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Abstract: The Expected Shortfall or Conditional Value-at-Risk (CVaR) has been playing the role of main risk measure in the recent years and paving the way for an enormous number of applications in risk management due to its very intuitive form and important coherence properties. This work aims to explore this measure as a probability-dependent utility functional, introducing an alternative view point for its Choquet Expected Utility representation. Within this point of view, its main preference properties will be characterized and its utility representation provided through local utilities with an explicit dependence on the assessed revenue's distribution (quantile) function. Then, an intuitive interpretation for the related probability dependence and the piecewise form of such utility will be introduced on an investment pricing context, in which a CVaR maximizer agent will behave in a relativistic way based on his previous estimates of the probability function. Finally, such functional will be extended to incorporate a larger range of risk-averse attitudes and its main properties and implications will be illustrated through examples, such as the so-called Allais Paradox.

Key words: Conditional Value-at-Risk, Probability-dependent utility function, Choquet Expected Utility, Certainty Equivalent, Allais Paradox.

1. INTRODUCTION

In the past years, the Expected Shortfall or, as it has been called by the main risk management recent works, the Conditional Value-at-Risk (CVaR), has been paving the way for many financial and engineering applications that used to be very hard or even intractable. Some of these applications, such as investment and portfolio decision problems [7][8][15], energy trade [1][2][18][22][28] and integrated Gas-Energy portfolios [3] are at the top of the list of today's market economic problems. Two important and relevant references that triggered the worldwide use of such measure by both theoretical and quantitative researchers were Artzner et al. (1999) in [10] – which stated the coherence background for risk measures – and Rockafellar et al. (2000) in [26] – which introduced an efficient CVaR formulation by means of a convex expected value minimization that can be performed by a simple linear programming (LP) problem. While the latter has provided the analytical formulation and theoretical background that allowed this measure to be straightforwardly implemented and solved by existing efficient algorithms, the former has stated the coherence background and shown its main implications, which have been widely explored.

Generally, the CVaR risk measure has been defined for a loss distribution, since it has been used to control financial losses. Therefore, in this framework it is commonly defined as

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the conditioned expectation of the loss distribution's value, worse (greater) than a given α quantile. Alternatively, in a net revenue or financial profit context, for which agents or decisions makers generally express their preferences, the CVaR can be conveniently redefined as the conditioned expectation of the revenue left-side worst distribution scenarios, below a given $(1-\alpha)$ quantile – typically 1% to 10% (or α from 0.99 to 0.90). Once defined in a revenue context, the CVaR measure will be explored as a preference functional, with which a probability-dependent utility function will be associated as well as interpreted.

In [16], section 2, the Conditional Value-at-Risk was written as particular case of a Choquet Expected Utility (CEU) functional and in [12] as a particular case of the *Recourse Certainty Equivalent* (RCE), proposed by Ben-Tal et al. (1991) in [11]. While the latter representation is based on the stochastic programming with recourse problem idea, providing the optimal present and future consumption of the value of the random revenue according to an utility function, the former computes expected utilities based on capacities (non-additive probability measures), which can be modeled as a distortion on the probabilities that "(…) *acts to accentuate the implicit likelihood of the least-favorable outcomes and depress the likelihood of the most favorable ones"* (G.W. Basset Jr., R. Koenker, G. Kordas, 2004 – [16]). Thus, the CVaR measure shares the virtues of such functionals and, as will be shown in this work, the proposed alternative representation (based on a probability-dependent utility function) will let us to explore some additional interesting interpretations, which will rise as a consequence of such point of view. The main properties analyzed in this work are: risk aversion, convexity in probabilities and the equivalence between the Expected Utility (EU) form and its induced Certainty Equivalent (CE).

Finally, the CVaR preference functional will be extended through its convex combination with the unconditional expected value, which will provide a more general – named Extended CVaR Preference (ECP) – functional capable to model a larger range of risk-averse behaviors. Thus, the associated probability-dependent utility of this functional will be characterized in the same way as done before. An illustrative example of the consequences of convexity in the probabilities will be given by means of the so-called Allais Paradox.

The main contribution of this work is to provide a probability-dependent utility representation for a measure widely used in many types of engineering and finance applications. This representation will also provide intuitive interpretations for CVaR maximizers' behavior and an alternative viewpoint for the Choquet Expected Utility form ([16]). In this sense, by analyzing the implicit probability-dependent utility we will be able to conclude that firms or agents that use this measure in, e.g., portfolio selection problems, will behave in a relativistic way with their estimated probabilities[2].

The ECP functional has been adopted by a large range of researchers and investment firms in order to explore the market's opportunities and develop strategic decision policies in risky environment, as mentioned before. Thus, it is beyond the scope of this work to judge the plausibility and applicability of such functional; instead, the focus of this work will aim at the related properties and insights for those agents who have already chosen to be Extended CVaR maximizers.

The rest of this work is structured as follows: Section 2 characterizes the CVaR risk measure as a preference functional in the revenue context, Section 3 characterizes its implicit probability-dependent utility function and provide some properties for such preference functional, Section 4 extends the CVaR preference functional to the ECP functional and also the established properties and remarks made in Section 3. Finally, Section 5 provides some empirical implications on the use of the ECP by means of two examples and then, Section 6 concludes this work.

2. CVAR MEASURE IN THE REVENUE CONTEXT

As mentioned before, the CVaR needs to be conveniently redefined for the revenue context in order to let us to use it as preference functional. In this sense, a probability space $(\Omega,\Im,P)$ is assumed and the stochastic revenues will be defined as $\Im$-measurable functions $R:\Omega\rightarrow Q$, which map elements from the set of all possible states of nature (Ω) to the compact set of all possible revenue outcomes $Q\subset\Re$. Thus, the induced cumulative probability function $F_R(.)$ of a given random revenue R can be defined as:

$$F_R(r) = P\{\omega\in\Omega \mid R(\omega) \leq r\} \qquad \forall\, r \in Q \tag{1}$$

In this work we will concentrate on random variables that have cumulative probability functions in Ψ, which denotes the set of all probability functions with compact support (Q).

[2] It is considered that all distributions are known and so, the absence of ambiguity [14] is supposed. For a recent implication of ambiguity on CEU functionals reference [24] is proposed.

In addition, the generalized inverse of the right-continuous function (1) provides the quantile function as well as help us in the Value-at-Risk (VaR) and CVaR definitions. For this purpose, let VaR_α denote the quantile function defined as:

$$VaR_\alpha(R) = F_R^{-1}(1-\alpha) = \inf\{r \in Q \mid F_R(r) \geq 1-\alpha\} \qquad (2)$$

And the left-tail conditional expectation, for revenue values up to the (1-α) quantile (or $VaR_\alpha(R)$ as stated in (2)), will be written as:

$$CVaR_\alpha(R) = E[R \mid R \leq VaR_\alpha(R)] \qquad (3)$$

Where, E(.|.) is the mathematical expectation operator conditioned to a given set. Thus, for a set $A \subseteq Q$, it is defined as:

$$E(R \mid R \in A) = \int_{\{r \in A\}} r \frac{dF_R(r)}{P\{\omega \in \Omega \mid R(\omega) \in A\}}$$

Then, unconditioned expectations are assumed to be taken over the whole support set (A=Q).

Figure 1 illustrates the $CVaR_\alpha$ and VaR_α configuration for a given smooth cumulative probability function F_R.

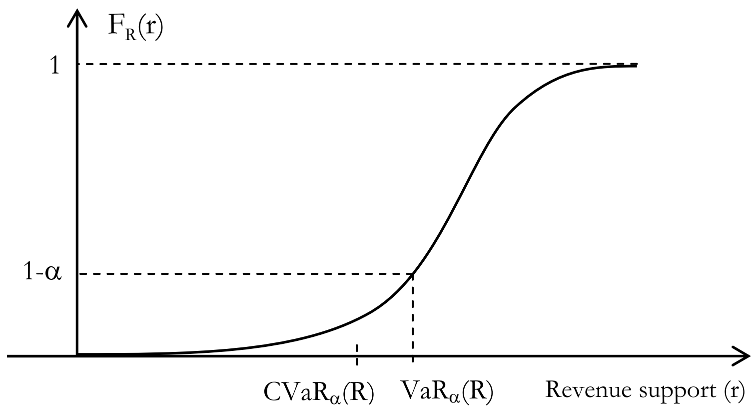

Figure 1 – $CVaR_\alpha$ and VaR_α configuration for a smooth revenue cumulative probability function.

If R is a discrete random variable, with a finite number of scenarios S and known probabilities $\{P(R_s)>0\}_{s=1,\ldots,S}$, R can be represented as a set of its ordered scenarios: $R=\{R_s,P(R_s)\}_{s=1,\ldots,S}$, in which R_s is the s^{th}-smaller revenue scenario. Then, expression (2) will provide the superior (1-α) quantile, which in the discrete representation is defined as:

$$VaR_\alpha(R)^+ = R_{s^*(\alpha,R)} \qquad (4)$$

where $s^*(\alpha,R) \in \{1,\ldots,S\}$ is the unique scenario index for which

$$\Sigma_{\{s\leq s^*(\alpha,R)-1\}} P(R_s) < 1-\alpha \leq \Sigma_{\{s\leq s^*(\alpha,R)\}} P(R_s). \quad (5)$$

In accordance with (4) and (5), the Conditional Value-at-Risk, in the presence of a probability atom, can be defined as the following weighted average (6), which will take into account the proportion of the superior (1-α) quantile and the rest of the inferior tail scenarios, if they exist (see [27] for a entire description of the CVaR and VaR for a general random variable):

$$CVaR_\alpha(R)=(1-\alpha)^{-1}\cdot\{ \Sigma_{\{s\leq s^*(\alpha,R)-1\}} P(R_s)\cdot R_s + R_{s^*(\alpha,R)}\cdot[1 - \alpha - \Sigma_{\{s\leq s^*(\alpha,R)-1\}} P(R_s)] \} \quad (6)$$

Note that $s^*(\alpha,R)$ depends on the 1-α percentile (the reliability level α) and also on the revenue distribution, represented by the set $\{P(R_s)\}_{s=1,\ldots,S}$ of the random variable R.

Remark 1 If $s^*(\alpha,R) = 1$, then

$$\{s\leq s^*(\alpha,R)-1\} = \varnothing \text{ and } \Sigma_{\{s\leq s^*(\alpha,R)-1\}} P(R_s) = 0,$$

which will imply in expression (6) to be reduced to R_1. In the case of a CVaR maximizer it is equivalent to the so-called MaxMin approach, in which the decision maker maximizes the worst revenue scenario.

Remark 2 The VaR_α definition in (2) should provide definition (4) stated for the discrete case in which $F_R(.)$ is a stepwise right-continuous function.

In the stochastic optimization context[3], where agents seek for the random variable $R(\mathbf{x},\xi)$[4] that maximizes their preferences by tuning a controlling vector $\mathbf{x}\in\mathbf{X}$ (usually called as decision variables, where $\mathbf{X}\subset\Re^n$), expression (3) may increase the complexity of any simple application, even for a *Carathéodory* function[5] $R:\mathbf{X}\times\Xi\rightarrow Q$.

Generally, in practical applications, the random revenue R is characterized as a function not only of the set of the agent's decisions ($\mathbf{X}$), but also of a more primitive multidimensional random variable ($\xi \in \Xi$), which models the exogenous risk factors. For

[3] [17] is referred as a standard handbook.

[4] $\mathbf{x}$ generally is a decision maker controlling variable defined over a compact subset $\mathbf{X}$ in the $\Re^n$ that parameterizes the selection of random variables with distribution functions in Ψ. In many classical applications in which the set of feasible decision variables can be defined as a set of linear inequalities, $\mathbf{X}$ assumes a polyhedral shape.

[5] $R(\mathbf{x},\xi):\mathbf{X}\times\Xi\rightarrow Q$ *Carathéodory* means that for all $\xi\in\Xi$, $R(.,\xi)$ is continuous and for all $\mathbf{x}\in\mathbf{X}$, $R(\mathbf{x},.)$ is measurable on the adopted sigma-field of Ξ.

not so high dimensions of Ξ, the assessment of expression (3) will require hard and time-consuming multidimensional integration methods to assess the expectation conditioned to the subset of Ξ that provides $R(\mathbf{x},\xi) \leq r$. In this sense, Rockafellar et al., in [26] and [27], have shown that expression (3) could be assessed through (8), which is the maximum of the unconditional expectation-based functional $H:\Psi\times Q\to Q$, presented in (7), reaching its maximum point at $z = z^*(\alpha,R) = VaR_\alpha(R)$.

$$H(R,z) = z - E[\,(R-z)|^-]/(1-\alpha) \tag{7}$$

$$CVaR_\alpha(R) = Max_{(z)}\, H(R, z), \tag{8}$$

where $(\,.\,)|^-$ is the negative truncation function, which is equivalent to $(.)|^- \equiv -\min(\,.\,,0)$.

The optimality proof of (8) is provided for a general random variable in the original work [27], which is a well known result of convex analysis [25][29].

Since (8) does not depend on any constraints in the outcomes set, this approach exchanges the difficulty on computing the conditional expectation on (3) for a maximum calculation over a convex family of unconditional expectations defined on (7). In other words, it replaces difficult integrals for maximization problems. However, generally, the latter has the advantage of some well known convergence results that are provided for finite sampled scenarios, e.g., $Max_{(z)}S^{-1}\cdot\Sigma_s H(R_s,z) \xrightarrow{S\to\infty} Max_{(z)}E[H(R_s,z)]$. Thus, (8) has paved the way for the use of many different simulation schemes to easily estimate the $CVaR_\alpha[R(\mathbf{x},\xi)]$ through a finite sample of the exogenous risk variables $\{\xi_s\}_s$ (the so-called *Sampled Average Approximation* (SAA) – see [1], chapter 4), by means of a convex maximization problem, which is suitable to be coupled in risk-constrained or risk-averse optimization problems. In [27] some examples are provided for portfolio problems and in [1][7] some aspects on the sampling convergence of such measure are discussed[6]. In addition to this benefit, the form provided by (8) will also allow us to find an implicit utility for such measure when it is used as a preference functional, which is the main idea of this work.

[6] Reference [13] also provides nice convergence results for the so-called chance constrained problems.

3. THE CVAR UTILITY PREFERENCE FUNCTIONAL FORM

In order to obtain an utility maximization functional form for the CVaR, in the revenue or profit context, expression (8) can be read without the maximization operator by substituting the optimality result $z^*(\alpha,R)$ into it:

$$CVaR_\alpha(R) = E\{z^*(\alpha,R) - [R - z^*(\alpha,R)]|^-/(1-\alpha)\} \tag{9}$$

Expression (9) shows us the piecewise linear form of such utility that depends on the α parameter and on the probability distribution function of R, due to its explicit dependence on the revenue quantile function $z^*(\alpha,R) = VaR_\alpha(R) = F_R^{-1}(1-\alpha)$. The next expression presents the analytical form of the $CVaR_\alpha$'s implicit probability (or quantile) dependent utility function found in (9).

$$U_\alpha[r, F_R(.)] = F_R^{-1}(1-\alpha) - [r - F_R^{-1}(1-\alpha)]|^-/(1-\alpha) \tag{10}$$

It is clear that such utility is not a classical von Neumann-Morgenstern utility function [30], since it depends on the revenue-induced probabilities, by means of the inverse cumulative distribution function (or quantile function) $F_R^{-1}:[0,1]\to Q$ assessed on the $1-\alpha$ value. Then, we can state that:

Theorem 1 (The CVaR preference functional probability-dependent utility function representation)

For any $\alpha\in[0,1)$, $U_\alpha: Q\times\Psi\to\Re$

I. *$U_\alpha: Q\times\Psi\to\Re$ is a two-segment piecewise concave on Q (then risk-averse) probability (quantile) dependent utility function.*

II. *$U_\alpha: Q\times\Psi\to\Re$ has a compact range in $\Re$.*

III. *$U_\alpha[r, F_R(.)] \le r$ for all $r\in Q$, with a unique fixed point on $r = F_R^{-1}(1-\alpha)$.*

Proof. First of all, let expression (10) be rewritten to evidence its probability dependence:

$$U_\alpha[r, F_R(.)] = G_\alpha[F_R(.)] + \min\{0, r - G_\alpha[F_R(.)]\}/(1-\alpha),$$

where, $G_\alpha:\Psi\to Q$ is defined as $G_\alpha[F_R(.)] = F_R^{-1}(1-\alpha)$, according to (2), for any distribution $F_R(.)\in\Psi$.

I. It is sufficient to verify that for all $F_R(.)\in\Psi$, $U_\alpha[\ .\ , F_R(.)]$ is the sum of an affine function (the constant function $G_\alpha[F_R(.)]$) and the minimum of two other affine functions, which is clearly a continuous piecewise concave function, with an unique break point (slope change) on the quantile $G_\alpha[F_R(.)]$ (see Figure 2 for a graphical illustration of $U_\alpha[\,.\,, F_R(.)]$).

II. Range compactness can be shown through the following rationale: since *range*(G_α)=Q (by definition of quantile function) and $r \in Q$, then,

$U_\alpha[r, F_R(.)] = U_\alpha\{r, G_\alpha[F_R(.)]\}: Q \times Q \rightarrow \Re$

Because $\Re \times \Re$, in which $Q \times Q$ is embedded, and $\Re^2$ are isomorphic, it is possible to redefine U_α as:

$U_\alpha\{r, G_\alpha[F_R(.)]\} = U_\alpha(\mathbf{v}) = f_1(\mathbf{v}) + f_2(\mathbf{v})$

Where,

$\mathbf{v} = [r, G_\alpha[F_R(.)]]^T \in V$

V is a compact subset of $\Re^2$, being the isomorphic image of $Q \times Q$ in this space;

$f_1(\mathbf{v}) = \mathbf{a}^T \cdot \mathbf{v}$ and $f_2(\mathbf{v}) = \min(0, \mathbf{b}^T \cdot \mathbf{v})/(1-\alpha)$;

with, $\mathbf{a}^T = [0, 1]$ and $\mathbf{b}^T = [1, -1]$.

By verifying that f_1 and f_2 are both concave, then continuous, it follows that U_α is a continuous map between a compact set ($Q \times Q$) and the real line ($\Re$). Then, *range*(U_α) is compact[7].

III. For all $r < F_R^{-1}(1-\alpha)$,

$U_\alpha[r, F_R(.)] = (r - G_\alpha[F_R(.)] \cdot \alpha)/(1-\alpha) < r$ if and only if $r < G_\alpha[F_R(.)] = F_R^{-1}(1-\alpha)$, which is the previous hypothesis.

For all $r > F_R^{-1}(1-\alpha)$,

$U_\alpha[r, F_R(.)] = G_\alpha[F_R(.)] = F_R^{-1}(1-\alpha) < r$

which is again the previous hypothesis.

For $r = F_R^{-1}(1-\alpha)$

$U_\alpha[F_R^{-1}(1-\alpha), F_R(.)] = F_R^{-1}(1-\alpha)$. □

For the case of a smooth (continuous) cumulative probability function $F_R(.)$, the following Figure 2 illustrates the utility probability dependence.

[7] For further details see [20], Theorem 3.64 (a)

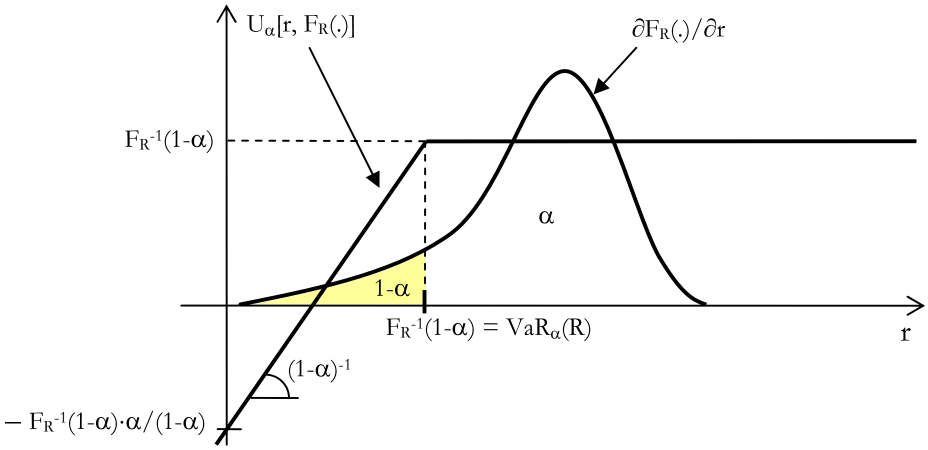

Figure 2 –The $CVaR_\alpha$ local piecewise linear utility function, for a fixed revenue distribution F_R.

The next figure shows the utility *locus* due to a probability movement. This effect is illustrated for a generic move of the revenue distribution, e.g., from F_R to F_{R*}. In this sense, the corresponding *locus* of such probability-dependent utility function is totally characterized by the revenue 1-α quantile movement.

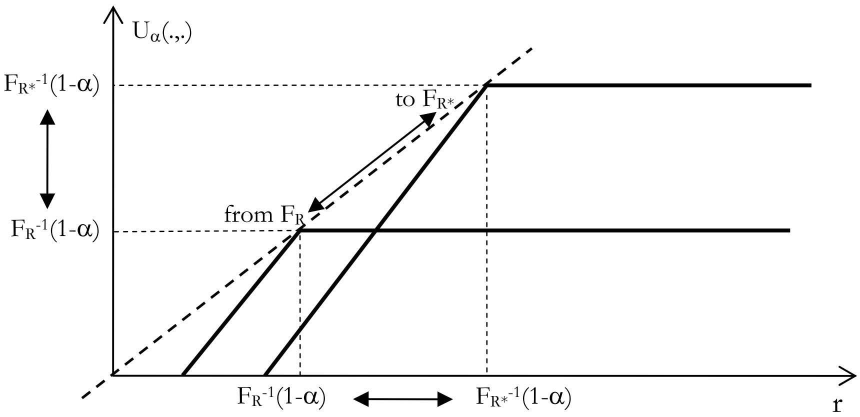

Figure 3 – Locus of the local utility functions of the $CVaR_\alpha$ preference functional.

The effect of such movement is that U_α slides its fixed point (on r) over the identity function I(r) = r, leaving $F_R^{-1}(1\text{-}\alpha)$ to reach the new quantile $F_{R*}^{-1}(1\text{-}\alpha)$. Then, U_α is a local piecewise linear concave utility function with a probability dependence that provides a quantile shift effect when moving from one distribution to another. This fact can be understood as a relativistic accommodation of the agent's perspective of future possible outcomes. Then we can state the following remark:

Remark 3 An interesting interpretation for such probability dependence may rise in the investment under uncertainty point of view, in which a CVaR maximizer investor will only "regret" if a given project (with F_R distribution) provides an improbable downsize realization $R(\omega_0)$, with $F_R[R(\omega_0)] \leq 1-\alpha$, based on his previous[8] estimated probability function $F_R(.)$. In this sense, $F_R^{-1}(1-\alpha)$ – or $VaR_\alpha(R)$ – turns to be the critical, and generally pessimistic, point for which this agent exhibits a change in his marginal utility. In terms of utility shape, a CVaR maximizer probability-dependent utility function does not discriminate surplus scenarios above the critical value (zero marginal utility), but penalizes violations to this value, in deficit scenarios, with a $(1-\alpha)^{-1}$ marginal utility decrease. Thus, the risk-profile of such agent is characterized by: (i) the α risk-aversion parameter, which determines the utility loss per unit of quantile violation, and (ii) the critical value $F_R^{-1}(1-\alpha)$, which places the utility in accordance with each faced distribution function.

Once the $CVaR_\alpha$ measure has been characterized as a preference functional with a probability-dependent utility representation (10), it is possible to explore its properties and implications.

Properties 1

The following properties hold for the $CVaR_\alpha$ preference functional:

(a) **Translation invariance**

For any real valued random variable R with $F_R \in \Psi$ and $t \in \Re$,

$CVaR_\alpha(R + t) = CVaR_\alpha(R) + t.$

(b) **Positive homogeneity**

For any real valued random variable R with $F_R \in \Psi$ and $t \in \Re$,

$CVaR_\alpha(t \cdot R) = t \cdot CVaR_\alpha(R).$

(c) **Superadditivity**

For any pair of real valued random variables R_1 and R_2 with F_{R_1} and $F_{R_2} \in \Psi$,

$CVaR_\alpha(R_1 + R_2) \geq CVaR_\alpha(R_1) + CVaR_\alpha(R_2).$

(d) **Monotonicity**

[8] The word previous is referred to the occurrence of the outcome realization ω_o

For any pair of real valued random variables $R_1 \leq R_2$ with F_{R_1} and $F_{R_2} \in \Psi$,

$CVaR_\alpha(R_1) \leq CVaR_\alpha(R_2)$.

(e) **Consistency**

For any deterministic random variable *t*, with S = 1 and $t_1 = t \in \Re$, $CVaR_\alpha(t) = t$.

(f) **Equivalence between Expected Utility and Certainty Equivalent**

If $CE_\alpha(R) \in Q$ is the $CVaR_\alpha$ induced Certainty Equivalent of a given random revenue R and $EU_\alpha(R) \equiv E\{ U_\alpha[R, F_R(.)] \}$, then,

$$CE_\alpha(R) = EU_\alpha(R). \tag{11}$$

(g) **Convexity in the probabilities**

If R_ϕ, R_1 and R_2, with F_{R_ϕ}, F_{R_1} and $F_{R_2} \in \Psi$ are such that $F_{R_\phi} = \phi \cdot F_{R_1} + (1-\phi) \cdot F_{R_2}$, then,

$CVaR_\alpha(R_\phi) \leq \phi \cdot CVaR_\alpha(R_1) + (1-\phi) \cdot CVaR_\alpha(R_2) \quad \forall\ \phi \in [0,1]$.

(h) **Risk Averse**

If $CE_\alpha(R) \in Q$ is the $CVaR_\alpha$ induced Certainty Equivalent of R, then,

$CE_\alpha(R) \leq E(R)$ for all $\alpha \in (0,1)$.

Alternatively, following the expected utility characterization of risk-aversion, for all $F_R \in \Psi$, $U_\alpha[\ .\ , F_R(.)]$ is concave (see [11] Remark 4.1).

Proof. (a) to (e) are the coherence properties verified for the CVaR risk measure in [10] and also verified in [11] for the RCE general form.

Since (e) will play a key role in property (f), it will be conveniently repeated here. We use the VaR_α definition in (2) for a stepwise probability function $F_t(r) = \delta_{\{r \geq t\}}(r)$, where $\delta_A(r) = 1$ if $r \in A$, with $A \subseteq Q$, and zero for all other cases. For this specific probability function,

$VaR_\alpha(t) = \min\{r \mid \delta_{\{r \geq t\}}(r) \geq 1-\alpha\} = t$

which applied in definition (3) will lead to

$CVaR_\alpha(t) = E_\xi\{t \mid t \leq t\} = t$,

or alternatively, use Remark 1 with S = 1.

(f) The Certainty Equivalent ($CE_\alpha(R) \in Q$) of a CVaR maximizer agent for a given random revenue R can be understood as the real value for which such agent becomes indifferent to the stochastic revenue outcome R. This statement implies in $CVaR_\alpha[CE_\alpha(R)] = CVaR_\alpha(R)$.

But from consistence property (e), the left-hand side of this equality is exactly the $CE_\alpha(R)$, once it is a deterministic real value. Then,

$$CE_\alpha(R) = CVaR_\alpha(R) \tag{12}$$

Finally, it is possible to verify (11) through (12), (9) and (10):

$$CE_\alpha(R) = CVaR_\alpha(R) = E\{U_\alpha[R, F_R(.)]\} = EU_\alpha(R).$$

(g) For each point z in Q, $H(. , z){:}\Psi \rightarrow Q$ is an affine function of $dF_R(.)$ (see expression (13)).

$$H(R,z) = H(F_R(.),z) = z - (1-\alpha)^{-1} \cdot \int (r - z)\,|^{-} \cdot dF_R(r) \tag{13}$$

The previous statement combined with the fact that Ψ is a convex set and $z^*(\alpha,R)$ always exists (proof provided in [27]), allows us to conclude that: since the CVaR is a maximum over a set $\{H(F_R(.),z)\}_{z\in Q}$ of affine functions in Ψ, it is convex.

(h) Due to (12) and (3) we have that

$$CE_\alpha(R) = CVaR_\alpha(R) = E[R \mid R \le VaR_\alpha(R)] = g(\alpha)$$

where,

$$g(t) \equiv E\{R \mid R \le F_R^{-1}(1-t)\} \qquad \forall\ t \in [0,1] \tag{14}$$

g(t) is a non-increasing function of t, since $F_R^{-1}(1-t)$ is a cumulative distribution and thus, non-increasing in $t\in[0,1]$. Due to that:

$$E(R) = \lim_{(t\rightarrow 0+)} g(t) \ge g(\alpha) = CE_\alpha(R) \quad \forall\ \alpha \in [0,1].$$

Alternatively, following the expected utility characterization of risk-aversion, for all $F_R\in\Psi$, $U_\alpha[\ .\ ,F_R(.)]$ is concave (see [11] Remark 4.1), which has been verified in Theorem 1 (see also [23]). □

Properties (a) to (e) are known in the recent literature as coherence properties and constitute a set of desirable preference properties that are backed up by financial interpretations. For a whole discussion of the relevance of these properties and its implications, references [7][8][9][10][11] are suggested.

Remark 4 An important intermediary result obtained on proving property (f) is the following: the $CVaR_\alpha$ preference index of a given random variable is its certainty equivalent. This may be of great interest in many different fields such as Investment and Management Science, Actuarial Science, etc.

Remark 5 In addition to Remark 4, the existence of a probability-dependent expected utility form for such functional, with the equivalence property stated in (f), provides a starting point to solve some practical disagreements, e.g., in the case of multi-period preferences based on per-period separable expected utility functionals and the related estimation of their impatience discount factors[9]. In this sense, as the EU index meets exactly the CE, by measuring the preference of a multi-period cash flow through the discounted expected utilities or through the related certainty equivalents, it will result the same numerical indexes and preferences. In this sense, the estimation of a timing discount factor turns to be an easier task, since in this case, we are dealing with a monetary and deterministic equivalent flow.

4. EXTENDING THE PREFERENCE FUNCTIONAL

As shown before, the $CVaR_\alpha$ preference index has shown to be the Certainty Equivalent of the assessed random variable. In this context, the agent CE ignores the whole probability distribution above the (1-α) quantile. It seems reasonable that for two different random outcomes with the same $CVaR_\alpha$ metric, but different expected values, a rational agent would prefer the one which provides the greatest unconditioned expectation.

The convex combination of this functional with the unconditioned expectation, which now will be referred to as the Extended CVaR Preference (ECP), would let the decision maker to express its preference for both "worst cases" (for α greater than 0.5) and distribution average, as follows:

$$\Phi_{\alpha,\lambda}(R) = \lambda \cdot CVaR_\alpha(R) + (1-\lambda)\cdot E(R) \qquad \text{with } \lambda \in [0,1] \tag{15}$$

For such functional, the same approach used before on expressions (9) and (10) can be used, in order to provide its probability-dependent expected utility form:

$$\Phi_{\alpha,\lambda}(R) = \lambda \cdot E\{z^*(\alpha,R) - [R - z^*(\alpha,R)]|^-/(1-\alpha)\} + (1-\lambda)\cdot E(R) \tag{16}$$

Expression (16) will let us to access $U_{\alpha,\lambda}:Q\times\Psi\rightarrow\Re$ for any $\alpha\in[0,1)$ and $\lambda\in[0,1]$ as follows:

$$U_{\alpha,\lambda}[r, F_R(.)] = \lambda\cdot\{ F_R^{-1}(1-\alpha) - [r - F_R^{-1}(1-\alpha)]|^-/(1-\alpha) \} + (1-\lambda)\cdot r \tag{17}$$

Thus, by means of expression (17) we finally reach the probability-dependent expected utility form of the ECP functional:

[9] See [21] for an entire discussion about this topic and for further references.

$\Phi_{\alpha,\lambda}(R) = E\{ U_{\alpha,\lambda}[R, F_R(.)] \}$ (18)

Theorem 2

For all $\lambda \in (0,1]$ and $\alpha \in (0,1)$ that provide risk-aversive attitudes, Theorem 1 will hold for the ECP functional with probability-dependent utility function $U_{\alpha,\lambda}$.

Proof. Thus, let (17) be rewritten in order to evidence the probability dependence in $U_{\alpha,\lambda}$, as done before:

$$U_{\alpha,\lambda}[r, F_R(.)] = \lambda \cdot G_\alpha[F_R(.)] + (1 - \lambda) \cdot r + \lambda \cdot \min\{0, r - G_\alpha[F_R(.)]\}/(1-\alpha)$$

Where, $G_\alpha : \Psi \rightarrow Q$ is defined as $G_\alpha[F_R(.)] = F_R^{-1}(1-\alpha)$, according to (2), for any distribution $F_R(.) \in \Psi$.

I. It is sufficient to verify that for all $F_R(.) \in \Psi$, $U_{\alpha,\lambda}[\ .\ , F_R(.)]$ is the sum of an affine function ($\lambda \cdot G_\alpha[F_R(.)] + (1 - \lambda) \cdot r$) and the minimum of two other affine functions, which is clearly a continuous piecewise concave function, with an unique break point (slope change) on the quantile $G_\alpha[F_R(.)]$ (see Figure 4 for graphical illustration of $U_{\alpha,\lambda}[\ .\ , F_R(.)]$).

II. Range compactness can be shown through the same rationale stated before: since *range*(G_α)=Q (by definition of quantile function) and $r \in Q$, then,

$$U_{\alpha,\lambda}[r, F_R(.)] = U_{\alpha,\lambda}\{r, G_\alpha[F_R(.)]\} : Q \times Q \rightarrow \Re$$

Because $\Re \times \Re$, in which $Q \times Q$ is embedded, and $\Re^2$ are isomorphic, it is possible to redefine $U_{\alpha,\lambda}$ as:

$$U_{\alpha,\lambda}\{r, G_\alpha[F_R(.)]\} = U_{\alpha,\lambda}(\mathbf{v}) = f_1(\mathbf{v}) + f_2(\mathbf{v})$$

Where,

$\mathbf{v} = [r, G_\alpha[F_R(.)]] \in V$

V is a compact subset of $\Re^2$, being the isomorphic image of $Q \times Q$ in this space;

$f_1(\mathbf{v}) = \mathbf{a}^T \cdot \mathbf{v}$ and $f_2(\mathbf{v}) = \min(0, \mathbf{b}^T \cdot \mathbf{v})/(1-\alpha)$;

with, $\mathbf{a}^T = [(1-\lambda), \lambda]$ and $\mathbf{b}^T = [\lambda, -\lambda]$.

By verifying that f_1 and f_2 are both concave and continuous, it follows that $U_{\alpha,\lambda}$ is a continuous map between a compact set ($Q \times Q$) and the real line $\Re$. Then, *range*($U_{\alpha,\lambda}$) is compact[10].

III. For all $r < F_R^{-1}(1-\alpha)$,

$$U_{\alpha,\lambda}[r, F_R(.)] = \lambda \cdot G_\alpha[F_R(.)] + (1-\lambda)\cdot r + \lambda \cdot \{r - G_\alpha[F_R(.)]\}/(1-\alpha)$$
$$= \{ r\cdot[\lambda + (1-\lambda)\cdot(1-\alpha)] - G_\alpha[F_R(.)]\cdot\alpha\cdot\lambda \}/(1-\alpha)$$
$$= \{ r\cdot[(1-\alpha) + \alpha\cdot\lambda] - G_\alpha[F_R(.)]\cdot\alpha\cdot\lambda \}/(1-\alpha) < r$$

if and only if $r < G_\alpha[F_R(.)] = F_R^{-1}(1-\alpha)$, which is the previous hypothesis.

For all $r > F_R^{-1}(1-\alpha)$,

$$U_{\alpha,\lambda}[r, F_R(.)] = \lambda \cdot G_\alpha[F_R(.)] + (1-\lambda)\cdot r < r$$

if and only if $r > G_\alpha[F_R(.)] = F_R^{-1}(1-\alpha)$, which is again the previous hypothesis.

For $r = F_R^{-1}(1-\alpha)$

$$U_{\alpha,\lambda}[F_R^{-1}(1-\alpha), F_R(.)] = F_R^{-1}(1-\alpha). \square$$

In order to provide a graphical visualization of (17), the key is to analyze this function for each segment, $r \leq F_R^{-1}(1-\alpha)$ and $r > F_R^{-1}(1-\alpha)$, verifying that on $r = F_R^{-1}(1-\alpha)$, $U_{\alpha,\lambda}[\,.\,, F_R(.)]$ is continuous, as done in Theorem 1's proof, extension (III). The next figure shows the shape of such utility.

[10] Again, for further details see [20], Theorem 3.64 (a)

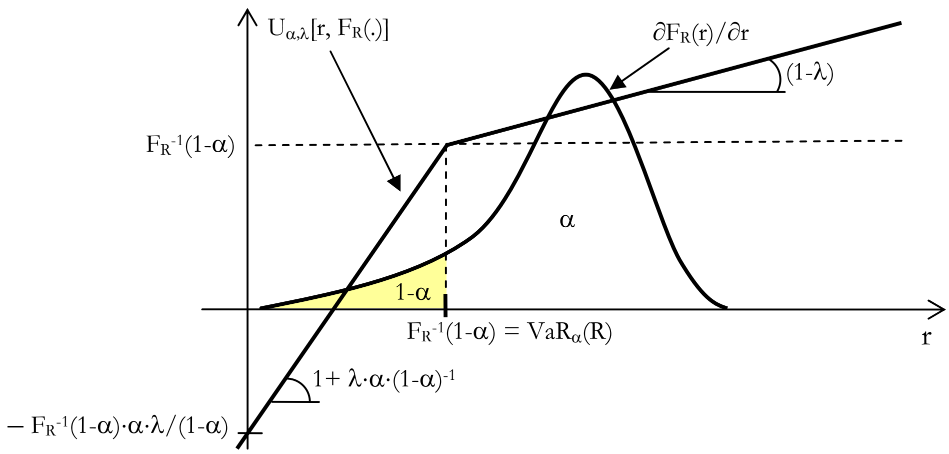

Figure 4 – The $\Phi_{\alpha,\lambda}$ local piecewise linear utility function, for a fixed revenue distribution $F_R(.)$.

Properties 1 can be straightforwardly verified for the extended functional since the mathematical expectation operator E(.) is linear, homogeneous and risk-neutral, providing (depending on the α and λ risk-averse parameters) preferences that can be risk-averse or neutral, but never risk-taker.

Properties 2

Properties 1 can be extended to the ECP functional ($\Phi_{\alpha,\lambda}$).

Proof.

Properties (a) to (e) can be demonstrated separately for both operators, $CVaR_\alpha(.)$ and E(.), since the extended functional is a weighted average of them. Thus, as E(.) is known to attend these properties, so will $\Phi_{\alpha,\lambda}$.

Property (f) follows the same approach used before:

$\Phi_{\alpha,\lambda}[CE_{\alpha,\lambda}(R)] = \Phi_{\alpha,\lambda}(R)$

$\lambda\cdot CVaR_\alpha[CE_{\alpha,\lambda}(R)] + (1-\lambda)\cdot E[CE_{\alpha,\lambda}(R)] = \Phi_{\alpha,\lambda}(R)$.

By property (e) – consistency, we have that $CE_{\alpha,\lambda}(R) = \Phi_{\alpha,\lambda}(R)$. Thus, due to (18) we conclude that $CE_{\alpha,\lambda}(R) = E\{U_{\alpha,\lambda}[R, F_R(.)]\}$, providing the equivalence between the expected utility form and the associated certainty equivalent of such functional.

For property (g), again the same approach used before can be followed. Since the unconditioned expectation (the last term of expression (15)) is linear in the probabilities,

$\Phi_{\alpha,\lambda}$ being a sum of a convex function ($CVaR_\alpha$) with an affine function (E(.)), will also be convex in the probabilities.

The risk-averse property will depend on the λ and α parameters. If $\lambda = 0$, or $\alpha = 0$, $\Phi_{\alpha,\lambda}$ will be reduced to the unconditional expectation and then, a $\Phi_{\alpha,\lambda}$ maximizer will behave as a risk-neutral agent. For $\lambda \in (0,1]$ and $\alpha \in (0,1)$, the convex combination between the $CVaR_\alpha$ and the unconditioned expectation will be smaller or equal to the latter, due to property (h). □

Before exploring some implications in the use of this functional, a slight revision on remark 3 must be done in order to adjust the investment interpretation for the ECP's implicit local utility. In this case, by choosing $\lambda \in (0,1)$ and $\alpha \in (0,1)$, $\Phi_{\alpha,\lambda}$ maximizers will do differentiate surplus outcomes (greater than the critical point $VaR_\alpha(R)$) with a marginal utility of $(1-\lambda)$.

5. IMPLICATIONS OF USING $\Phi_{\alpha,\lambda}$ AS A PREFERENCE

The well known Allais Paradox (see [5][6][23]) has raised from the nonlinear characteristic (in the probability set) of the related choices presented in the Paradox instance. In order to provide an illustrative implication of the convexity property verified on the functional $\Phi_{\alpha,\lambda}$, it will be submitted to a given instance of the aforementioned paradox. After that, by means of a simple example of a mixture between two lotteries, the convexity in probabilities will be graphically represented.

Consider a pair of two alternative lotteries (discrete random variables) with fixed support and different probabilities, as shown in the next figure, in which arrows indicates the existence of a probability atom.

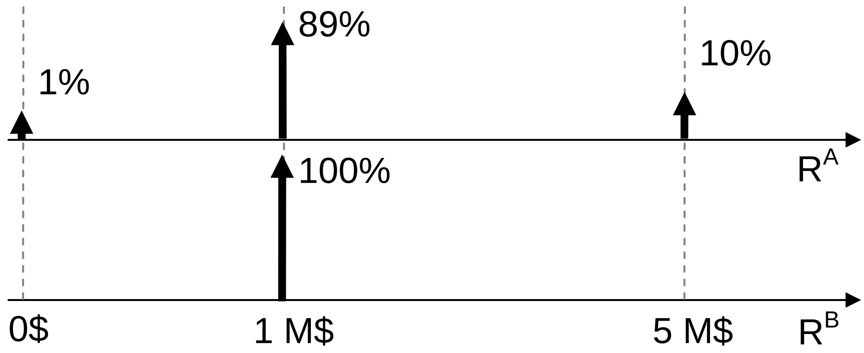

Figure 5 – Two alternative lotteries (R^A and R^B) – part of the Allais Paradox.

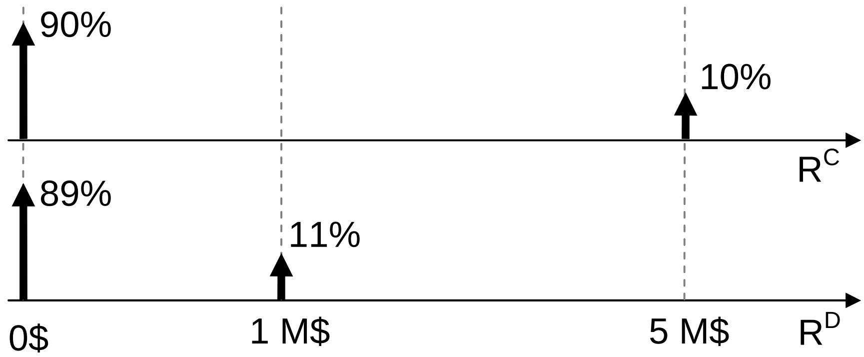

Figure 6 – Two alternative lotteries (R^C and R^D) – part of the Allais Paradox.

The so-called "Allais Paradox" was first proposed as an empirical experiment in which individuals were asked to rank two pairs of lotteries: (A *vs.* B) and then (C *vs.* D). The majority of questioned individuals shown to prefer to bet in R^B rather than in R^A, and for the second options, the preferred ordering was R^C rather than R^D. Since such ordering is impossible to be recovered by any classical von Neumann-Morgenstern (vN-M) utility function $U:\mathfrak{R}\rightarrow\mathfrak{R}$, the classical utility axioms were criticized by Maurice Allais in [5][6]. For further explanations and related implications of this paradox see [23].

(Attending the First Choice): ($R^B \rangle_{EU} R^A$):

$$E[U(R^B)] > E[U(R^A)] \Rightarrow 100\%\cdot U(1) > 10\%\cdot U(5) + 89\%\cdot U(1) + 1\%\cdot U(0)$$
$$\Rightarrow \underline{11\%\cdot U(1)} > 10\%\cdot U(5) + 1\%\cdot U(0)$$

(Attending the Second Choice): ($R^C \rangle_{EU} R^D$):

$$E[U(R^C)] > E[U(R^D)] \Rightarrow 10\%\cdot U(5) + 90\%\cdot U(0) > 11\%\cdot U(1) + 89\%\cdot U(0)$$
$$\Rightarrow \underline{11\%\cdot U(1)} < 10\%\cdot U(5) + 1\%\cdot U(0)$$

These two choices point out a contradiction for any vN-M utility function. In this sense, it is possible to find out the region in which the risk-aversive parameters of $\Phi_{\alpha,\lambda}$ should be in order to avoid this paradox. For instance, let $\alpha = 98\%$ and $\lambda = ½$. The resultant CE's for the first two lotteries can be calculated through expression (15), using the CVaR definition in (6), as follows:

$$\Phi_{98\%,½}(R^A) = ½\cdot(2\% - 1\%)\cdot 1/2\% + ½\cdot(1\%\cdot 0 + 89\%\cdot 1 + 10\%\cdot 5) = 0.945$$

$$\Phi_{98\%,½}(R^B) = ½\cdot 1 + ½\cdot 1 = 1.$$

providing the demanded ordering for the first pair of lotteries, $R^B \rangle_{\Phi_{98\%,½}} R^A$. The same can be done for the second pair,

$$\Phi_{98\%,½}(R^C) = ½\cdot 0 + ½\cdot(90\%\cdot 0 + 10\%\cdot 5) = 0.250$$

$$\Phi_{98\%,½}(R^D) = ½\cdot 0 + ½\cdot(89\%\cdot 0 + 11\%\cdot 2500) = 0.055$$

which will attend to the second demanded ordering on the illustrated Paradox instance.

Essentially, the independence axiom is being violated by such pair of choices, which ultimately let us to verify the effect of convex preferences in probabilities, captured by the $\Phi_{98\%,½}$. It is important to emphasize that this will not be verified for all plausible risk-averse parameters $(\alpha,\lambda)\in(0,1)\times[0,1]$, which is indeed desirable, making it possible to accommodate both classes of behavior. For this case, the set of pairs for which this paradox will be captured by $\Phi_{\alpha,\lambda}$ can be obtained through the following two constraints on the parameters set: $\Phi_{\alpha,\lambda}(R^B) > \Phi_{\alpha,\lambda}(R^A)$ and $\Phi_{\alpha,\lambda}(R^C) > \Phi_{\alpha,\lambda}(R^D)$. Figure 7 shows on the Cartesian space $(\alpha,\lambda)\in(0,1)\times[0,1]$ the pairs that attend to the above condition.

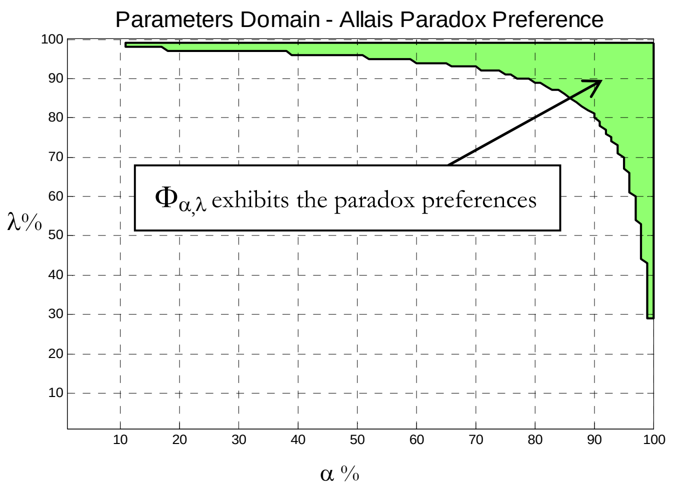

Figure 7 – Risk Averse parameters (of the functional $\Phi_{\alpha,\lambda}$) that capture the Allais Paradox Preferences.

Figure 7 shows that the aforementioned instance of the Allais Paradox is preferentially captured by more risk aversive attitudes, since it concentrates the majority of the paradox area on the set (α>50%, λ>50%), but also reveals that for λ=100% (in which the functional $\Phi_{\alpha,\lambda}$ ignores the expected value term) the paradox strict preferences are not verified. This happens because the preference between R^A and R^B changes at the same point (α=10%) that it changes for R^C and R^D, as a consequence of the symmetry present on such instance. Despite of this singularity, for greater values of α, the $CVaR_\alpha$ exhibits indifference between R^C and R^D, since these are the cases in which the worst value is assigned to the $CVaR_\alpha$.

In fact, for discrete random variables, whenever $P(R_1) \geq 1\text{-}\alpha$ the case outlined in Remark 1 will be observed and the $CVaR_\alpha$ response will be constant and equal to R_1. Then, as $P(R_1)$ crosses down the 1-α threshold, the remaining probability 1 - α - $P(R_1)$ will weigh R_2 and provide a different response as $P(R_1)$ decreases. As a consequence, a convex response (preference) can be drawn by varying the probabilities assigned to a set of scenarios according to Properties 1(g).

In order to provide a visualization of the CVaR convexity in the probability set, consider a convex probability mixture $F_{R_\phi} = \phi \cdot F_{R_1} + (1\text{-}\phi) \cdot F_{R_2}$, with $\phi \in [0,1]$, of the following lotteries (discrete random variables):

R_1 = {(10, 1%); (100, 99%)} and R_2 = {(10, 99%); (100, 1%)}

The mixture distribution will combine the individual probabilities, resulting in a third distribution:

R_ϕ = {(10, $\phi \cdot 0.01 + (1\text{-}\phi) \cdot 0.99$); (100, $\phi \cdot 0.99 + (1\text{-}\phi) \cdot 0.01$)}

Then, by selecting α=95% and λ=100%, $\Phi_{95\%,1}(R_\phi) = CVaR_{95\%}(R_\phi)$. The specified preference locus is illustrated in the next figure, parameterized on ϕ, on the two-dimensional probability space. According to (6),

$$\Phi_{95\%,1}(R_\phi) = (5\%)^{-1} \cdot \{ \Sigma_{\{s \leq s^*(\alpha,R)-1\}} P(R_s) \cdot R_s + R_{s^*(\alpha,R)} \cdot [1 - \alpha - \Sigma_{\{s \leq s^*(\alpha,R)-1\}} P(R_s)] \}$$

$$= \begin{cases} \text{if } \phi \cdot 0.01 + (1-\phi) \cdot 0.99 \geq 0.05, \text{ then: } 10 \\ \text{if } \phi \cdot 0.01 + (1-\phi) \cdot 0.99 < 0.05, \text{ then: } \dfrac{5 - 90 \cdot (\phi \cdot 0.01 + (1-\phi) \cdot 0.99)}{0.05} \end{cases}$$

$$= \begin{cases} \text{if } 0 \leq \phi \leq 94/98, \text{ then: } 10 \\ \text{if } 1 \geq \phi > 94/98, \text{ then: } \phi \cdot 1764 - 1682 \end{cases}.$$

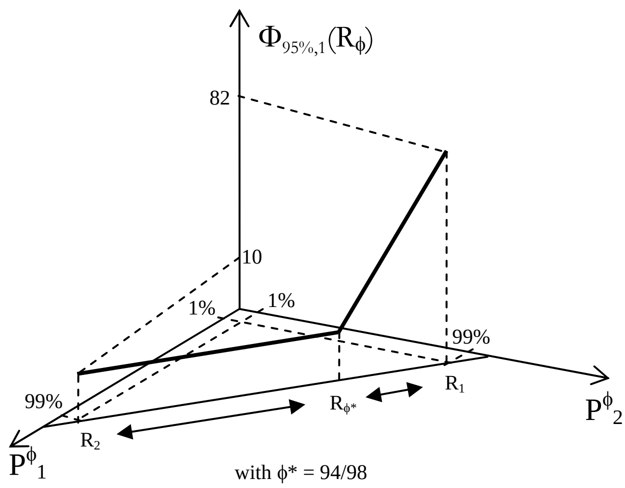

Figure 8 – The $CVaR_{95\%}$ in the probability space – for a mixture of two lotteries

In the above figure, the bold line is the [ϕ, $CVaR_{95\%}(R_\phi)$] pair, for all possible values of $\phi \in [0,1]$. On the sum one probability line (at the $P^\phi_1 x P^\phi_2$ plane) there are three marked points in terms of ϕ values. On $\phi = 0$ the mixture distribution meets exactly R_2 and as ϕ grows it moves to R_1, passing through point $\phi^* = 94/98$. For all $\phi \in [0, 94/98]$, the $CVaR_{95\%}(R_\phi)$ provides the same constant value, 10 \$, since $s^*(\alpha,R) = 1$ (see Remark 1). But for $\phi \in [94/98, 1]$ it rapidly increases at a marginal rate of 1764 \$, reaching the $CVaR_{95\%}(R_1) = 82$ \$ on $\phi = 1$ due to a change in $s^*(\alpha,R)$ from the first to the second scenario. Figure 8 has shown a simple illustration of the convexity of $\Phi_{\alpha,\lambda}$ for a specific case in which λ=100%. By varying ϕ it was possible to track the CVaR outcome for a given lottery path in the probability space and visualize its piecewise form due to the discrete nature of such lotteries.

6. CONCLUSIONS

This work has explored the Conditional Value-at-Risk and its extension ECP as a preference functional, which has been the choice of many financial institutions and investors to take decisions under risk. In this sense, an alternative for the Choquet Expected Utility representation was provided through a probability-dependent utility function based on Rockafellar and Uryasev's developments ([25]). Such utility has shown to be a two-segment and piecewise concave function with a fixed point on the assessed distribution (1-α) Value-at-Risk (the left-tail 1-α quantile), which has also allowed the relativistic investment interpretation.

Finally, the CVaR preference, which is a particular case of the RCE (or OCE as in [12]) and CEU, has shown interesting properties such as the verified equivalence between the expected utility form and the associated certainty equivalent. This property can support different types of pricing applications and, as argued in remark 5, some possible simplifications in multi-period decision problems. An example for the convexity in probabilities implication on preferences was provided by means of the so-called Allais Paradox prospects and also graphically illustrated for a simple bi-dimensional parameterized lottery.

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