3
Equilibrium

We proceed by solving the game backwards. To do so, we first derive the communication protocol that a Central Banker must adopt in any equilibrium. Then, taken as given an arbitrary statement made by the Central Banker, we solve for the equilibrium of the pricing game between the price setters. Finally, we solve for the optimal communication policy for the Central Banker.

3.1
The Communication Protocol

The misalignment of incentives bound from above the amount of information that a Central Bank can credibly convey to the price setters through its statements. In fact, it is easy to see that there is no equilibrium in which the Bank completely and truthfully reveals the state of the economy: if there were such an equilibrium, the price setters could ignore their private signals, and set

$$p_i = \theta \text{ for all } i.$$

so that

$$\overline{p}|s_n = \theta, \text{ and } Var(p_n|s_n, \theta) = 0.$$

In such a case, however, the Central Banker would have an incentive to report $\theta + c$ rather than θ, misleading the price setters towards his objectives. The above discussion shows that, if there is a misalignment of interest between the Central Banker and the price-setters, communication must involve *noise*, that is, the state *cannot* be perfectly inverted from the message sent.

Now, fix a given noisy statement about the state made by the Central Banker. In any equilibrium of the price setting game induced by such statement, there will be a single average price, $\bar{p} \equiv \int_i p_i di$, and a single price dispersion, σ_p^2. Note that $\bar{p}$ and σ_p^2 depend on θ only through the noisy statement made by the Central Banker. **Since we fixed that statement, we have:**

$$\frac{\partial \overline{p}}{\partial \theta} = \frac{\partial \sigma_p^2}{\partial \theta} = 0$$

Hence, as the Central Banker's preferences are strictly concave and such that:

$$\frac{\partial^2 U^{CB}\left(\overline{p}, \sigma_p^2; \theta\right)}{\partial \overline{p} \partial \theta} > 0,$$

there can be at most one state θ for which the Central Banker is indifferent between any two **statements, which implies two combinations of average price and price dispersion**. Moreover, the set of states $\{\theta\}$ for which a given average price and dispersion are best must be an interval. Since the Central Banker's payoffs are continuous in the states, those intervals will form a partition of the possible states.

It follows from the above discussion that the communication between the Central Banker and the firms takes the form of intervals (partitions). Formally,

Proposition 1: If there is a misalignment of incentives between the Central Banker and the price-setter ($c > 0$), then communication must involve noisy signaling. The set of possible states is partitioned into intervals, and a message is sent only if the actual state lies in the interval associated with it.

In any equilibrium, the Central Banker states in which interval the actual state lies in, and the price setters update their beliefs about the state in setting their prices. Such communication protocol allows the Central Banker to disclose some information (and consequently influence the actions of the price setters), and, at the same time, withhold enough information so to make such communication policy credible to firms.

3.2 The Pricing Game

A statement s made by the Central Bank in period zero, and private signals $\{x_i\}_i$ received in period 1 define a game of incomplete information among the price setters in period 2. We now derive the equilibrium set of such pricing game.

After observing the message s and the signal x_i, firm i chooses prices p_i to solve

$$\max_{p_i} -E\left[(1-r)(p_i - \theta)^2 + r(p_i - \overline{p})^2 \,|s, x_i\right] \tag{3-1}$$

where the overall price level is taken as given.

The solution to 3-1 is

$$p_i = (1 - r)\, E\,[\theta|s, x_i] + rE\,[\overline{p}|s, x_i] \tag{3-2}$$

Given the definition of $\overline{p}$, we can iterate (3-2) to get:

$$p_i = (1 - r) \sum_{k=0}^{\infty} E[\overline{E}^k(\theta)|s, x_i] \tag{3-3}$$

where $\overline{E}(x_i) = \int_j E[\theta|s, x_j]dj$ and $\overline{E}^k$ is defined recursively as:

$$\overline{E}^k(\theta) = \int_j E[\overline{E}^{k-1}(\theta)|s, x_j]dj \tag{3-4}$$

As shown in the previous section, in any equilibrium, the communication protocol is such that the set of states $[0, 1]$ is partitioned into intervals $\{[\theta_{n-1}, \theta_n]\}_n$, and, whenever $\theta \in [\theta_{n-1}, \theta_n]$, a message s_n is sent. Hence, after observing a message s_n, the firms can infer that the state lies in $[\theta_n, \theta_{n+1}]$ so that, if $x_i \notin s_n$ firm i can infer $I_i = 0$, so $E[\theta|s_n, x_i] = E[\theta|\theta \in s_n]$. On the other hand, if $x_i \in s_n$ we have:

$$E\,[\theta|s_n, x_i] = Pr(I_i = 1)E(\theta|\theta \in s_n, x_i, I_i = 1) + Pr(I_i = 0)E(\theta|\theta \in s_n, x_i, I_i = 0). \tag{3-5}$$

Hence

$$E\,[\theta|s_n, x_i] = \begin{cases} y_n & \text{if } x_i \notin s_n \\ qx_i + (1 - q)y_n & \text{if } x_i \in s_n \end{cases} \tag{3-6}$$

where $y_n \equiv E(\theta|s_n) = \frac{\theta_n + \theta_{n-1}}{2}$.

Given (3-6), we have : $\overline{E}[\theta] = g_n q^2\theta + (1 - g_n q^2)y_n$, where $g_n \equiv Pr(x_i \in s_n)$. The following lemma is useful to characterize the equilibria of the pricing game

Lemma 1: Let $\mu^k \equiv g_n^k q^{2k}$. We then have:

1. $\overline{E}^k[\theta] = \mu^k\theta + (1 - \mu^k)y_n$

2. $E[\overline{E}^k[\theta]|s_n, x_i] = \begin{cases} y_n & \text{if } x_i \notin s_n \\ q\mu^k x_i + (1 - q\mu^k)y_n & \text{if } x_i \in s_n \end{cases}$

Proof: As seen above, the Lemma is valid for k=1. Suppose that:

$$\overline{E}^{k-1}[\theta] = \mu^{k-1}\theta + (1 - \mu^{k-1})y_n$$

so $E[\overline{E}^{k-1}[\theta]|s_n, x_i] = \mu^{k-1}E[\theta|s_n, x_i] + (1 - \mu^{k-1})y_n$. If $x_i \notin s_n$ then $E[\theta|s_n, x_i] = E[\theta|s_n] = y_n$. On the other hand, if $x_i \in s_n$ then $E[\theta|s_n, x_i] = qx_i + (1 - q)y_n$. Therefore:

$$E[\overline{E}^{k-1}[\theta]|s_n, x_i] = \begin{cases} y_n & \text{if } x_i \notin s_n \\ q\mu_{k-1}x_i + (1 - q\mu_{k-1})y_n & \text{if } x_i \in s_n \end{cases}$$

Now, since

$$\int_i E[\overline{E}^{k-1}[\theta]|s_n, x_i]di = g_n q^2 \mu^{k-1}\theta + (1 - g_n q^2 \mu^{k-1})y_n$$

we have $\overline{E}^k[\theta] = \mu^k\theta + (1 - \mu^k)y_n$. Finally:

$$E[\overline{E}^k[\theta]|s_n, x_i] = \begin{cases} y_n & \text{if } x_i \notin s_n \\ q\mu_k x_i + (1 - q\mu_k)y_n & \text{if } x_i \in s_n \end{cases}$$

where $\tilde{k} \equiv \frac{(1-r)q}{(1-rq^2 g_n)}$

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From Lemma 1 , it follows that

$$\begin{aligned} p_i(s_i, x_i) &= (1 - r)\sum_{k=0}^{\infty} E\left[\overline{E}^k(\theta)|s, x_i\right] \\ &= \begin{cases} (1 - r)\sum_{k=0}^{\infty} r^k y_n & \text{if } x_i \notin s_n \\ (1 - r)\sum_{k=0}^{\infty} r^k \left[q\mu^k x_i + \left(1 - q\mu^k\right)y_n\right] & \text{if } x_i \in s_n \end{cases} \\ & \begin{cases} y_n & \text{if } x_i \notin s_n \\ \widetilde{k}x_i + \left(1 - \widetilde{k}\right)y_n & \text{if } x_i \in s_n \end{cases} \end{aligned}$$

where $\widetilde{k} \equiv \frac{(1-r)q}{(1-rq^2 g_n)}$.

We then have

Proposition 2: Given a statement s_n made by the Central Bank when $\theta \in [\theta_{n-1}, \theta_n]$, and private signals $\{x_i\}$, an equilibrium in the pricing game takes the form of

$$p_i = \left(\frac{q(1-r)}{1-rq^2g_n}\right)x_i + \left(1 - \frac{q(1-r)}{1-rq^2g_n}\right)y_n \quad \forall\, i \in [0,1],$$

where

$$y_n \equiv E(\theta|s_n) = \frac{\theta_n + \theta_{n-1}}{2}$$

3.3 The Communication Game

Having characterized the firms' equilibrium pricing policy for a *given* statement made by the Central Bank, we now move on to derive the set of equilibrium communication policies.

Given Proposition 3.2, the average price level is

$$\begin{aligned} \overline{p}|(\theta, s_n) &= g_n\tilde{k}q\theta + (1 - g_n\tilde{k}q)y_n \\ &\equiv \overline{k}\theta + (1 - \overline{k}y_n), \end{aligned} \tag{3-7}$$

while the price variability is

$$\begin{aligned} Var(p_i|s_n,\theta) &= g_n\tilde{k}_n^2[(1-q)1/3\left(\tfrac{\theta_n-\theta_{n-1}}{2}\right)^2 + q(1-q)(\theta-y_n)^2] \\ &+ g_n(1-g_n)\tilde{k}_n^2q^2(\theta-y_n)^2 \end{aligned} \tag{3-8}$$

Using these expressions, one has that the Central Bank's expected payoff, given a state $\theta \in [\theta_{n-1}, \theta_n]$ and the correspondent statement s_n, is

$$(\overline{p}|(\theta, s_n) - \theta + c)^2 + \beta Var(p_n|s_n, \theta).$$

In equilibrium, a Central Banker who faces a "cut-off" state θ_n must be indifferent between making the statements s_n and s_{n+1} (which is the statement made when the state lies in $[\theta_n, \theta_{n+1}]$). Hence, one must have:

$$(\bar{p}|(\theta_n, s_n) - \theta_n + c)^2 + \beta Var(p_n|s_n, \theta_n) = (\bar{p}|(\theta_n, s_{n+1}) - \theta_n + c)^2 + \beta Var(p_n|s_{n+1}, \theta_n).$$

These indifference conditions define the set of partitions $\{[\theta_{n-1}, \theta_n]\}_n$ that can be part of a communication equilibrium. The next result states the amount of informativeness that can prevail in any communication equilibrium.

Proposition 3: Given (c, q, r, β), there exists an integer $M(c, q, r, \beta)$ such that $\forall 1 \leqslant N \leqslant M(\cdot)$, $N \in \mathbb{N}$, there is at least one equilibrium with N different messages been sent. Further, message s_n is sent if, and only if, $\theta \in [\theta_{n-1}, \theta_n]$.

It follows that the key parameters to determine the informativeness of a Central Banker policy are related to his preferences – though the bias c, and the parameter that measures his aversion to price dispersion, β –, the precision of the price setters signal, q, and the degree of complementarity in their pricing decisions.

We first shows that, if the Central Banker is only concerned with price dispersion, there exists an equilibrium in which he can perfectly communicate the state in which the economy is.

Proposition 4: The more the Central Banker cares about the price dispersion (larger β), the more informative an equilibrium can be. In the limit, if the Central Banker cares only about the price dispersion, $(\beta \to \infty)$, the the communication can be fully informative.

Proof: Let $\{\theta\}_{n=1}^{N}$ be a set of limiting partitions of an equilibrium. Given θ_{n-1}, θ_n and θ_{n+1}, let: $V(\theta_{n-1}, \theta_n, \theta_{n+1}) = 0$ be the indifference condition that defines those partitions. For all $\beta > 0$, $1/\beta V(\cdot) = 0$, as $\beta \to +\infty$, $V(\cdot)$ tends to:

$$\begin{array}{l} g_{n+1}\tilde{k}_{n+1}\left(\frac{\Delta\theta_{n+1}}{2}\right)^2 [(1-q)(1/3+q) + (1-g_{n+1})q^2] \\ -g_n\tilde{k}_n\left(\frac{\Delta\theta_n}{2}\right)^2 [(1-q)(1/3+q) + (1-g_n)q^2] \end{array}$$

which is equal to zero when $\Delta\theta_n = \Delta\theta_{n+1}$. So, any equal partition of $[0, 1]$ can be an equilibrium.

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The intuition behind this result follows immediately from an analysis of the Central Bank's incentives. On the one hand, it would like the price setters to set $p_i = \theta + c$; such a force is, given the bias c, partly in dissonance with the agents' interests. On the other hand, the Bank would like to reduce the variability of prices; such a force is in consonance with the agents' interests.The larger β, the larger the consonance between the Bank, and the firms. As $\beta \to \infty$, we move toward full alignment of interests.

In the previous result, we have shown that *all* relevant information can be transmitted if the Central Banker just cares about price dispersion. The next result, in turn, illustrates the importance of the Central Banker's aversion to price dispersion for the possibility of conveyance of *some* information

Proposition 5: If the Central Banker does not care about the price dispersion ($\beta = 0$), and the price-setters care almost only about the actual state ($r \sim 0$), then, as their information becomes almost precise ($q \to 1$,), the unique communication equilibrium leads to a *single* statement s being made irrespective of the state θ.

Proof: Let $V(\theta)$ be the indifference condition of the Central Banker at a 2 partition equilibrium. So θ such that $V(\theta) = 0$ defines the partition. If $r \sim 0$, then $\tilde{k}_n \sim q$ and $\overline{k}_n \sim g_n q^2$. Therefore:

$$V(\theta) \sim (c-(1-g_1q^2)\theta/2+(1-q_2q^2)(1-\theta)/2+c)(-(1-g_1q^2)\theta/2-(1-q_2q^2)(1-\theta)/2)$$

since the second term of the above equation is always negative, $V(\theta) = 0$ only if:

$$(c - (1 - g_1q^2)\theta/2 + (1 - q_2q^2)(1 - \theta)/2 + c) = 0$$

which implies:

$$\theta = \left(\frac{1}{1-q^2}\right)[4c + (1 - g_2)q^2)]$$

$\Rightarrow \lim_{q \to 1} \theta = \infty$ which proves that there is no equilibrium in two partitions. ■

The non-informative equilibrium characterized in 3.3 is called the "Babbling Equilibrium" in the cheap talk literature. Such an equilibrium always exist in any cheap talk game.[1] The reason why, when $\beta = 0$ and $q \to 1$, this is the *unique* equilibrium in our setting is simple. When the agents learn x_i, it is almost as if they learned the state. This means that all price setters will coordinate their pricing decisions almost perfectly, so there will be no variability in prices. It follows that the Central Bank's only concern will be its price level target. Hence, its incentives to deceive the agents are magnified, making any informative statement about the economy not credible. Note that, apart from taking that some bias is present, this result holds irrespective of any other assumption regarding c. This contrasts with what is found in, for example, (5). In their model, for moderate values of c, a partially infor-

[1]As, for all parameter values, the following is always an equilibrium: the price setters ignore whatever message the Central Banker sends, and the Central Banker sends a single statement for all possible states.

mative equilibrium always exist. In addition to the lack of aversion to price dispersion by the Central Bank, what is key for the above result is the fact that the price setters, through their private signals, almost learn the true state.

The results in propositions 3.3 and 3.3 cover two extreme cases of informativeness. We can also show that, when the price setters are more concerned with relative prices than with the actual state of the economy ($r \sim 1$), the Central Banker can always convey some information about the state, irrespective of β. More precisely,

Proposition 6: If firms care more about the average price than the actual state of the economy ($r \sim 1$), there always exists an equilibrium with at least two statements being made by the Central Banker, for all $q \in (0,1)$, $\beta \in [0,+\infty)$; given $c \in (0,\bar{c})$.

Proof: Let $V(\theta)$ be the indifference condition of the Central Banker at a 2 partition equilibrium. So θ such that $V(\theta) = 0$ defines the partition. If $r \sim 1$, then $\tilde{k}_n \sim 0$ and $\overline{k}_n \sim 0$. Therefore:

$$V(\theta) = (-\theta/2 - (1-\theta)/2)(c - \theta/2 + c + (1-\theta)/2)$$

The first term of the above equation is always negative, so $V(\theta) = 0$ implies $\theta = 2c + 1/2$, which does not depend on q or β and $\theta \in [0,1]$ if $c \in (0, 1/4]$. ■

The interpretation for the result is as follows. Irrespective of the statement made by the Central Banker, price setters will coordinate on the same price. Moreover, as $r \sim 1$, they won't care if this price is different than the state. Hence, they will not be concerned with how informative the statement is. The only important thing is that a statement is made and it can serve as a coordination device.

Although we are able to derive some properties of the set of equilibria for the general model, it is quite difficult to fully characterize it. To make some progress, we specialize further the informational structure in the next section.