied at the axis of axisymmetry must produce no displacements in the body, except for rigid body displacements in the z-direction).

The corresponding orthogonal projector and the complementary orthogonal projector are, respectively,

$$\mathbf{P}_{\mathrm{A}} = \mathbf{A}\,\mathbf{A}^{\mathrm{T}} \tag{3-21}$$

$$\mathbf{P}_{\mathrm{A}}^{\perp} = \mathbf{I} - \mathbf{P}_{\mathrm{A}} \tag{3-22}$$

The inverse of $\mathbf{G}$ can be obtained in the frame of the Bott-Duffin inverse [76, 75] for the solution of $\mathbf{t}$ in the following restricted system

$$\begin{cases} \mathbf{G}\,\mathbf{t} = \mathbf{d}^* \\ \mathbf{P}_{\mathrm{A}}\,\mathbf{t} = \mathbf{0} \end{cases} \tag{3-23}$$

which refers to the transformation of traction forces $\mathbf{t} = \mathbf{P}_{\mathrm{A}}{}^{\perp}\,\mathbf{t}$ into displacements $\mathbf{d}^*$. Since $\mathbf{G}$ does not transform traction forces of the space spanned by $\mathbf{P}_{\mathrm{A}}$, i.e., $\mathbf{G}\,\mathbf{P}_{\mathrm{A}} = \mathbf{0}$, its inverse can be written as

$$\mathbf{G}^{(-1)} = \mathbf{P}_{\mathrm{A}}^{\perp}\,(\mathbf{G} + \lambda\,\mathbf{P}_{\mathrm{A}})^{-1} \tag{3-24}$$

The term in brackets is by construction a non-singular matrix. The projection $\mathbf{P}_{\mathrm{A}}$ is multiplied by a arbitrary constant λ to make sure the summands have approximately the same order of magnitude []. In this case, one may adopt the order $1/\mu$, where μ is the elastic shear modulus introduced by Eq. (2-10).

where the sum between parenthesis is non-singular and λ is a constant of order $1/\mu$ to assure well-conditioning.

The inverse of $\mathbf{G}$ given by the above equation may appear more cumbersome to calculate than the simple inverse presented in Case 1. However, its computational implementation comprises only the following steps: a) prior to the inversion, make $G_{ml} = \lambda$ for each element where m and l refer to the nodes on the axis of axisymmetry and the r-direction; b) invert the modified matrix; c) after the inversion, make $G^{(-1)}_{lm} = 0$ for each element modified in step b).

Case 3: $n_u < n_t$ and $\Gamma(r, z)$ does not intersect the axis of axisymmetry

The matrix $\mathbf{G}$ is rectangular and full rank. Its generalized right-inverse is given by [58]

$$\mathbf{G}^{(-1)} = \mathbf{L}\,(\mathbf{G}\,\mathbf{L})^{-1} \tag{3-25}$$

Case 4: $n_u < n_t$ and $\Gamma(r, z)$ intersects the axis of axisymmetry

Let n_a be the number of point at which $\Gamma(r, z)$ intersects the axis of axisymmetry. The matrix $\mathbf{G}$ is rectangular with rank $\mathbf{G}) = n_u - n_a$. Its generalized inversion can be obtained by combining Cases 2 and 3, leading to

$$\mathbf{G}^{(-1)} = \mathbf{L}\,\mathbf{P}^{\perp}_{\mathrm{A}}\,(\mathbf{G}\,\mathbf{L} + \lambda\,\mathbf{P}_{\mathbf{A}})^{-1} \tag{3-26}$$

Its computational implementation can be performed in a similar manner to the procedure presented in Case 2, modifying specific elements of matrix $\mathbf{G}\,\mathbf{L}$.

Notice that for the Cases 2 and 4, the stiffness matrix $\mathbf{K}$ has its rank reduced by -1 for each node on the axis of axisymmetry. In this case, prescription of zero radial displacements at these nodes provides the additional conditions for the equation system to be solved.

More elaborated developments on the spectral properties of the matrix $\mathbf{G}$, its inverses and the spectral transformations and algebraic spaces that arise in the frame of the boundary element methods may be found in Dumont [71, 43], Dumont et al. [57], Oliveira [58] and Oliveira et al. [77].

3.1.5 Discontinuity constants $\mathbf{c}^{\mathrm{f}}_{mn}$

The constants c^{f}_{mn} account for the discontinuous part of the first integral in Eq. (3-6) and contribute only to the elements of H_{mn} in which m and n refer to the same node. In this cases, they can be expressed as

$$c^{\mathrm{f}}_{mn} = \delta^{*}_{mn} + 2\pi \lim_{\epsilon\to 0} \int_{\Gamma_\epsilon} t^{*\mathrm{f}}_{im}(\mathrm{P},\mathrm{Q})\, u_{in}(\mathrm{Q}) r\, \mathrm{d}\Gamma = 2\pi \lim_{\epsilon\to 0} \int_{\bar{\Gamma}_\epsilon} t^{*\mathrm{f}}_{im}(\mathrm{P},\mathrm{Q})\, u_{in}(\mathrm{Q})\, r\, \mathrm{d}\Gamma \tag{3-27}$$

where Γ_ϵ and $\bar{\Gamma}_\epsilon$ are portions of the circumference of radius ϵ, as depicted in Fig. 3.3 for P(ξ, z') placed either outside or on the axis of axisymmetry z.

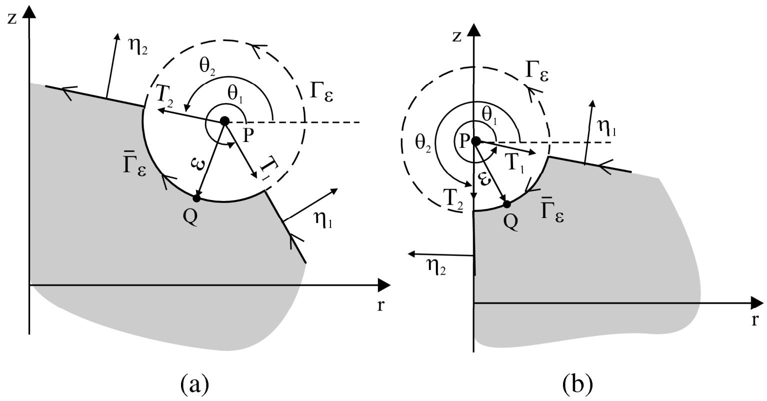


Figure 3.3: Integration paths of constants c^{f}_{mn} for (a) $\xi > 0$ and (b) $\xi = 0$.

For the traction forces $t^{*\mathrm{f}}_{im}$ in the fullspace, presented in Section 2.2, when the distance ρ between P(ξ, z') and Q(r, z) tends to zero, i.e. $\rho \rightarrow 0$, the modulus m of the complete elliptic integrals tends to unity. Accordingly, $E(m) \rightarrow 1$ and $K(m) \rightarrow \infty$ in the integrals $\bar{I}_{pq\lambda}$. One can expand $K(m)$ by an infinity series for $m < 1$ [68] as

$$K(m) = \frac{1}{2\pi}\left[1 + \left(\frac{1}{2}\right)^2 m + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 m^2 + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 m^3 + \ldots\right] \tag{3-28}$$

Then, the integral in Eq. (3-27) can be simplified by replacing $r = \xi + \epsilon \cos\theta$, $z = z' + \epsilon \sin\theta$, $n_r = -\cos\theta$, $n_z = -\sin\theta$, $\mathrm{d}\bar{\Gamma}_\epsilon = -\epsilon\, \mathrm{d}\theta$. As a result, when m and n do not refer to the same node,

$$c^{\mathrm{f}}_{mn} = 0 \tag{3-29}$$

On the other hand, when m and n refer to the same node and $\xi > 0$, one obtains

$$\begin{aligned} c^{\mathrm{f}}_{(r\,r)} &= \frac{1}{4\pi(1-\nu)}\left[\frac{\sin 2\theta_1 - \sin 2\theta_2}{2} + 2(1-\nu)\,\Delta\theta\right] \\ c^{\mathrm{f}}_{(r\,z)} = c^{\mathrm{c}}_{(z\,r)} &= \frac{1}{4\pi(1-\nu)}\left[\sin\theta_1{}^2 - \sin\theta_2{}^2\right] \\ c^{\mathrm{f}}_{(z\,z)} &= \frac{1}{4\pi(1-\nu)}\left[-\frac{\sin 2\theta_1 - \sin 2\theta_2}{2} + 2(1-\nu)\,\Delta\theta\right] \end{aligned} \tag{3-30}$$

where $\theta_2 = \theta_1 - \Delta\theta$ and $\Delta\theta$ the internal angle beteween θ_1 and θ_2. Finally, when m

and n refer to the same node and $\xi = 0$,

$$
\begin{aligned}
c^{\mathrm{f}}_{(r\,r)} &= 1 \\
c^{\mathrm{f}}_{(r\,z)} &= 0 \\
c^{\mathrm{f}}_{(z\,r)} &= \frac{1}{4\pi\,(1-\nu)}\left[-\cos\theta_1{}^3 + \cos\theta_2{}^3\right] \\
c^{\mathrm{f}}_{(z\,z)} &= \frac{1}{4\pi\,(1-\nu)}\left[\sin\theta_1\,[2(1-\nu) - \cos\theta_1{}^2] - \sin\theta_2\,[2(1-\nu) - \cos\theta_2{}^2\right]
\end{aligned}
\tag{3-31}
$$

The expression given by Eq. (3-30) was obtained by Cruse et al. [15] and coincide with the constants for plane strain elasticity [78]. Correspondingly, the expression given by Eq. (3-31) can be derived by integrating the constants for three-dimensional elasticity [78] over the axis of axisymmetry. Also, these constants can be computed in an indirect manner, by applying analytical solutions to the final system of equations [79], as it is presented in detail in Section 5.1.

3.1.6 Displacements and stresses in the domain

From the solution u_i and t_i along the boundary, displacements at a point $\mathrm{P}(\xi, z')$ in the domain can be obtained by the Somigliana's identity, expressed in Eq. (3-6). Stresses in the domain can be evaluated by applying the Somigliana's identity to the constitutive relations given by Eqs. (2-12) to (2-17), leading to

$$
\sigma_{mn}(\mathrm{P}) = 2\pi \int_\Gamma \bar{t}^{*\mathrm{f}}_{imn}(\mathrm{P},\mathrm{Q})\, u_i(\mathrm{Q})\, r\, \mathrm{d}\Gamma(r,z) + 2\pi \int_\Gamma \bar{u}^{*\mathrm{f}}_{imn}(\mathrm{P},\mathrm{Q})\, t_i(\mathrm{Q})\, r\, \mathrm{d}\Gamma(r,z) \tag{3-32}
$$

where u_i and t_i are displacements and traction forces along the boundary interpolated from nodal values, as expressed in Eq. (3-7). The evaluation of $\bar{u}^{*\mathrm{f}}_{imn}$ and $\bar{t}^{*\mathrm{f}}_{imn}$ is a cumbersome task since it involves the derivatives of the fundamental solutions $u^{*\mathrm{f}}_{im}$ and $t^{*\mathrm{f}}_{im}$. Their expressions were tabulated by Tan [80, 81] in terms of complete elliptic integrals. However, they can be written in a more compact form in terms of integrals of Lipschitz-Hankel type as developed in the frame of the present theoretical investigations. Thus, for $\xi > 0$, one has

$$\bar{u}^{*f}_{r(rr)} = \frac{1}{8\pi(1-\nu)}\left\{\frac{1}{\xi}\left[-P_4\,\bar{I}_{110} + |\bar{z}|\,\bar{I}_{111}\right] + P_3\,\bar{I}_{001} - |\bar{z}|\,\bar{I}_{012}\right\} \tag{3-33}$$

$$\bar{u}^{*f}_{z(rr)} = \frac{1}{8\pi(1-\nu)}\left\{-\frac{\bar{z}\,\bar{I}_{101}}{\xi} - sign(\bar{z})\,2\nu\,\bar{I}_{001} + \bar{z}\,\bar{I}_{002}\right\} \tag{3-34}$$

$$\bar{u}^{*f}_{r(rz)} = \frac{1}{8\pi(1-\nu)}\left\{-sign(\bar{z})\,2\,P_1\,\bar{I}_{111} + \bar{z}\,\bar{I}_{112}\right\} \tag{3-35}$$

$$\bar{u}^{*f}_{z(rz)} = \frac{1}{8\pi(1-\nu)}\left\{-P_2\,\bar{I}_{101} - |\bar{z}|\,\bar{I}_{102}\right\} \tag{3-36}$$

$$\bar{u}^{*f}_{r(zz)} = \frac{1}{8\pi(1-\nu)}\left\{-P_2\,\bar{I}_{011} + |\bar{z}|\,\bar{I}_{012}\right\} \tag{3-37}$$

$$\bar{u}^{*f}_{z(zz)} = \frac{1}{8\pi(1-\nu)}\left\{-sign(\bar{z})\,2\,P_1\,\bar{I}_{001} - \bar{z}\,\bar{I}_{002}\right\} \tag{3-38}$$

and

$$\bar{\sigma}^{*f}_{rr(rr)} = \frac{\mu}{4\pi(1-\nu)}\left\{\frac{1}{\xi}\left[P_3\,\bar{I}_{101} - |\bar{z}|\,\bar{I}_{102}\right] + \frac{1}{\xi\,r}\left[-P_4\,\bar{I}_{110} + |\bar{z}|\,\bar{I}_{111}\right] + \right.$$
$$\left. \frac{1}{r}\left[P_3\,\bar{I}_{011}|\bar{z}|\,\bar{I}_{012}\right] - 3\,\bar{I}_{002} - |\bar{z}|\,\bar{I}_{003}\right\} \tag{3-39}$$

$$\bar{\sigma}^{*f}_{rz(rr)} = \frac{\mu}{4\pi(1-\nu)}\left\{\frac{1}{\xi}\left[sign(\bar{z})\,2P_1\,\bar{I}_{111} - \bar{z}\,\bar{I}_{112}\right] - sign(\bar{z})\,2\,\bar{I}_{012} + \bar{z}\,\bar{I}_{013}\right\} \tag{3-40}$$

$$\bar{\sigma}^{*f}_{zz(rr)} = \frac{\mu}{4\pi(1-\nu)}\left\{\frac{1}{\xi}\left[-P_2\,\bar{I}_{101} + |\bar{z}|\,\bar{I}_{102}\right] + \bar{I}_{002} - |\bar{z}|\,\bar{I}_{003}\right\} \tag{3-41}$$

$$\bar{\sigma}^{*f}_{rr(rz)} = \frac{\mu}{4\pi(1-\nu)}\left\{\frac{1}{r}\left[-sign(\bar{z})\,2P_1\,\bar{I}_{111} + \bar{z}\,\bar{I}_{112}\right] + sign(\bar{z})\,2\,\bar{I}_{102} - \bar{z}\,\bar{I}_{103}\right\} \tag{3-42}$$

$$\bar{\sigma}^{*f}_{rz(rz)} = \frac{\mu}{4\pi(1-\nu)}\left\{\bar{I}_{112} - |\bar{z}|\,\bar{I}_{113}\right\} \tag{3-43}$$

$$\bar{\sigma}^{*f}_{zz(rz)} = \frac{\mu\,\bar{z}\,\bar{I}_{103}}{4\pi(1-\nu)} \tag{3-44}$$

$$\bar{\sigma}^{*f}_{rr(zz)} = \frac{\mu}{4\pi(1-\nu)}\left\{\frac{1}{r}\left[-P_2\,\bar{I}_{011} + |\bar{z}|\,\bar{I}_{012}\right] + \bar{I}_{002} - |\bar{z}|\,\bar{I}_{003}\right\} \tag{3-45}$$

$$\bar{\sigma}^{*f}_{rz(zz)} = -\frac{\mu\,\bar{z}\,\bar{I}_{013}}{4\pi(1-\nu)} \tag{3-46}$$

$$\bar{\sigma}^{*f}_{zz(zz)} = \frac{\mu}{4\pi(1-\nu)}\left\{\bar{I}_{002} + |\bar{z}|\,\bar{I}_{003}\right\} \tag{3-47}$$

where

$$\bar{t}^{*f}_{imn} = \bar{\sigma}^{*f}_{ijmn}\,\eta_j = \bar{\sigma}^{*f}_{jimn}\,\eta_j \tag{3-48}$$

and

$$P_1 = 1-\nu, \quad P_2 = 1-2\nu, \quad P_3 = 3-2\nu, \quad P_4 = 3-4\nu \tag{3-49}$$

For $\xi = 0$, one can derive $\bar{u}^{*f}_{imn}$ and $\bar{t}^{*f}_{imn}$ by taking the limit of the above equations. The expressions for $\lim_{\xi\to 0}\bar{I}_{pq\lambda}$ are listed in Appendix A.

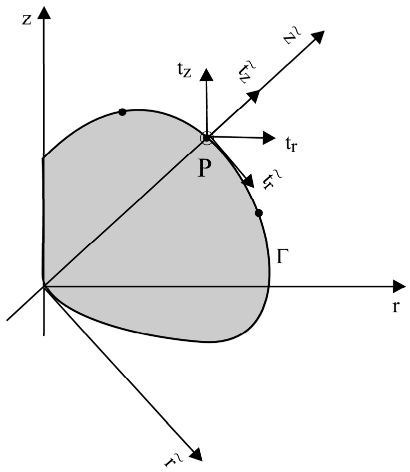
3.1.7
Stresses on the boundary

Stresses at a point P(ξ, z') on the boundary can be obtained by substituting for $u_m(P)$ according to the Somigliana's identity of Eq. (3-6) in the constitutive relations for axisymmetry, given by Eqs. (2-12) to (2-17). As a result, the integral equation becomes hypersingular. This integral was first presented by Lacerda & Wrobel [82], with contributions by Mukherjee [83] regarding its numerical integration.

Due to the complexity in evaluating these hypersingular integrals, this work adopts the approach of interpolating the nodal results in a local coordinate system [73]. The nodal values of traction forces and displacements can be rotated to the local coordinate system $(\tilde{r}, \tilde{z})$ at P(ξ, z'), as depicted in Fig. 3.4, to arrive at

$$\tilde{\sigma}_{zz} = \tilde{t}_z \qquad \text{and} \qquad \tilde{\sigma}_{rz} = \tilde{t}_r \tag{3-50}$$

where $\tilde{\sigma}_{ij}$ and $\tilde{t}_i$ are stresses and traction forces in the local coordinate system.



(a)

Figure 3.4: Local coordinate system at point P(ξ, z')

The remaining stresses are given by

$$\tilde{\sigma}_{rr} = \frac{1}{(1-\nu)} \left[2\mu\left(\tilde{\epsilon}_{rr} + \nu\,\tilde{\epsilon}_{\theta\theta}\right) + \nu\,\tilde{\sigma}_{zz}\right] \tag{3-51}$$

$$\tilde{\sigma}_{\theta\theta} = \frac{1}{(1-\nu)} \left[2\mu\left(\tilde{\epsilon}_{\theta\theta} + \nu\,\tilde{\epsilon}_{rr}\right) + \nu\,\tilde{\sigma}_{zz}\right] \tag{3-52}$$

in which, for $r \neq 0$, $\tilde{\epsilon}_{rr} = \epsilon_{rr} = \frac{u_r}{r}$ and $\tilde{\epsilon}_{\theta\theta} = \frac{\partial \tilde{u}_r}{\partial \tilde{r}}$; and for $r = 0$, $\tilde{\epsilon}_{tt} = \tilde{\epsilon}_{rr}$. Displacements $\tilde{u}_r$ are evaluated by interpolating the nodal displacement values along the element, as expressed in Eq. (3-7). Finally, stresses in the global coordinate system are determined by back rotating the tensor $\tilde{\sigma}_{ij}$.

3.2
Formulation for the axisymmetric halfspace problem

3.2.1
Boundary integral equation

In a manner similar to that used for the fullspace, displacements $u_m(\mathrm{P})$ in the domain Ω of an elastic halfspace can be expressed by the Somigliana's identity

$$u_m(\mathrm{P}) = -2\pi \int_{\Gamma} t^{*\mathrm{h}}_{im}(\mathrm{P},\mathrm{Q})\, u_i(\mathrm{Q})\, r\, \mathrm{d}\Gamma(r,z) + 2\pi \int_{\Gamma} u^{*\mathrm{h}}_{im}(\mathrm{P},\mathrm{Q})\, t_i(\mathrm{Q})\, r\, \mathrm{d}\Gamma(r,z) \quad \text{(3-53)}$$

where $\Gamma(r,z) = \Gamma_i \cup \Gamma_s \cup \Gamma_0$ is the boundary of the meridian plane shown in Fig. 3.3. In this figure, Γ_i, Γ_s and Γ_∞ refer to the internal boundary, the loaded portion of the boundary at $z = 0$ and the traction free portion of the boundary at $z = 0$, respectively.

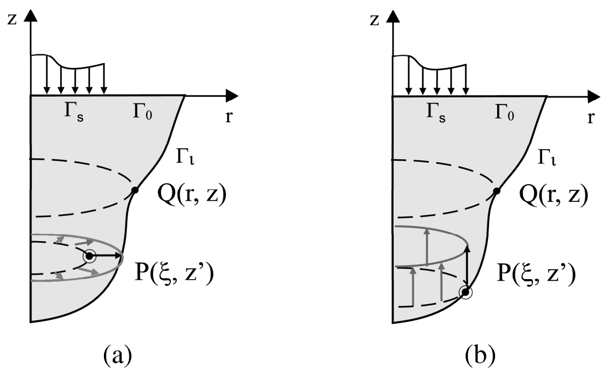


Figure 3.5: Axisymmetric halfspace submitted to: a) radial ring load; b) axial ring load.

The fundamental solutions $u^{*\mathrm{h}}_{im}$ and $t^{*\mathrm{h}}_{im}$ are displacements and traction forces in the halfspace that satisfy in advance the boundary conditions at $z = 0$. Since there are by definition no traction forces $t_i(\mathrm{Q})$ on Γ_0, Eq. (3-53) simplifies to

$$u_m(\mathrm{P}) = -2\pi \int_{\Gamma_i} t^{*\mathrm{h}}_{im}(\mathrm{P},\mathrm{Q})\, u_i(\mathrm{Q})\, r\, \mathrm{d}\Gamma(r,z) + 2\pi \int_{\bar{\Gamma}} u^{*\mathrm{h}}_{im}(\mathrm{P},\mathrm{Q})\, t_i(\mathrm{Q})\, r\, \mathrm{d}\Gamma(r,z) \quad \text{(3-54)}$$

where $\bar{\Gamma} = \Gamma_i \cup \Gamma_s$. From the above expression on may obtain a system of boundary integral equations and the matrix governing equation, provided that the fundamental solution for the halfspace is available, as presented in Section 2.3.

3.2.2
Matrix governing equation

Taking the point $\mathrm{P}(\xi, z')$ to the boundary in the Somigliana's identity given by Eq. (3-54), the following integral equation is obtained

$$2\pi \int_{\Gamma_i} t^{*\mathrm{h}}_{im}(\mathrm{P},\mathrm{Q})\, u_i(\mathrm{Q})\, r\, \mathrm{d}\Gamma(r,z) + \delta^*_{im}\, u_i(\mathrm{Q}) = 2\pi \int_{\bar{\Gamma}} u^{*\mathrm{h}}_{im}(\mathrm{P},\mathrm{Q})\, t_i(\mathrm{Q})\, r\, \mathrm{d}\Gamma(r,z) \quad (3\text{-}55)$$

in which the first integral is singular. Displacements and traction forces can be approximated by the relations given by Eq. (3-7), arriving at

$$H_{mn}\, u_n = G_{ml}\, t_l \quad \text{or} \quad \mathbf{H\,u} = \mathbf{G\,t} \qquad (3\text{-}56)$$

where

$$H_{mn} = 2\pi \int_{\Gamma_i} t^{*\mathrm{h}}_{im}(\mathrm{P},\mathrm{Q})\, u_{in}(\mathrm{Q})\, r\, \mathrm{d}\Gamma(r,z) + \delta^*_{mn} = \widehat{H}_{mn} + c^{\mathrm{h}}_{mn} \qquad (3\text{-}57)$$

$$G_{ml} = 2\pi \int_{\bar{\Gamma}} u^{*\mathrm{h}}_{im}(\mathrm{P},\mathrm{Q})\, t_{il}(\mathrm{Q})\, r\, \mathrm{d}\Gamma(r,z) \qquad (3\text{-}58)$$

Eq. (3-57) is expressed in terms of the Cauchy principal value of the singular integral

$$\widehat{H}_{mn} = 2\pi\, \fint_{\Gamma_i} t^{*\mathrm{h}}_{im}(\mathrm{P},\mathrm{Q})\, u_{in}(\mathrm{Q})\, r\, \mathrm{d}\Gamma(r,z) \qquad (3\text{-}59)$$

and of the discontinuity term

$$c^{\mathrm{h}}_{mn} = \begin{cases} \delta^*_{mn} & \text{if } \mathrm{P}(r,z) \in \Gamma_s \\ \bar{c}^{\mathrm{h}}_{mn} & \text{if } \mathrm{P}(r,z) \in \Gamma_i \end{cases} \qquad (3\text{-}60)$$

The constants $\bar{c}^{\mathrm{h}}_{mn}$ can be obtained similarly to the procedure presented in Section 3.1.5 for the fullspace. For the halfspace, the fundamental solution can be decomposed as $t^{*\mathrm{h}}_{im} = t^{*\mathrm{f}}_{im} + t^{*\mathrm{d}}_{im}$, as presented in Section 2.3, where term $t^{*\mathrm{f}}_{im}$ is the fundamental solution for the fullspace. Since the difference term $t^{*\mathrm{d}}_{im}$ has no singularity if $\mathrm{P}(\xi, z') \in \Gamma_i$, i.e. $z \neq 0$, the constants $\bar{c}^{\mathrm{h}}_{mn}$ are identical to those derived for the fullspace in Section Seção 3.1.5 and thus

$$\bar{c}^{\mathrm{h}}_{mn} = c^{\mathrm{f}}_{mn} \qquad (3\text{-}61)$$

Equation (3-56) can be rearranged in a system of equations, as mentioned in Section 3.1.3 for the fullspace formulation. Alternatively, one may also find a stiffness matrix ax expressed by Eq. (3-19), as presented in Section 3.1.4.

Notice that the submatrices of **H** becomes the identity matrix when the integration refers to the boundary Γ_s. In this case, one has $z = 0$ and as a consequence $\widehat{H}_{mn} = 0$ and $c^{\text{h}}_{mn} = \delta^{*}_{mn}$.

3.2.3
Displacements and stresses in the domain

Displacements at a point $\text{P}(\xi, z')$ in the domain can be directly evaluated by the Somigliana's identity given by Eq. (3-54), similarly to the procedure presented in Section 3.1.6 for the fullspace. Stresses in the domain are evaluated by applying this equation to the constitutive relation, arriving at

$$\sigma_{mn}(\text{P}) = 2\pi \int_{\Gamma_i} \bar{t}^{*\text{h}}_{imn}(\text{P},\text{Q})\, u_i(\text{Q})\, r\, \text{d}\Gamma(r,z) + 2\pi \int_{\bar{\Gamma}} \bar{u}^{*\text{h}}_{imn}(\text{P},\text{Q})\, t_i(\text{Q})\, r\, \text{d}\Gamma(r,z) \quad (3\text{-}62)$$

where u_i and t_i are displacements and traction forces interpolated from the nodal solutions along the boundary. The halfspace fundamental solutions can be decomposed as indicated in Eq. (2-61) and, as a consequence, $\bar{u}^{*\text{s}}_{imn}$ and $\bar{t}^{*\text{s}}_{imn}$ can be expressed as

$$\bar{u}^{*\text{h}}_{imn} = \bar{u}^{*\text{f}}_{imn} + \bar{u}^{*\text{d}}_{imn} \quad (3\text{-}63)$$

$$\bar{t}^{*\text{h}}_{imn} = \bar{t}^{*\text{f}}_{imn} + \bar{t}^{*\text{d}}_{imn} \quad (3\text{-}64)$$

where $\bar{u}^{*\text{f}}_{imn}$ and $\bar{t}^{*\text{f}}_{imn}$ are the functions listed in Eqs. (3-33) to (3-47) for the fullspace.

The functions $\bar{u}^{*\text{d}}_{imn}$ and $\bar{t}^{*\text{d}}_{imn}$ for $\xi \neq 0$ can be written in terms of integrals of Lipschitz-Hankel type as

$$\bar{u}^{*\text{d}}_{r(rr)} = \frac{1}{8\pi(1-\nu)} \left\{ \frac{1}{\xi} \left[-P_6\, \hat{I}_{110} + P_4\, |\hat{z}|\, \hat{I}_{111} - 2zz'\, \hat{I}_{112} \right] + P_5\, \hat{I}_{011} - sign(\hat{z})\, (P_4\, z' + 3z)\, \hat{I}_{012} + 2zz' \hat{I}_{013} \right\} \quad (3\text{-}65)$$

$$\bar{u}^{*\text{d}}_{z(rr)} = \frac{1}{8\pi(1-\nu)} \left\{ \frac{1}{\xi} \left[sign(\hat{z})\, 4P_1\, P_2\, \hat{I}_{100} - P_4\, \bar{z}\, \hat{I}_{101} - sign(\hat{z})\, 2zz'\, \hat{I}_{102} \right] - sign(\hat{z})\, 2\, P_3\, \hat{I}_{001} + (P_4\, z' - 3z)\, \hat{I}_{002} + sign(\hat{z})\, 2zz'\, \hat{I}_{003} \right\} \quad (3\text{-}66)$$

$$\bar{u}^{*\text{d}}_{r(rz)} = \frac{1}{8\pi(1-\nu)} \left\{ -sign(\hat{z})\, 2P_1\, \hat{I}_{111} + (P_4\, z' + z)\, \hat{I}_{112} - sign(\hat{z})\, 2zz'\, \hat{I}_{113} \right\} \quad (3\text{-}67)$$

$$\bar{u}^{*\text{d}}_{z(rz)} = \frac{1}{8\pi(1-\nu)} \left\{ P_2\, \hat{I}_{101} - sign(\hat{z})\, (P_4\, z' - z)\, \hat{I}_{102} - 2zz'\, \hat{I}_{103} \right\} \quad (3\text{-}68)$$

$$\bar{u}^{*\text{d}}_{r(zz)} = \frac{1}{8\pi(1-\nu)} \left\{ P_2\, \hat{I}_{011} + sign(\hat{z})\, (P_4\, z' - z)\, \hat{I}_{012} - 2zz'\, \hat{I}_{013} \right\} \quad (3\text{-}69)$$

$$\bar{u}^{*\text{d}}_{z(zz)} = \frac{1}{8\pi(1-\nu)} \left\{ -sign(\hat{z})\, 2\, P_1\, \hat{I}_{001} - (P_4\, z' + z)\, \hat{I}_{002} - 2\, sign(\hat{z})\, zz' \hat{I}_{003} \right\} \quad (3\text{-}70)$$

and

$$\bar{\sigma}^{*d}_{rr(rr)} = \frac{\mu}{4\pi(1-\nu)}\left\{\frac{1}{\xi}\left[P_5\,\hat{I}_{101} - sign(\hat{z})\,(P_4\,z + 3z')\,\hat{I}_{102} + 2zz'\,\hat{I}_{103}\right] + \frac{1}{\xi\,r}\left[-P_6\,\hat{I}_{110} - 2zz'\,\hat{I}_{112} + P_4\,|\hat{z}|\,\hat{I}_{111}\right] + \frac{1}{r}\left[P_5\,\hat{I}_{011} - sign(\hat{z})\,(P_4\,z' + 3z)\,\hat{I}_{012} + 2zz'\,\hat{I}_{013}\right] - 5\hat{I}_{002} + 3|\hat{z}|\,\hat{I}_{003} - 2zz'\,\hat{I}_{004}\right\} \quad (3\text{-}71)$$

$$\bar{\sigma}^{*d}_{rz(rr)} = \frac{\mu}{4\pi(1-\nu)}\left\{\frac{1}{\xi}\left[-sign(\hat{z})\,2P_1\,\hat{I}_{111} + (P_4\,z + z')\,\hat{I}_{112} - sign(\hat{z})\,2zz'\,\hat{I}_{113}\right] + sign(\hat{z})\,2\,\hat{I}_{012} - (3z + z')\,\hat{I}_{013} + sign(\hat{z})\,2zz'\,\hat{I}_{014}\right\} \quad (3\text{-}72)$$

$$\bar{\sigma}^{*d}_{zz(rr)} = \frac{\mu}{4\pi(1-\nu)}\left\{\frac{1}{\xi}\left[P_2\,\hat{I}_{101} + sign(\hat{z})\,(P_4\,z - z')\,\hat{I}_{102} - 2zz'\,\hat{I}_{103}\right] - \hat{I}_{002} + sign(\hat{z})\,(-3z + z')\,\hat{I}_{003} + 2zz'\,\hat{I}_{004}\right\} \quad (3\text{-}73)$$

$$\bar{\sigma}^{*d}_{rr(rz)} = \frac{\mu}{4\pi(1-\nu)}\left\{\frac{1}{r}\left[-sign(\hat{z})\,2P_1\,\hat{I}_{111} + (P_3\,z' + z)\,\hat{I}_{112} - sign(\hat{z})\,2zz'\,\hat{I}_{113}\right] + sign(\hat{z})\,2\,\hat{I}_{102} - (z + 3z')\,\hat{I}_{103} + sign(\hat{z})\,2zz'\,\hat{I}_{104}\right\} \quad (3\text{-}74)$$

$$\bar{\sigma}^{*d}_{rz(rz)} = \frac{\mu}{4\pi(1-\nu)}\left\{-\hat{I}_{112} + |\hat{z}|\,\hat{I}_{113} - 2zz'\,\hat{I}_{114}\right\} \quad (3\text{-}75)$$

$$\bar{\sigma}^{*d}_{zz(rz)} = \frac{\mu}{4\pi(1-\nu)}\left\{-\bar{z}\,\hat{I}_{103} - sign(\hat{z})\,2zz'\,\hat{I}_{104}\right\} \quad (3\text{-}76)$$

$$\bar{\sigma}^{*d}_{rr(zz)} = \frac{\mu}{4\pi(1-\nu)}\left\{\frac{1}{r}\left[P_2\,\hat{I}_{011} + sign(\hat{z})\,(P_4\,z' - z)\,\hat{I}_{012} - 2zz'\,\hat{I}_{013}\right] - \hat{I}_{002} + sign(\hat{z})\,(z - 3z')\,\hat{I}_{003} + 2zz'\hat{I}_{004}\right\} \quad (3\text{-}77)$$

$$\bar{\sigma}^{*d}_{rz(zz)} = \frac{\mu}{4\pi(1-\nu)}\left\{\bar{z}\,\hat{I}_{013} - sign(\hat{z})\,2zz'\,\hat{I}_{014}\right\} \quad (3\text{-}78)$$

$$\bar{\sigma}^{*d}_{zz(zz)} = \frac{\mu}{4\pi(1-\nu)}\left\{-\hat{I}_{002} - |\hat{z}|\,\hat{I}_{003} - 2zz'\hat{I}_{004}\right\} \quad (3\text{-}79)$$

where

$$\begin{array}{lll} P_1 = 1-\nu & P_2 = 1-2\nu & P_3 = 2-3\nu \\ P_4 = 3-4\nu & P_5 = 5-6\nu & P_6 = 5-12\nu+8\nu^2 \end{array} \quad (3\text{-}80)$$

For $\xi = 0$, these functions can be obtained in terms of limits. The expressions for $\lim_{\xi\to 0}\hat{I}_{pq\lambda}$ are listed in Appendix A.

Notice that these equations are also valid to evaluate displacements and stresses on the non-discretized boundary Γ_∞ at the free surface, for $z = 0$.

Stresses at a point $\mathrm{P}(\xi, z')$ on the boundary can be evaluated in a similar manner to that presented in Section 3.1.7 for the fullspace.