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Introduction

In elasticity, axisymmetric problems can be found in the analysis of circular footings and settlement of soils in geomechanics [1], indentation of cylinders and spheres in contact mechanics [2, 3, 4], and fracture evolution and intensity stress factors of penny shaped cracks and cylindrical inclusions in fracture mechanics [2, 5, 6, 7], to name just a few examples. Some analytical solutions of these problems can be found in the literature, such as those outlined by Selvadurai [1] in geomechanics. However, in some cases, the solutions are given only either for some specific location of the body, or for specific boundary conditions, or for specific material properties.

For the cases not supported by analytical solutions, a possible approach is the use of fundamental solutions to find the result of the problem either semi-analytically or numerically. For axisymmetric problems, a variety of fundamental solutions can be found in elasticity depending on the material (isotropic or anisotropic, homogeneous or non-homoegeneous), the boundary condition (fullspace, halfspace or layered-media), the load type (point load, strip load or ring load), and location (embedded in the medium or on the surface). Extensive surveys on the existing solutions are given by Wang & Liao [8, 9], Wang et al. [10] and Wideberg & Benitez [11].

In particular, the boundary element formulations are advantageous for axisymmetric problems, since they reduce the analysis of the three-dimensional body to a one-dimensional mesh discretization and the evaluation of linear integrals. However, the fundamental solutions involved are of more complexity, requiring special considerations on their manipulation and integration to correctly evaluate the influence coefficients arising from the boundary integral equations.

This work presents the conventional boundary element method as well as the simplified-hybrid boundary element method for axisymmetric elasticity problems in the fullspace and halfspace. One employs the fundamental solutions due to radial and axial ring loads enbedded in a fullspace and halfspace derived by Kermanidis [12] and Hasegawa [13, 14], respectively. By expressing the fundamental solutions by integrals of Lipschitz-Hankel, as adopted by Selvadurai & Rajapakse [2], one has managed to manipulate the equations in an easier way, writing explicit equations for the expression of results at internal points. Moreover, the integrals that arise in the formulation could be accurately evaluated via some appropriate numerical schemes. The formulation for torsional loads, not addressed in this work, involve simpler fundamental solutions and may be dealt with in a similar manner.

1.1 The boundary element method for axisymmetric elasticity

The boundary element method for axisymmetric elasticiy was first formulated by Cruse et al. [15], using the fullspace fundamental solution derived by Kermanidis [12]. One may cite several contributions regarding the axisymmetric formulation, as the expansion of non-symmetric boundary conditions by Fourier series, as suggested by Mayr [16] and Rizzo & Shippy [17, 18], and the assessment of body forces by means of particular integrals incorporated by Park [19]. Also, axisymmetric formulations have been developed for transversely isotropy by Ishida & Ochiai [20], thermoelasticity by Bakr & Fenner [21], elastoplasticity by Cathie & Banerjee [22] and viscoplasticity by Sarihan & Mukherjee [23]. In elastodynamics, one may cite the works by Wang & Banerjee [24, 25], Tsinopoulos et al. [26] and Yang & Zhou [27] in the frequency domain. The method has also been successfully applied to contact problems [28] and fracture mechanics [29].

For axisymmetric problems in the halfspace, the boundary element formulation employed with the fullspace fundamental solution requires the discretization of the free infinite surface. In this case, to take advantage of the reduction by one dimension provided by the axisymmetric formulation, one needs to truncate the surface at a reasonable distance from the axis of axisymmetry and the region of interest [30]. The disadvantage of such scheme is that a large number of boundary elements may be needed to model the remote surface satisfactorily.

An alternative way to deal with this problem is to use infinite boundary elements, as suggested by Watson [31]. These infinite elements, which simulate the decay of the displacement and stress fields in the far field, are mapped into a finite region of intrinsic coordinate system to facilitate the integration. A variety of infinite elements can be found in the literature for three-dimensional elasticity, depending on the mapping scheme used and the application [32, 33, 34]. However, for axisymmetry, such elements are not available, probably because treating the integration of the singular kernels over the mapped infinite elements is not straightforward for the fullspace fundamental solutions. Therefore, Kelvin's three-dimensional fundamental solutions are usually employed together with the available surface infinite elements for axisymmetric applications in the halfspace [35, 36, 37], thus requiring the boundary surfaces to be discretized by three-dimensional elements.

Another way to handle this problem is to implement the fundamental solutions that satisfy in advance the traction free boundary condition on the free surface, which circumvents its numerical discretization. In elasticity, this approach was used by Telles & Brebbia [38] and Dumir & Mehta [39] to deal with problems in the isotropic and orthotropic halfplane, respectively. This work presents the boundary element formulation using the axisymmetric fundamental solution by Hasegawa [13, 14] for the halfspace [40]. By identifying an embedded term due to the fullspace fundamental solution in the halfspace fundamental solution, the proposed formulation can be implemented by applying few modifications in the existing axisymmetric computational codes.

1.2
The simplified-hybrid boundary element method for axisymmetric elasticity

The hybrid boundary element method was introduced by Dumont [41, 42, 43] about two decades ago on the basis of the Hellinger-Reissner potential, as a generalization of Pian's hybrid finite element method [44]. The formulation requires evaluation of integrals only along the boundary and makes use of fundamental solutions to interpolate fields in the domain. Accordingly, an elastic body of arbitrary shape may be treated as a single finite macro-element with as many boundary degrees of freedom as desired. In the meantime, the formulation has evolved to several application possibilities, including time-dependent problems, fracture mechanics, sensitivity analysis, and non-homogeneous materials [45, 46, 47, 48, 49]. The original method makes use of a flexibility matrix **F**, for which evaluation of integrals along the entire boundary is required.

A simplified, although equally accurate, version of the hybrid boundary element method was proposed by Dumont & Chaves [50] about a decade ago [51, 52]. This simplified-hybrid boundary element method makes use of a displacement matrix $\mathbf{U}^*$ that is obtained directly from the fundamental solution, with which the time-consuming evaluation of **F** is circumvented. Since it lacks a variational basis, however, the method leads to a non-symmetric stiffness matrix. The manner of evaluating results at internal points was inherited from the hybrid boundary element method, which requires no integrals along the boundary. The ease of post-processing results, the need for only integrating the well known influence matrix **H** and its accuracy make the simplified-hybrid boundary element method of great simplicity and applicability. The formulation is particularly attractive in the case of problems for which the corresponding fundamental solutions are difficult to manipulate, such as axisymmetry and gradient elasticity.

The first application of the simplified-hybrid boundary element method was presented by Dumont & Chaves [50] and Chaves [51] for steady state potential and elasticity. Later on, the formulation was extended to general time-dependent problems [53, 54, 47], functionally graded materials [55, 49] and sensitivity analysis [56]. In the successive applications, the method has undergone several modifications and important theoretical aspects could be consolidated. The orthonormal basis and the spectral properties of the matrices involved in the formulation play an essential role and have been extensively investigated [57, 58].

In either the hybrid or the simplified-hybrid boundary element formulation, submatrices about the main diagonal of matrices $\mathbf{F}^*$ or $\mathbf{U}^*$, respectively, cannot be obtained in the same way as the coefficients referring to different nodes are evaluated: their evaluation requires the use of spectral properties that are related to rigid-body displacements, for a bounded domain (or for the complementary domain, in the case of an unbounded region) [41, 42]. For some specific topological configurations, however, as in the case of notches, for axisymmetric problems or for some spectral abnormalities related to material non-homogeneity, this procedure may lead to local mathematical indefinitions (approximate zero by zero divisions) [59] and the diagonal submatrices can only — if ever — be obtained by interpolation of values from adjacent coefficients.

This work presents new theoretical developments that provide a definitive solution to the issue [60]. The simplified-hybrid boundary element method relies basically on a virtual work statement and on a displacement compatibility equation. The key improvement consists in correctly applying a contragradient theorem to derive simple relations that are generally valid and can successfully substitute for the spectral properties. Actually, an underlying hybrid virtual work principle was known since the onset of the formulation, but its application had been precluded by some until recently not well understood theoretical subtleties. With the new developments, once some simple stress or strain cases are identified as inherent to a given problem, it is always possible to find a set of linearly independent analytical solutions to provide sufficient equations for the evaluation of the submatrices about the main diagonal, regardless of topology and spectral properties.

The development of this new version of the simplified-hybrid boundary element method has been motivated by the need to solve axisymmetric problems, for which the domain to be analyzed is usually non-convex. For such topological configurations, some submatrices about the main diagonal of $\mathbf{U}^*$ cannot be determined by the procedure proposed in the original formulation. Moreover, an orthonormal basis $\mathbf{A}$ was introduced to formally take into account the fact that axisymmetric radial loads applied on the axis of axisymmetry generate no displacements.

In this work, one presents the complete formulation of the simplified-hybrid boundary element method for the axisymmetric fullspace. For the halfspace, some questions regarding the analytical solutions needed for the evaluation of submatrices about the main diagonal of $\mathbf{U}^*$ still remain unsolved.

1.3
Volume composition

This thesis comprises eight chapters and two appendices, as described in the following.

Chapter 2 treats the axisymmetric fundamental solutions for the elastic fullspace and halfspace. The governing equations for an axisymmetric elastic medium are presented in cylindrical coordinates. The fundamental solutions for the fullspace and halfspace are derived from Muki's solution of the Navier-Cauchy equilibrium equation in terms of integrals of Lipschitz-Hankel type.

Chapter 3 presents the boundary element method for fullspace and halfspace axisymmetric problems. The boundary integral equations and their corresponding matrix governing equations are derived by employing the fundamental solutions presented in Chapter 2. Also, the evaluation of a stiffness matrix and the manipulation of some generalized inverses are discussed. The expressions for assessing displacements and stresses in the domain and along the boundary are provided in explicit manner.

Chapter 4 discusses the simplified-hybrid boundary element method for fullspace and halfspace axisymmetric problems. A new version of the method is introduced, in which the governing equations are derived from a displacement virtual work, a nodal displacement compatibility statement and a hybrid contragradient theorem. The procedure of evaluating the unknown coefficients of $\mathbf{U}^*$ is described in detail for the fullspace, including the necessary analytical solutions. The evaluation of a stiffness matrix as well as of displacements and stresses at internal points is also discussed. The orthonormal basis, projectors and generalized inverses involved in the formulation are commented through the whole chapter.

Chapter 5 deals with the numerical schemes to evaluate the integrals arising in the boundary element method and the simplified-hybrid boundary element method for the fullspace and halfspace axisymmetric problems. The integration cases are grouped according to the position of the load source of the fundamental solution relative to the axis of axisymmetry and to the segment of the boundary along which the integration is carried out.

Chapter 6 presents a few numerical, validating examples of finite, infinite and halfspace axisymmetric problems solved by both the boundary element method and the simplified-hybrid boundary element method. The results of displacements and stresses are compared with analytical solutions on the boundary and at some points in the domain.

Chapter 7 provides the conclusions of each aspect discussed in this work, emphasizing the advantages and disadvantages of the boundary element method and the hybrid boundary element method, for the fullspace and halfspace axisymmetric problems. Moreover, one outlines the contributions as well as the questions that still remain unsolved in these formulations.

Finally, Appendix A refers to Lipschitz-Hankel integrals in terms of products of Bessel functions, giving their explicit expressions in terms of complete elliptic integrals. Appendix B provides an overview of the numerical schemes to evaluate regular, weakly singular integrals of logarithmic terms and finite part of singular integrals.