2
Fundamental solutions for an axisymmetric isotropic elastic medium

An usual approach for solving problems in elasticity is the use of fundamental solutions, constituted of displacements and stresses due to either distributed or point loads in a fullspace, a halfspace or a layered media. These solutions are used for the evaluation of influence coefficients in analytical, semi-analytical or numerical analyses of problems of more complex geometry and boundary conditions.

The boundary element method for axisymmetric problems of elasticity makes use of fundamental solutions, in cylindrical coordinates (r, θ, z), due to ring loads of intensity $\frac{1}{2\pi\xi}$ applied in the radial, tangential and axial directions, as depicted in Fig. 2.1. The complete formulation of the method requires the expressions for displacements and stresses at a point $Q(r, z)$ due to ring loads applied at any other point $P(\xi, z')$ of the medium. These solutions are denoted by u^*_{im} and σ^*_{ijm}, in which the index m refers to the direction of the load applied at $P(\xi, z')$. The indexes i and j stand for the displacements and stresses components measured at $Q(r, z)$.

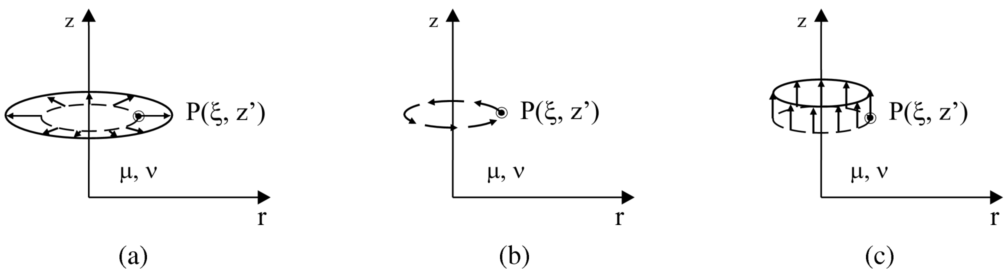

Figure 2.1: Ring loads: (a) in the radial direction; (b) in the tangential direction; (c) in the axial direction.

This work addresses axisymmetric applications in homogeneous fullspace and halsfspace media involving radial and axial loads. Accordingly, the fundamental solutions due to ring loads in the radial and axial directions are necessary.

For the fullspace, displacements due to ring loads were first derived by Kermanidis [12], by applying Betti's theorem to Papkovich-Neuber solution [61] for an infinite elastic medium. Later, Cruse et al. [15] and Bakr & Fenner [21] solved Navier's equilibrium equations by expressing displacements as Galerkin vectors [61] and considering ring loads as body forces. Also, Shippy et al. [62]

integrated Kelvin's solution [61] for the three-dimensional infinite medium along a circular path centered in the axisymmetric axis.

For the halfspace, Hasegawa [13, 14] deduced displacements and stresses from stress functions [63] obtained by means of Fourier and Hankel transforms and considering ring loads as body forces. Later on, Selvadurai & Rajapakse [2] imposed boundary conditions and continuity conditions to displacements and stresses expressed by Muki's solution [64, 65] and arrived at the same solutions. These solutions were also obtained by Hanson & Wang [66] as a particular case of the transversaly isotropic medium.

Both axisymmetric fundamental solutions for fullspace and halfspace can be expressed by means of either integrals of Lipschitz-Hankel type involving products of Bessel functions [67], or complete elliptic integrals of the first and second types [68], or Legendre functions [68]. In this section, the approach presented by Selvadurai & Rajapakse [2] is adopted. Expressions are written in terms of integrals of Lipschitz-Hankel type [67]

$$I_{pq\lambda}(\xi, r; c) = \int_0^\infty J_p(\xi t)\, J_q(rt)\, e^{-ct}\, t^\lambda \,\mathrm{d}t \tag{2-1}$$

in which p, q and λ are integers, $J_p(\xi t)$ and $J_q(rt)$ are Bessel functions of the first kind of order p and q, respectively. The integrals occurring in the axisymmetric fundamental solutions are convergent [67] and their closed form expressions are listed in Appendix A in terms of complete elliptic integrals of the first, second and third kinds [68].

2.1 Governing equations

Strains $\underset{\approx}{\epsilon}$ due to displacements $\underset{\sim}{u}$ in a isotropic elastic medium are given by

$$\underset{\approx}{\epsilon} = \frac{1}{2}\left(\underset{\sim}{\nabla}\,\underset{\sim}{u} + \underset{\sim}{u}\,\underset{\sim}{\nabla}\right) \tag{2-2}$$

in which $\underset{\approx}{\delta}$ is the Kronecker delta and $\underset{\sim}{\nabla}$ is the Del operator. For axisymmetric problems, displacements do not depend on θ. Thus, applying the Del operator in cylindrical coordinates,

$$\underset{\sim}{\nabla} = \hat{\imath}_r \frac{\partial}{\partial r} + \hat{\imath}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\imath}_z \frac{\partial}{\partial z} \tag{2-3}$$

the expressions for strains in Eq. (2-2) may be written as

$$\epsilon_{rr} = u_{r,r} \tag{2-4}$$

$$\epsilon_{r\theta} = \frac{1}{2}\left(-\frac{u_\theta}{r} + u_{\theta,r}\right) \tag{2-5}$$

$$\epsilon_{rz} = \frac{1}{2}\left(u_{r,z} + u_{z,r}\right) \tag{2-6}$$

$$\epsilon_{\theta\theta} = \frac{u_r}{r} \tag{2-7}$$

$$\epsilon_{\theta z} = \frac{1}{2}u_{\theta,z} \tag{2-8}$$

$$\epsilon_{zz} = u_{z,z} \tag{2-9}$$

Stresses $\underset{\approx}{\sigma}$ are given by the constitutive relation

$$\underset{\approx}{\sigma} = 2\mu\,\underset{\approx}{\epsilon} + \lambda\,\underset{\approx}{\delta}\,(\underset{\sim}{\nabla}\cdot\underset{\approx}{\epsilon}) \tag{2-10}$$

where μ and λ are the Lamé coefficients, also expressed as

$$\mu = \frac{E}{2(1+\nu)} \qquad \text{e} \qquad \lambda = \frac{2\mu\,\nu}{1-2\nu} \tag{2-11}$$

in which μ is the elastic shear modulus, E is the Young's modulus and ν is the Poisson's ratio. Substituting for strain and displacements given by Eqs. (2-4) to (2-9) leads to

$$\sigma_{rr} = 2\mu\left(u_{r,r} + \frac{\nu}{1-2\nu}\,\Delta\right) \tag{2-12}$$

$$\sigma_{r\theta} = \mu\left(-\frac{u_\theta}{r} + u_{\theta,r}\right) \tag{2-13}$$

$$\sigma_{rz} = \mu\,(u_{r,z} + u_{z,r}) \tag{2-14}$$

$$\sigma_{\theta\theta} = 2\mu\left(\frac{u_r}{r} + \frac{\nu}{1-2\nu}\,\Delta\right) \tag{2-15}$$

$$\sigma_{\theta z} = \mu\,u_{\theta,z} \tag{2-16}$$

$$\sigma_{zz} = 2\mu\left(u_{z,z} + \frac{\nu}{1-2\nu}\,\Delta\right) \tag{2-17}$$

where

$$\Delta = u_{r,r} + \frac{u_r}{r} + u_{z,z} \tag{2-18}$$

Traction forces are given by

$$\underset{\sim}{t} = \underset{\sim}{\eta}\cdot\underset{\approx}{\sigma} \tag{2-19}$$

in which $\underset{\sim}{\eta}$ is the surface unit outward normal.

The equilibrium equation in the domain, for applied body forces $\underset{\sim}{b}$,

$$\underset{\sim}{\nabla} \cdot \underset{\approx}{\sigma} + \underset{\sim}{b} = 0 \quad (2\text{-}20)$$

is written in cylindrical coordinates as

$$\sigma_{rr,r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + \sigma_{rz,z} + b_r = 0 \quad (2\text{-}21)$$

$$\sigma_{r\theta,r} + \frac{2\sigma_{r\theta}}{r} + \sigma_{\theta z,z} + b_\theta = 0 \quad (2\text{-}22)$$

$$\sigma_{rz,r} + \frac{\sigma_{rz}}{r} + \sigma_{zz,z} + b_z = 0 \quad (2\text{-}23)$$

Substituting the strain and stress relations given by Eqs. (2-4) to (2-9) and Eqs. (2-12) to (2-17) into the equilibrium equation leads to the Navier-Cauchy equation

$$\mu\nabla^2\underset{\sim}{u} + (\mu + \lambda)\,\underset{\sim}{\nabla}\,(\underset{\sim}{\nabla} \cdot \underset{\sim}{u}) + \underset{\sim}{b} = 0 \quad (2\text{-}24)$$

where ∇^2 is the Laplacian operator. In cylindrical coordinates, the Laplacian operator becomes

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \quad (2\text{-}25)$$

and the equilibrium equations are given by

$$(1 - 2\nu)\left(\nabla^2 u_r - \frac{u_r}{r^2}\right) + \Delta_{,r} + b_r = 0 \quad (2\text{-}26)$$

$$(1 - 2\nu)\left(\nabla^2 u_\theta - \frac{u_\theta}{r^2}\right) + b_\theta = 0 \quad (2\text{-}27)$$

$$(1 - 2\nu)\,\nabla^2 u_z + \Delta_{,z} + b_z = 0 \quad (2\text{-}28)$$

The above expressions can be solved as two uncoupled problems, one referring to Eq. (2-27) considering only torsional loads and the other one in terms of Eqs. (2-26) and (2-28) for applied radial and axial loads. In the fundamental solutions for radial and axial loads, according to Eqs. (2-26) and (2-28), $u_\theta = 0$ and $\sigma_{r\theta} = \sigma_{\theta z} = 0$.

2.2 Fundamental solution for the axisymmetric fullspace

The fundamental solution can be derived from Muki's asymmetric solution [64, 65] of Eq. (2-24) in the absence of body forces. Muki represented displacements by means of harmonic and bi-harmonic functions and used Hankel transforms and their correspondence to Fourier transforms to arrive at the following solution

$$u_r = \frac{1}{2}\sum_{m=0}^{\infty}[U_{m+1}(r,z) - V_{m-1}(r,z)]\cos m\theta \tag{2-29}$$

$$u_\theta = \frac{1}{2}\sum_{m=0}^{\infty}[U_{m+1}(r,z) + V_{m-1}(r,z)]\sin m\theta \tag{2-30}$$

$$u_z = \sum_{m=0}^{\infty}\left[\int_0^\infty \left\{(1-2\nu)\frac{\mathrm{d}^2 G_m}{\mathrm{d}z^2} - 2(1-\nu)t^2 G_m\right\} t\, J_m(rt)\,\mathrm{d}t\right]\cos m\theta \tag{2-31}$$

where

$$U_{m+1}(r,z) = \int_0^\infty \left(\frac{\mathrm{d}G_m}{\mathrm{d}z} + 2H_m\right) t^2 J_{m+1}(rt)\,\mathrm{d}t \tag{2-32}$$

$$V_{m-1}(r,z) = \int_0^\infty \left(\frac{\mathrm{d}G_m}{\mathrm{d}z} - 2H_m\right) t^2 J_{m-1}(rt)\,\mathrm{d}t \tag{2-33}$$

$$G_m(t,z) = (A_m + B_m\, z)\, e^{tz} + (C_m + D_m\, z)\, e^{-tz} \tag{2-34}$$

$$H_m(t,z) = E_m\, e^{tz} + F_m\, e^{-tz} \tag{2-35}$$

in which $A_m(t), B_m(t), \ldots, F_m(t)$ are unknown functions to be determined from boundary conditions.

If axisymmetry about the z-axis is considered, them $m = 0$ and the radial and axial displacements may be simplified to

$$u_r = \frac{1}{2}\int_0^\infty \left(\frac{\mathrm{d}G}{\mathrm{d}z} + 2H\right)[J_1(rt) - J_{-1}(rt)]\; t^2\,\mathrm{d}t \tag{2-36}$$

$$u_z = \int_0^\infty \left[(1-2\nu)\,\frac{\mathrm{d}^2 G}{\mathrm{d}z} - 2(1-\nu)\, t^2\, G\right] J_0(rt)\,\mathrm{d}t \tag{2-37}$$

where

$$G(t,z) = (A + Bz)\, e^{zt} + (C + Dz)\, e^{-zt} \tag{2-38}$$

$$H(t,z) = E\, e^{zt} + F\, e^{-zt} \tag{2-39}$$

in which $A(t), B(t), \ldots, F(t)$ are unknown functions. The above equations constitute the general axisymmetric solution of Eqs. (2-26) and (2-28) in cylindrical coordinates, in the absence of body forces.

Consider a fullspace separated into two parts, I and II, by a plan normal to z at $z = z'$, as shown in Fig. 2.2. Applying Eqs. (2-36) and (2-37) to parts I and II leads to a total of 10 unknown functions. These functions can be obtained by applying regularity conditions of displacements and stresses as $z \to \pm\infty$,

$$u_i^{\mathrm{I,II}}(r, \pm\infty) = 0, \qquad \sigma_{ij}^{\mathrm{I,II}}(r, \pm\infty) = 0 \tag{2-40}$$

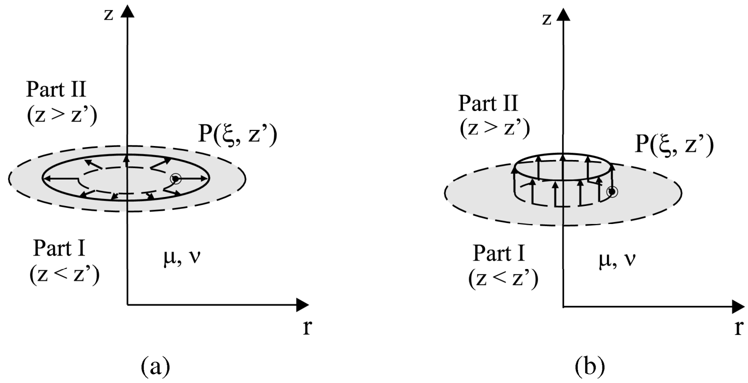

Figure 2.2: Ring loads in a fullspace: (a) in the radial direction; (b) in the axial direction.

compatibility conditions of displacements,

$$u_i^{\mathrm{I}}(r, z') = u_i^{\mathrm{II}}(r, z') \tag{2-41}$$

and equilibrium conditions for radial and axial unit ring loads applied at (ξ, z')

$$\begin{aligned}\sigma_{rr}^{\mathrm{I}}(r, z') - \sigma_{rr}^{\mathrm{II}}(r, z') &= \frac{\delta(r-\xi)}{2\pi\xi} = \frac{\mathscr{H}_1^{-1}[\mathscr{H}_1[\delta(r-\xi)]]}{2\pi\xi} \\ &= \frac{1}{2\pi\xi}\int_0^{\infty} J_1(\xi t)\, J_1(rt)\, \xi\, t\, \mathrm{d}t \end{aligned} \tag{2-42}$$

$$\sigma_{rz}^{\mathrm{I}}(r, z') - \sigma_{rz}^{\mathrm{II}}(r, z') = 0 \tag{2-43}$$

and

$$\sigma_{zr}^{\mathrm{I}}(r, z') - \sigma_{zr}^{\mathrm{II}}(r, z') = 0 \tag{2-44}$$

$$\begin{aligned}\sigma_{zz}^{\mathrm{I}}(r, z') - \sigma_{zz}^{\mathrm{II}}(r, z') &= \frac{\delta(r-\xi)}{2\pi\xi} = \frac{\mathscr{H}_0^{-1}[\mathscr{H}_0[\delta(r-\xi)]]}{2\pi\xi} \\ &= \frac{1}{2\pi\xi}\int_0^{\infty} J_0(\xi t)\, J_0(rt)\, \xi\, t\, \mathrm{d}t \end{aligned} \tag{2-45}$$

In these equations, δ is the Dirac delta [69], and $\mathscr{H}_m[f(r), r \to t] = \bar{f}(t)$ and $\mathscr{H}_m^{-1}[\bar{f}(t), t \to r] = \mathscr{H}_m[\bar{f}(t), t \to r] = f(r)$ are the Hankel transform of order m and its corresponding inverse, respectively [70]. The orders $m = 0$ and $m = 1$ of the Hankel transforms lead to the most simple solutions of the proposed problem.

The final expressions for displacements $u_i^{\mathrm{I}}(r, z)$ and $u_i^{\mathrm{II}}(r, z)$ can be combined, leading to the following equations for displacements $u_i^{\mathrm{I}}(r,z)$ e $u_i^{\mathrm{II}}(r,z)$ of a fullspace

$$u^{*\mathrm{f}}_{r(r)} = \frac{1}{16\pi\mu(1-\nu)}\left\{(3-4\nu)\,\bar{I}_{110} - |\bar{z}|\,\bar{I}_{111}\right\} \tag{2-46}$$

$$u^{*\mathrm{f}}_{z(r)} = \frac{\bar{z}\,\bar{I}_{101}}{16\pi\mu(1-\nu)} \tag{2-47}$$

$$u^{*\mathrm{f}}_{r(z)} = -\frac{\bar{z}\,\bar{I}_{011}}{16\pi\mu(1-\nu)} \tag{2-48}$$

$$u^{*\mathrm{f}}_{z(z)} = \frac{1}{16\pi\mu(1-\nu)}\left\{(3-4\nu)\,\bar{I}_{000} + |\bar{z}|\,\bar{I}_{001}\right\} \tag{2-49}$$

where

$$\bar{z} = z' - z \qquad \text{and} \qquad \bar{I}_{pq\lambda} = I_{pq\lambda}(\xi, r; c = |\bar{z}|) \tag{2-50}$$

and the superscript f stands for fullspace fundamental solution.

Applying the elastic constitutive relations of Eqs. (2-12) to (2-17), one obtains stresses $\sigma^{*\mathrm{f}}_{ijm}(\mathrm{P},\mathrm{Q})$ as

$$\sigma^{*\mathrm{f}}_{rr(r)} = \frac{1}{8(1-\nu)}\left\{-\frac{(3-4\nu)\,\bar{I}_{110}}{r} + \frac{|\bar{z}|\,\bar{I}_{111}}{r} + (3-2\nu)\,\bar{I}_{101} - |\bar{z}|\,\bar{I}_{102}\right\} \tag{2-51}$$

$$\sigma^{*\mathrm{f}}_{rz(r)} = \frac{1}{8(1-\nu)}\left\{sign(\bar{z})\,2(1-\nu)\,\bar{I}_{111} - \bar{z}\,\bar{I}_{112}\right\} \tag{2-52}$$

$$\sigma^{*\mathrm{f}}_{zz(r)} = \frac{1}{8(1-\nu)}\left\{-(1-2\nu)\,\bar{I}_{101} + |\bar{z}|\,\bar{I}_{102}\right\} \tag{2-53}$$

$$\sigma^{*\mathrm{f}}_{rr(z)} = \frac{1}{8(1-\nu)}\left\{\frac{\bar{z}\,\bar{I}_{011}}{r} + sign(\bar{z})\,2\nu\,\bar{I}_{001} - \bar{z}\,\bar{I}_{002}\right\} \tag{2-54}$$

$$\sigma^{*\mathrm{f}}_{rz(z)} = \frac{1}{8(1-\nu)}\left\{-(1-2\nu)\,\bar{I}_{011} - \bar{z}\,\bar{I}_{012}\right\} \tag{2-55}$$

$$\sigma^{*\mathrm{f}}_{zz(z)} = \frac{1}{8(1-\nu)}\left\{sign(\bar{z})\,2(1-\nu)\,\bar{I}_{001} + \bar{z}\,\bar{I}_{002}\right\} \tag{2-56}$$

where

$$sign(\bar{z}) = \begin{cases} 1 & \text{if } \bar{z} \geq 0 \\ -1 & \text{if } \bar{z} < 0 \end{cases} \tag{2-57}$$

In the above expressions, the indexes i and j refer to displacement and stress components measured at $\mathrm{Q}(r,z)$. The index m refers to both the direction and the location $\mathrm{P}(\xi, z')$ at which the ring load is applied, its value indicated between parenthesis.

If the ring load is applied at the axis of axisymmetry, i.e. $\xi = 0$, the load in the radial direction is naturally void and, consequently, $u^{*\mathrm{f}}_{i(r)}|_{\xi=0} = 0$ and $\sigma^{*\mathrm{f}}_{ij(r)}|_{\xi=0} = 0$. In such a case, the fundamental solution for the axial load simplifies to Kelvin's three-dimensional solution [61]. The expressions for $u^{*\mathrm{f}}_{i(z)}|_{\xi=0}$ and $\sigma^{*\mathrm{f}}_{ij(z)}|_{\xi=0}$ can be derived by taking the limit as $\xi \to 0$ in Eqs. (2-46) to (2-49) and Eqs. (2-51) to

(2-56). Appendix A presents the limits of the integrals of Lipschitz-Hankel type as $\xi \rightarrow= 0$.

2.3
Fundamental solution for the axisymmetric halfspace

An analogous procedure can be carried out for the axisymmetric halfspace. Consider a plan normal to z, at $z = z'$, separating the halfspace defined for $z \leq 0$ into two parts, as depicted in Fig. 2.3. Applying Eqs. (2-36) and (2-37) to each part

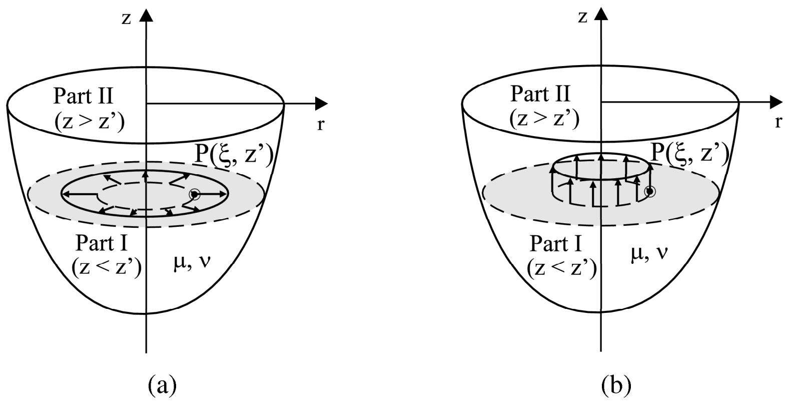

Figure 2.3: Ring loads in a halfspace: (a) in the radial direction; (b) in the axial direction.

of the halfspace leads to 10 unknown functions, as in the fullspace problem. These functions can be evaluated by applying regularity conditions of displacements and stresses at $z \rightarrow -\infty$ in part I,

$$u_i^{\mathrm{I}}(r, -\infty) = 0, \qquad \sigma_{ij}^{\mathrm{I}}(r, -\infty) = 0 \tag{2-58}$$

traction free boundary condition on part II,

$$\sigma_{zj}^{\mathrm{II}}(r, 0) = 0 \tag{2-59}$$

displacement compatibility conditions and equilibrium conditions for the radial and axial ring loads expressed in Eqs. (2-41) to (2-45).

The expressions of displacements for parts I and II can be combined and a similar procedure can also be applied to the complementary halsfspace $z \geq 0$. The final expressions of displacements $u_{im}^{*\mathrm{h}}(\mathrm{P}, \mathrm{Q})$ and their corresponding stresses $\sigma_{ijm}^{*\mathrm{h}}(\mathrm{P}, \mathrm{Q})$ are given by

$$u_{im}^{*\mathrm{h}}(\mathrm{P}, \mathrm{Q}) = u_{im}^{*\mathrm{f}}(\mathrm{P}, \mathrm{Q}) + u_{im}^{*\mathrm{d}}(\mathrm{P}, \mathrm{Q}) \tag{2-60}$$

$$\sigma_{ijm}^{*\mathrm{h}}(\mathrm{P}, \mathrm{Q}) = \sigma_{ijm}^{*\mathrm{f}}(\mathrm{P}, \mathrm{Q}) + \sigma_{ijm}^{*\mathrm{d}}(\mathrm{P}, \mathrm{Q}) \tag{2-61}$$

in which $u^{*\mathrm{f}}_{im}(\mathrm{P},\mathrm{Q})$ and $\sigma^{*\mathrm{f}}_{ijm}(\mathrm{P},\mathrm{Q})$ are the fullspace fundamental solution given by Eqs. (2-46) to (2-49) and Eqs. (2-51) to (2-56). The index d in the remaining terms $u^{*\mathrm{d}}_{im}(\mathrm{P},\mathrm{Q})$ and $\sigma^{*\mathrm{d}}_{ijm}(\mathrm{P},\mathrm{Q})$ refers to the difference between the halfspace and fullspace fundamental solutions; and for the halfspace defined either for $z \leq\!< 0$ or $z \geq\!> 0$ are

$$u^{*\mathrm{d}}_{r(r)} = \frac{1}{16\pi\mu(1-\nu)}\left\{(5-12\nu+8\nu^2)\,\hat{I}_{110} - (3-4\nu)\,|\hat{z}|\,\hat{I}_{111} + 2zz'\,\hat{I}_{112}\right\} \tag{2-62}$$

$$u^{*\mathrm{d}}_{z(r)} = \frac{1}{16\pi\mu(1-\nu)}\left\{-4(1-\nu)(1-2\nu)\,sign(\hat{z})\,\hat{I}_{100} + (3-4\nu)\,\bar{z}\,\hat{I}_{101} + 2zz'\,sign(\hat{z})\,\hat{I}_{102}\right\} \tag{2-63}$$

$$u^{*\mathrm{d}}_{r(z)} = \frac{1}{16\pi\mu(1-\nu)}\left\{-4(1-\nu)(1-2\nu)\,sign(\hat{z})\,\hat{I}_{010} - (3-4\nu)\,\bar{z}\,\hat{I}_{011} + 2zz'\,sign(\hat{z})\,\hat{I}_{012}\right\} \tag{2-64}$$

$$u^{*\mathrm{d}}_{z(z)} = \frac{1}{16\pi\mu(1-\nu)}\left\{(5-12\nu+8\nu^2)\,\hat{I}_{000} + (3-4\nu)\,|\hat{z}|\,\hat{I}_{001} + 2zz'\,\hat{I}_{002}\right\} \tag{2-65}$$

and

$$\sigma^{*\mathrm{d}}_{rr(r)} = \frac{1}{8\pi(1-\nu)}\left\{\frac{1}{r}\left[-(5-12\nu+8\nu^2)\,\hat{I}_{110} + (3-4\nu)\,|\hat{z}|\,\hat{I}_{111} - 2zz'\hat{I}_{112}\right] + (5-6\nu)\,\hat{I}_{101} - sign(\hat{z})[(3-4\nu)z + 3z']\,\hat{I}_{102} - 2zz'\hat{I}_{112} + 2zz'\hat{I}_{103}\right\} \tag{2-66}$$

$$\sigma^{*\mathrm{d}}_{rz(r)} = \frac{1}{8\pi(1-\nu)}\left\{-sign(\hat{z})\,2(1-\nu)\,\hat{I}_{111} + [(3-4\nu)z + z']\,\hat{I}_{112} - sign(\hat{z})\,2zz'\,\hat{I}_{113}\right\} \tag{2-67}$$

$$\sigma^{*\mathrm{d}}_{zz(r)} = \frac{1}{8\pi(1-\nu)}\left\{(1-2\nu)\,\hat{I}_{101} + sign(\hat{z})\,[(3-4\nu)z - z']\,\hat{I}_{102} - 2zz'\,\hat{I}_{103}\right\} \tag{2-68}$$

$$\sigma^{*\mathrm{d}}_{rr(z)} = \frac{1}{8\pi(1-\nu)}\left\{\frac{1}{r}\left[sign(\hat{z})\,4(1-\nu)\,(1-2\nu)\,\hat{I}_{010} + (3-4\nu)\,\bar{z}\hat{I}_{011} - sign(\hat{z})2zz'\hat{I}_{012}\right] - sign(\hat{z})\,(2-3\nu)\,\hat{I}_{001} + [(3-4\nu)z - 3z']\,\hat{I}_{002} + sign(\hat{z})\,2zz'\,\hat{I}_{003}\right\} \tag{2-69}$$

$$\sigma^{*\mathrm{d}}_{rz(z)} = \frac{1}{8\pi(1-\nu)}\left\{(1-2\nu)\,\hat{I}_{011} - sign(\hat{z})\,[(3-4\nu)z - z']\,\hat{I}_{012} - 2zz'\hat{I}_{013}\right\} \tag{2-70}$$

$$\sigma^{*\mathrm{d}}_{zz(z)} = \frac{1}{8\pi(1-\nu)}\left\{-sign(\hat{z})\,2(1-\nu)\,\hat{I}_{001} - [(3-4\nu)z + z']\,\hat{I}_{002} - sign(\hat{z})2zz'\,\hat{I}_{003}\right\} \tag{2-71}$$

where

$$\hat{z} = z' + z \qquad \text{and} \qquad \hat{I}_{pq\lambda} = I_{pq\lambda}(\xi, r; c = |\hat{z}|) \tag{2-72}$$

If the ring load is applied at the axis of axisymmetry, i.e. $\xi = 0$, $u^{*\mathrm{h}}_{i(r)}|_{\xi=0} = 0$ and $\sigma^{*\mathrm{h}}_{ij(r)}|_{\xi=0} = 0$. In the case of axial load, $u^{*\mathrm{h}}_{i(z)}|_{\xi=0}$ and $\sigma^{*\mathrm{h}}_{ij(z)}|_{\xi=0}$ can be derived by taking the limit as $\xi \rightarrow 0$ in Eqs. (2-62) to (2-65) and Eqs. (2-66) to (2-71). The terms $u^{*d}_{im}(P,Q)$ and $\sigma^{*d}_{ijm}(P,Q)$ are singular only if $z = 0$. One may notice that the implementation of the halfspace fundamental solution requires little change in codes where the fullspace solution is already used.

2.4
Properties of the fundamental solution

In sections 2.2 and 2.3, the fundamental solutions satisfying the equilibrium equations expressed in Eq. (2-20) were derived considering no body forces, i.e, $\underset{\sim}{b} = \underset{\sim}{0}$. The effect of the radial and axial loads was taken into account by prescribing adequate boundary conditions. Alternatively, these loads can be represented as body forces and one checks that the fundamental solution satisfies the following equilibrium equation

$$\underset{\sim}{\nabla} \cdot \underset{\approx}{\sigma}^* + \underset{\approx}{\Delta}^* = 0 \tag{2-73}$$

where $\underset{\approx}{\Delta}^*$ is a generalization of Dirac's delta function and is defined as

$$\Delta^*_{im} = \begin{cases} \delta(r-\xi) & \text{if both } m \text{ and } i \text{ refer to } r\text{-direction} \\ \delta(z-z') & \text{if both } m \text{ and } i \text{ refer to } z\text{-direction} \\ 0 & \text{if } m \text{ and } i \text{ refer to different coordinate directions} \end{cases} \tag{2-74}$$

Also, given an analytical function $f(r,z)$,

$$\int_\Omega \underset{\approx}{\Delta}^* f \, d\Omega = \underset{\approx}{\delta}^* f \tag{2-75}$$

The tensor $\underset{\approx}{\delta}^*$ is a generalization of the Krockecker delta, given as

$$\delta^*_{im} = \begin{cases} 1 & \text{if } i \text{ and } m \text{ refer to the same coordinate direction} \\ 0 & \text{otherwise} \end{cases} \tag{2-76}$$

From Eqs. (2-73) and (2-75), one arrives at the relation

$$\int_\Omega \underset{\sim}{\nabla} \cdot \underset{\approx}{\sigma}^* \, \mathrm{d}\Omega = -\underset{\approx}{\delta}^* \tag{2-77}$$

which is used in chapters 3 and 4 when deriving the boundary element formulation.